



QCD–Gravity Duality and Discrete BFKL Evolution in Multi-Regge Kinematics



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Outline

- 1 Double copy structure
 - QCD process
 - Gravity process
 - Double-copy structure in the MRK
- 2 Discrete BFKL
 - Introduction to the discrete BFKL
 - Walk expansion and Reggeization
 - Anomalous dimension
- 3 Conclusions and outlook



Procedure - Mathematica libraries

- FeynArts draws the Feynman Diagrams¹

¹Hahn 2001.

²Shtabovenko et al. 2025.

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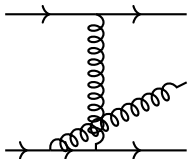
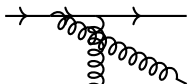


QCD diagrams: $qq' \rightarrow qq'g$

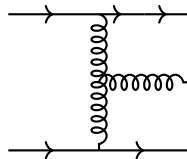
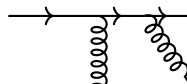
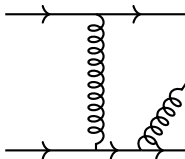
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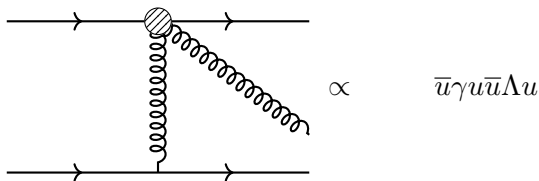


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And with them, an effective vertex pops up





By combining the two GI vertices one obtains the full amplitude

$$\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) \Bigg|_{\text{MRK}} = \begin{array}{c} \text{Diagram 3} \end{array}$$

The diagram on the left shows two Feynman diagrams for a QCD process, enclosed in large parentheses and separated by a plus sign. Each diagram consists of two horizontal lines representing incoming and outgoing particles, with a vertical wavy line (gluon) connecting them. The left diagram has a shaded circle at the top vertex, and the right diagram has a shaded circle at the bottom vertex. The vertical wavy line is labeled k_1 . The diagrams are separated by a vertical line labeled MRK.

The diagram on the right shows the full amplitude, which is a single Feynman diagram. It consists of two horizontal lines representing incoming and outgoing particles, with a vertical wavy line (gluon) connecting them. The left diagram has a shaded circle at the top vertex, and the right diagram has a shaded circle at the bottom vertex. The vertical wavy line is labeled k_1 . The diagram is labeled τ, α .



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The diagrams represent Feynman-like diagrams for a QCD process. The left side shows two diagrams in parentheses, separated by a plus sign. Each diagram has two horizontal lines with arrows, representing incoming and outgoing particles. A vertical wavy line connects the two horizontal lines. In the first diagram, a diagonal wavy line with a shaded circle at its top end extends from the top horizontal line. In the second diagram, a diagonal wavy line with a shaded circle at its bottom end extends from the bottom horizontal line. The right side of the equation, after the MRK label, shows a single diagram where the two horizontal lines are connected by a vertical green wavy line. A horizontal wavy line with a black dot at its left end and a label τ, α extends from the right side of the vertical green wavy line. Momenta are labeled: p_a and k_0 on the top line, p_b and k_2 on the bottom line, and k_1 on the vertical green wavy line.

Where we can define the effective vertex

$$\begin{array}{c} \text{Diagram 4} \end{array} \propto \Gamma_{\chi\sigma}^{\tau} = i\alpha_s g_{\chi\sigma} \Omega^{\tau}$$

The diagram on the left shows a vertical green wavy line connected to a horizontal wavy line with a black dot at the vertex and a label τ .

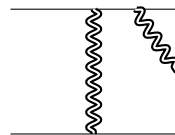
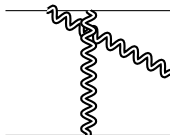
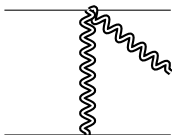


Gravitational diagrams

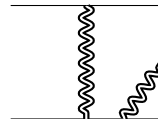
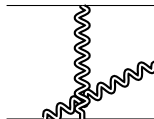
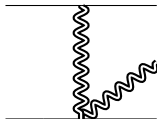
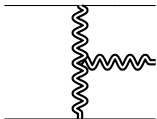
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Gravitational diagrams



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We now define a new GI amplitudes, in the same way as before

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$$M_{\uparrow} = M_2 + M_3 + \frac{t}{t - t'}(M_1 + M_4)$$

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By applying

- 1 Check Steinmann relations⁴
- 2 On-shell conditions
- 3 Apply MRK limit

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We obtain the total amplitude

$$\mathcal{M} = \mathcal{M}^{\mu\nu} \epsilon_{\mu\nu}$$

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With this we can see that we do not obtain

$$\mathcal{M}^{\mu\nu} = \Omega^\mu \Omega^\nu.$$

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$$\mathcal{M}^{\mu\nu} = \Omega^\mu \Omega^\nu.$$

By checking the Steinmann relations we can see that this should not be valid, and we need to use⁵

$$\mathcal{M}^{\mu\nu} = \Omega^\mu \Omega^\nu - \mathcal{N}^\mu \mathcal{N}^\nu.$$

⁵Lipatov 1982.



Discrete Hamiltonian

As shown in the BFKL equation for forward scattering, i.e.,

$$\frac{\partial \phi(Q^2, Y)}{\alpha \partial Y} = \int_0^\infty \frac{dq^2}{|q^2 - Q^2|} \left(\phi(q^2, Y) - \frac{2 \min(q^2, Q^2)}{q^2 + Q^2} \phi(Q^2, Y) \right)$$

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The discretized version uses $q^2 = n\Delta$, $Q^2 = N\Delta^6$, which leaves,

$$\frac{\partial}{\alpha \partial Y} |\phi^{(N)}\rangle = \hat{\mathcal{H}}_N |\phi^{(N)}\rangle; \quad \hat{\mathcal{H}}_N = A - 2D,$$

where

$$A_{ij} = \begin{cases} 1/|i - j|, & i \neq j, \\ 0, & i = j \end{cases} \quad D_{ij} = H_{i-1} \delta_{ij}$$

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Norm bounds and Rayleigh quotient

We can now define the Rayleigh quotient

$$\mathcal{R} = \frac{\langle v^\gamma, \hat{\mathcal{H}}_N v^\gamma \rangle}{\langle v^\gamma, v^\gamma \rangle}$$



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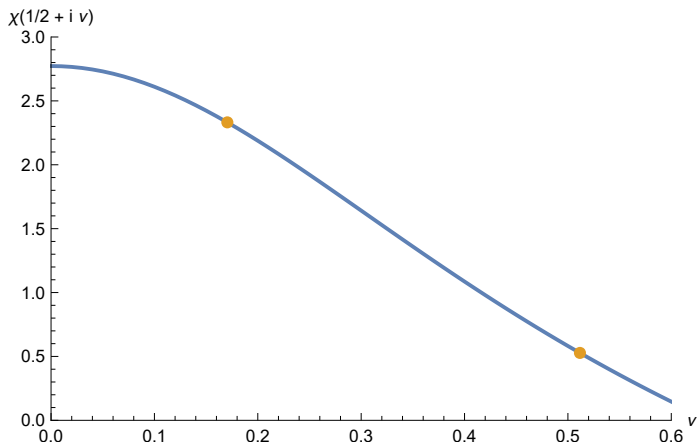
$$\mathcal{R}_{N,\epsilon} = \chi(\gamma) + \delta_N(\gamma)$$

By imposing hard-wall conditions on the boundary we obtain

$$\lambda_n(\hat{\mathcal{H}}_N) \approx 4 \log(2) - 14\zeta(3) \left(\frac{n\pi}{L} \right)^2$$



Introduction to the discrete BFKL





Vertex-dressed resolvent

To obtain the evolution we now want to compute

$$\begin{aligned}
 |\phi^{(N)}(Y)\rangle &= e^{\alpha Y \hat{\mathcal{H}}_N} |\phi^{(N)}(0)\rangle \\
 &= (1 + \alpha Y \hat{\mathcal{H}}_N + (\alpha Y \hat{\mathcal{H}}_N)^2 + \dots) |\phi^{(N)}(0)\rangle
 \end{aligned}$$



Vertex-dressed resolvent

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Starting at a site N_0 and looking at a site n , the evolution is given by

$$d_n^{(r)} = \langle e_n, (A - 2D)^r e_{N_0} \rangle.$$



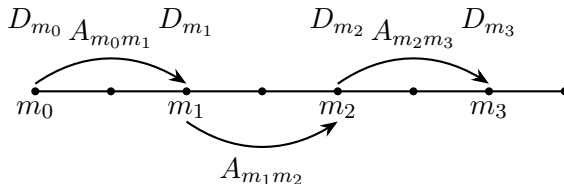
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Reggeization

With this, we obtain

$$d_n^{(r)} = \sum_{j=0}^r (-2)^{r-j} \sum_{(m_0, \dots, m_j)} \left[h_{r-j}(H_{m_0-1}, \dots, H_{m_j-1}) \prod_{i=0}^{j-1} \frac{1}{|m_i - m_{i+1}|} \right]$$

When the sum over m 's means for all the paths connecting n and N_0 .



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When the sum over m 's means for all the paths connecting n and N_0 .
By introducing a generating function, we can write

$$\mathcal{G}_n(z) = \sum_{j=0}^{\infty} z^j \sum_{(m_0, \dots, m_j)} \left(\prod_{i=0}^{j-1} \frac{1}{|m_i - m_{i+1}|} \right) \left(\prod_{l=0}^j \frac{1}{1 + 2zH_{m_l-1}} \right)$$



From discrete to continuum

Instead of writting

$$d_n^{(r)} = (A - 2D)d^{(r-1)}; \quad \mathcal{G}^{(r)}(m/n; x) = \frac{d_m^{(r)}}{n(m/n - x)}$$



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one can write

$$\mathcal{G}^{(r+1)}(1; x) = \frac{1}{x} \int_0^x \frac{\mathcal{G}^{(r)}(y; x)}{y} dy + \frac{1}{1-x} \int_x^1 \frac{\mathcal{G}^{(r)}(y; x)}{1-y} dy$$

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And from this, one can obtain the differential equation

$$\frac{d}{dx} \mathcal{F}^{(r+1)}(x) = \left(\frac{A_0}{x} + \frac{A_1}{1-x} \right) \mathcal{F}^{(r)}(x),$$

where $\mathcal{G}^{(r)}(1; x) \equiv \mathcal{F}^{(r)}(x)$.



Single log structure

Following the HPL structure, we can write

$$\mathcal{F}^{(r)}(x) = \sum_{|\tau|=r-1} c_{\tau}^{(r)} H_{\tau}(x)$$

⁷Del Duca et al. 2014.



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The Gluon's green function can be written as

$$\tilde{\phi}_n(\omega) = \sum_{r \geq 0} \frac{\alpha^r}{\omega^{r+1}} d_n^{(r)} \quad \tilde{\phi}_n(\omega) \propto \sum_{r \geq 0} \sum_{|\omega|=r-1} \frac{\alpha^r}{\omega^{r+1}} c_{\omega}^{(r)} H_{\omega}(x)$$

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The single-log and single-valued⁷ version is given by

$$\Pi_{SV}(\tilde{\phi}_n) \propto \log(x).$$

⁷Del Duca et al. 2014.



Anomalous dimension coefficients

By writting the anomalous dimension expansion as

$$\gamma_{\omega}(\alpha) = \sum_{s \geq 1} c_1^{(s+1)} \left(\frac{\alpha}{\omega} \right)^s$$



Anomalous dimension coefficients

By writing the anomalous dimension expansion as

$$\gamma_\omega(\alpha) = \sum_{s \geq 1} c_1^{(s+1)} \left(\frac{\alpha}{\omega} \right)^s$$

one finds the following coefficients

n	c_1^{n+1}
0	1
1	0
2	0
3	$2\zeta_3$
4	0
5	$2\zeta_5$
$\vdots$	$\vdots$



Results and conclusions

- Double copy structure
 - 1 Correct amplitudes with Mathematica
 - 2 Double-copy structure



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- Discrete BFKL
 - 1 Pomeron intercept
 - 2 Evolution and Reggeization
 - 3 Knizhnik–Zamolodchikov and Harmonic Polylogarithms
 - 4 Anomalous dimension expansion



Further research

- Double copy structure
 - 1 Reproduce other known amplitudes
 - 2 Extension to higher-point amplitudes



Further research

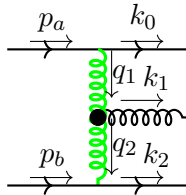
- Double copy structure
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 - 1 Include N²LO corrections
 - 2 Study the integrability structure
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Thank you for your attention!

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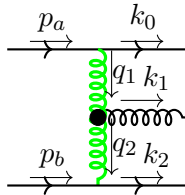
More on the MRK limit

By setting



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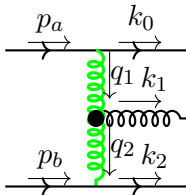


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And the generalized Mandestav variables as

$$\hat{s} = x_A x_B s; \quad \hat{s}_{ai} = -2p_a k_i; \quad \hat{s}_{bi} = -2p_b k_i; \quad \hat{s}_{ij} = 2k_i k_j,$$

we can obtain the following relations

More on the MRK limit

$$\hat{s}_{01} = -\hat{s}(\alpha_2 + \beta_2); \quad \hat{s}_{21} = \hat{s}(\alpha_1 + \beta_1); \quad \hat{s}_{a0} = -t = -\hat{s}\beta_1;$$

$$\hat{s}_{b2} = -t' = \hat{s}\alpha_2; \quad \hat{s}_{a1} = \hat{s}(\beta_1 - \beta_2); \quad \hat{s}_{b1} = \hat{s}(\alpha_1 - \alpha_2);$$

$$\hat{s}_{a2} = \hat{s}(1 + \beta_2); \quad \hat{s}_{b0} = \hat{s}(1 - \alpha_1);$$

$$\hat{s}_{02} = \hat{s}(1 - \alpha_1 + \alpha_2 - \beta_1 + \beta_2).$$

Steinmann relations

- We cannot have a double discontinuity
- No double poles
- Two channels cannot overlap

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I have not found yet a reference that is able to explain them in English.

More on duality relations

There are other works in the duality relation⁸

$$A = \sum_i \frac{n_i c_i}{d_i}; \quad \tilde{A} = \sum_i \frac{\tilde{n}_i c_i}{d_i}$$

$$n_i - n_j = n_k; \quad c_i - c_j = c_k;$$

$$A_{\text{grav}} = \sum_i \frac{n_i \tilde{n}_i}{d_i}$$

⁸Bern et al. 2019.

Rayleigh quotient

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where i is the index of the vector. While also staying away from the boundaries

$$I_\epsilon = \{i \in (1, \dots, N) | \epsilon N \leq i \leq (1 - \epsilon)N\}.$$

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With this, and being careful on the end points regularization, we obtain the BFKL characteristic function

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma).$$

Hard-wall boundary

Define the logarithmic coordinates

$$t = \log(i); \quad [T_-, T_+] = [\log(\epsilon N), \log((1 - \epsilon)N)]; \quad L = T_+ - T_-.$$

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$$\psi_j = A j^{-1/2} e^{-i\nu \log(j)} + B j^{1/2} e^{i\nu \log(j)}.$$

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Imposing boundary conditions means

$$\nu_n L + \delta(\nu) = \pi \left(\frac{1}{n} + n \right)$$

and the wave function becomes

$$\psi_i^{(n)} = 2B i^{-1/2} \cos(\nu(t - T_{\pm}))$$

Single Valued HPLS

The map from the HPLs to the SVHPLs reads, for the first ones

$$\mathcal{L}_0(z) = H_0 + \overline{H}_0; \quad \mathcal{L}(z) = H_1 + \overline{H}_1.$$

For the next weight two we have

$$\begin{aligned} \mathcal{L}_{0,0}(z) &= H_{0,0} + \overline{H}_{0,0} + H_0 \overline{H}_0; & \mathcal{L}_{0,1}(z) &= H_{0,1} + \overline{H}_{1,0} + H_0 \overline{H}_1; \\ \mathcal{L}_{1,0}(z) &= H_{1,0} + \overline{H}_{0,1} + H_1 \overline{H}_0; & \mathcal{L}_{1,1}(z) &= H_{1,1} + \overline{H}_{1,1} + H_1 \overline{H}_1; \end{aligned}$$

Anomalous dimension

We know that the Gluon Green Function can be written in Mellin momenta space as

$$f(\omega, \gamma) = \frac{\tilde{f}^0(\omega, \gamma)}{1 - (\bar{\alpha}_s/\omega)\chi(\gamma)},$$

where $t \rightarrow M$ and $\xi \rightarrow N$. There is a singularity around $\omega/\bar{\alpha}_s = \chi(\bar{\gamma})$. Inverting this we obtain

$$\bar{\gamma} = \frac{\bar{\alpha}_s}{\omega} + 2\zeta(3) \left(\frac{\bar{\alpha}_s}{\omega}\right)^4 + 2\zeta(5) \left(\frac{\bar{\alpha}_s}{\omega}\right)^6 + \mathcal{O}\left(\frac{\bar{\alpha}_s}{\omega}\right)^7$$

Gravity BFKL

There are works for the Gravity BFKL equation⁹

- We can ask if we can do something similar
- Would the KZ structure also appear?
- Work in progress

⁹Raj et al. 2025.