

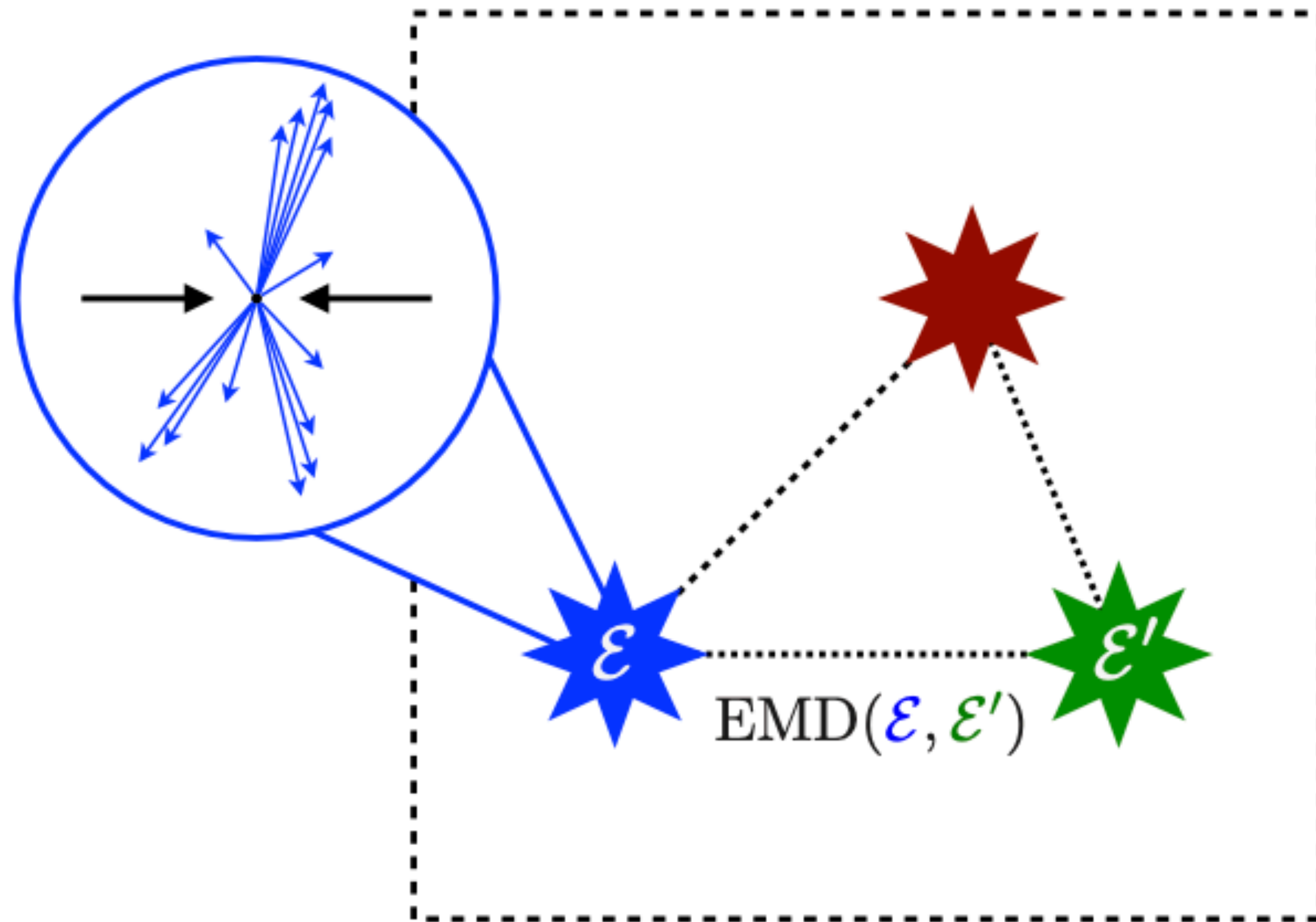
Event Isotropy in perturbative QCD

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2606.23924
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Event geometry

Komiske, Metodiev, Thaler, 2004.04159



Define a metric between events

Compare them according to their distance

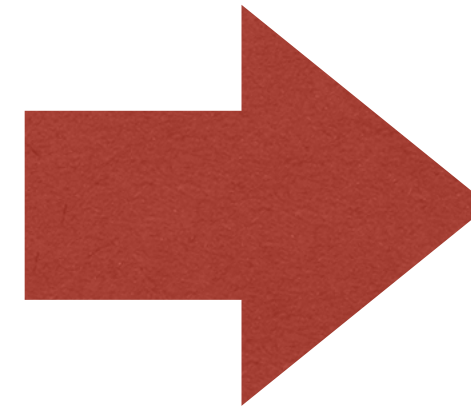
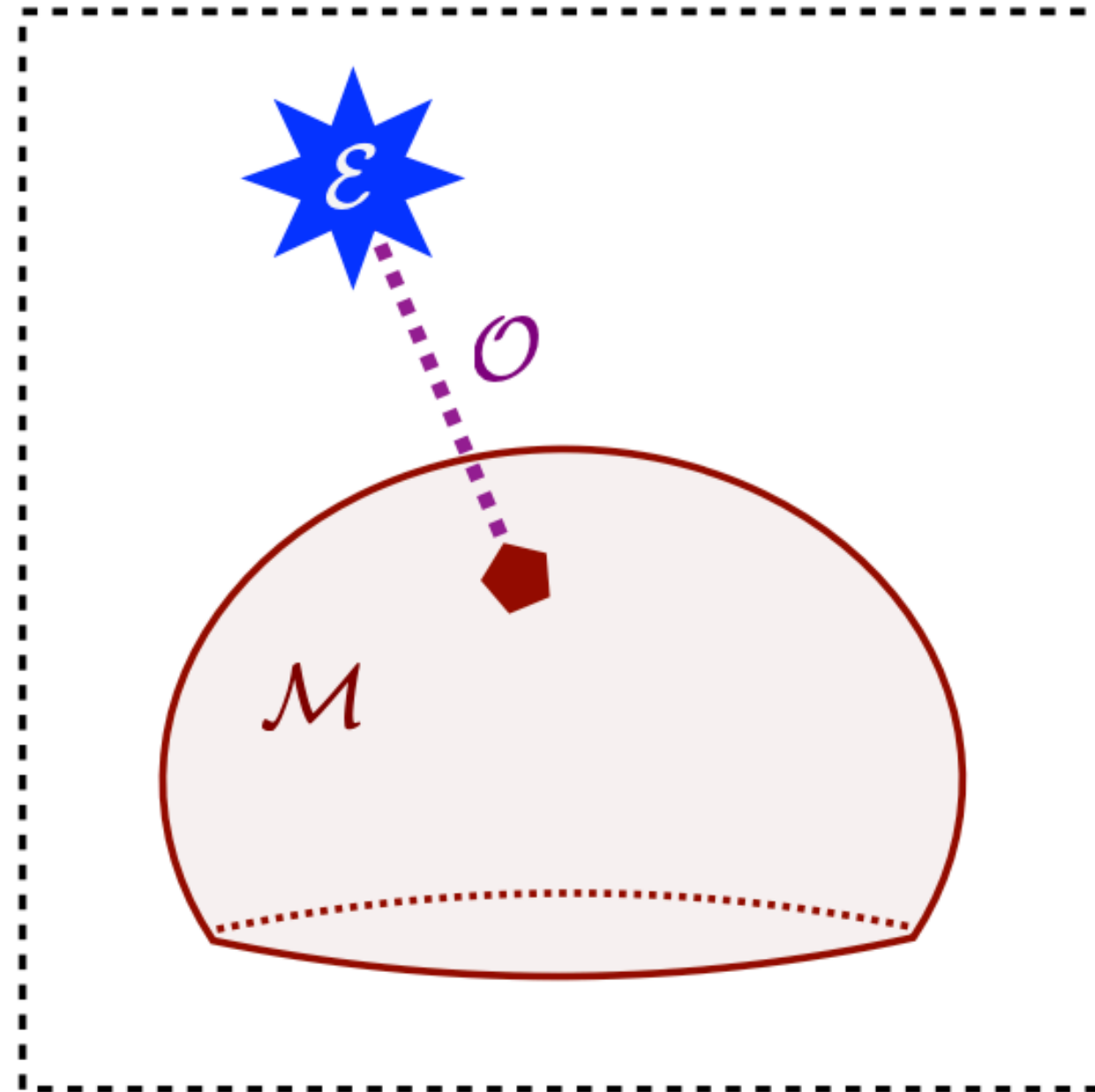
$$\text{EMD}_{\beta, R}(\mathcal{E}, \mathcal{E}') = \min_{\{f_{ij} \geq 0\}} \sum_{i=1}^M \sum_{j=1}^{M'} f_{ij} \left(\frac{d_{ij}}{R} \right)^{\beta} + \left| \sum_{i=1}^M E_i - \sum_{j=1}^{M'} E'_j \right|$$

Energy Mover's Distance
(~ Wasserstein distance)

Event geometry

New language for old observables

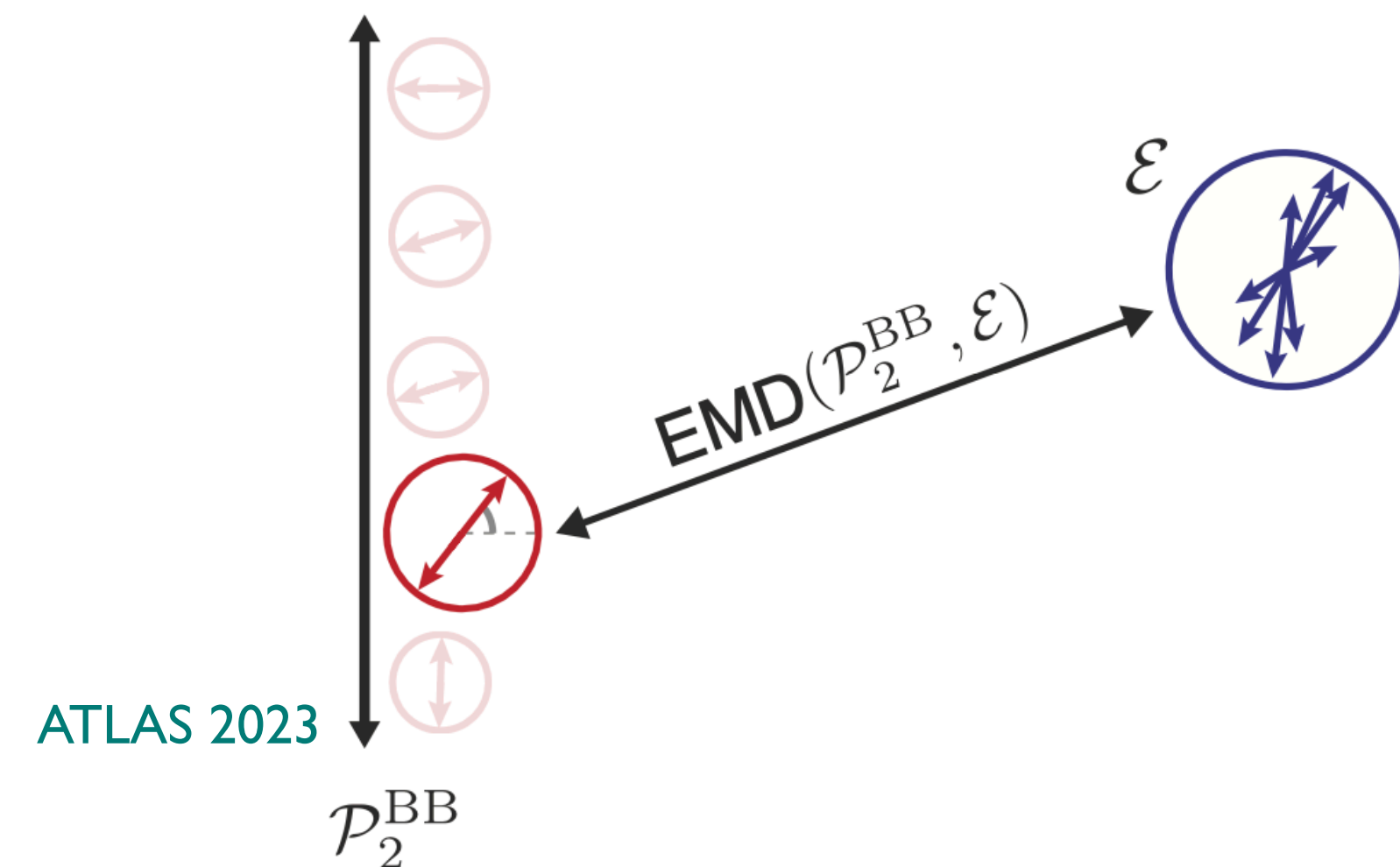
Komiske, Metodiev, Thaler, 2004.04159



e.g. **thrust** is given by the **minimum EMD** between an event and a manifold of **two back-to-back particles**

$$\mathcal{O}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}_{\beta, R}(\mathcal{E}, \mathcal{E}')$$

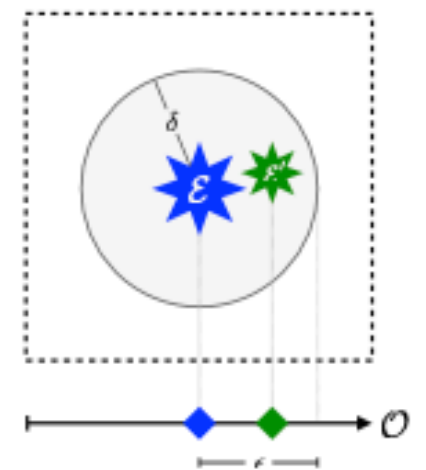
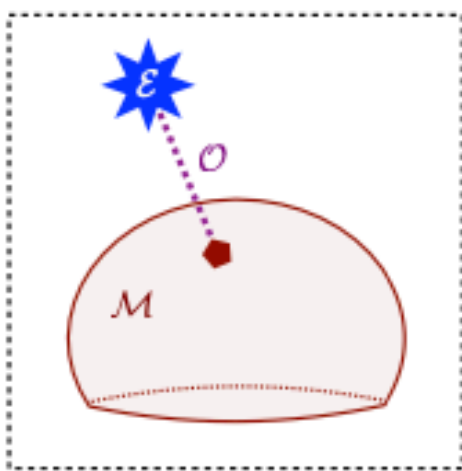
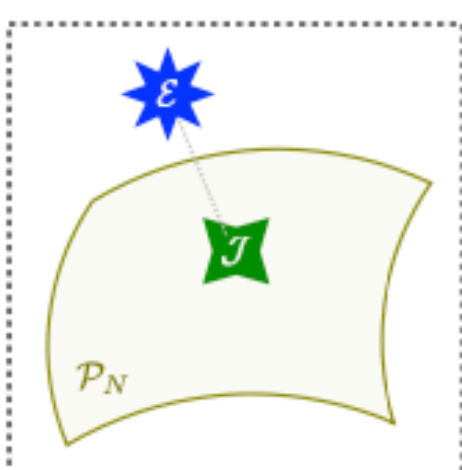
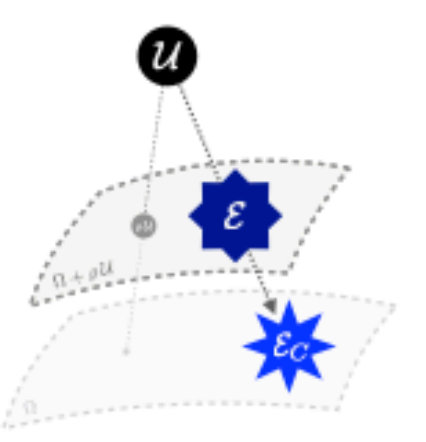
Minimum EMD between an event and a given manifold



Event geometry

Komiske, Metodiev, Thaler, 2004.04159

Many classical concepts and observables used in collider physics can be rewritten with this geometrical language

Sec.	Concept	Equation	Illustration
2	Infrared and Collinear Safety [27–39]	$\text{EMD}(\mathcal{E}, \mathcal{E}') < \delta \implies \mathcal{O}(\mathcal{E}) - \mathcal{O}(\mathcal{E}') < \epsilon$	
3 3.1 3.2	Observables Event Shapes [40–45] Jet Shapes [46–48]	$\mathcal{O}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}(\mathcal{E}, \mathcal{E}')$	
4 4.1 4.2	Jets Cone Finding [49, 50] Seq. Rec. [51–54]	$\mathcal{J}(\mathcal{E}) = \arg \min_{\mathcal{J} \in \mathcal{P}_N} \text{EMD}(\mathcal{E}, \mathcal{J})$	
5	Pileup Subtraction [55–61]	$\mathcal{E}_C(\mathcal{E}, \rho) = \arg \min_{\mathcal{E}' \in \Omega} \text{EMD}(\mathcal{E}, \mathcal{E}' + \rho \mathcal{U})$	

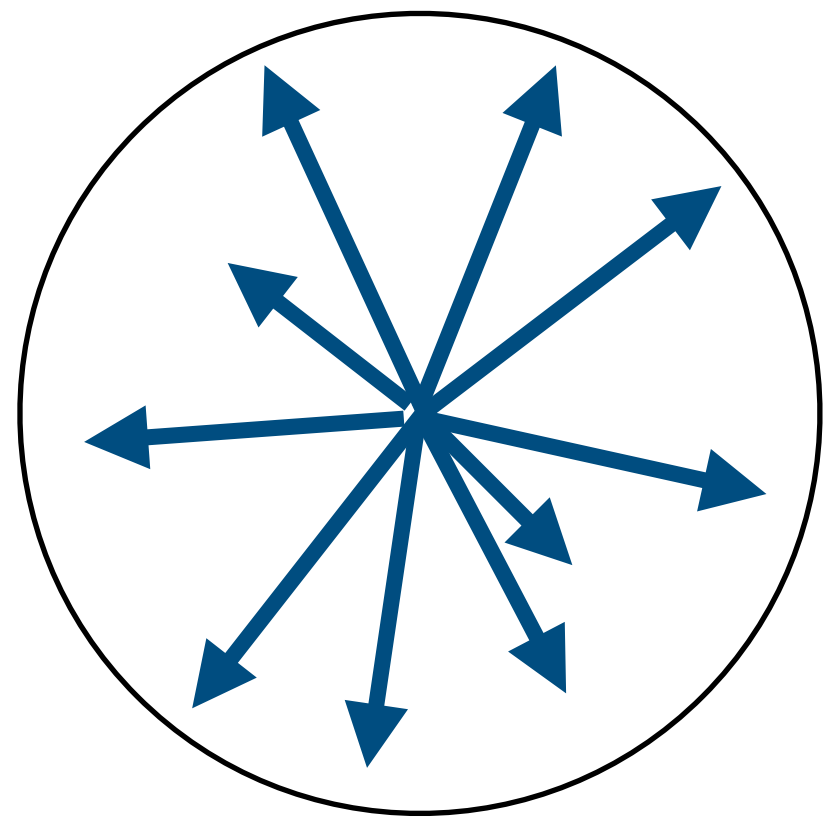
“Dimensionless distance between a collider event $\mathcal{E}$ and a **uniform** radiation pattern $\mathcal{U}$ of the same energy”

$$\mathcal{I}(\mathcal{E}) \equiv \text{EMD}(\mathcal{U}, \mathcal{E})$$

NB. Multiple choices are possible for the geometry of $\mathcal{U}$

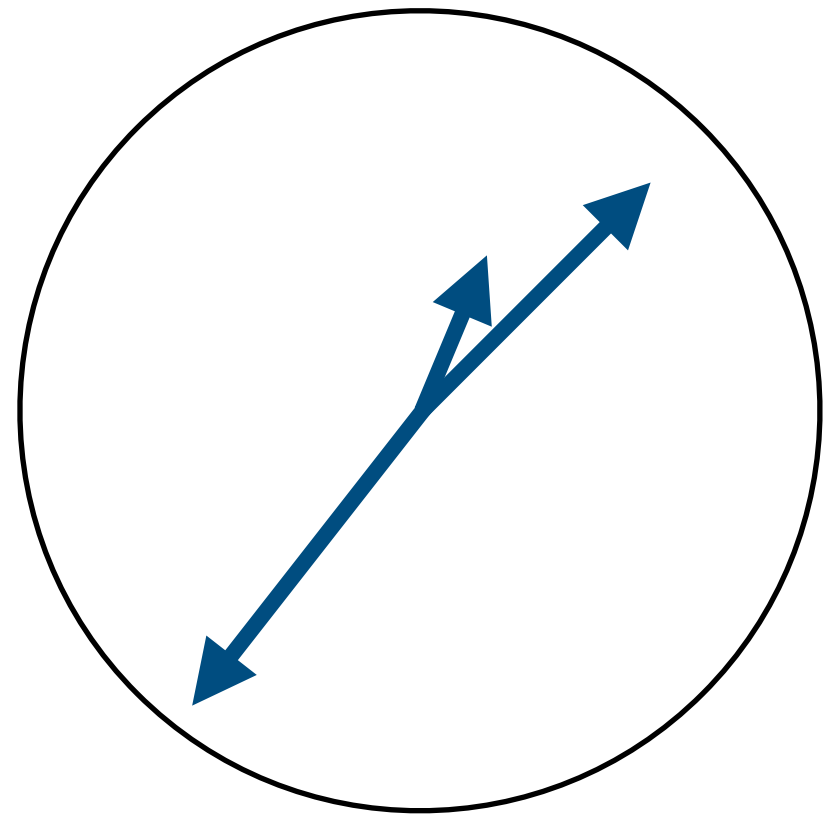
Isotropy

Unfortunately the original definition of isotropy can be confusing



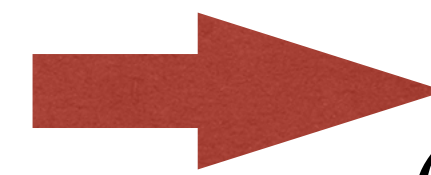
$$I \simeq 0$$

$$l \equiv 1 - I \simeq 1$$



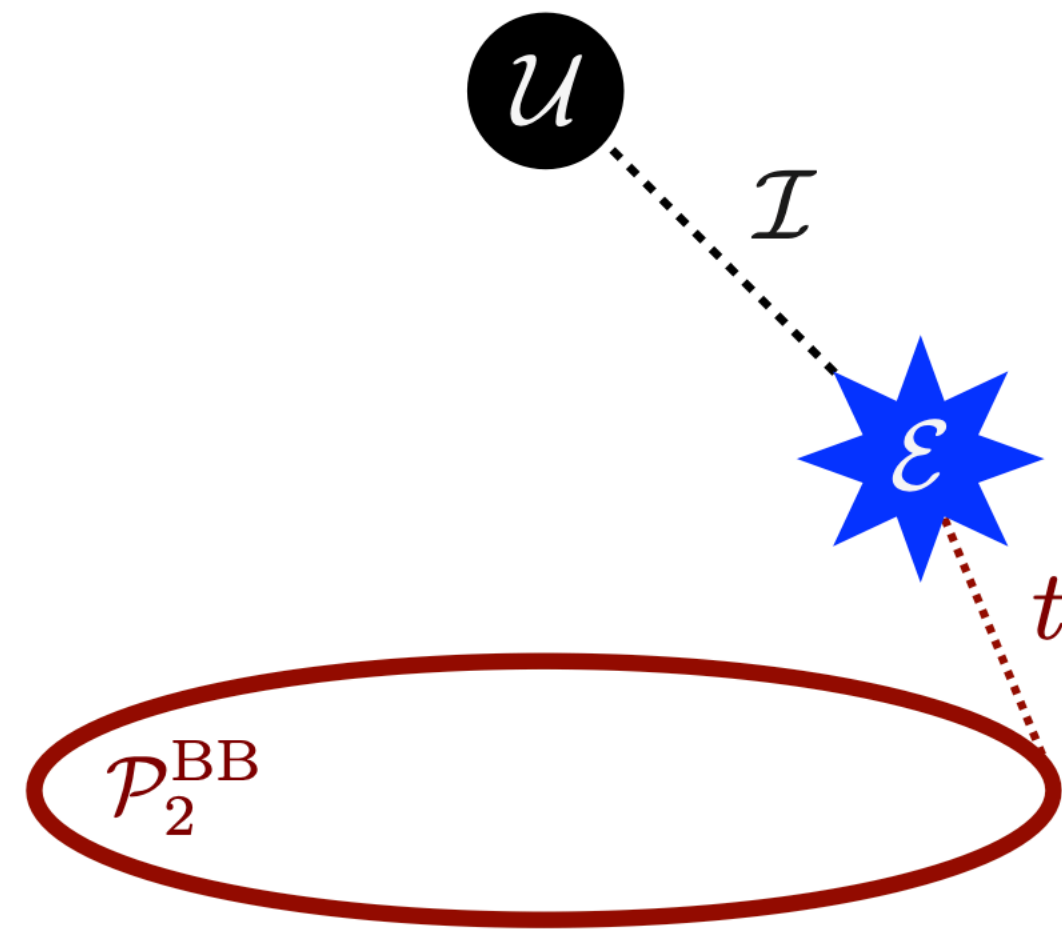
$$I \simeq 1$$

$$l \equiv 1 - I \simeq 0$$



We'll use
(and sometimes call 'isotropy')

Somewhat complementary to thrust

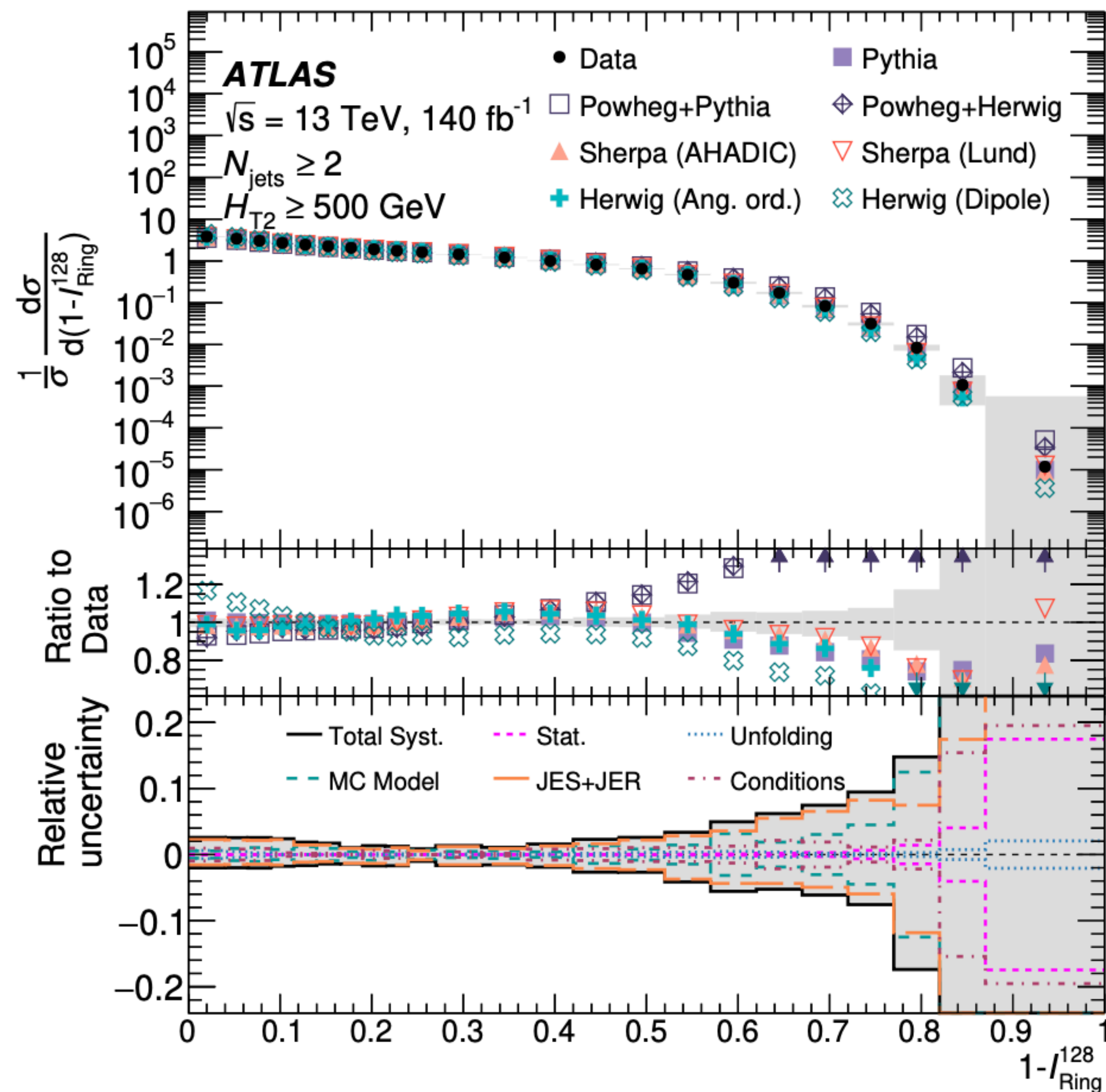


Thrust: closest distance to a 2-particle back-to-back event
Isotropy: distance from the uniform event (there's only one)

Isotropy has a **larger dynamic range**
in the region of high-multiplicity events:
it can more easily characterise scenarios
whose signatures are expected to be isotropic

Isotropy measurement

ATLAS Collaboration, 2023



The uniform radiation pattern chosen as a reference is a **ring in the transverse plane**

The “128” in I_{Ring}^{128} refers to the concrete choice made to describe the “uniform radiation”: 128 particles on the ring

Isotropy on a sphere

We choose to work with a sphere because we are interested in e^+e^-

With a uniform reference event of equal and constant energy,
the isotropy definition for n particles becomes

$$I_n^{\text{sph}} = \text{EMD}(\mathcal{E}_n, \mathcal{U}^{\text{sphere}}) = \min_{\{f_i\}} \sum_{i=1}^n \int_{S^2} d\Omega' f_i(\Omega') d_i^2(\Omega')$$

**transportation
plan**

**ground
distance**

with the constraints

$$\int_{S^2} d\Omega' f_i(\Omega') = \omega_i$$

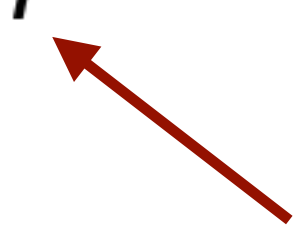
$$\sum_{i=1}^n f_i(\Omega') = \omega'(\Omega') = \frac{1}{4\pi}$$

Optimal Transport

Isotropy is a **semi-discrete Optimal Transport** problem:
continuous source (sphere) and discrete target (particles)

Well known problem in mathematics:
the solution of a semi-discrete OT problem with a uniform continuous distribution is a **piece-wise transportation plan on the Laguerre cells C_i** (weighted Voronoi cells)

$$\Rightarrow \mathcal{I}_n^{\text{sph}}(\{p_i\}) = \frac{1}{4\pi} \sum_{i=1}^n \int_{C_i} d\Omega' d_i^2(\Omega')$$

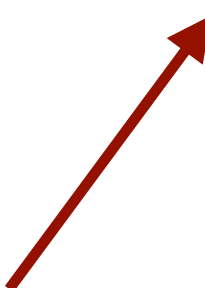
 **Laguerre cells**

Laguerre Cells

Voronoi: $V_i = \{y \in X \mid d(x_i, y) - d(x_j, y) \leq 0\}$

Set of points y closest to point x_i

Laguerre: $C_i = \{y \in X \mid d(x_i, y) - d(x_j, y) \leq c_{ij}\}$



weights

The Laguerre weights are such that the areas of the cells are proportional to the energies of the particles

[Claude and Opus 5.5]

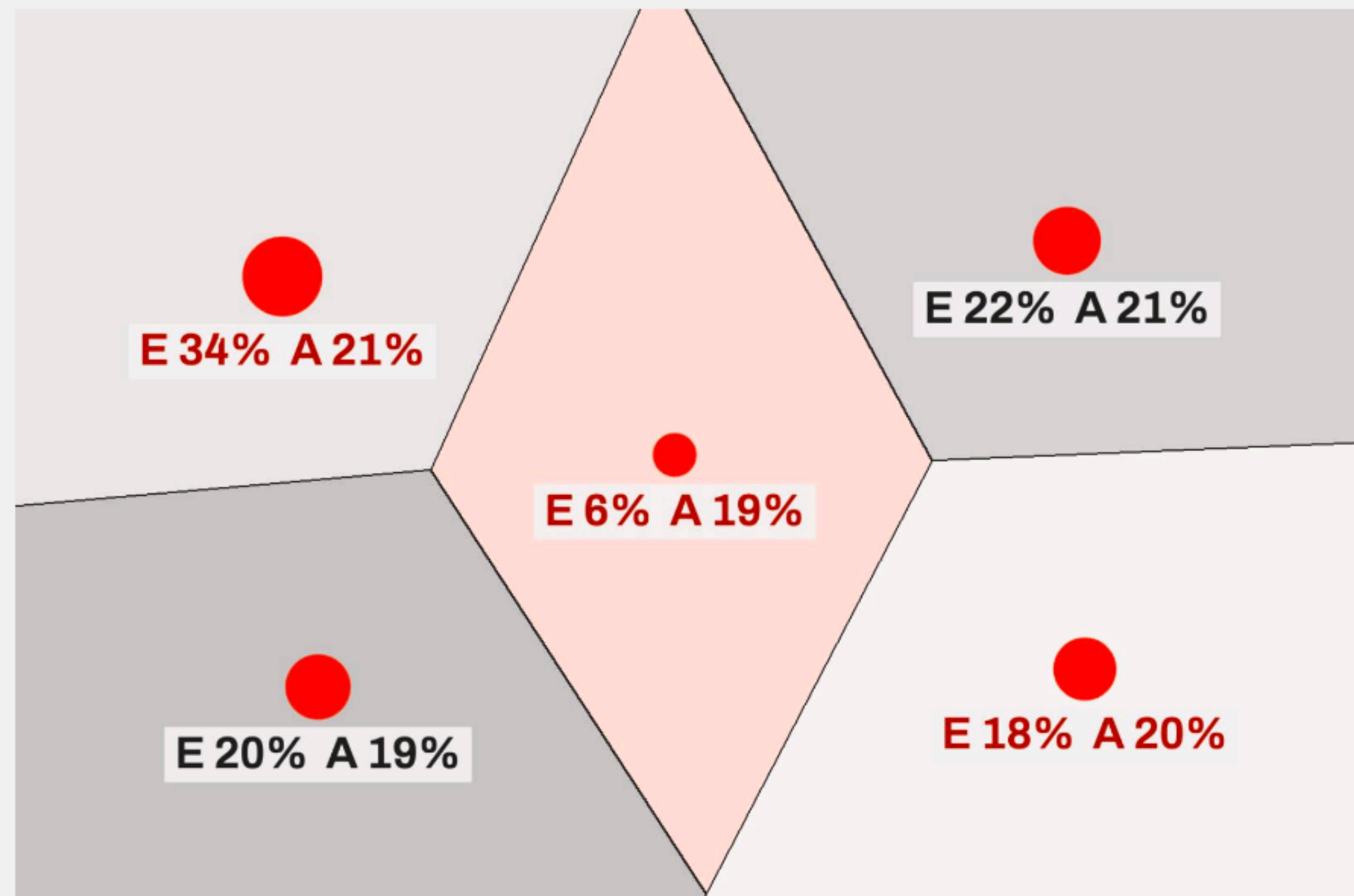
02 · SOLVING THE TRANSPORT PROBLEM

EVENT ISOTROPY IN PQCD

Voronoi and Laguerre cells

Voronoi

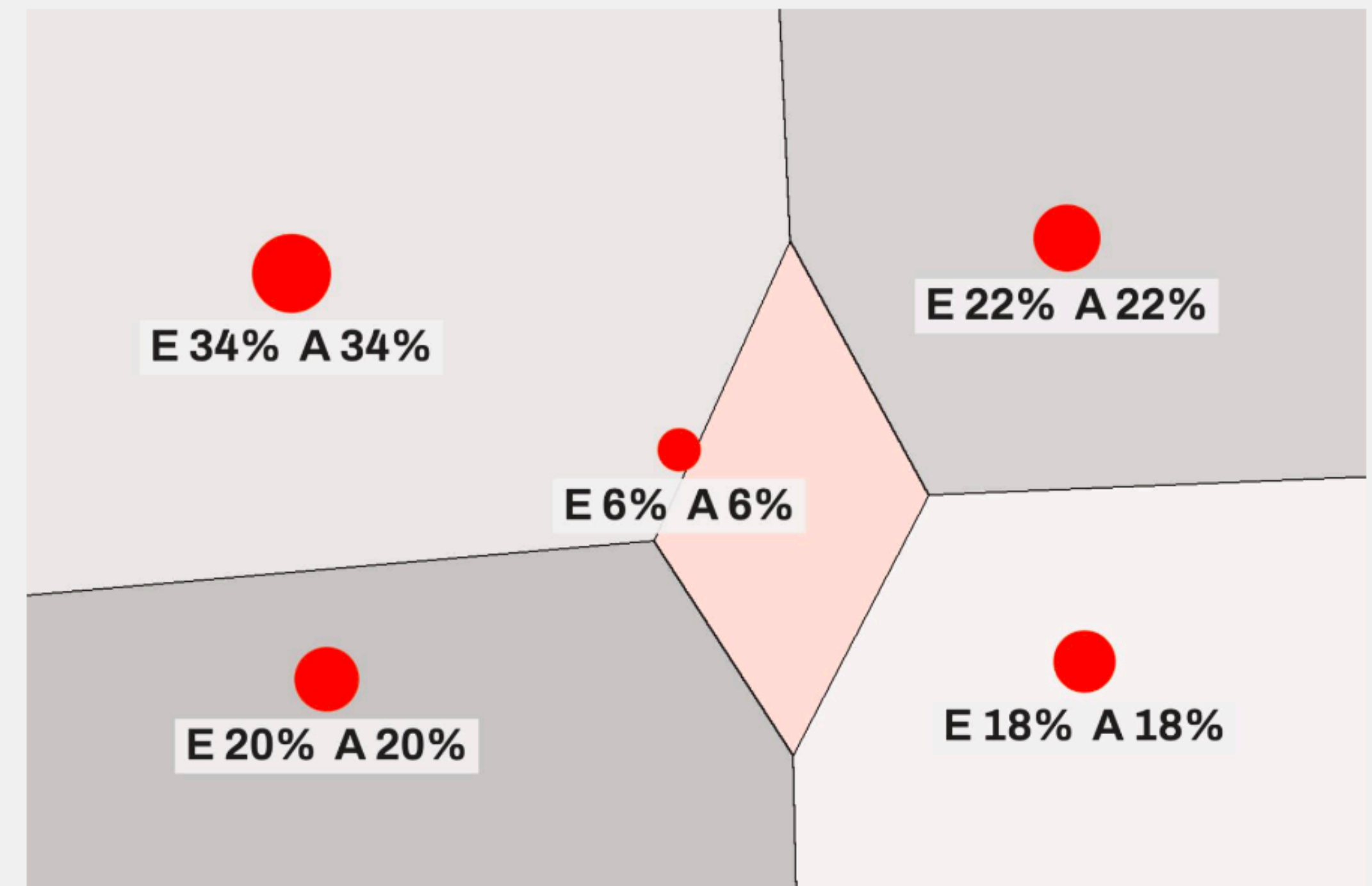
$$d_i^2(y) \leq d_j^2(y) \quad \forall j$$



Nearest site wins. Cell areas ignore the energies.

Laguerre

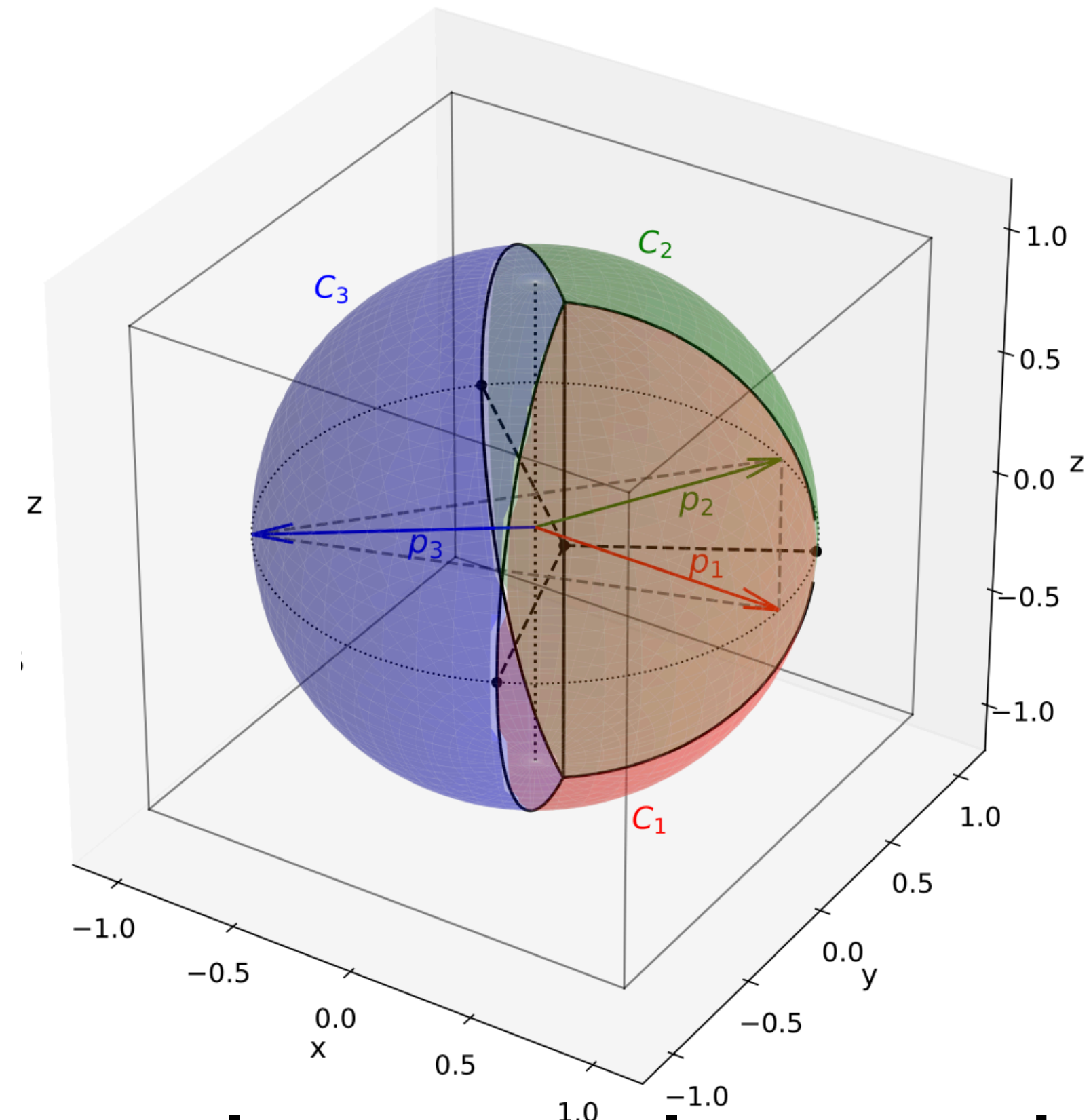
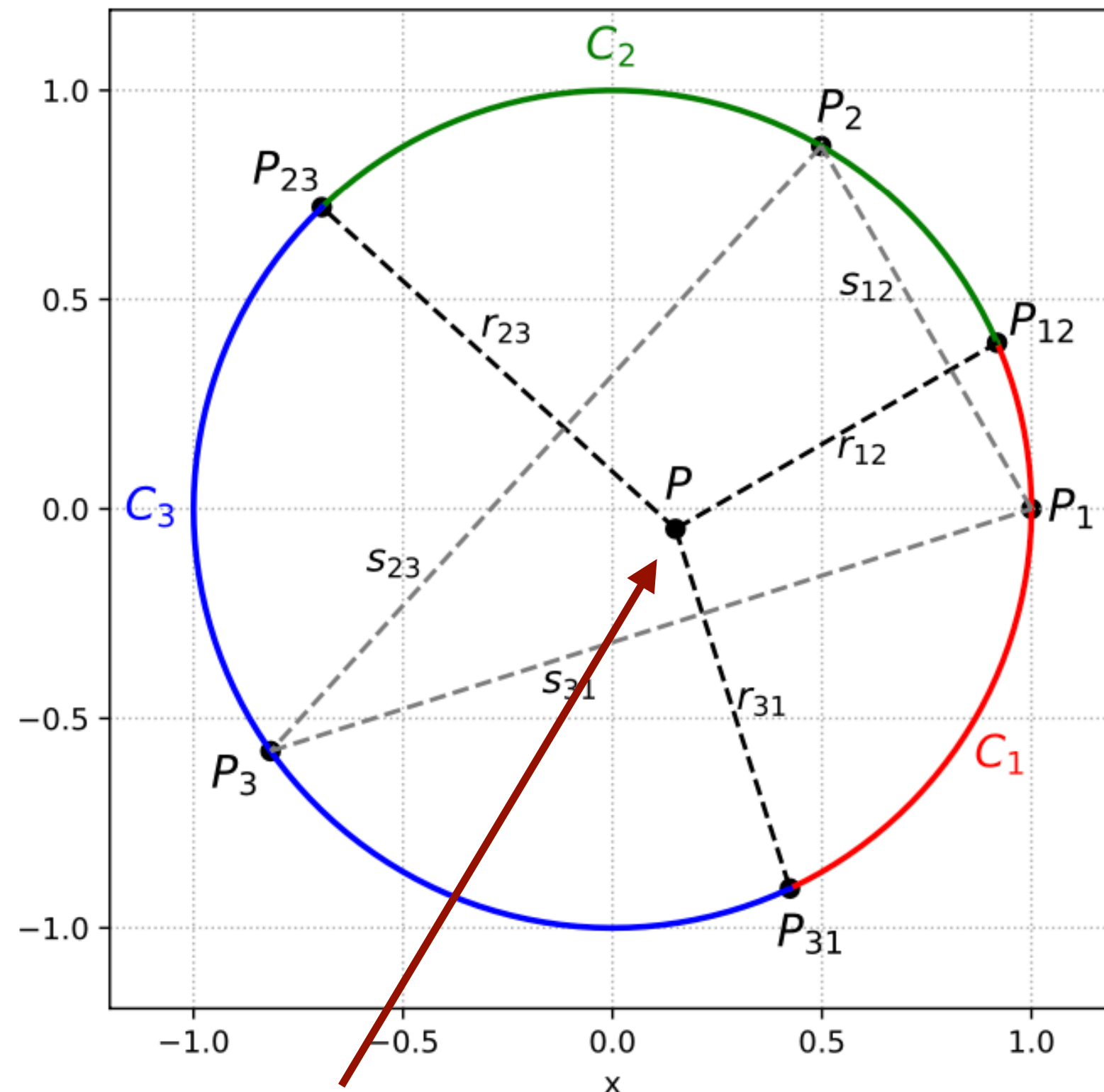
$$d_i^2(y) - w_i \leq d_j^2(y) - w_j$$



Weighted Voronoi: tune w_i until each area equals its energy share.

Isotropy of 3 particles

Three particles with momentum conservation lie on a plane.
The Laguerre cells are spherical wedges on a sphere.



The point P is determined (a numerical step is also needed)
by setting **wedge areas proportional to the energies**

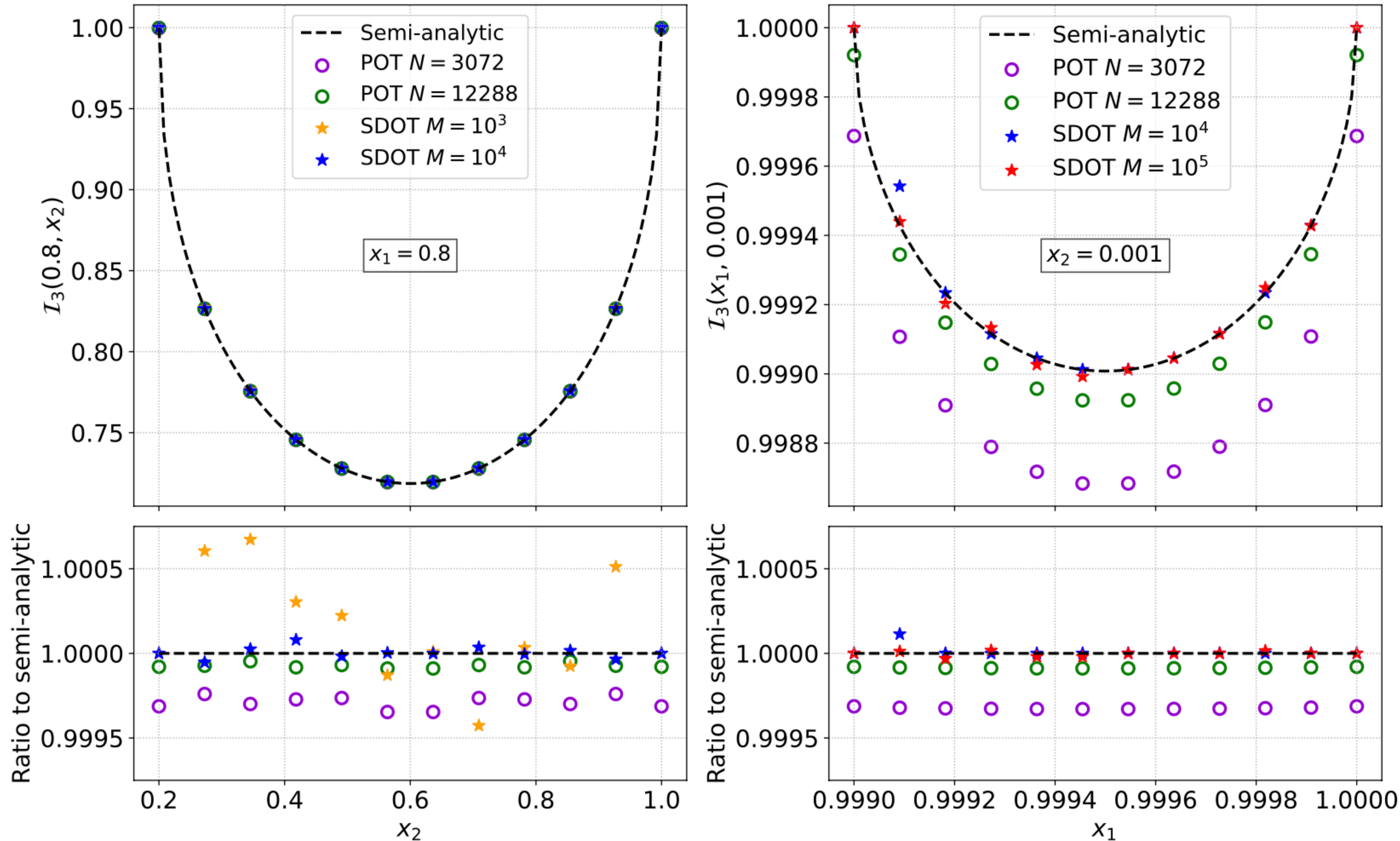
Isotropy of n particles

For $n > 3$ we do not have a semi-analytical solution.
A **numerical** approach is needed.

POT	Semi-analytical	SDOT
Transportation plan	Laguerre	Laguerre
Discretisation of uniform distribution on sphere and exact OT solution	Equations for wedges. Numerical step for areas (solution of system of equations)	Areas of cells determined numerically and iteratively with quadrature on the sphere
Becomes quickly slow at fine-grained discretisation	Fast but limited to $n=3$	Faster than POT at equivalent precision. Allows for high precision when particles are soft or collinear
“Exact solution of an approximate (discretised) problem”		“Approximate solution of the exact, semi-discrete problem”

Does it work?

Atzori, MC, Marzani, Soyez, 2606.23924 Isotropy $I_3(x_1, x_2)$

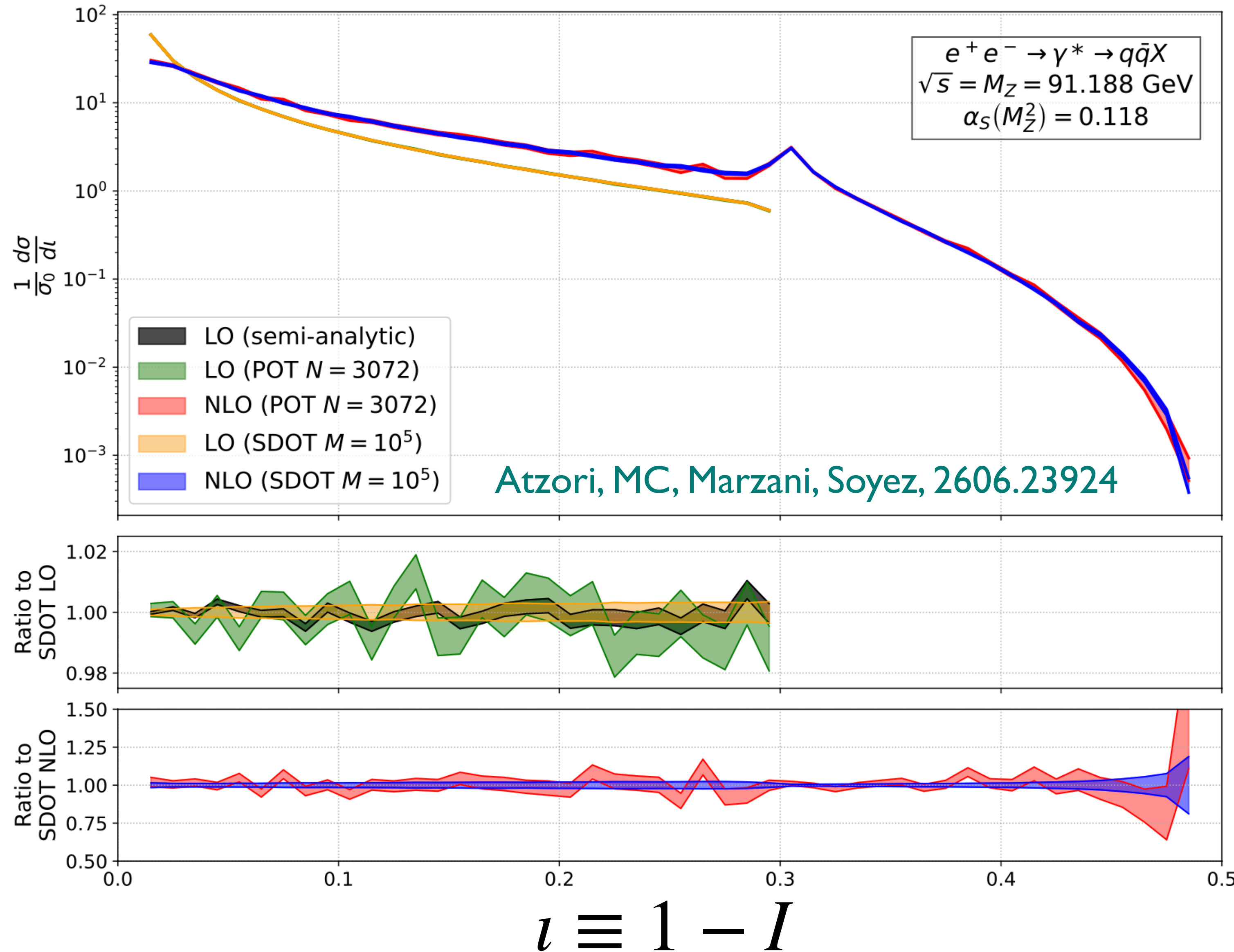


Isotropy of a 3-particle configuration.

Comparisons of POT, SDOT, semi-analytical

10^{-4} relative agreement

Isotropy in e^+e^- collisions



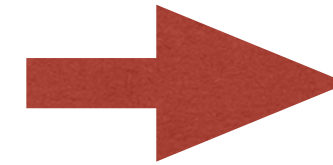
Fixed order calculation up to $\mathcal{O}(\alpha_s^2)$ using EVENT2

- Large logarithms at small ι . To be resummed
- Fairly sizeable perturbative corrections at intermediate ι .
- Likely Sudakov shoulder at 3-particle endpoint ($\iota_3^{\max} = 3\sqrt{3}/4 - 1 \simeq 0.299$)
- Really isotropic events are at large ι

Resummation

Resummation at NLL level needs three ingredients:

- scaling of ι in the soft/collinear limit
- understanding of recoil effects
- effect of multiple soft emissions



Our analysis shows that:

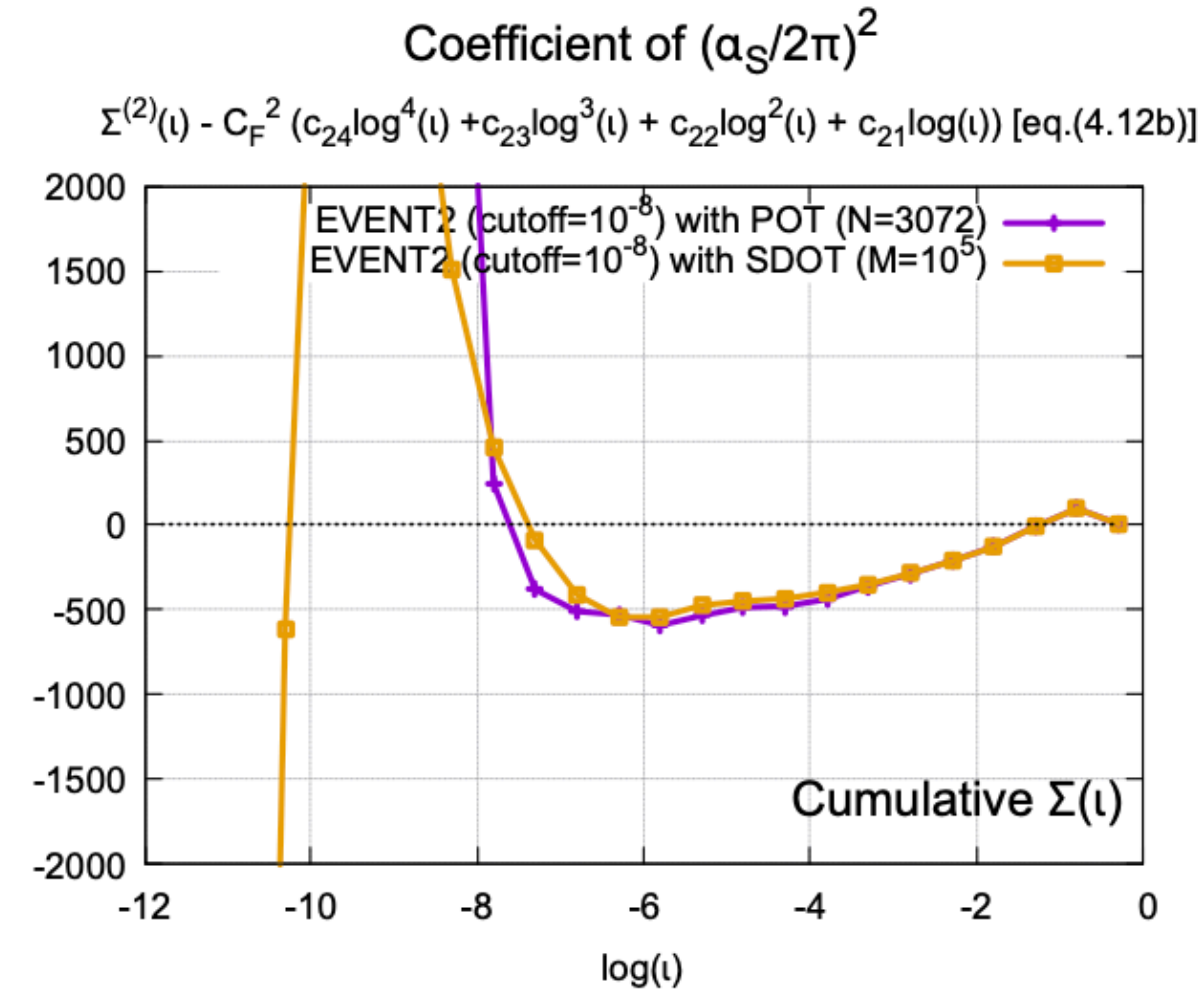
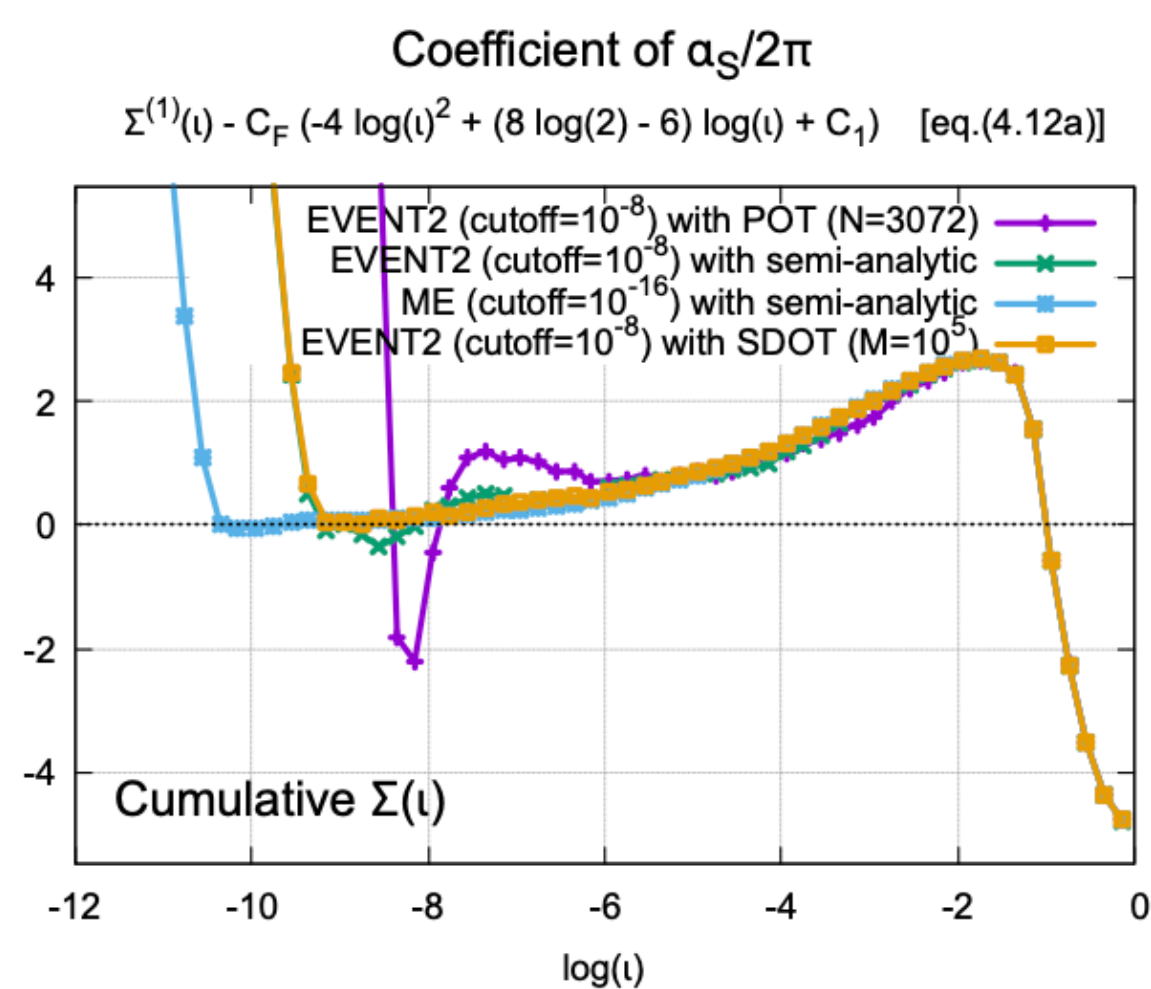
- the scaling is $\iota^{sc} \simeq z\theta = 2k_t/Q$
- there are no recoil effect at first order
- iota is additive in the soft limit: $\iota = \sum \iota_i$

The resummed cumulative function is given by the CAESAR formula:

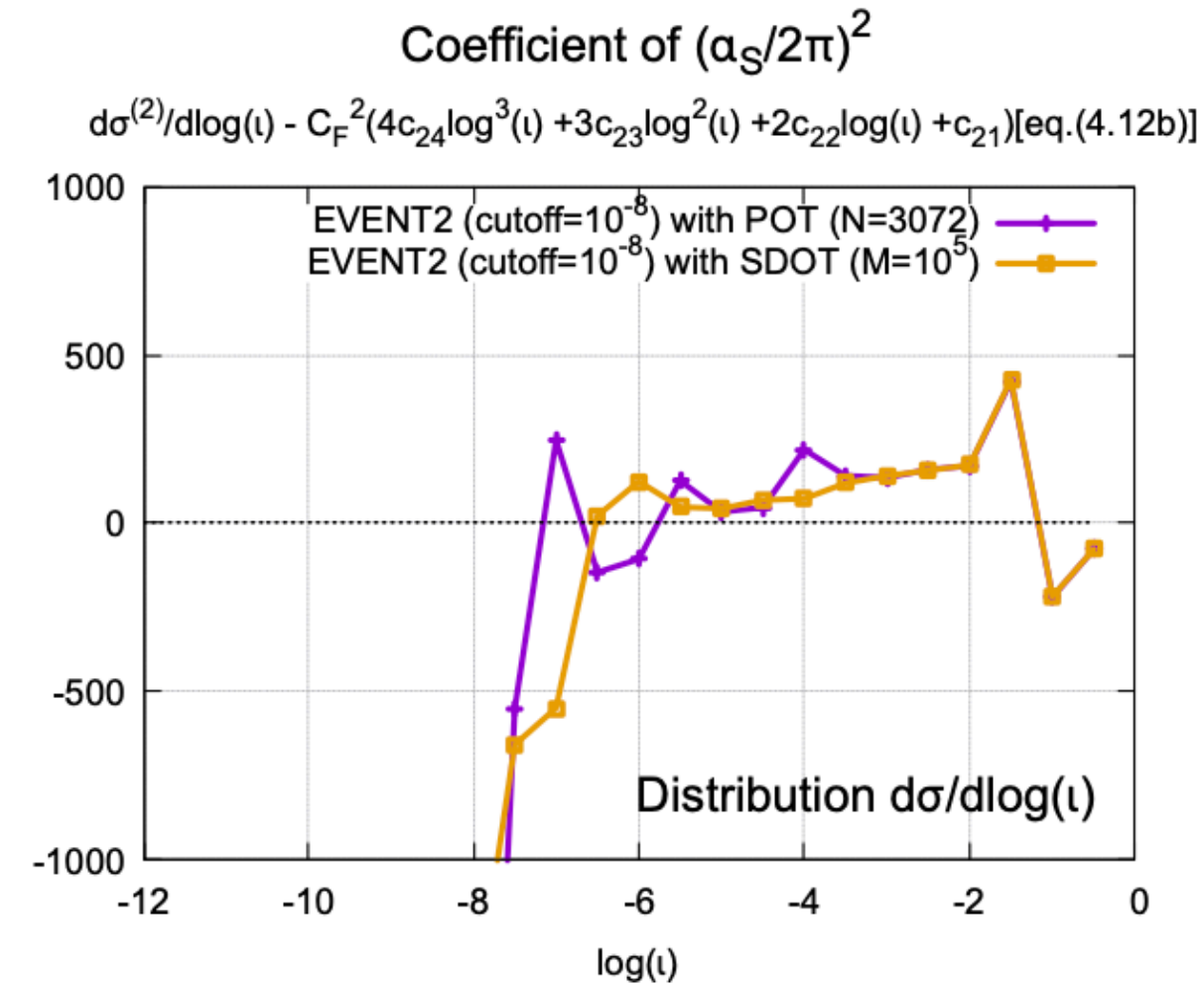
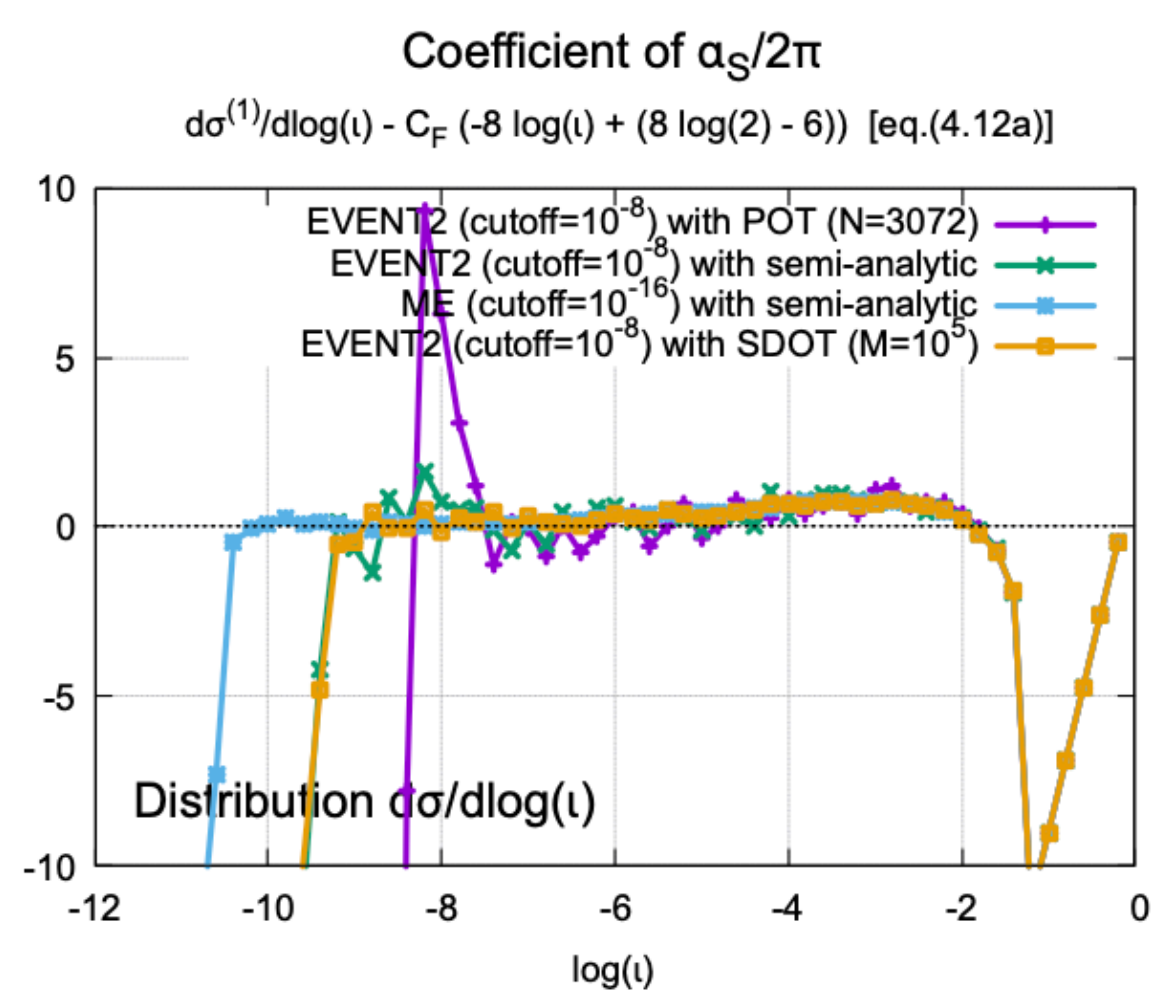
$$\Sigma_{\text{res}}(\iota) = \frac{1}{\sigma_0} \int_0^\iota d\iota' \frac{d\sigma_{\text{res}}}{d\iota'} = \left(1 + \frac{\alpha_S C_F}{2\pi} C_1 \right) \frac{e^{-\gamma_E R'_{LL}}}{\Gamma(1 + R'_{LL})} e^{-R(\iota)}$$

and we have also calculated $C_1 = \pi^2 - 7 - 4 \log^2 2 + 6 \log 2$

Checks



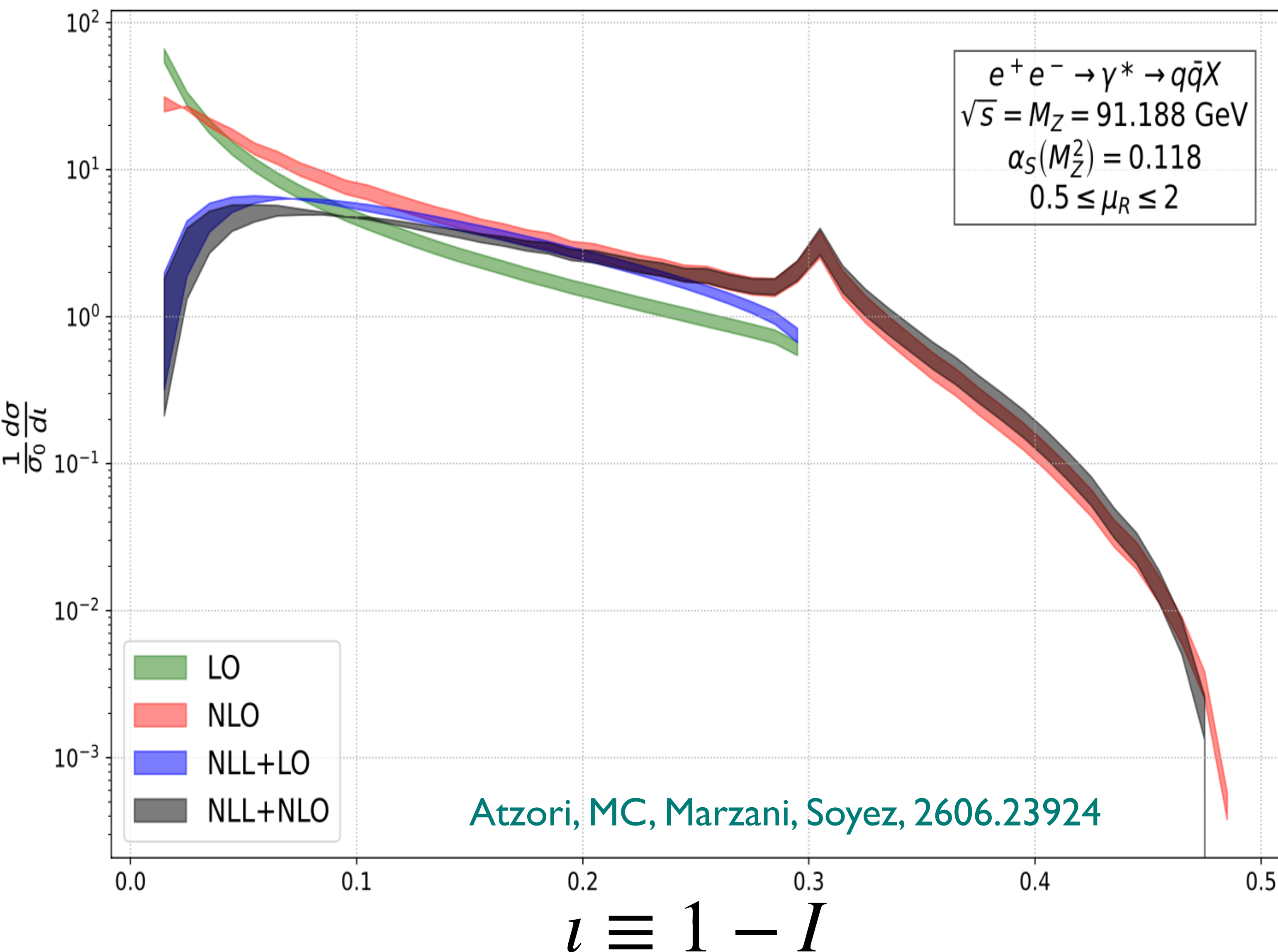
$$\Sigma_{\text{res}}(\iota) = \left[1 + \frac{\alpha_S C_F}{2\pi} \left(\underbrace{c_{12} \log^2 \iota}_{\text{green}} + \underbrace{c_{11} \log \iota}_{\text{green}} + \underbrace{c_{10}}_{\text{green}} \right) + \left(\frac{\alpha_S C_F}{2\pi} \right)^2 \left(\underbrace{c_{24} \log^4 \iota}_{\text{green}} + \underbrace{c_{23} \log^3 \iota}_{\text{green}} + \underbrace{c_{22} \log^2 \iota}_{\text{green}} + \underbrace{c_{21} \log \iota}_{\text{red}} \right) \right]$$



All coefficients of logs
up to NLL are correctly reproduced

Atzori, MC, Marzani, Soye, 2606.23924

Isotropy at NLO+NLL' in e^+e^-



Isotropy (iota) distribution,
resummed and matched

- Multiplicative matching of cumulative
- Endpoints forced to the value of fixed-order endpoints
- Resummation stabilises result in small ι region (expected)
- It also stabilises it in the intermediate ι region (unclear why)

- Isotropy in e^+e^- collisions has been considered for the first time in perturbative QCD at higher orders
- Numerical codes (semi-analytical and SDOT) have been developed to calculate isotropy with sufficient accuracy in the soft/collinear region
- We have calculated the resummation of soft emissions from first principles, and achieved NLO+NLL' accuracy

- Use of $\mathcal{O}(\alpha_S^3)$ calculation to reach NNLO accuracy
- Study of the Sudakov shoulder
- Study of the $\iota \rightarrow 1$ region of real isotropy, with many particles
- Impact of non-perturbative corrections
- Analysis of old LEP data with the e^+e^- Alliance 
- Calculation of isotropy for LHC (e.g. ring configuration used by ATLAS)