The Dynamical Algebra of the Generic Superintegrable Model on the Two-Sphere

and the Algebraic Interpretation of Two-Variable Jacobi Polynomials

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Superintegrable Systems

Definition

A quantum system in d dimensions is **maximally superintegrable** if it has $2d-1$ algebraically independent constants of motion $\{H_k\}$ including the Hamiltonian H, with $[H, H_k] = 0$ for all k.

Harmonic oscillator $(d \geq 2)$

Dynamical algebra: $\mathfrak{sp}(2\mathfrak{d})$
Symmetry: $\mathfrak{su}(d)$

Hydrogen atom

Dynamical algebra: $\mathfrak{so}(4,2)$
Symmetry: $\mathfrak{so}(4)$ [Pauli 1926]

Generic model on S^n

[this work]

Key distinction

- **Symmetry algebra**: commutes with H; explains *degeneracies*; finite-dimensional irreps per level
- **Dynamical algebra**: does *not* commute with H; connects *all* energy levels; full spectrum in **one** irreducible module

The Generic Superintegrable Model on S^2

Unit sphere $S^2 \subset \mathbb{R}^3$: $x_1^2 + x_2^2 + x_3^2 = 1$, restrict to octant $x_i > 0$.

$$\mathcal{H} = \hat{\mathcal{J}}_1^2 + \hat{\mathcal{J}}_2^2 + \hat{\mathcal{J}}_3^2 + \frac{a^2 - \frac{1}{4}}{x_1^2} + \frac{b^2 - \frac{1}{4}}{x_2^2} + \frac{c^2 - \frac{1}{4}}{x_3^2}, \qquad a, b, c \in \mathbb{R}$$

where $\hat{\mathcal{J}}_k = -i\varepsilon_{klm} x_l \partial_m$ are angular momentum operators.

Constants of motion (commute with $\mathcal{H}$):

$$\mathcal{S}_1 = \hat{\mathcal{J}}_1^2 + \frac{(b^2 - \frac{1}{4})x_3^2}{x_2^2} + \frac{(c^2 - \frac{1}{4})x_2^2}{x_3^2}$$

and cyclic permutations $\mathcal{S}_2, \mathcal{S}_3$.

$$\mathcal{H} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + a^2 + b^2 + c^2 - \tfrac{3}{4}$$

Universality

All second-order superintegrable models in 2D arise as **limits/contractions** of this 3-parameter system [Kalnins–Miller–Post 2013]

Quadratic Algebras

General structure (two generators K_1, K_2)

$$[[K_1, K_2], K_1] = f(K_1, K_2), \qquad [[K_2, [K_1, K_2]] = g(K_1, K_2)$$

where f, g are *quadratic* polynomials in K_1, K_2 plus central elements.

Racah algebra $\mathfrak{R}_1$ (rank 1)

$$[[K_1, K_2], K_1] = 2K_1^2 + 2\{K_1, K_2\} + \xi K_1 + \eta_1 K_2 + \zeta_1$$
$$[K_2, [K_1, K_2]] = 2K_2^2 + 2\{K_1, K_2\} + \xi K_2 + \eta_2 K_1 + \zeta_2$$

$\Rightarrow$ Symmetry algebra of $\mathcal{H}$
Racah / Wilson polynomials

Jacobi algebra $\mathfrak{J}_1$ (rank 1)

$$\begin{aligned}[][K_1, K_2], K_1] &= 2\{K_1, K_2\} - 2K_1 \\ &\quad - (\alpha+\beta)(\alpha+\beta+2)K_2 + (\alpha+\beta)(\alpha+1)\end{aligned}$$
$$[K_2, [K_1, K_2]] = 2K_2^2 - 2K_2$$

$\Rightarrow$ Bispectral algebra of $J_n^{(\alpha,\beta)}(x)$
Univariate Jacobi polynomials

Where Do Quadratic Algebras Come From?

Simplest case. If two operators K_1, K_2 share an eigenbasis $\{|n\rangle\}$ used to label states, then $K_1|n\rangle = \mu_n|n\rangle$, $K_2|n\rangle = \lambda_n|n\rangle$, and necessarily $[K_1, K_2] = 0$.

Next level of complexity. Suppose instead there are *two* natural bases: in one, K_1 is diagonal and K_2 tridiagonal; in the other, the roles are reversed. This is exactly the **bispectral** situation for the Askey scheme (Jacobi, Hahn, Racah, . . .): one basis diagonalizes the differential/difference operator, the other the recurrence operator.

Generically, such a pair does *not* commute, but satisfies a **quadratic algebra**: the double commutators $[[K_1, K_2], K_1]$ and $[[K_2, K_1], K_2]$ close back onto quadratic polynomials in K_1, K_2. This is how Jacobi, Hahn and Racah algebras encode bispectrality.

The $\mathfrak{su}(1,1)^{\otimes 3}$ Framework [Genest–Vinet–Zhedanov 2014]

$\mathfrak{su}(1,1)$: generators $J_0, J_\pm$ with $[J_0, J_\pm] = \pm J_\pm$, $[J_+, J_-] = -2J_0$. Casimir $C = J_0^2 - J_0 - J_+J_-$.
Differential realization (singular oscillator in x_i, $i = 1, 2, 3$)

$$\mathcal{J}_0^{(i)} = \tfrac{1}{4}\left(-\partial_i^2 + x_i^2 + \frac{4\nu_i(\nu_i - 1) + \frac{3}{4}}{x_i^2}\right), \quad \mathcal{J}_\pm^{(i)} = \tfrac{1}{4}\left(\partial_i^2 \mp 2x_i\partial_i + (x_i^2 \mp 1) - \frac{4\nu_i(\nu_i - 1) + \frac{3}{4}}{x_i^2}\right)$$

$$\nu_1 = \tfrac{a+1}{2},\ \nu_2 = \tfrac{b+1}{2},\ \nu_3 = \tfrac{c+1}{2}, \qquad C\,|\nu;\cdot\rangle = \nu(\nu-1)\,|\nu;\cdot\rangle \text{ on each irrep}$$

Three-fold tensor product and total Casimir:

$$C^{(123)} = C^{(12)} + C^{(23)} + C^{(13)} - C^{(1)} - C^{(2)} - C^{(3)}$$

On S^2 $(x_1^2 + x_2^2 + x_3^2 = 1)$:

$$\mathcal{H} = 4\,C^{(123)} + \tfrac{3}{4}$$

Spectrum

$$E_n = \big(2(n+1) + a + b + c\big)^2 - \tfrac{1}{4}$$

$n \in \mathbb{N}$, degeneracy $= n + 1$.

$C^{(ij)}$ commute with $C^{(123)}$
(they built it) ⇒ constants of motion

The $C^{(ij)}$ do *not* commute among themselves: their mutual commutators close on the **Racah algebra** $\mathfrak{R}_1$, which is therefore the symmetry algebra of $\mathcal{H}$ — its $(n+1)$-dim. irreps account for the degeneracy above.

From Symmetry to Dynamical: What Is Missing?

$\mathfrak{R}_1 = \langle L, L_1, L_3 \rangle$ commutes entirely with $\mathcal{H} \propto L$: it reshuffles states *within* a level but never changes n. **What operator could change n?** The position-like operators $X_1 = x_1^2$ and $X_3 = x_3^2$ do *not* commute

with L — exactly the kind of generator a dynamical algebra needs.

Adjoin X_1, X_3 to $\{L, L_1, L_3\}$.

The five generators $\{L, L_1, L_3, X_1, X_3\}$ close into a quadratic algebra of rank two — the rank-two **Jacobi algebra** $\mathfrak{J}_2$, our candidate dynamical algebra.

The Rank-Two Jacobi Algebra $\mathfrak{J}_2$

Five generators embedded in $\mathfrak{su}(1,1)^{\otimes 3}$

$$\{L,\ L_1,\ L_3,\ X_1,\ X_3\}\ \subset\ \mathfrak{su}(1,1)^{\otimes 3}$$

$$L = -C^{(123)} + \text{const} \propto \mathcal{H}, \qquad L_1 = -C^{(23)} + \text{const}, \qquad L_3 = -C^{(12)} + \text{const}$$

$$X_1 = [2J_0 + J_+ + J_-] \otimes 1 \otimes 1\ (= x_1^2), \quad X_3 = 1 \otimes 1 \otimes [2J_0 + J_+ + J_-]\ (= x_3^2)$$

Vanishing commutators:

$$[L, L_1] = [L, L_3] = [L_1, X_1] = [L_3, X_3] = [X_1, X_3] = 0$$

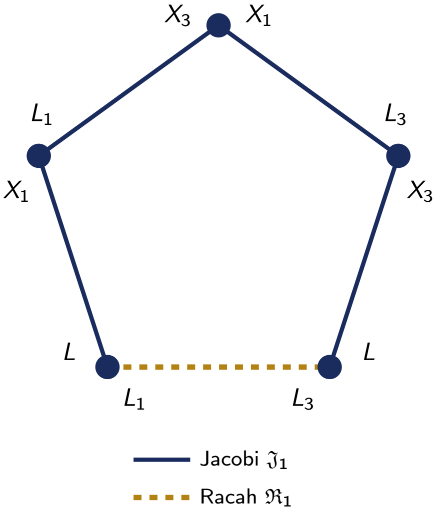
Sample non-trivial relation:

$$\begin{aligned}[][[L, X_1], L] = 2\{X_1, L\} - 2L + 2L_1 \\ - (a{+}b{+}c{+}1)\big((a{+}b{+}c{+}3)X_1 - (a{+}1)\big)\end{aligned}$$

Subalgebra structure

Centralizing of	Type
L_1	Jacobi $\mathfrak{J}_1$ (via L, X_1)
X_1	Jacobi $\mathfrak{J}_1$ (via L_1, X_3)
X_3	Jacobi $\mathfrak{J}_1$ (via L_3, X_1)
L_3	Jacobi $\mathfrak{J}_1$ (via L, X_3)
L	**Racah** $\mathfrak{R}_1$ (via L_1, L_3)

The Pentagonal Subalgebra Structure



Reading the pentagon

- Each **vertex** = commuting pair = representation basis
- **Solid edges**: rank-1 Jacobi algebra $\Rightarrow$ overlaps = **Jacobi polynomials**
- **Dashed edge**: rank-1 Racah algebra $\Rightarrow$ overlaps = **Racah polynomials**

Three representation bases:

- $\{|x, y\rangle\}$: eigenbasis of X_1, X_3 (position)
- $\{|n, k; a, b, c\rangle\}$: eigenbasis of L, L_1 (energy + first symmetry)
- $\{|n, k; a, b, c\rangle^{\pi}\}$: eigenbasis of L, L_3 (energy + second symmetry)

Physical States: Dirac Quantization

Constraint: $X_1 + X_2 + X_3 = 1 \;\Leftrightarrow\; x_1^2 + x_2^2 + x_3^2 = 1$

Step 1 — Generalized eigenvectors of $X = 2J_0 + J_+ + J_-$ in $\mathfrak{su}(1,1)$:

$$|\nu;r\rangle = r|\nu;r\rangle \to |\nu;r\rangle = \sum_{i=0}^{\infty} q_i^{(\nu)}(r)|\nu;i\rangle, \qquad q_i^{(\nu)}(r) = \frac{(-1)^i}{(2\nu)_i} \cdot \frac{e^{-r/2} r^{\nu-1/2}}{\Gamma(2\nu)^{1/2}} L_i^{(2\nu-1)}(r)$$

Step 2 — Joint eigenvectors of $C^{(123)}, C^{(23)}, X^{(123)}$ via Clebsch–Gordan coefficients (= Hahn polynomials):

$$|r,n,k;a,b,c\rangle = \frac{1}{c_n} \sum_{m\geq 0} q_m^{(\nu_{123}+n)}(r)|\nu_{123}+n, \nu_{23}+k; m\rangle$$

Step 3 — Dirac constraint:

$$|n,k;a,b,c\rangle \;:=\; |r{=}1,\; n,k;a,b,c\rangle \qquad \text{(freeze radial d.o.f.)}$$

Orthogonality of the Physical Basis

Following the Dirac quantization on the previous slide, the basis $\{|n, k; a, b, c\rangle\}$ is genuinely discrete and normalizable. **Orthogonality** (computed exactly from the Laguerre resolution of the identity):

$$\langle n', k'; a, b, c | n, k; a, b, c\rangle = N_{n-k}^{(a,\, b+c+2k+1)}\, N_k^{(b,c)}\, \delta_{n,n'}\, \delta_{k,k'}$$

with the standard univariate normalization constant

$$N_n^{(a,b)} = \frac{\Gamma(n+a+1)\Gamma(n+b+1)}{(2n+a+b+1)\, n!\, \Gamma(n+a+b+1)}$$

The orthogonality factorizes into a product of two univariate weights — a first hint that two-variable Jacobi polynomials are about to appear.

Representation of the Symmetry Algebra $\mathfrak{R}_1$

Basis vectors $|n, k; a, b, c\rangle$ diagonalize L and L_1:

$$L|n,k\rangle = -n(n+a+b+c+2)|n,k\rangle, \qquad L_1|n,k\rangle = -k(k+b+c+1)|n,k\rangle$$

Action of L_3 (tridiagonal in k, fixed n)

$$L_3|n,k\rangle = \gamma^1_{nk}|n,k+1\rangle + \gamma^2_{nk}|n,k\rangle + \gamma^3_{nk}|n,k-1\rangle$$

$$\gamma^1_{nk} = \frac{(k+1)(n-k+a)(k+b+c+1)(n+k+a+b+c+2)}{(2k+b+c+1)(2k+b+c+2)}$$

$$\gamma^3_{nk} = \frac{(n-k+1)(k+b)(k+c)(n+k+b+c+1)}{(2k+b+c+1)(2k+b+c)}$$

Proof: Expand $|r,n,k\rangle$ over $C^{(12)}$-eigenbasis via **Racah coefficients** (= Racah polynomials), then apply their 3-term recurrence relation.

Action of the Position Operator X_1 — Connecting Energy Levels

X_1 is the key **spectrum-generating** operator: it changes n by ±1.

$$X_1|n,k\rangle = \alpha^1_{nk}|n+1,k\rangle + \alpha^2_{nk}|n,k\rangle + \alpha^3_{nk}|n-1,k\rangle$$

$$\alpha^1_{nk} = -\frac{(n-k+1)(n+k+a+b+c+2)}{(2n+a+b+c+2)(2n+a+b+c+3)},$$

$$\alpha^3_{nk} = -\frac{(n+k+b+c+1)(n-k+a)}{(2n+a+b+c+1)(2n+a+b+c+2)}$$

Proof strategy:

- Work in auxiliary $\mathfrak{su}(1,1)$ module
- Use **contiguity relations for Laguerre polynomials** to shift parameter $\nu_f \to \nu_f \pm 1$
- Use **contiguity relations for Hahn polynomials** (Clebsch–Gordan coefficients)

Action of X_3 — The Nine-Term Relation

X_3 changes both n and k simultaneously:

$$\begin{aligned} X_3|n,k\rangle = {} & \beta^1_{nk}|n+1,k-1\rangle + \beta^2_{nk}|n+1,k\rangle + \beta^3_{nk}|n+1,k+1\rangle \\ & + \beta^4_{nk}|n,k-1\rangle + \beta^5_{nk}|n,k\rangle + \beta^6_{nk}|n,k+1\rangle \\ & + \beta^7_{nk}|n-1,k-1\rangle + \beta^8_{nk}|n-1,k\rangle + \beta^9_{nk}|n-1,k+1\rangle \end{aligned}$$

Sample coefficients:

$$\beta^1_{nk} = \frac{(k+b)(k+c)(n-k+1)(n-k+2)}{(2k+b+c)(2k+b+c+1)(2n+a+b+c+2)(2n+a+b+c+3)}$$

$$\beta^3_{nk} = \frac{(k+1)(k+b+c+1)(n+k+a+b+c+2)(n+k+a+b+c+3)}{(2k+b+c+1)(2k+b+c+2)(2n+a+b+c+2)(2n+a+b+c+3)}$$

Proof: Uses B_2- and B_2'-type **contiguity relations for Hahn polynomials** (classified in [Crampé et al., 2025]). $\Rightarrow (1-x-y)\cdot J^{(a,b,c)}_{n,k}(x,y) = \sum_i \beta^i J_{\dots}$ (9-term recurrence)

Algebraic Solution: Wavefunctions on S^2

Position basis (barycentric coordinates $x + y + z = 1$, $z = 1 - x - y$):

$$|x, y; a, b, c\rangle = |\nu_1; x\rangle \otimes |\nu_2; y\rangle \otimes |\nu_3; z\rangle$$

Wavefunctions = overlaps between energy basis and position basis:

$$\varphi_{n,k}^{(a,b,c)}(x, y) = \langle x, y; a, b, c | n, k; a, b, c\rangle$$

The 3-term and 9-term recurrences allow to recognize bivariate Jacobi polynomials and to write:

$$\varphi_{n,k}^{(a,b,c)}(x, y) = \varphi_{0,0}^{(a,b,c)}(x, y) \cdot J_{n,k}^{(a,b,c)}(x, y)$$

Ground state (computed via Laguerre addition formula + Dirichlet integral):

$$\varphi_{0,0}^{(a,b,c)}(x, y) = x^{a/2}\, y^{b/2}\, (1 - x - y)^{c/2}$$

$$\boxed{\varphi_{n,k}^{(a,b,c)}(x, y) = x^{a/2}\, y^{b/2}\, (1 - x - y)^{c/2} \cdot J_{n,k}^{(a,b,c)}(x, y)}$$

Two-Variable Jacobi Polynomials on the Triangle

Definition [Proriol 1957, Koornwinder 1975]

$$J_{n,k}^{(a,b,c)}(x,y) = J_{n-k}^{(a,\, b+c+2k+1)}(x)\cdot(1-x)^k\cdot J_k^{(b,c)}\left(\frac{y}{1-x}\right), \qquad n\geq k\geq 0$$

with $J_n^{(a,b)}(x) = \dfrac{(a+1)_n}{n!}\,{}_2F_1\left({-n,\, n+a+b+1 \atop a+1}\,;\,x\right)$

Orthogonality on the simplex:

$$\int_{0\leq x\leq 1-y\leq 1} x^a y^b (1-x-y)^c\, J_{n,k}^{(a,b,c)} J_{n',k'}^{(a,b,c)}\,dx\,dy = N_{n-k}^{(a,\, b+c+2k+1)}\, N_k^{(b,c)}\,\delta_{nn'}\delta_{kk'}$$

Properties obtained algebraically from the $\mathfrak{J}_2$ representation:

- **PDEs:** $\tilde{\mathcal{L}}\,[J_{n,k}] = -n(n+a+b+c+2)J_{n,k}, \quad \tilde{\mathcal{L}}_1[J_{n,k}] = -k(k+b+c+1)J_{n,k}$
- **3-term recurrence in n:** from action of $X_1 = x$
- **9-term recurrence in (n,k):** from action of $X_3 = 1-x-y$ [new]

Two-Variable Jacobi Polynomials as Overlaps: The Pentagon

Theorem (Crampé–Labriet–Morey–Tsujimoto–Vinet–Zhedanov 2026)

$$\langle n,k;a,b,c|x,y;a,b,c\rangle = \sqrt{\frac{x^a y^b (1-x-y)^c}{N_k^{(b,c)}\,N_{n-k}^{(a,b+c+2k+1)}}}\; J_{n,k}^{(a,b,c)}(x,y)$$

Proof via convolution along the left path of the pentagon:

$$\langle n,k;a,b,c|x,y\rangle = \sum_{k'} \int_0^1 \langle n,k;a,b,c|x',k';b,c\rangle\langle x',k';b,c|x,y\rangle\,dx'$$

$$\begin{tikzcd}
\underset{(X_1,X_3)}{\{|x,y\rangle\}} \arrow[r, "\mathfrak{J}_1\colon\ J_k^{(b,c)}"] & \underset{(L_1,X_1)}{\{|x,k;b,c\rangle\}} \arrow[r, "\mathfrak{J}_1\colon\ J_{n-k}^{(a,\dots)}"] & \underset{(L,L_1)}{\{|n,k\rangle\}}
\end{tikzcd}$$

- Each segment corresponds to a rank-1 Jacobi subalgebra where the *common generator* is central
- Overlaps within each segment = **univariate Jacobi polynomials**
- Convolution gives the **bivariate** polynomial

Racah Polynomials as Expansion Coefficients

Bottom edge of the pentagon: L_1 and L_3 generate $\mathfrak{R}_1$ (Racah algebra), centrally extended by L.

Overlap between the two bottom bases

$$^{\pi}\langle n,\ell;a,b,c|n,m;a,b,c\rangle=(-1)^m\sqrt{\frac{w^{(b,c,-n-1,n+1+a+b)}(\ell)}{M_m^{(b,c,-n-1,n+1+a+b)}}}\,R_m\Big(\lambda^{(a,b)}(\ell);\;b,c,-n-1,n+1+a+b\Big)$$

Inserting the Racah overlap into the pentagon convolution gives the change-of-basis between the two families of bivariate Jacobi polynomials:

The Resulting Expansion Formula

Expansion formula [new algebraic proof of Dunkl 1984]

$$J_{n,\ell}^{(c,b,a)}(1-x-y,\,y) = \mathfrak{F}(\ell)\sum_{m=0}^{n}\frac{(-1)^m}{\mathfrak{G}(m)}\,R_m\Big(\lambda^{(a,b)}(\ell);\ b,c,-n-1,n+1+a+b\Big)\ J_{n,m}^{(a,b,c)}(x,y)$$

where $R_m(\lambda(\ell)) = {}_4F_3\left({-m,\ m+b+c+1,\ -\ell,\ \ell+a+b+1 \atop b+1,\ n+a+b+c+2,\ -n};1\right)$

This is a new, transparent algebraic proof of a formula first obtained by Dunkl (1984) by direct computation. The pentagon is not just a mnemonic: each of its features corresponds to a precise algebraic and analytic statement about the bivariate Jacobi polynomials.

Analogy: This is a *continuum* version of the **Biedenharn–Elliott identity** for Racah/6j-symbols:

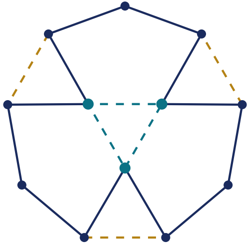
$$\text{bivariate poly} = \sum_m \text{Racah poly} \times (\text{univariate Jacobi})^2$$

Order-3 Symmetry of the Triangle (D_3)

The simplex has D_3 symmetry. Three families of bivariate Jacobi polynomials:

Transformation	Polynomial
Identity	$J_{n,k}^{(a,b,c)}(x,y)$
π: $a \leftrightarrow c$, $x \leftrightarrow 1-x-y$	$J_{n,k}^{(c,b,a)}(1-x-y,y)$
σ: $a \leftrightarrow b$, $x \leftrightarrow y$	$J_{n,k}^{(b,a,c)}(y,x)$

Three-pentagon picture:



Key facts

- τ (exchange $b \leftrightarrow c$):
  $J_{n,k}^{(a,c,b)}(x,1-x-y) = (-1)^k J_{n,k}^{(a,b,c)}(x,y)$
- Rotations $R(\frac{2\pi}{3})$, $R(\frac{4\pi}{3})$ give σ- and π-families up to $(-1)^k$

Each pentagon ↔ one coordinate direction.
Change-of-basis matrices ↔ Racah polynomials.

Summary of Main Results

1. **Dynamical algebra identified:** $\mathfrak{J}_2 = \{L, L_1, L_3, X_1, X_3\}$ is the spectrum-generating algebra of the generic superintegrable model on S^2 — *first identification*.
2. **Physical representation constructed:** via $\mathfrak{su}(1,1)^{\otimes 3}$ embedding + Dirac quantization; basis orthogonality proved exactly.
3. **Algebraic solution:** $\varphi_{n,k}^{(a,b,c)}(x,y) = x^{a/2} y^{b/2} (1-x-y)^{c/2}\, J_{n,k}^{(a,b,c)}(x,y)$
4. **Characterization of the bivariate Jacobi polynomials from the representation of the Jacobi algebra of rank 2**
5. **Racah polynomials as change-of-basis:** new algebraic proof of the Dunkl (1984) expansion formula.

Thank You

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