SubTropica

or quantum field theory for babies

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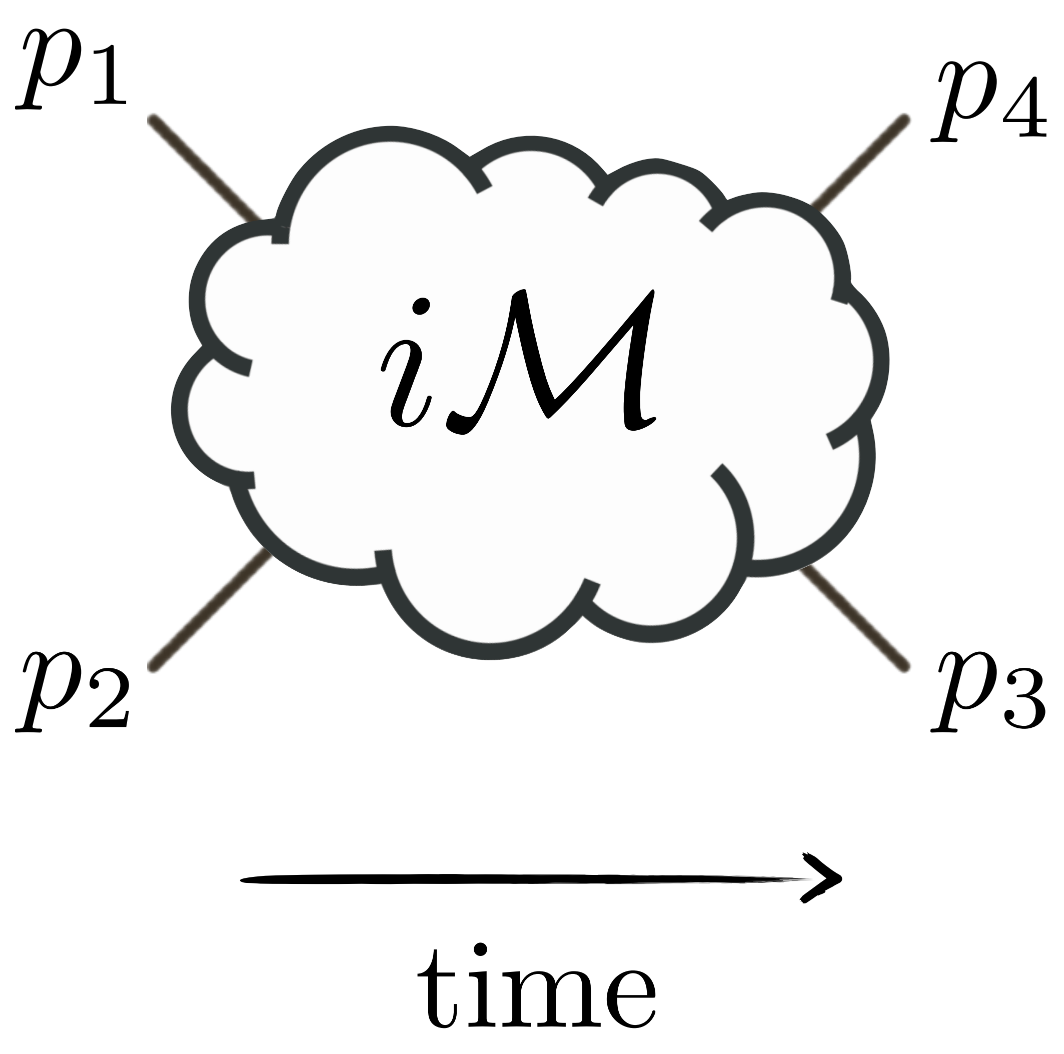
Try it yourself!

Based on [arXiv:2604.20954] & ongoing work with S. Mizera and G. Salvatori

Try it yourself!

While not limited to those integrals, my talk will focus on *Feynman integrals*

Feynman integrals arise in perturbative amplitudes

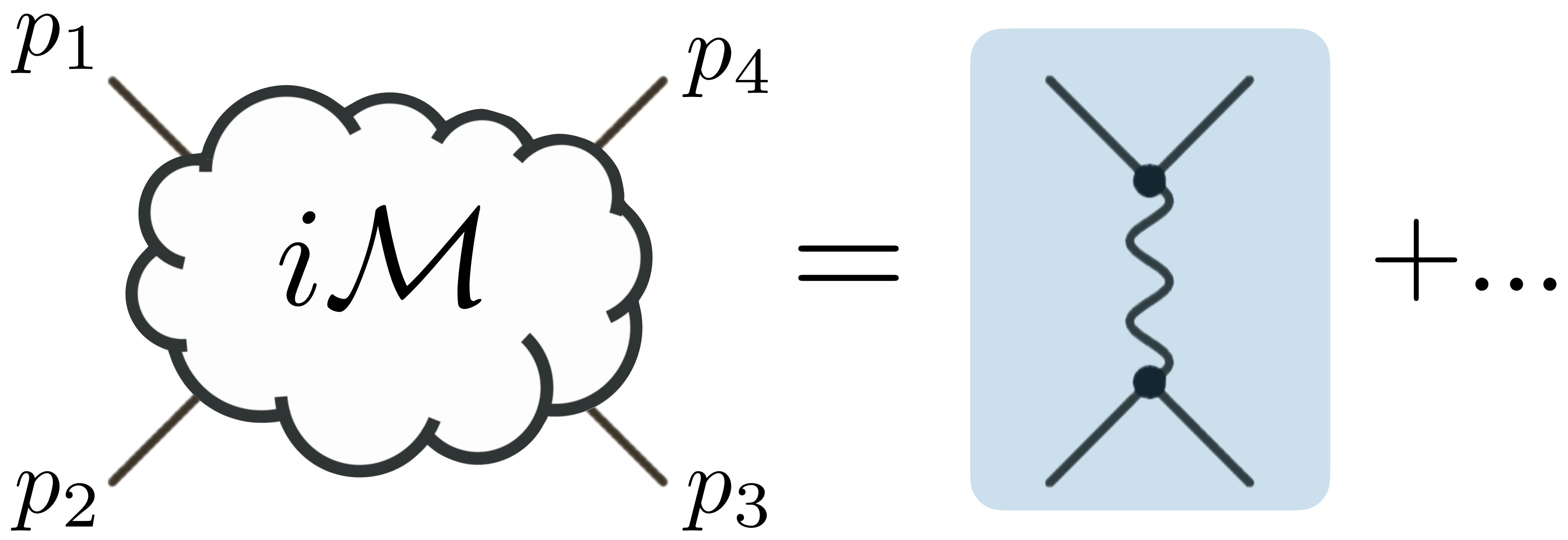


$$S = 1 + i\mathcal{M}$$

Connected S-matrix elements

we typically organize them as sums of *Feynman diagrams* illustrating all that could happen in a process

Arise in perturbative amplitudes organized as a sum of Feynman diagrams allowed by a theory

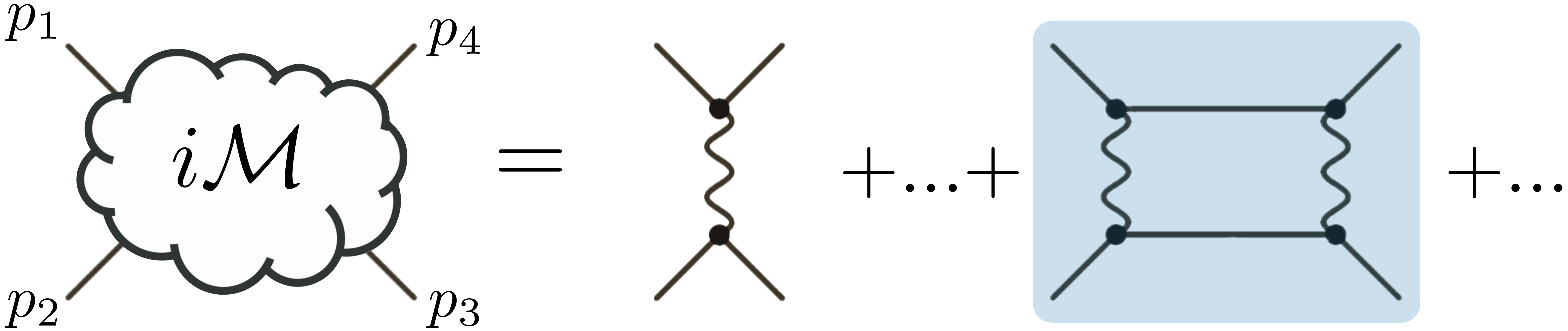


Tree-level: “simple” rational function

$$\left[\bar{u}(-p_1 - p_2 - p_3)(-ie\gamma^{\mu})u(p_1)\right] \frac{-i\,\eta_{\mu\nu}}{(p_2 + p_3)^2 + i\epsilon} \left[\bar{u}(p_3)(-ie\gamma^{\nu})u(p_2)\right]$$

$(+, -, \ldots, -)$

Arise in perturbative amplitudes organized as a sum of Feynman diagrams allowed by a theory

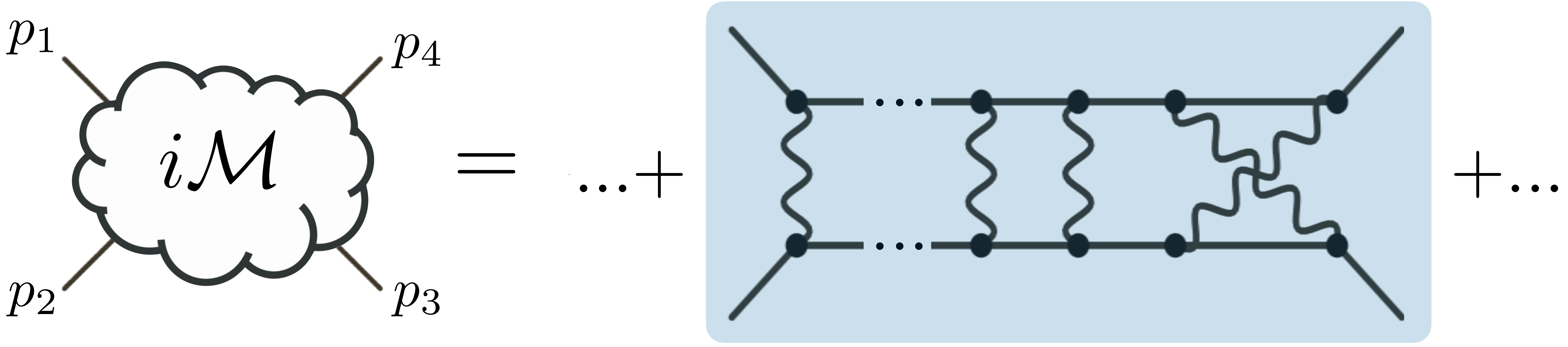


One-loop: more complicated integral of a rational function of the unconstrained loop momentum

$$-ie^4\int \mathrm{d}^{4-2\varepsilon}\ell\,\frac{\left[\bar{u}(-p_1-p_2-p_3)\,\gamma^\nu(\not{\ell}+\not{p}_1+\not{p}_2+\not{p}_3+m)\,\gamma^\mu\,u(p_1)\right]\left[\bar{u}(p_2)\,\gamma_\mu(\not{\ell}+\not{p}_3+m)\,\gamma_\nu\,u(p_3)\right]}{\left[(\ell+p_1+p_2+p_3)^2-m^2+i\epsilon\right]\left[(\ell+p_3)^2-m^2+i\epsilon\right]\left[(\ell+p_2+p_3)^2+i\epsilon\right]\left[\ell^2+i\epsilon\right]}$$

$(+,-,\ldots,-)$

Arise in perturbative amplitudes organized as a sum of Feynman diagrams allowed by a theory

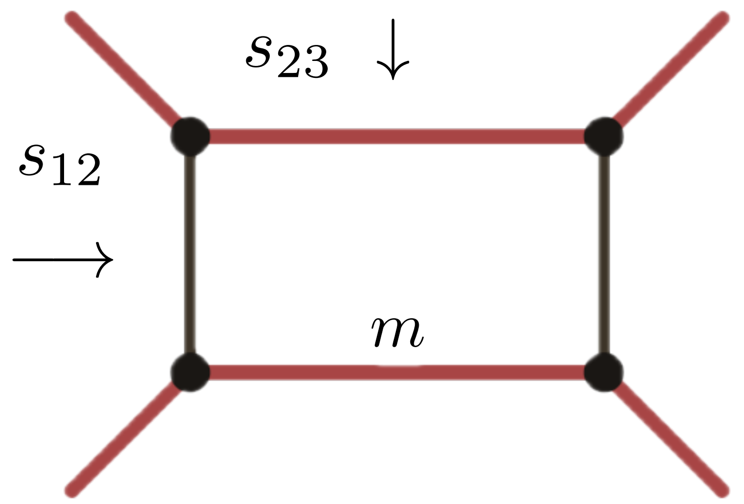


L-loop: comes extremely difficult integrals to evaluate…

$$G \xrightarrow{\text{Feynman rules}} I_G(z) = \int \frac{N(\ell,p)\,\mathrm{d}^{\mathrm{D}\times L}\ell}{\prod_{e=1}^{E}[q_e^2(p,\ell)-m_e^2+i\varepsilon]}$$

Integration over *L non-integer* dimensional vectors

What does "evaluate" means ?



Scalar *massive box*

$$= \mathrm{e}^{\varepsilon\gamma_E}\int\frac{\mathrm{d}^{4-2\varepsilon}\ell}{i\pi^{2-\varepsilon}}\frac{1}{\left[(\ell+p_2+p_3)^2\right]\left[(\ell+p_3)^2-m^2\right]\left[\ell^2\right]\left[(\ell+p_1+p_2+p_3)^2-m^2\right]}$$

It means: represent the integral as a Laurent series in the external kinematic up to the finite piece*

$$\frac{\log\left(-W_1^+\right)-\log\left(-W_1^-\right)}{\varepsilon m^2 s_{23}\left(W_1^- - W_1^+\right)}+\frac{\left(\log(m^2)-\log\left(-\frac{m^2}{s_{23}}\right)\right)\left(\log\left(-W_1^-\right)-\log\left(-W_1^+\right)\right)}{m^2 s_{23}\left(W_1^- - W_1^+\right)}+O\left(\varepsilon^1\right)$$

$$s_{i_1,\ldots,i_n}=(p_{i_1}+\ldots+p_{i_n})^2$$

$$W_1^- = \tfrac{1}{2}m^{-2}\left(-\sqrt{-s_{12}\left(4m^2-s_{12}\right)}-2m^2+s_{12}\right)$$

$$W_1^+ = \tfrac{1}{2}m^{-2}\left(+\sqrt{-s_{12}\left(4m^2-s_{12}\right)}-2m^2+s_{12}\right)$$

* e.g., *in the Euclidean region*

Motivating question

How *fantastic* would it be to be able to compute *analytically* those integrals with a *single*

“`Shift + Enter`” ?

(and as a byproduct be able to push the state-of-the art)

That’s what we are aiming for! Possible user inputs

Literally just draw the diagram (demo soon)

```
STIntegrate[]
```

List of edges and external legs

```
STIntegrate[{{{1,2},0}, {{2,3},0}, {{3,4},0}, {{4,1},0},
             {{1,0}, {2,0}, {3,0}, {4,0}}}]
```

Loop momentum form

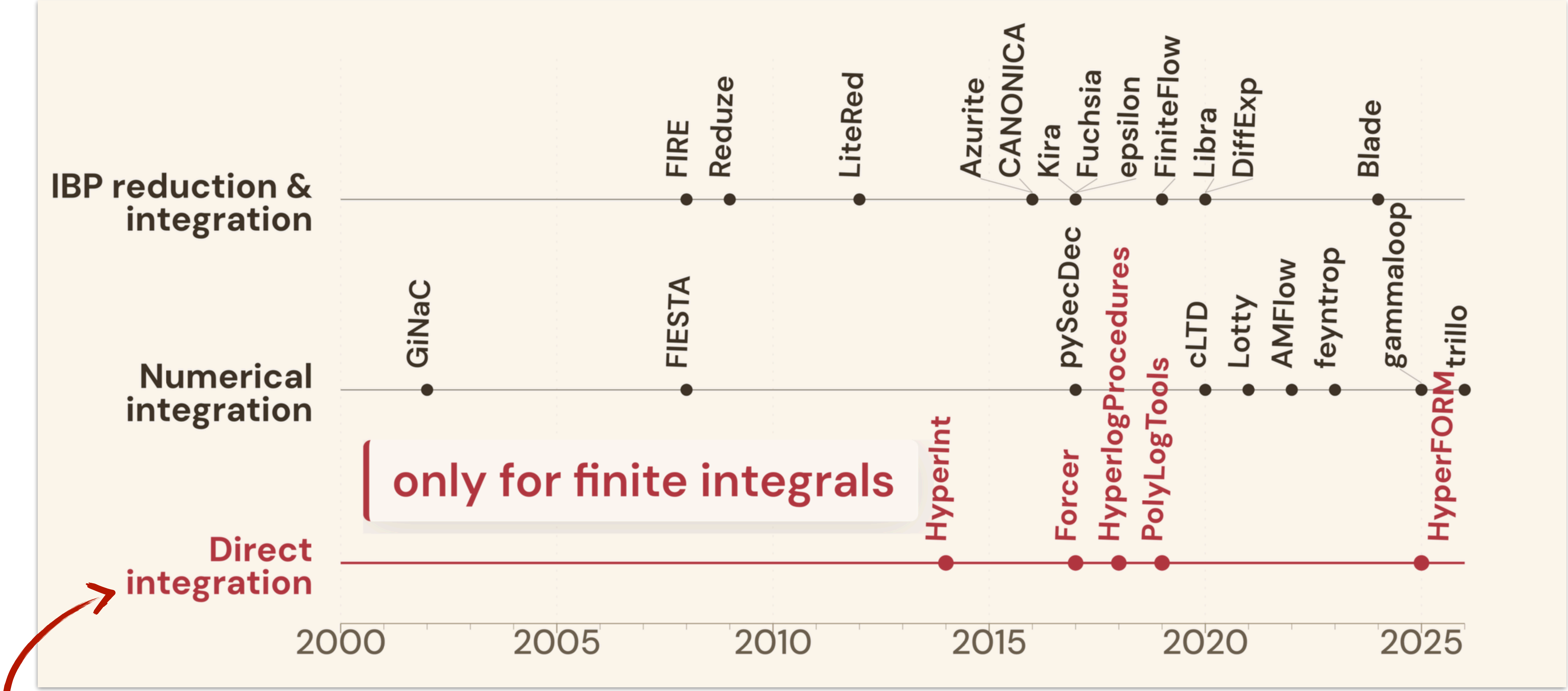
```
STIntegrate[{l[1]^2, (l[1]+p[1])^2, (l[1]+p[1]+p[2])^2, (l[1]-p[4])^2}]
```

Your favorite Euler integral

```
STIntegrate[x^(1+eps) y^eps (x^2+y+xy)^(-2-eps), {x, 0, ∞}, {y, 0, ∞}]
```

then `Shift + Enter`

The toolbox as of today & and which direction is more geared toward automation ?



only for finite integrals

`SubTropica` will solve this

*What are the possible roadblocks to **automated** direct symbolic integration?*

⚠️ Divergences require to work in $D = 4 - 2\varepsilon$, where $0 < |\varepsilon| \ll 1$

💭 Make sense of integrating non-integer dimensional vectors ?

💭 How do we deal & organize divergences at $\varepsilon = 0$?

⚠️ A priori no control over the zoo of functions these evaluate to

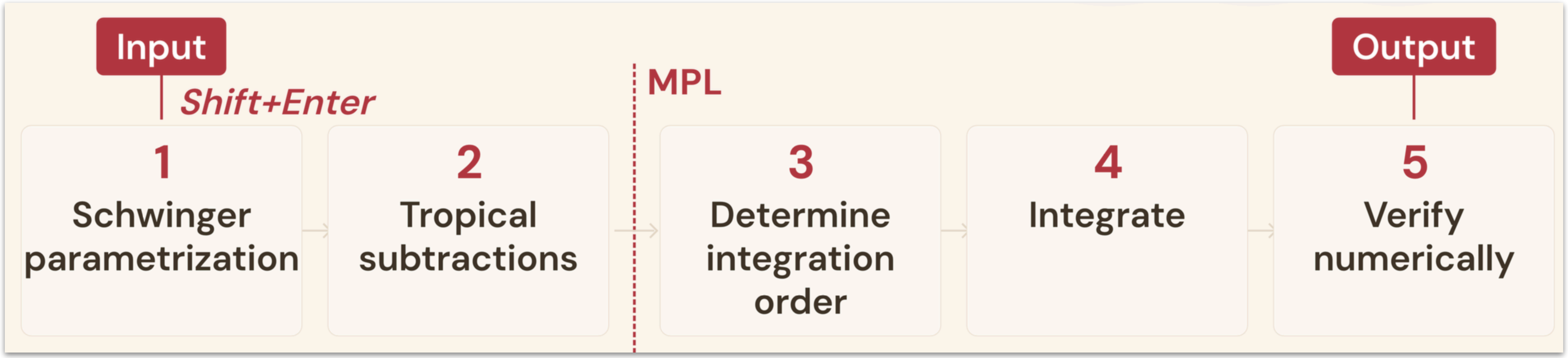
💭 Which restriction(s) on the function space to impose?

The development of `SubTropica` *necessitated a (partial) answer to all these questions*

* *Ask me later for the details*

Summary of the `SubTropica` *pipeline*

Each main step of the program takes care separately of each roadblock



Input

Shift+Enter

MPL

Output

1 Schwinger parametrization

2 Tropical subtractions

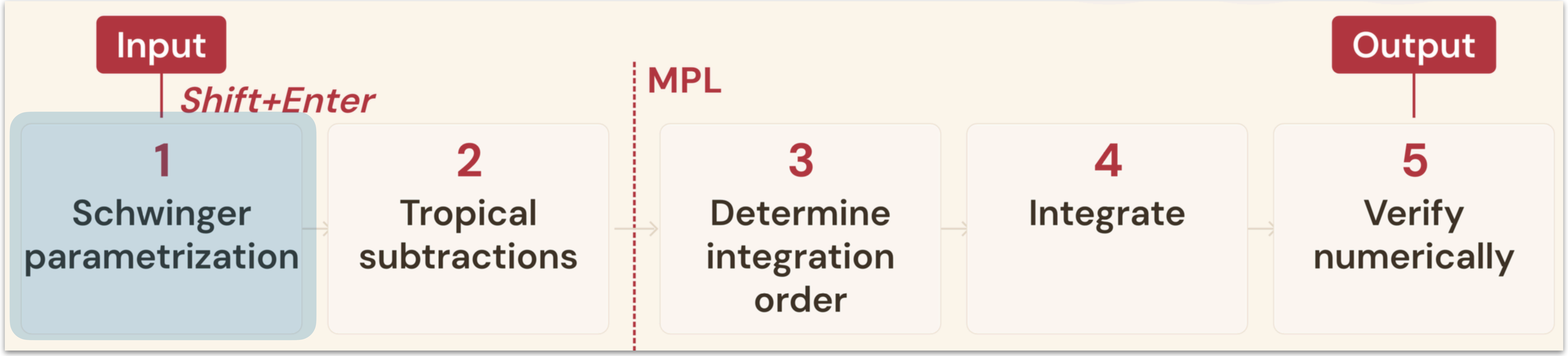
3 Determine integration order

4 Integrate

5 Verify numerically

Summary of the `SubTropica` *pipeline*

💭 Make sense of integrating non-integer dimensional vectors ?



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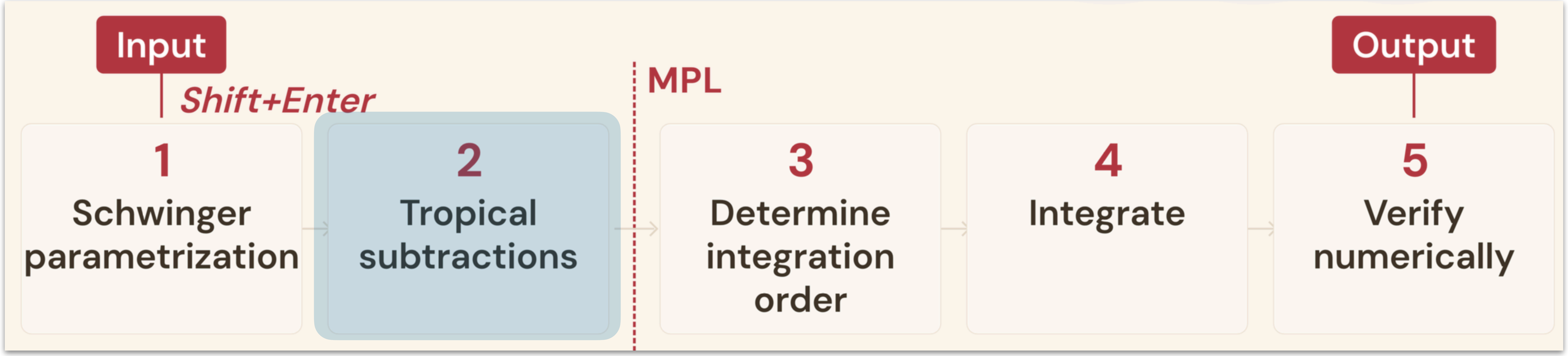
$$\frac{i}{(q_e^2 - m_e^2 + i\varepsilon)^{\nu_e}} = \frac{1}{\Gamma(\nu_e)} \int_0^\infty \mathrm{d}\alpha_e \, \alpha_e^{\nu_e - 1} \, \exp\left[i(q_e^2 - m_e^2 + i\varepsilon)\alpha_e\right]$$

* *Ask me later for the details*

Summary of the `SubTropica` *pipeline*

💭 How do we deal & organize divergences at $\varepsilon = 0$?



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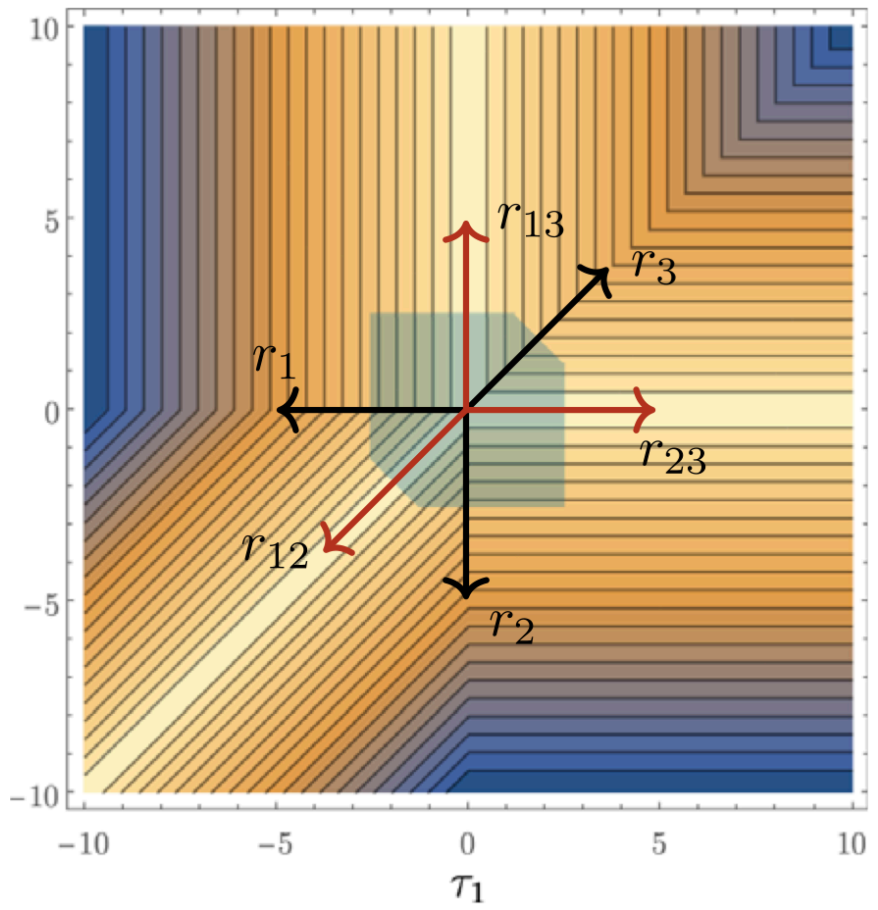
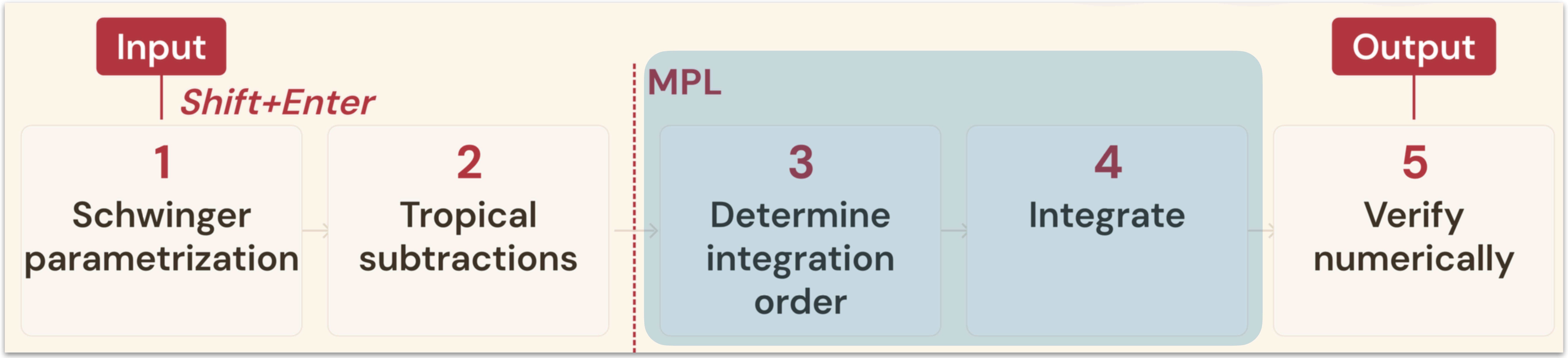


“ Controlled by the *Newton polytope* ”
of the Schwinger integrand
(i.e., Symanzik polynomials)

* *Ask me later for the details*

Summary of the `SubTropica` *pipeline*

💭 Which restriction(s) on the function space to impose?

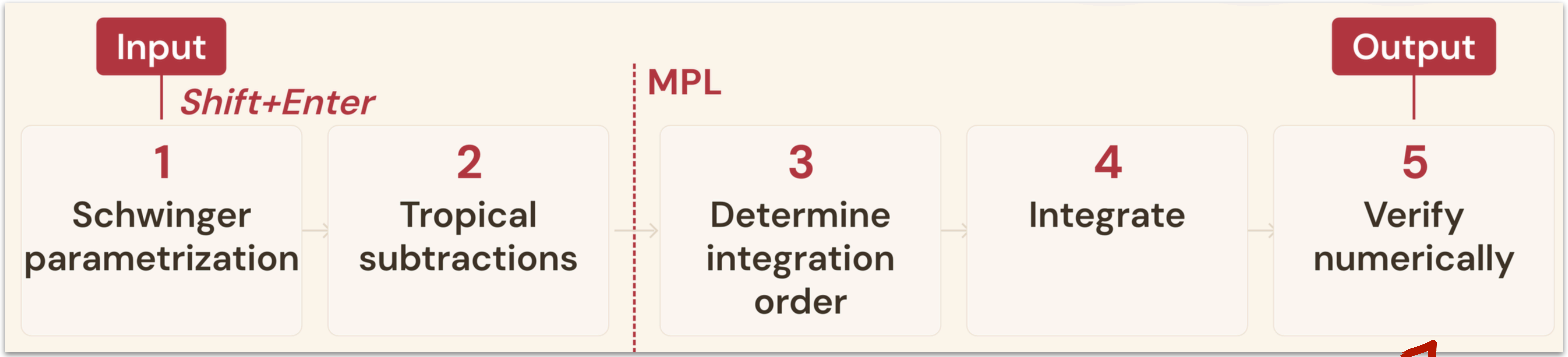


$$\mathcal{L}_{x_1} = \bigcup_{P\in\mathcal{L}} \{\mathrm{Disc}_{x_1}(P)\} \cup \bigcup_{P,Q\in\mathcal{L}} \{\mathrm{Res}_{x_1}(P,Q)\}$$

$$\mathrm{H}(z;\sigma_1,\ldots,\sigma_w) \equiv \int_0^z \frac{\mathrm{d}t_1}{t_1-\sigma_1}\int_0^{t_1}\frac{\mathrm{d}t_2}{t_2-\sigma_2}\cdots\int_0^{t_{w-1}}\frac{\mathrm{d}t_w}{t_w-\sigma_w}, \quad \mathrm{H}(z;\varnothing)=1$$

* *Ask me later for the details*

Summary of the `SubTropica` *pipeline*



Input

Shift+Enter

MPL

Output

1. Schwinger parametrization
2. Tropical subtractions
3. Determine integration order
4. Integrate
5. Verify numerically

`SubTropica` *can optionally stress-test the symbolic output by comparing independently with* `PySecDec` *at a point*

How to use `SubTropica`

and how it can accelerate your research?

Literally just draw the diagram (demo soon)

```
STIntegrate[]
```

List of edges and external legs

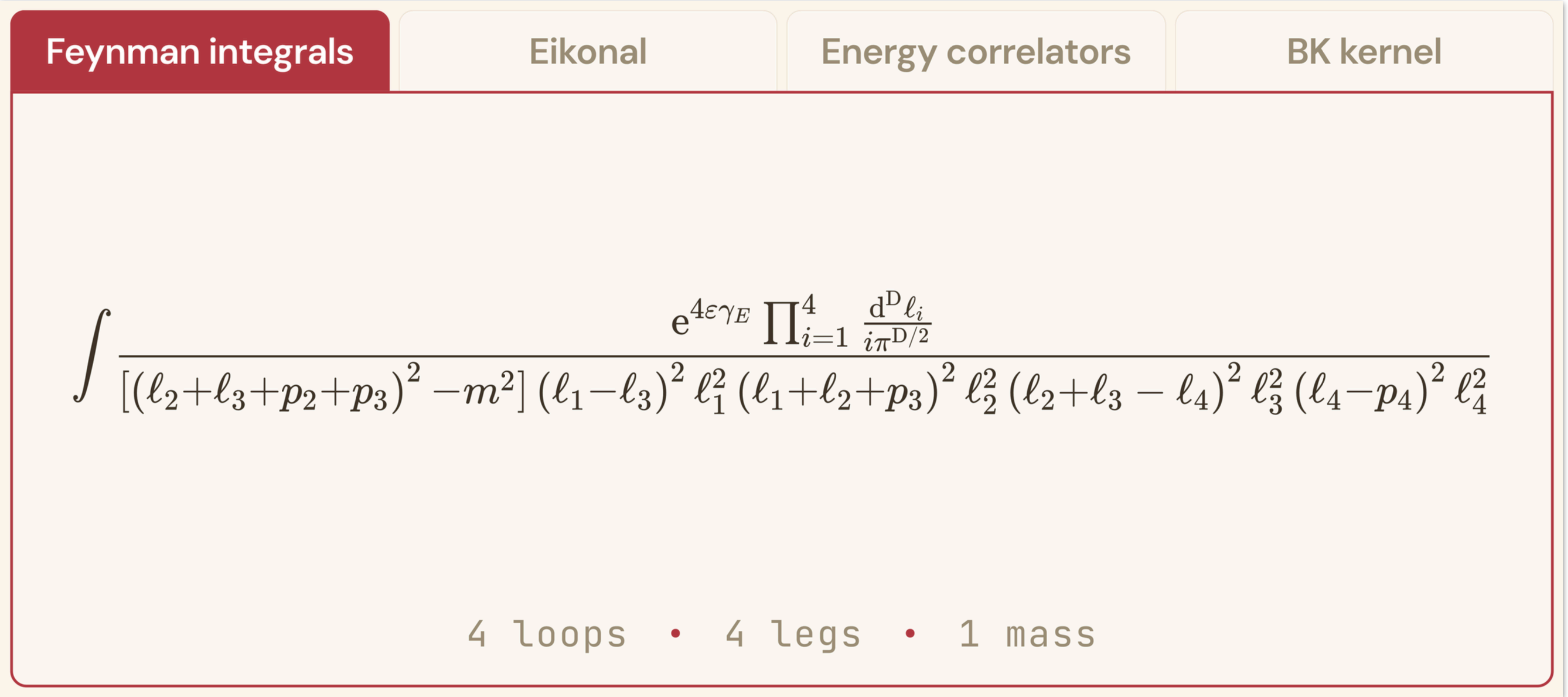
```
STIntegrate[{{{1,2},0}, {{2,3},0}, {{3,4},0}, {{4,1},0},
             {{1,0}, {2,0}, {3,0}, {4,0}}}]
```

Loop momentum form

```
STIntegrate[{l[1]^2, (l[1]+p[1])^2, (l[1]+p[1]+p[2])^2, (l[1]-p[4])^2}]
```

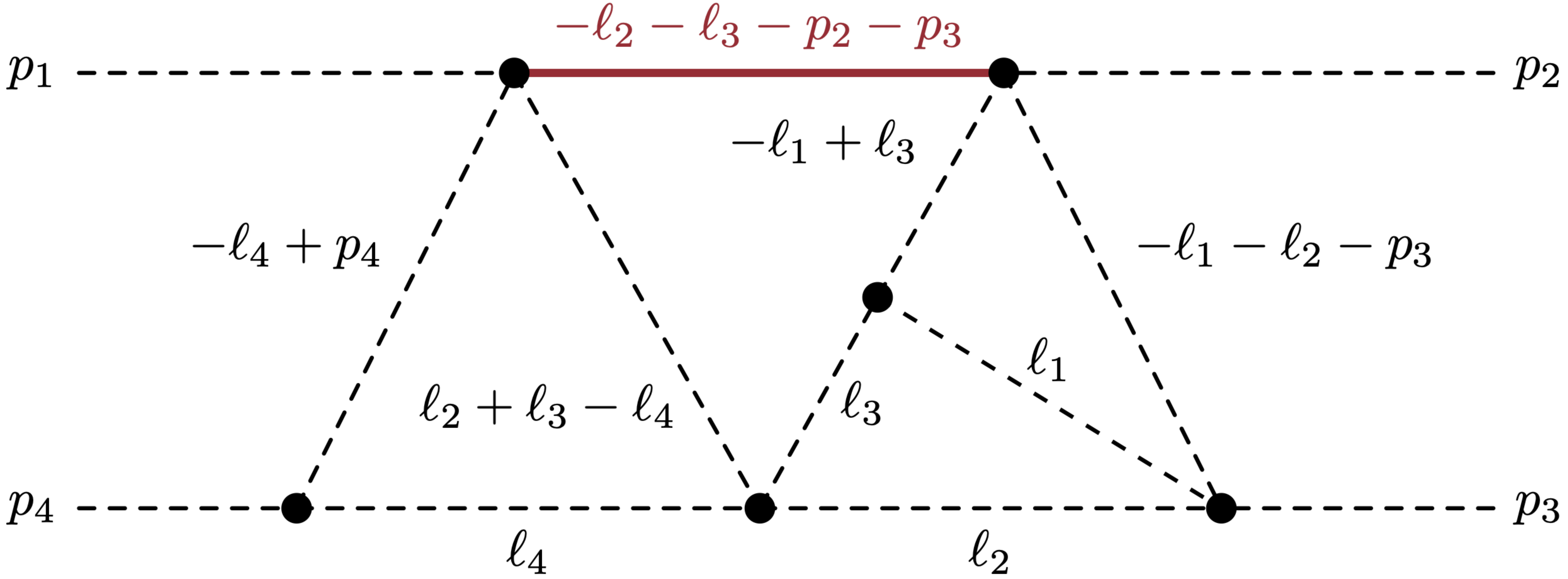
Your favorite Euler integral

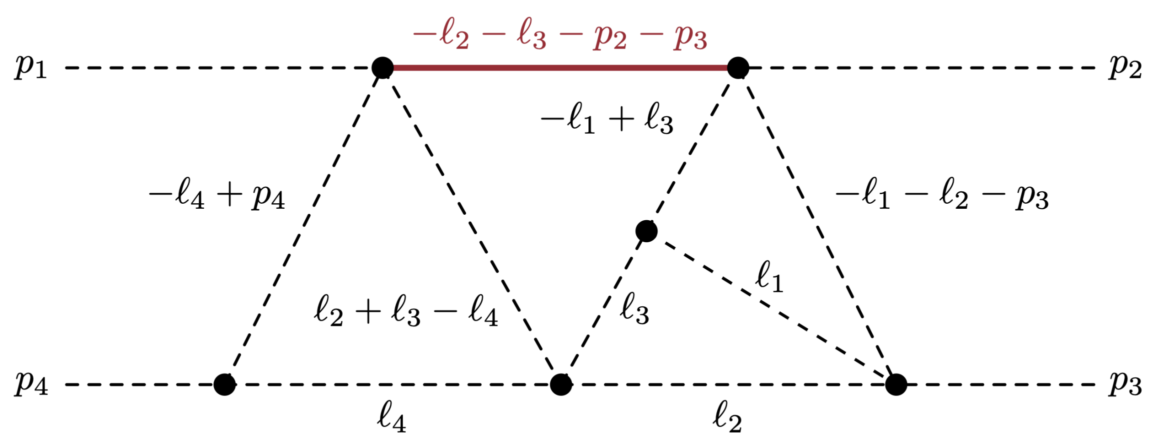
```
STIntegrate[x^(1+eps) y^eps (x^2+y+xy)^(-2-eps), {x, 0, ∞}, {y, 0, ∞}]
```

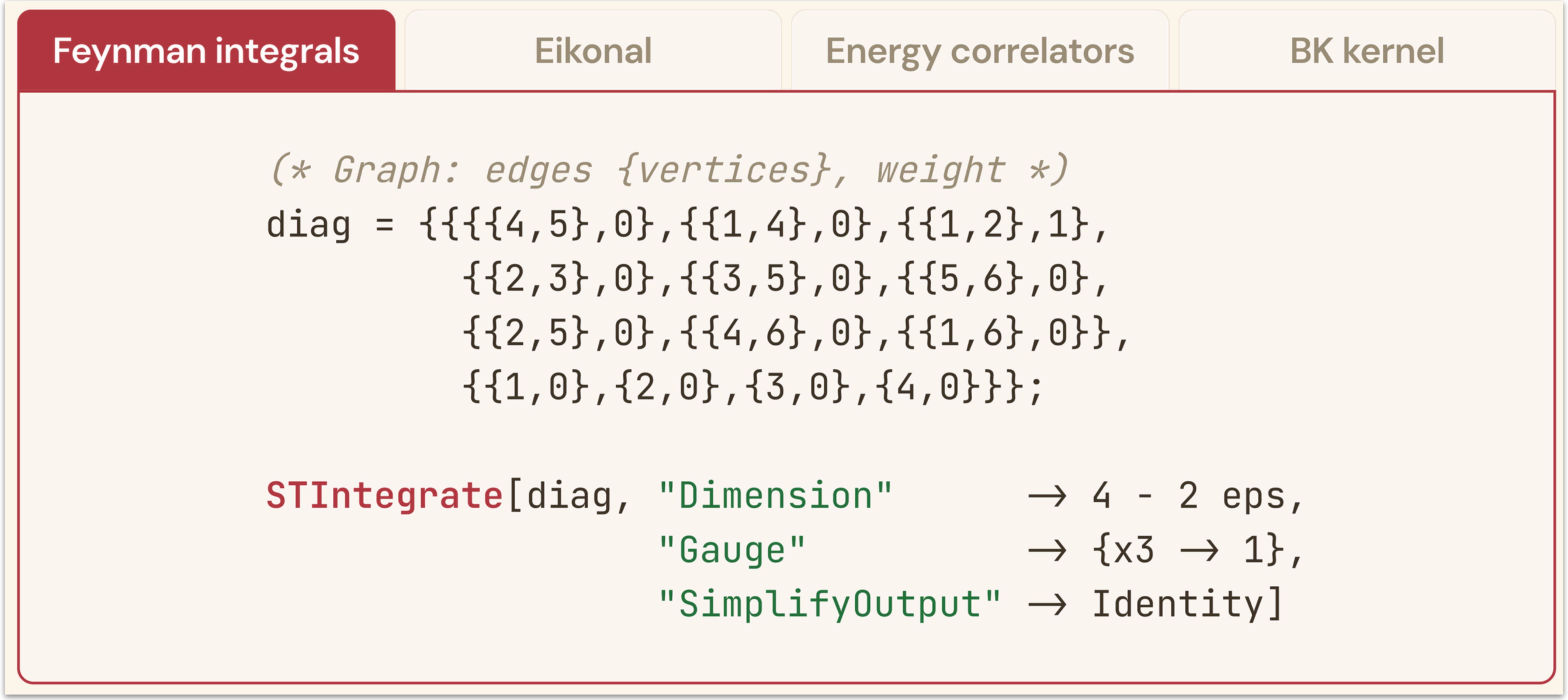


$$\int \frac{\mathrm{e}^{4\varepsilon\gamma_E}\prod_{i=1}^{4}\frac{\mathrm{d}^{\mathrm{D}}\ell_i}{i\pi^{\mathrm{D}/2}}}{\left[(\ell_2+\ell_3+p_2+p_3)^2-m^2\right](\ell_1-\ell_3)^2\,\ell_1^2\,(\ell_1+\ell_2+p_3)^2\,\ell_2^2\,(\ell_2+\ell_3-\ell_4)^2\,\ell_3^2\,(\ell_4-p_4)^2\,\ell_4^2}$$

4 loops • 4 legs • 1 mass

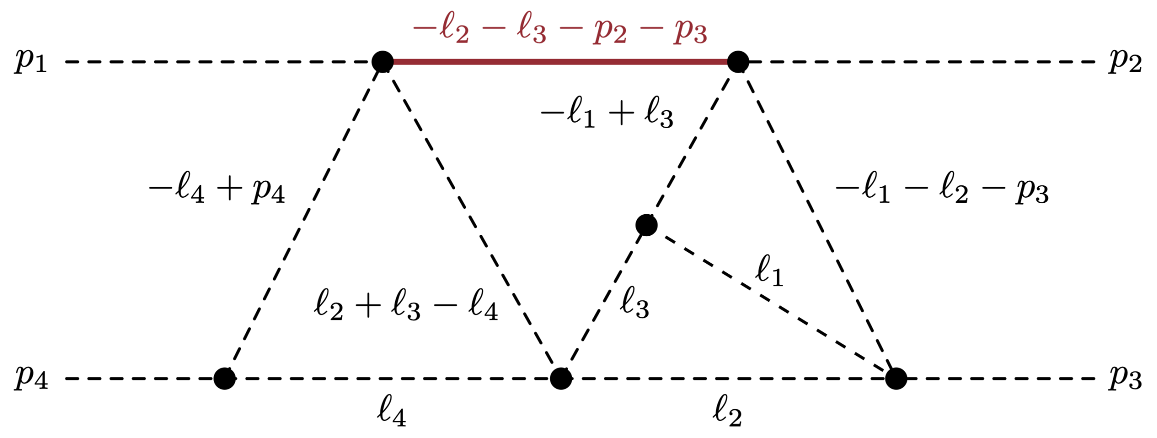

$-\ell_2-\ell_3-p_2-p_3$
p_1
p_2
$-\ell_1+\ell_3$
$-\ell_4+p_4$
$-\ell_1-\ell_2-p_3$
ℓ_1
$\ell_2+\ell_3-\ell_4$
ℓ_3
p_4
p_3
ℓ_4
ℓ_2

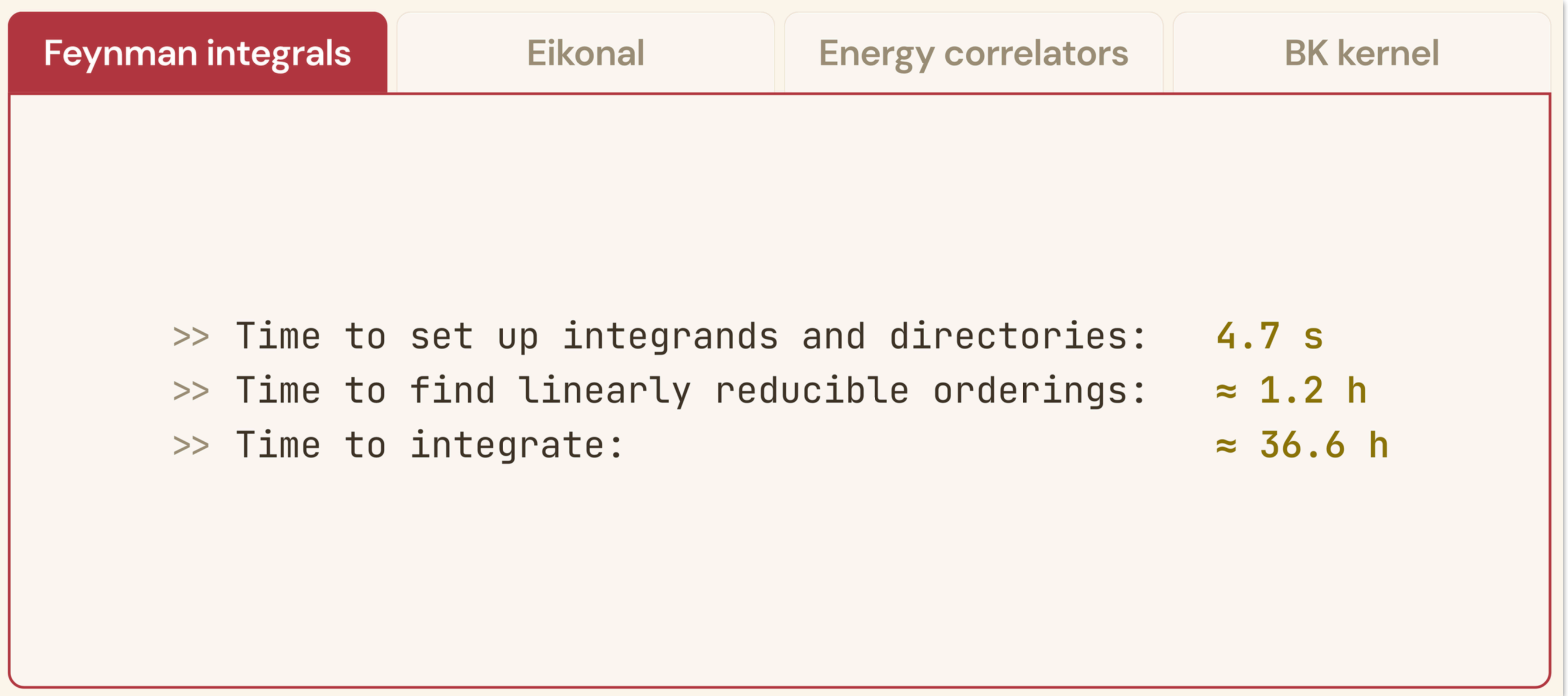




```
(* Graph: edges {vertices}, weight *)
diag = {{{{4,5},0},{{1,4},0},{{1,2},1},
         {{2,3},0},{{3,5},0},{{5,6},0},
         {{2,5},0},{{4,6},0},{{1,6},0}},
         {{1,0},{2,0},{3,0},{4,0}}};

STIntegrate[diag, "Dimension"      → 4 - 2 eps,
                  "Gauge"          → {x3 → 1},
                  "SimplifyOutput" → Identity]
```

then `Shift + Enter`





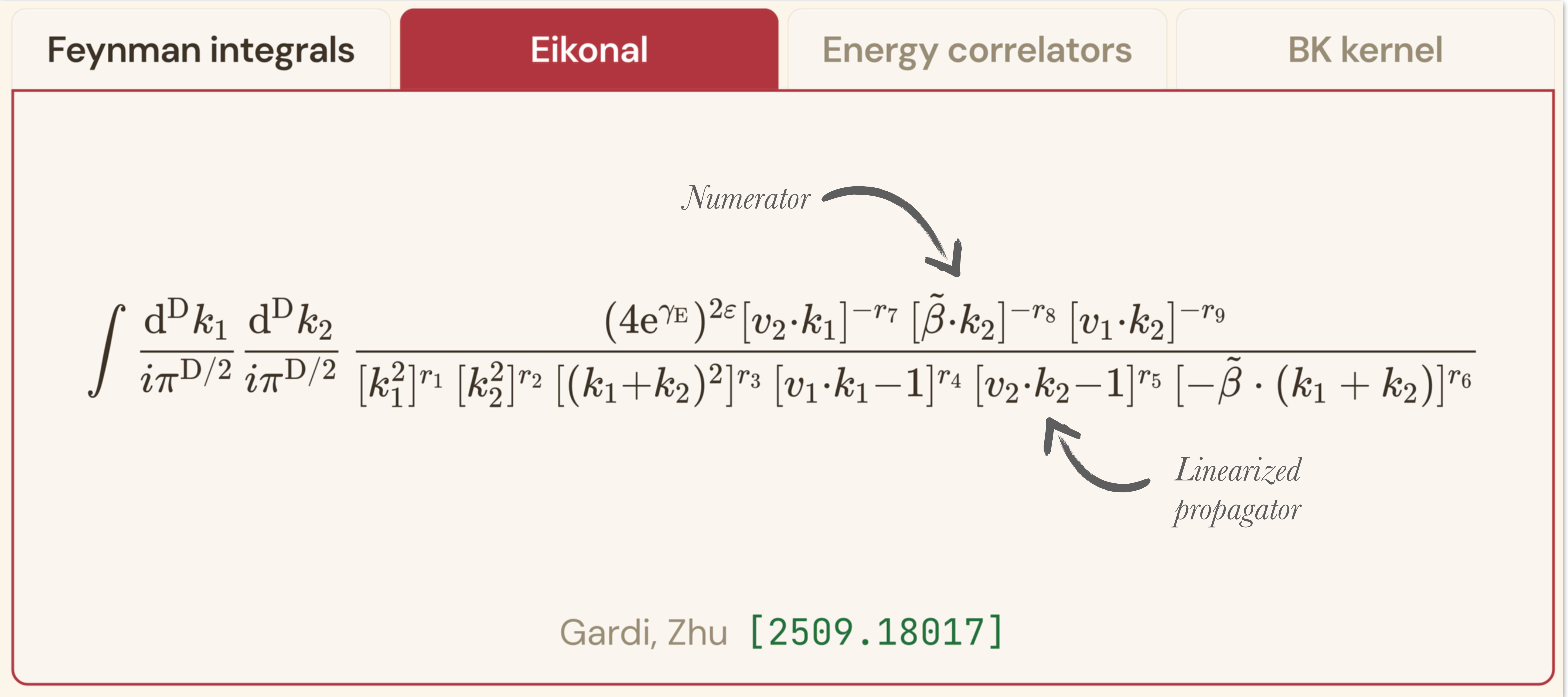
```
>> Time to set up integrands and directories:   4.7 s
>> Time to find linearly reducible orderings:   ≈ 1.2 h
>> Time to integrate:                           ≈ 36.6 h
```

Has not been computed before!

Result in terms of multi-polylogarithms on the GitHub

* *Timings from the April implementation, the unpublished version is now dramatically faster; ask me later for more details*

Integral appearing in calculation of 3-loop soft anomalous dimension

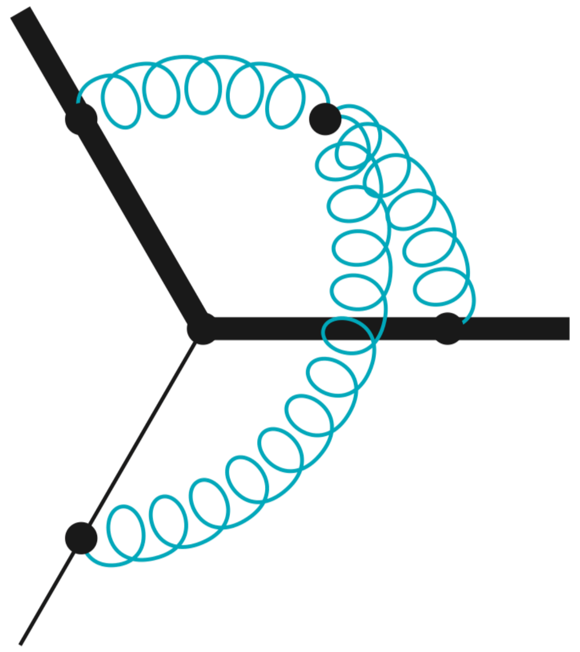


Numerator

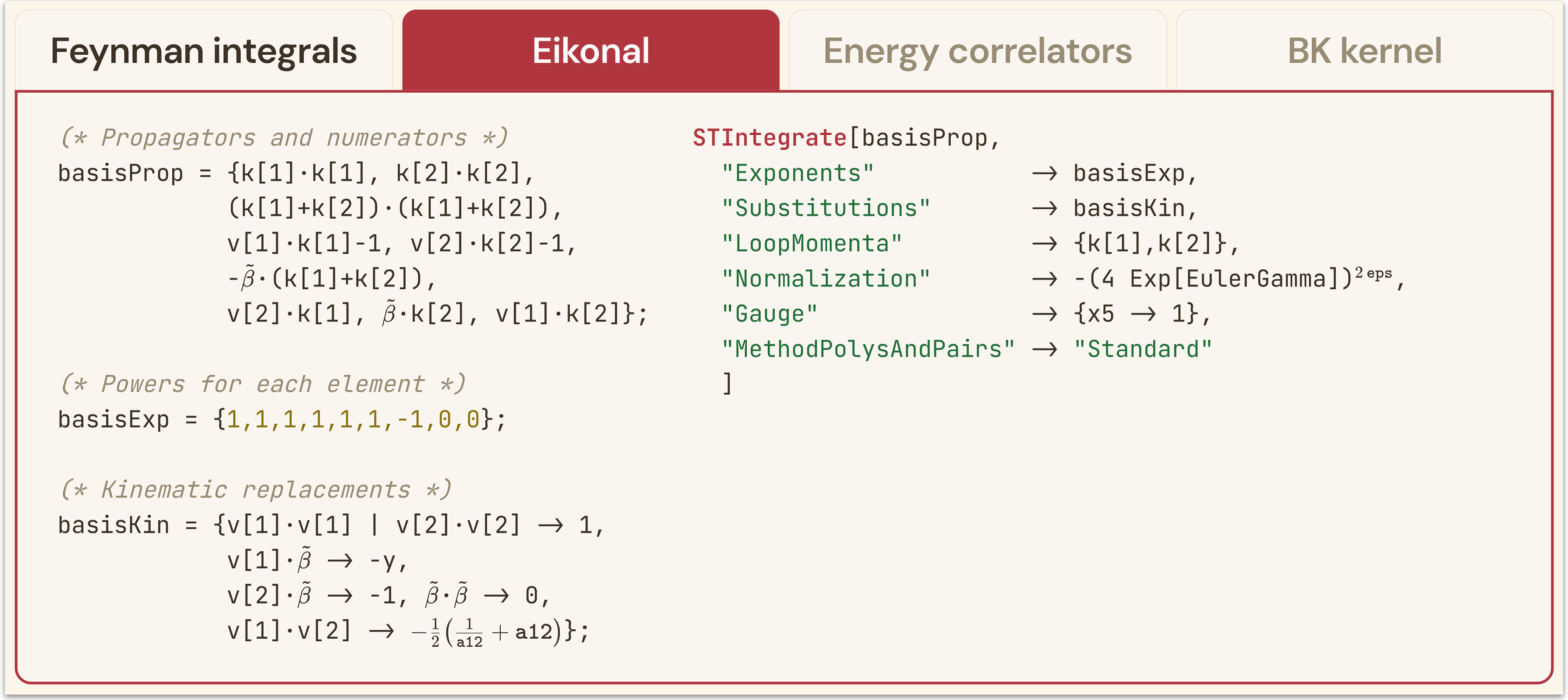
$$\int \frac{\mathrm{d}^{\mathrm{D}} k_1}{i\pi^{\mathrm{D}/2}} \frac{\mathrm{d}^{\mathrm{D}} k_2}{i\pi^{\mathrm{D}/2}} \frac{(4\mathrm{e}^{\gamma_{\mathrm{E}}})^{2\varepsilon}[v_2\cdot k_1]^{-r_7}\,[\tilde{\beta}\cdot k_2]^{-r_8}\,[v_1\cdot k_2]^{-r_9}}{[k_1^2]^{r_1}\,[k_2^2]^{r_2}\,[(k_1+k_2)^2]^{r_3}\,[v_1\cdot k_1-1]^{r_4}\,[v_2\cdot k_2-1]^{r_5}\,[-\tilde{\beta}\cdot(k_1+k_2)]^{r_6}}$$

Linearized propagator

Gardi, Zhu [2509.18017]

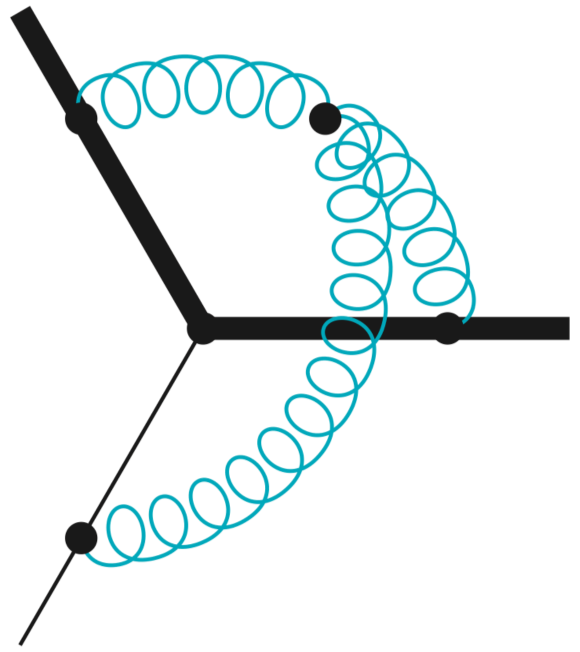


Integral appearing in calculation of 3-loop soft anomalous dimension



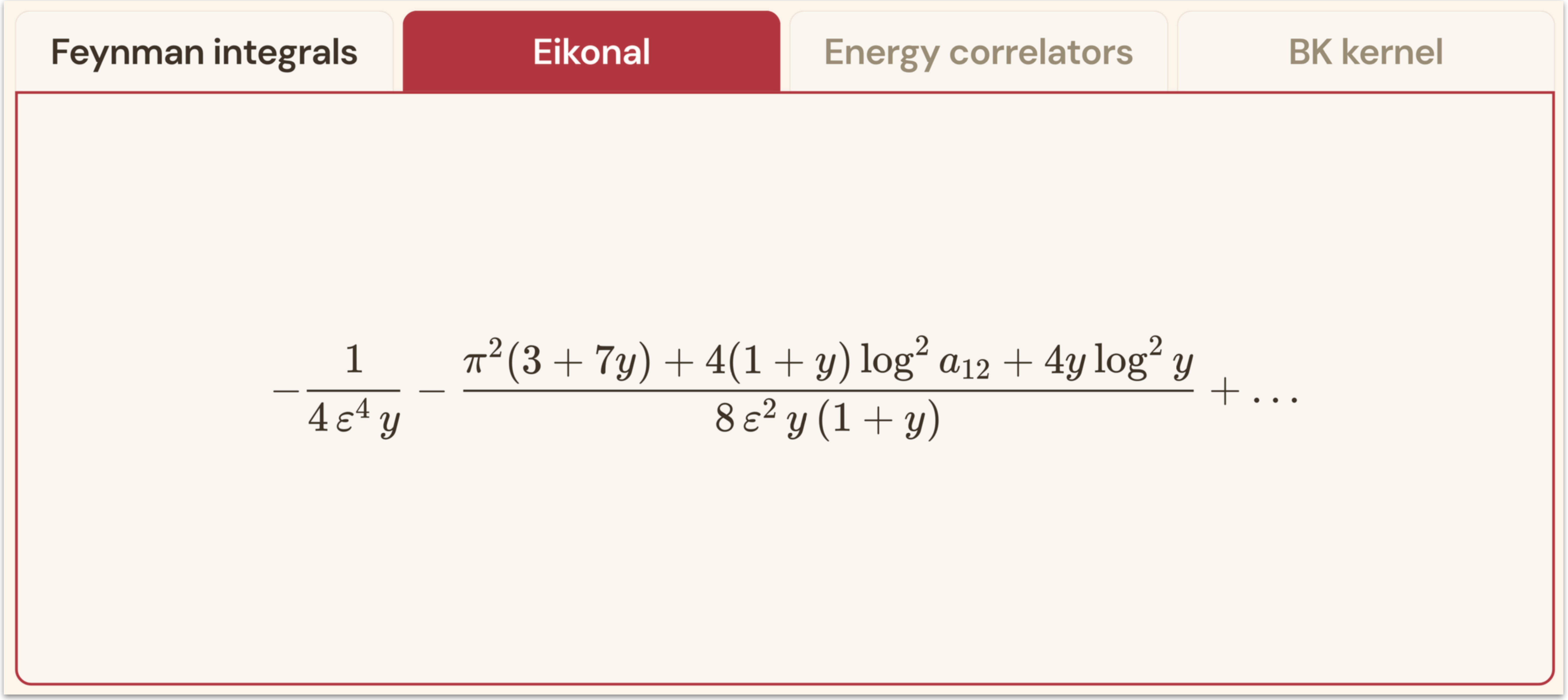
```
(* Propagators and numerators *)
basisProp = {k[1]·k[1], k[2]·k[2],
             (k[1]+k[2])·(k[1]+k[2]),
             v[1]·k[1]-1, v[2]·k[2]-1,
             -β̃·(k[1]+k[2]),
             v[2]·k[1], β̃·k[2], v[1]·k[2]};

(* Powers for each element *)
basisExp = {1,1,1,1,1,1,-1,0,0};

(* Kinematic replacements *)
basisKin = {v[1]·v[1] | v[2]·v[2] → 1,
             v[1]·β̃ → -y,
             v[2]·β̃ → -1, β̃·β̃ → 0,
             v[1]·v[2] → -1/2(1/a12 + a12)};
```

```
STIntegrate[basisProp,
  "Exponents"            → basisExp,
  "Substitutions"        → basisKin,
  "LoopMomenta"          → {k[1],k[2]},
  "Normalization"        → -(4 Exp[EulerGamma])^(2 eps),
  "Gauge"                → {x5 → 1},
  "MethodPolysAndPairs"  → "Standard"
  ]
```

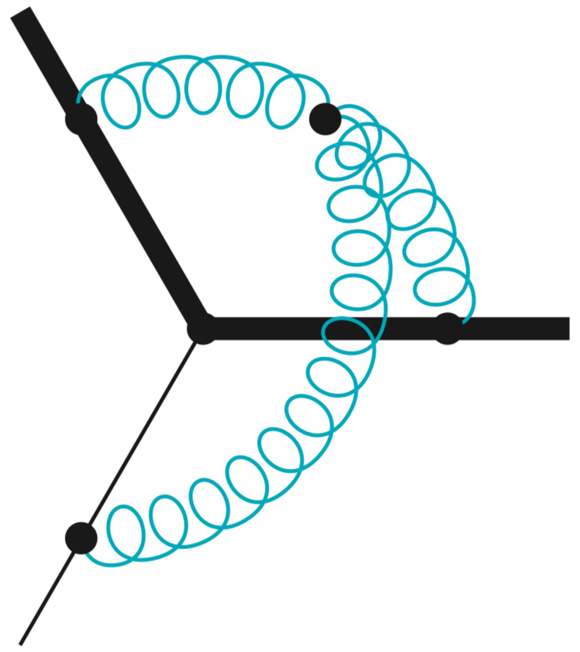


then **`Shift + Enter`**

Integral appearing in calculation of 3-loop soft anomalous dimension

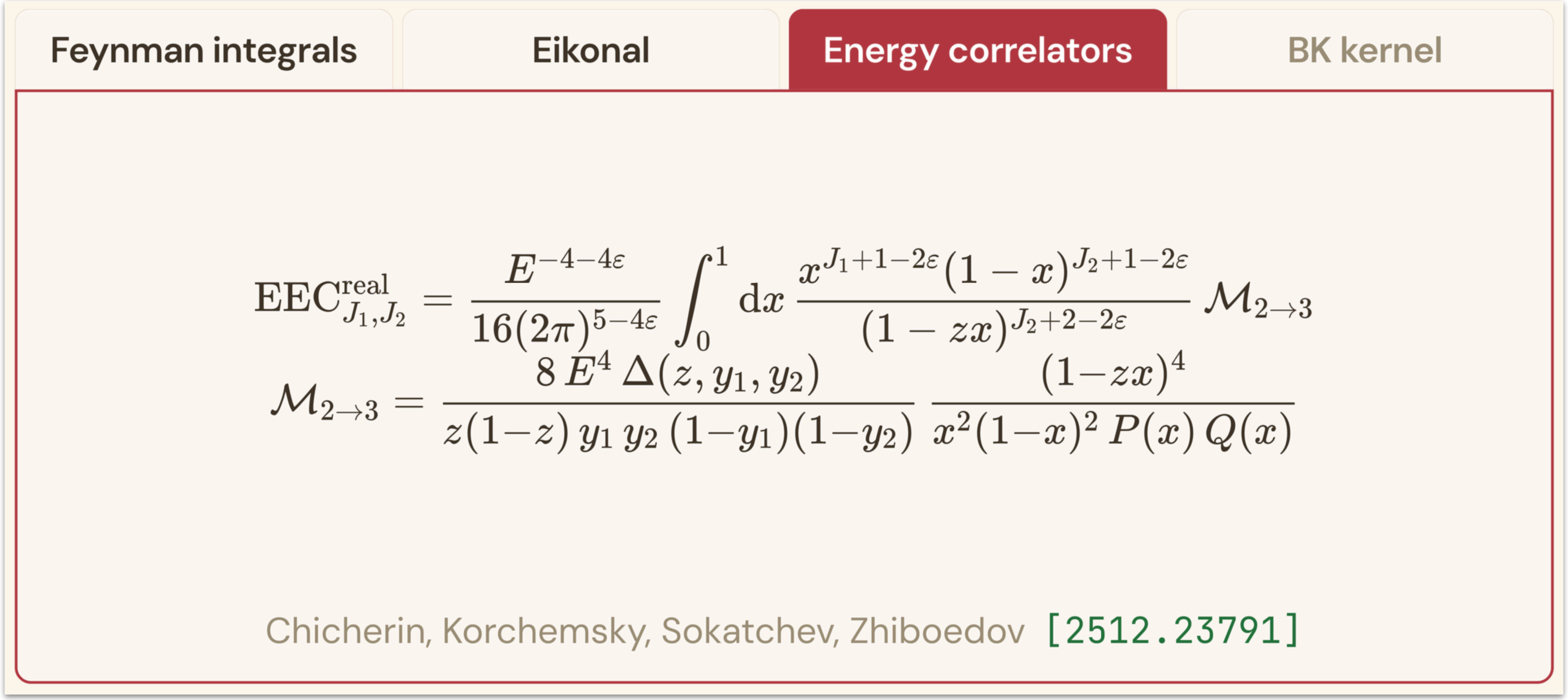


$$-\frac{1}{4\,\varepsilon^4\,y} - \frac{\pi^2(3+7y) + 4(1+y)\log^2 a_{12} + 4y\log^2 y}{8\,\varepsilon^2\,y\,(1+y)} + \ldots$$



Matches exactly known results from differential equations!

Integral appearing in calculation of 1-loop gravitational EEC

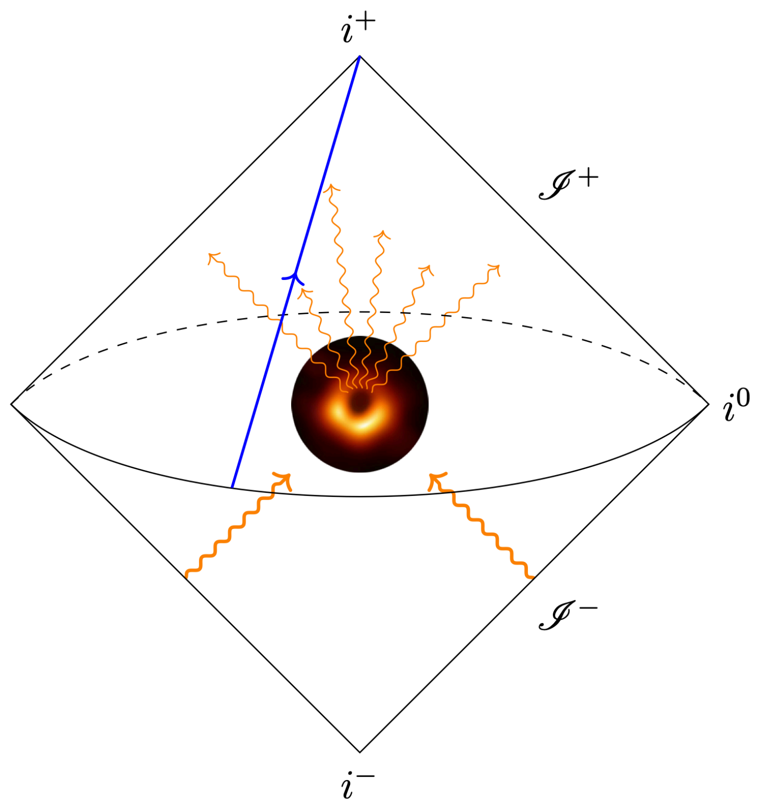


Feynman integrals | **Eikonal** | **Energy correlators** | **BK kernel**

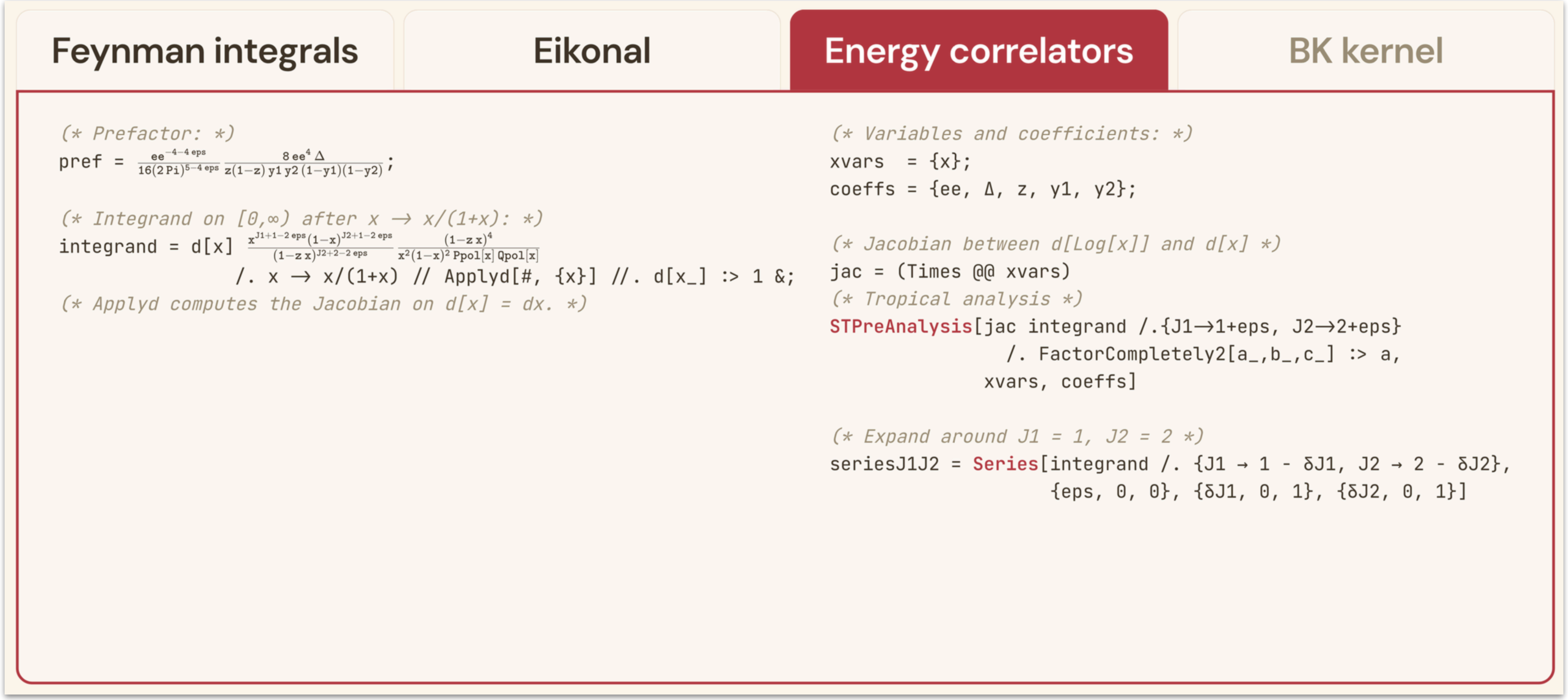
$$\mathrm{EEC}^{\mathrm{real}}_{J_1,J_2} = \frac{E^{-4-4\varepsilon}}{16(2\pi)^{5-4\varepsilon}} \int_0^1 \mathrm{d}x \, \frac{x^{J_1+1-2\varepsilon}(1-x)^{J_2+1-2\varepsilon}}{(1-zx)^{J_2+2-2\varepsilon}} \, \mathcal{M}_{2\to 3}$$

$$\mathcal{M}_{2\to 3} = \frac{8\,E^4\,\Delta(z,y_1,y_2)}{z(1-z)\,y_1\,y_2\,(1-y_1)(1-y_2)} \, \frac{(1-zx)^4}{x^2(1-x)^2\,P(x)\,Q(x)}$$

Chicherin, Korchemsky, Sokatchev, Zhiboedov [2512.23791]



Integral appearing in calculation of 1-loop gravitational EEC



```
(* Prefactor: *)
```

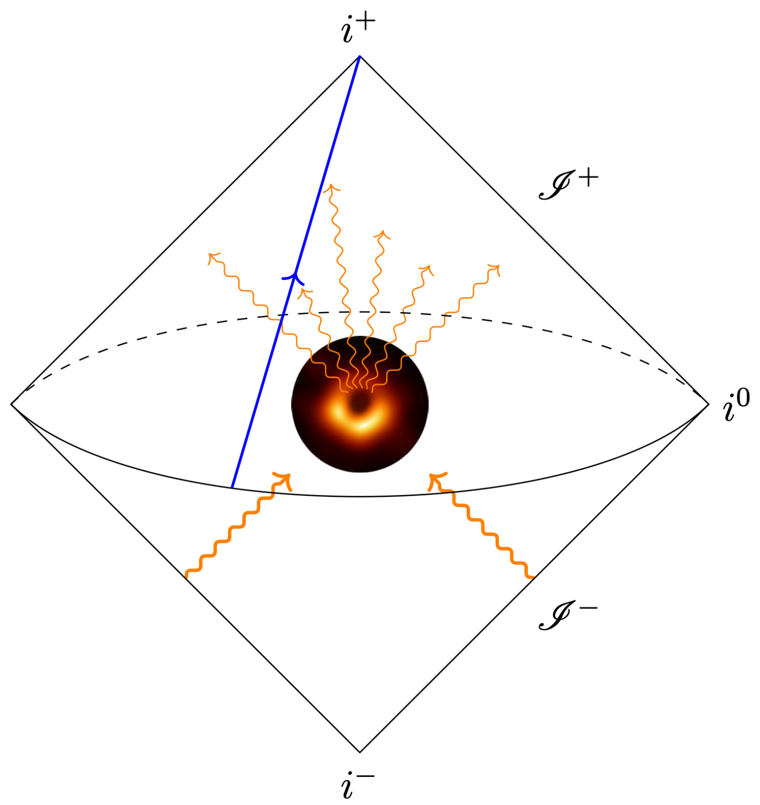
pref = $\frac{ee^{-4-4\,eps}}{16(2\,Pi)^{5-4\,eps}}\frac{8\,ee^{4}\,\Delta}{z(1-z)\,y1\,y2\,(1-y1)(1-y2)}$;

```
(* Integrand on [0,∞) after x ⟶ x/(1+x): *)
```

integrand = d[x] $\frac{x^{J1+1-2\,eps}(1-x)^{J2+1-2\,eps}}{(1-z\,x)^{J2+2-2\,eps}}\frac{(1-z\,x)^{4}}{x^{2}(1-x)^{2}\,Ppol[x]\,Qpol[x]}$

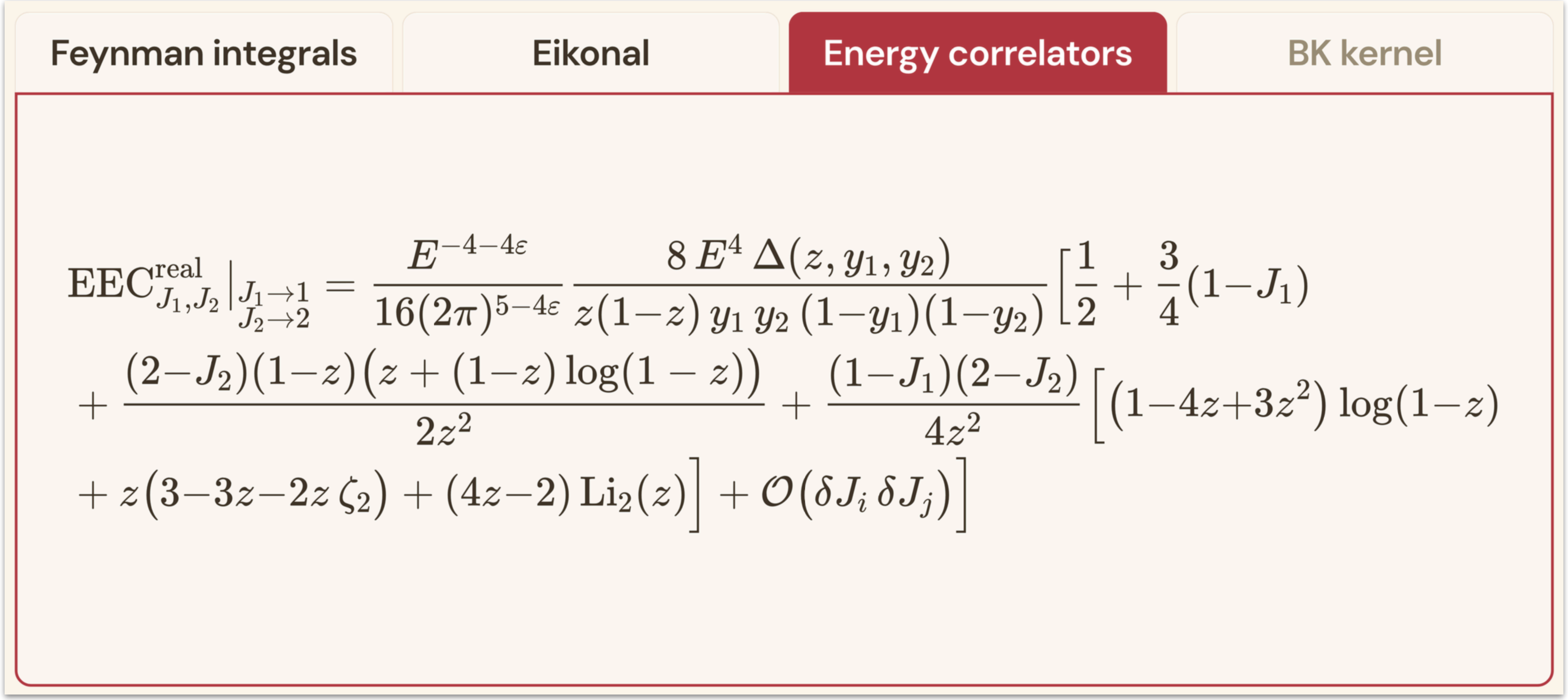
```
            /. x ⟶ x/(1+x) // Applyd[#, {x}] //. d[x_] :> 1 &;
(* Applyd computes the Jacobian on d[x] = dx. *)
```

```
(* Variables and coefficients: *)
xvars  = {x};
coeffs = {ee, Δ, z, y1, y2};

(* Jacobian between d[Log[x]] and d[x] *)
jac = (Times @@ xvars)
(* Tropical analysis *)
STPreAnalysis[jac integrand /.{J1⟶1+eps, J2⟶2+eps}
                  /. FactorCompletely2[a_,b_,c_] :> a,
                xvars, coeffs]

(* Expand around J1 = 1, J2 = 2 *)
seriesJ1J2 = Series[integrand /. {J1 → 1 - δJ1, J2 → 2 - δJ2},
                {eps, 0, 0}, {δJ1, 0, 1}, {δJ2, 0, 1}]
```



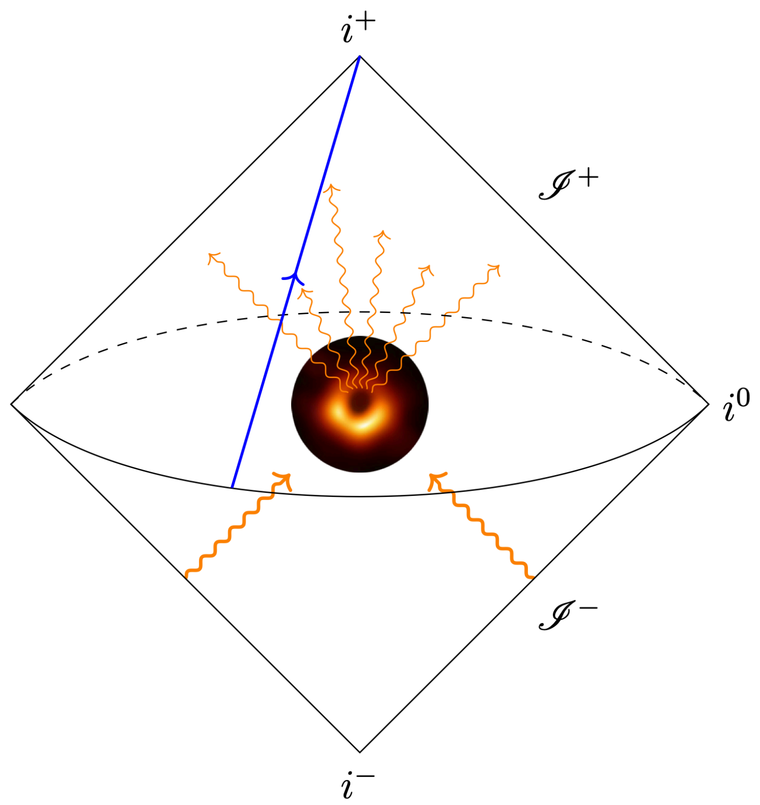
then **`Shift + Enter`**

Integral appearing in calculation of 1-loop gravitational EEC



Feynman integrals | Eikonal | **Energy correlators** | BK kernel

$$\mathrm{EEC}^{\mathrm{real}}_{J_1,J_2}\Big|_{\substack{J_1\to 1\\ J_2\to 2}} = \frac{E^{-4-4\varepsilon}}{16(2\pi)^{5-4\varepsilon}}\,\frac{8\,E^4\,\Delta(z,y_1,y_2)}{z(1-z)\,y_1\,y_2\,(1-y_1)(1-y_2)}\Bigg[\frac{1}{2}+\frac{3}{4}(1-J_1)$$
$$+\frac{(2-J_2)(1-z)\big(z+(1-z)\log(1-z)\big)}{2z^2}+\frac{(1-J_1)(2-J_2)}{4z^2}\Big[\big(1-4z+3z^2\big)\log(1-z)$$
$$+z\big(3-3z-2z\,\zeta_2\big)+(4z-2)\,\mathrm{Li}_2(z)\Big]+\mathcal{O}\big(\delta J_i\,\delta J_j\big)\Bigg]$$

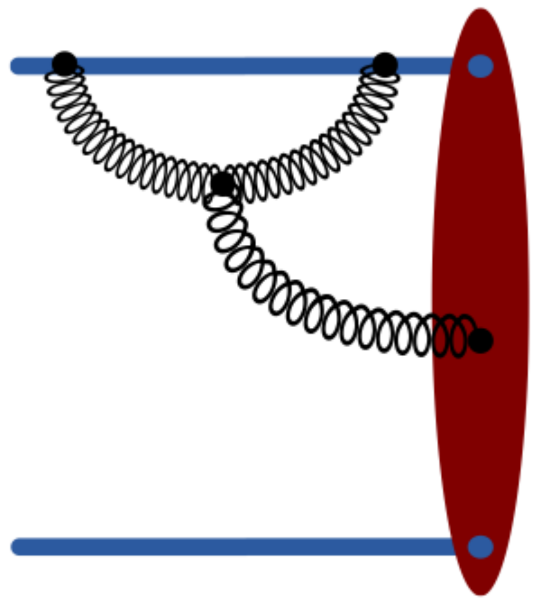


Integral appearing in the NNLO hadron rapidity evolution

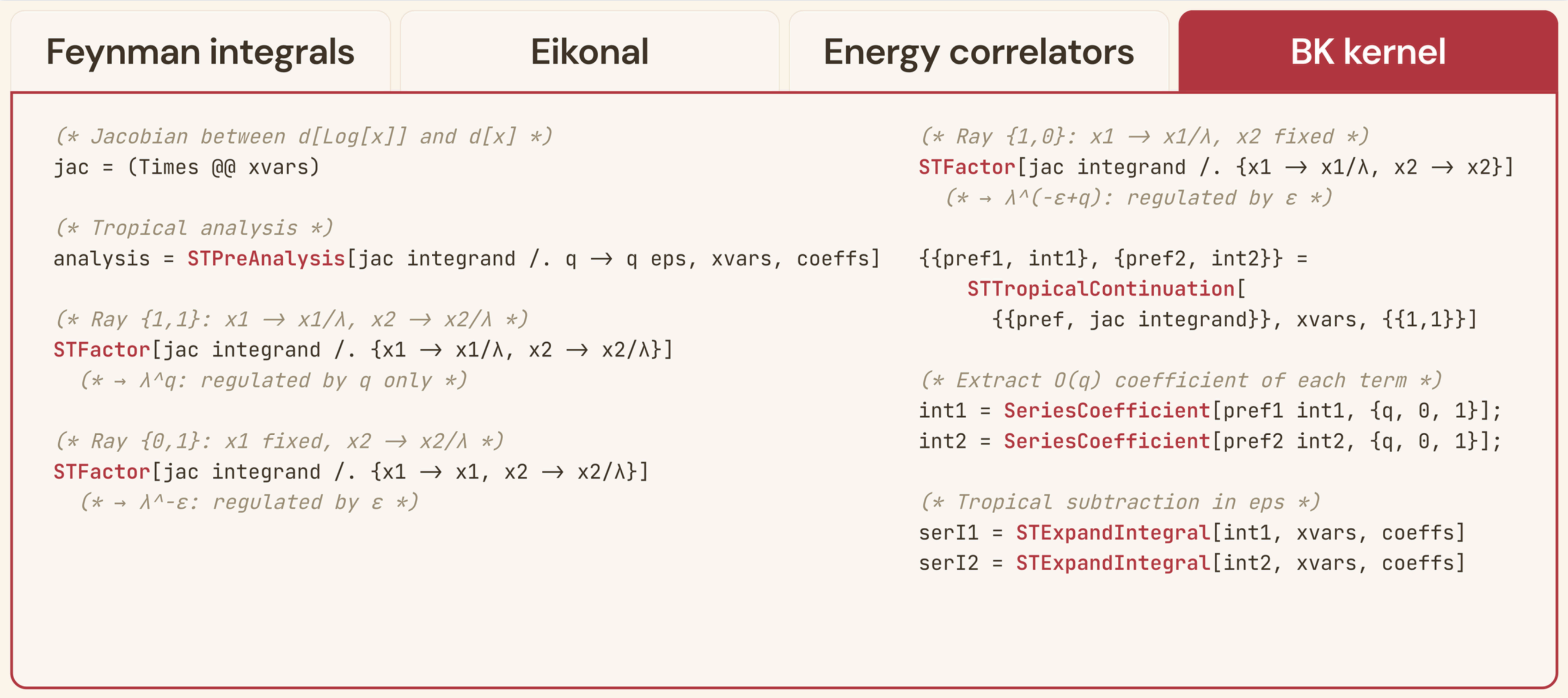
Feynman integrals | Eikonal | Energy correlators | **BK kernel**

$$I = \frac{\mathrm{e}^{-2\varepsilon\gamma_E}}{\pi^{\mathrm{D}}} \int \frac{\mathrm{d}^{\mathrm{D}}\ell_1\, \mathrm{d}^{\mathrm{D}}\ell_2}{\ell_1^2(\ell_1-\ell_2)^2} \log\left(\frac{\ell_1^2}{\ell_2^2}\right) \mathrm{e}^{i(\ell_1\cdot X_1+\ell_2\cdot X_2)}$$

Brunello, Caron-Huot, Crisanti, Giroux, Smith `[2508.03794]`

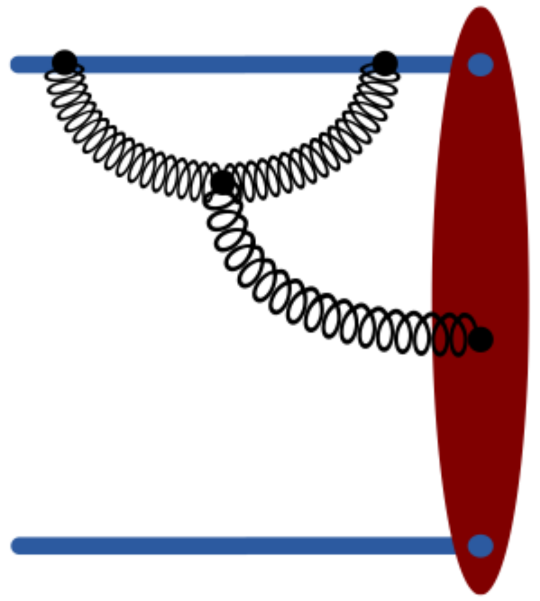


Integral appearing in the NNLO hadron rapidity evolution



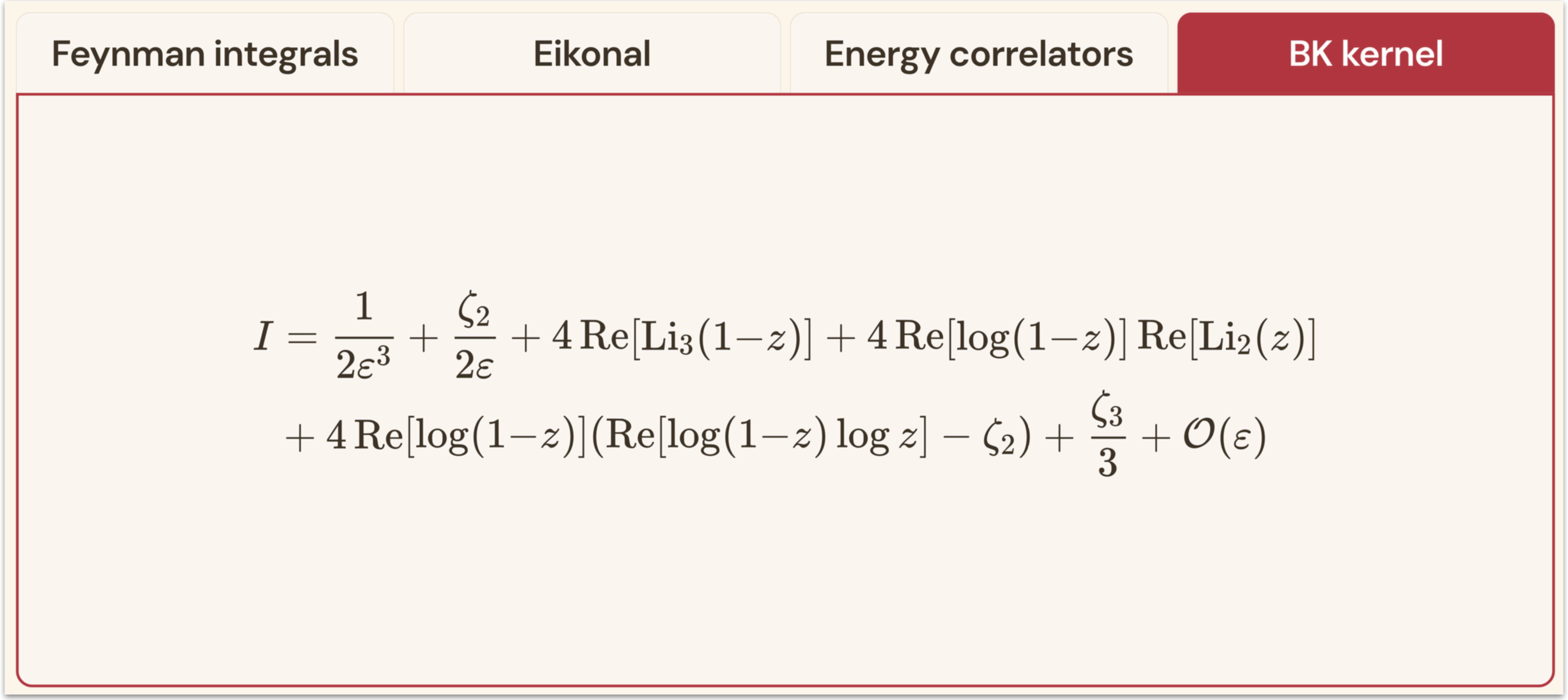
```
(* Jacobian between d[Log[x]] and d[x] *)
jac = (Times @@ xvars)

(* Tropical analysis *)
analysis = STPreAnalysis[jac integrand /. q → q eps, xvars, coeffs]

(* Ray {1,1}: x1 → x1/λ, x2 → x2/λ *)
STFactor[jac integrand /. {x1 → x1/λ, x2 → x2/λ}]
  (* → λ^q: regulated by q only *)

(* Ray {0,1}: x1 fixed, x2 → x2/λ *)
STFactor[jac integrand /. {x1 → x1, x2 → x2/λ}]
  (* → λ^-ε: regulated by ε *)
```

```
(* Ray {1,0}: x1 → x1/λ, x2 fixed *)
STFactor[jac integrand /. {x1 → x1/λ, x2 → x2}]
  (* → λ^(-ε+q): regulated by ε *)

{{pref1, int1}, {pref2, int2}} =
    STTropicalContinuation[
      {{pref, jac integrand}}, xvars, {{1,1}}]

(* Extract O(q) coefficient of each term *)
int1 = SeriesCoefficient[pref1 int1, {q, 0, 1}];
int2 = SeriesCoefficient[pref2 int2, {q, 0, 1}];

(* Tropical subtraction in eps *)
serI1 = STExpandIntegral[int1, xvars, coeffs]
serI2 = STExpandIntegral[int2, xvars, coeffs]
```

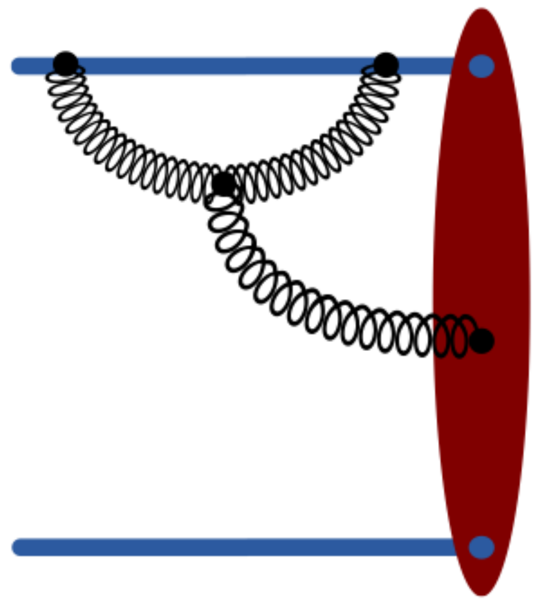


then `Shift + Enter`

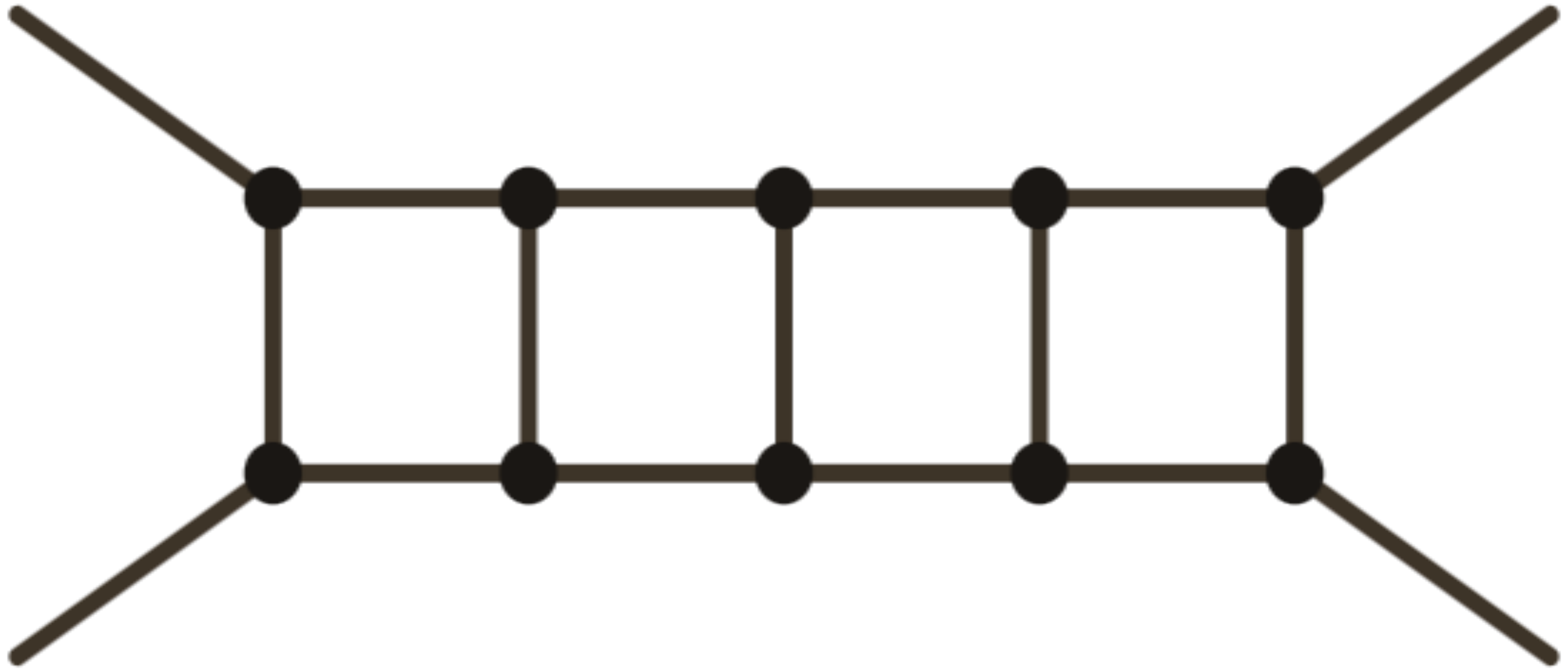
Integral appearing in the NNLO hadron rapidity evolution



$$I = \frac{1}{2\varepsilon^3} + \frac{\zeta_2}{2\varepsilon} + 4\,\mathrm{Re}[\mathrm{Li}_3(1-z)] + 4\,\mathrm{Re}[\log(1-z)]\,\mathrm{Re}[\mathrm{Li}_2(z)]$$
$$+\,4\,\mathrm{Re}[\log(1-z)](\mathrm{Re}[\log(1-z)\,\log z] - \zeta_2) + \frac{\zeta_3}{3} + \mathcal{O}(\varepsilon)$$



Catching up with what's beyond the state-of-the-art…



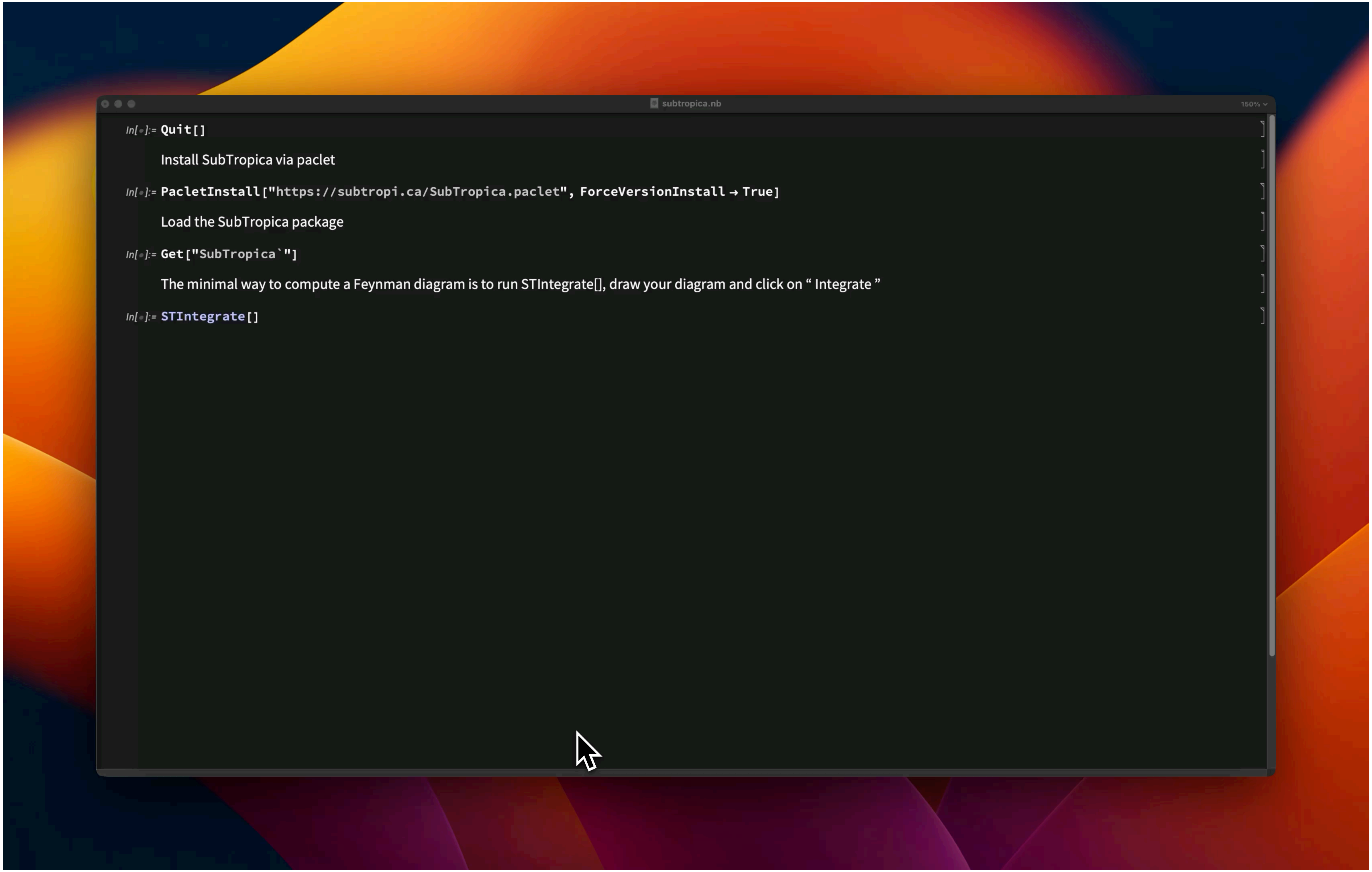
$$\int \frac{\mathrm{e}^{4\varepsilon\gamma_E}\,\frac{\mathrm{d}^D\ell_1}{i\pi^{D/2}}\,\frac{\mathrm{d}^D\ell_2}{i\pi^{D/2}}\,\frac{\mathrm{d}^D\ell_3}{i\pi^{D/2}}\,\frac{\mathrm{d}^D\ell_4}{i\pi^{D/2}}}{[(\ell_4-p_1-p_3)^2]^{\nu_1}\,[(\ell_3-p_1-p_3)^2]^{\nu_2}\,[(\ell_2-p_1-p_3)^2]^{\nu_3}\,[(\ell_1-p_1-p_3)^2]^{\nu_4}\,[(\ell_1-p_1-p_2-p_3)^2]^{\nu_5}\,[\ell_1^2]^{\nu_6}\,[\ell_2^2]^{\nu_7}\,[\ell_3^2]^{\nu_8}\,[\ell_4^2]^{\nu_9}\,[(\ell_4-p_3)^2]^{\nu_{10}}\,[(\ell_3-\ell_4)^2]^{\nu_{11}}\,[(\ell_2-\ell_3)^2]^{\nu_{12}}\,[(\ell_1-\ell_2)^2]^{\nu_{13}}}$$

Multi-polylogarithms

$$= \sum_{k=-8}^{0} \varepsilon^k\, c_k + O(\varepsilon)$$

(work in progress)

Installation, basic usage and submitting new results



```
In[∘]:= Quit[]
```

Install SubTropica via paclet

```
In[∘]:= PacletInstall["https://subtropi.ca/SubTropica.paclet", ForceVersionInstall → True]
```

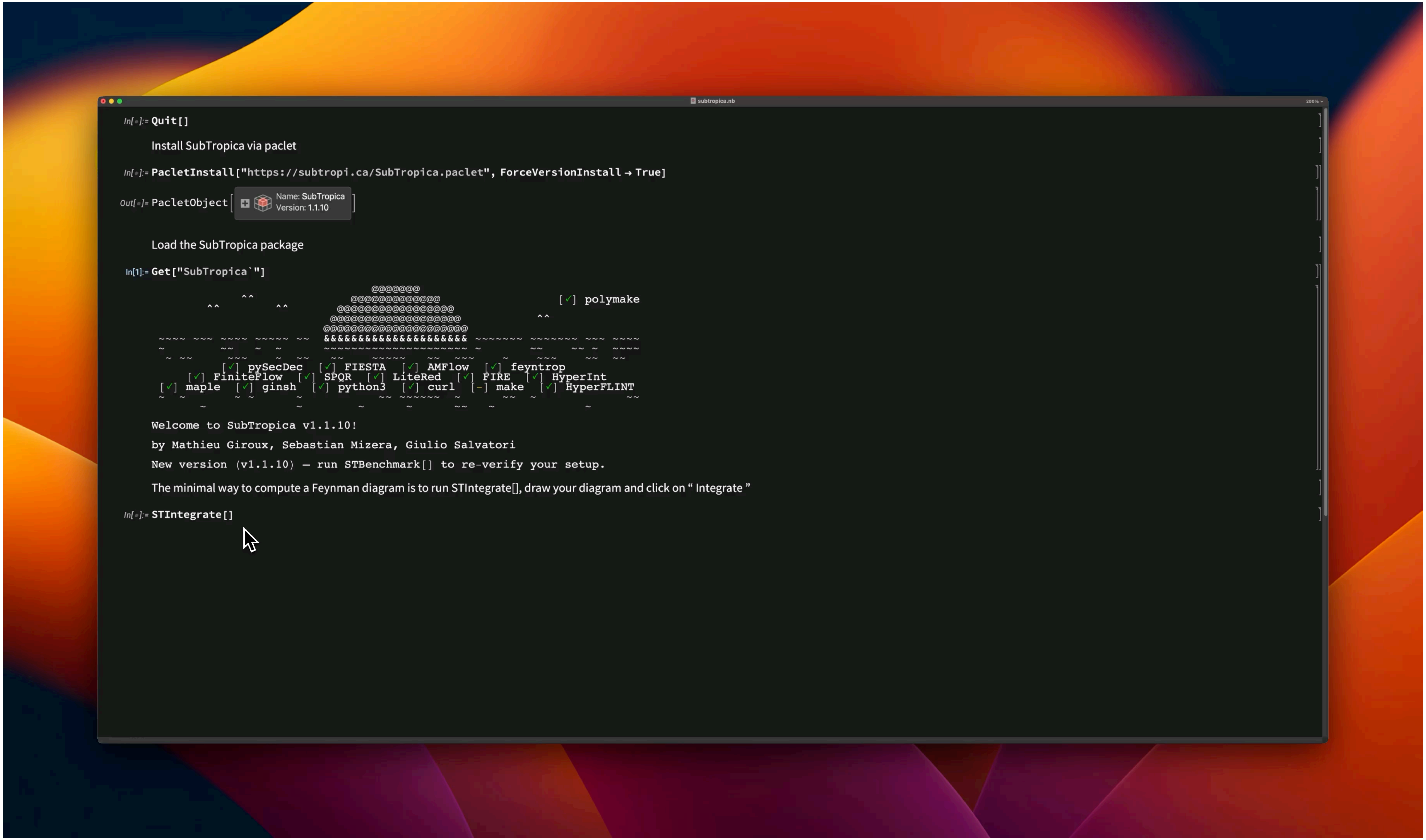
Load the SubTropica package

```
In[∘]:= Get["SubTropica`"]
```

The minimal way to compute a Feynman diagram is to run STIntegrate[], draw your diagram and click on “ Integrate ”

```
In[∘]:= STIntegrate[]
```

Drawing diagrams and inserting numerators



```
In[•]:= Quit[]

        Install SubTropica via paclet

In[•]:= PacletInstall["https://subtropi.ca/SubTropica.paclet", ForceVersionInstall → True]

Out[•]= PacletObject[ Name: SubTropica
                      Version: 1.1.10 ]

        Load the SubTropica package

In[1]:= Get["SubTropica`"]
```

```
                                @@@@@@@
             ^^               @@@@@@@@@@@@@                   [✓] polymake
        ^^        ^^        @@@@@@@@@@@@@@@@@
                          @@@@@@@@@@@@@@@@@@@            ^^
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  ~ ~~     ~~~    ~  ~~   ~~    ~~~~~   ~~   ~~~    ~   ~~~    ~~  ~~
          [✓] pySecDec  [✓] FIESTA   [✓] AMFlow  [✓] feyntrop
     [✓] FiniteFlow   [✓] SPQR  [✓] LiteRed  [✓] FIRE  [✓] HyperInt
 [✓] maple   [✓] ginsh  [✓] python3  [✓] curl  [-] make  [✓] HyperFLINT
 ~  ~        ~ ~    ~           ~~ ~~~~~~   ~      ~~   ~             ~~
       ~             ~        ~      ~      ~~    ~              ~

Welcome to SubTropica v1.1.10!

by Mathieu Giroux, Sebastian Mizera, Giulio Salvatori

New version (v1.1.10) — run STBenchmark[] to re-verify your setup.
```
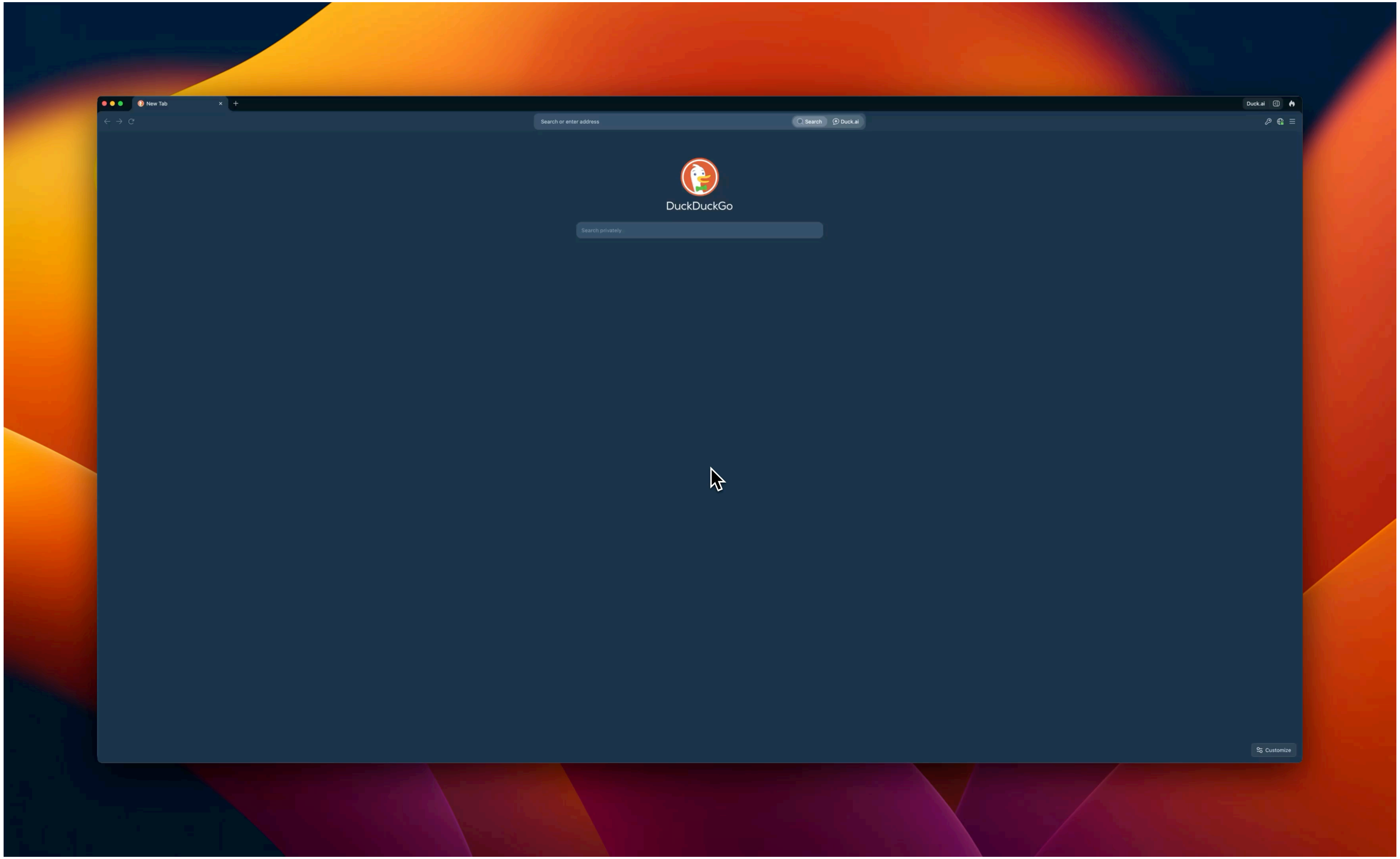
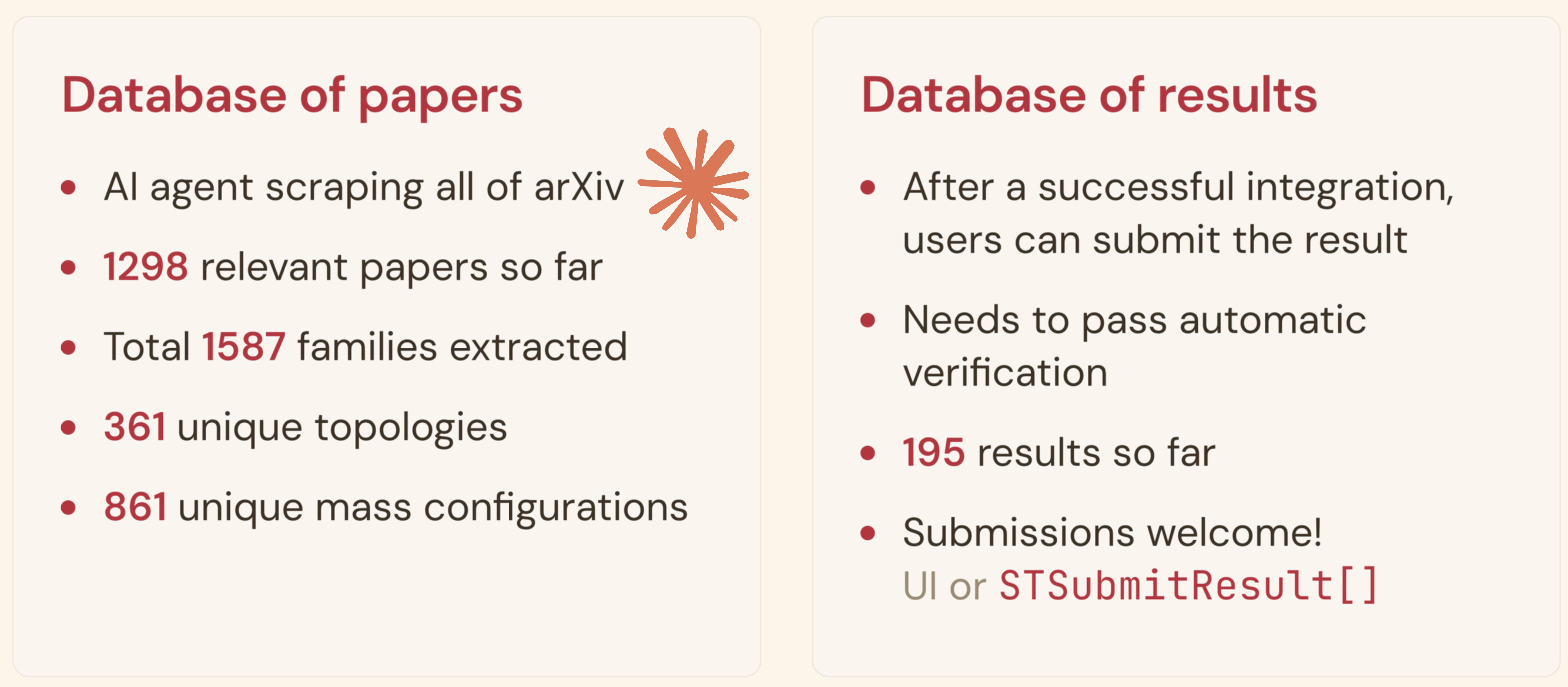

The minimal way to compute a Feynman diagram is to run STIntegrate[], draw your diagram and click on “ Integrate ”

```
In[•]:= STIntegrate[]
```

The database subtropi.ca 🇨🇦



The database subtropi.ca 🇨🇦



Database of papers

- AI agent scraping all of arXiv
- **1298** relevant papers so far
- Total **1587** families extracted
- **361** unique topologies
- **861** unique mass configurations

Database of results

- After a successful integration, users can submit the result
- Needs to pass automatic verification
- **195** results so far
- Submissions welcome!
  UI or `STSubmitResult[]`

Conclusion

Presented a `Mathematica` package to evaluate parametric integrals via

Graph, propagator + numerators or parametric inputs

Automatize tropical subtraction scheme

Initiate & maintain an online database of Feynman integrals

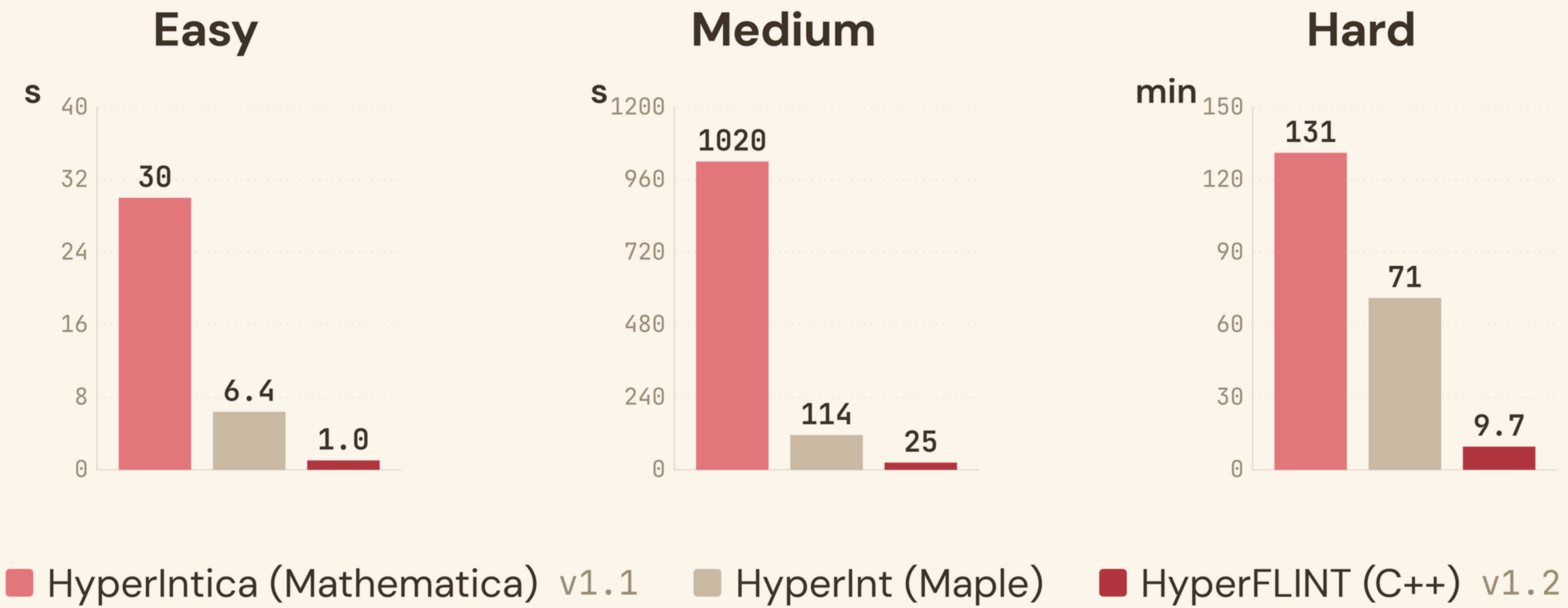
In progress

`C++/FLINT` *implementation*

Automatic rationalization

Expand class of integrals

Wall time, MacBook Pro M4 with 48 GB RAM
(all 5-variable integrals with a fixed integration order)



Extra slides I

💭 Make sense of integrating non-integer dimensional vectors ?

Simple and well-established solution.

Make a “change of variables” to end up integrating over an *integer number* of *scalar variables*

Many choices possible (Baikov, Feynman, …): let’s commit to the ***Schwinger representation***

$$I_{\vec{\nu}}(\mathbf{s}) \equiv \frac{\mathrm{e}^{\varepsilon L \gamma_{\mathrm{E}}}}{(i\pi^{\mathrm{D}/2})^L} \int \frac{\mathcal{N}\, \mathrm{d}^{\mathrm{D}}\ell_1\, \mathrm{d}^{\mathrm{D}}\ell_2 \cdots \mathrm{d}^{\mathrm{D}}\ell_L}{(q_1^2 - m_1^2)^{\nu_1}(q_2^2 - m_2^2)^{\nu_2} \cdots (q_N^2 - m_N^2)^{\nu_N}}$$

$$\frac{i}{(q_e^2 - m_e^2 + i\varepsilon)^{\nu_e}} = \frac{1}{\Gamma(\nu_e)} \int_0^\infty \mathrm{d}\alpha_e\, \alpha_e^{\nu_e - 1}\, \exp\big[i(q_e^2 - m_e^2 + i\varepsilon)\alpha_e\big]$$

$$= \mathrm{e}^{\varepsilon L \gamma_{\mathrm{E}}} \Gamma(\omega) \left(\prod_{e=1}^{N} \frac{(-1)^{\nu_e}}{\Gamma(\nu_e)} \right) \int \frac{\mathrm{d}^N x}{\mathrm{GL}(1)} \prod_{e=1}^{N} x_e^{\nu_e - 1} \frac{\tilde{\mathcal{N}}(\mathbf{s})}{\mathcal{U}^{\mathrm{D}/2-\omega}[-\mathcal{F}(\mathbf{s})]^{\omega}}$$

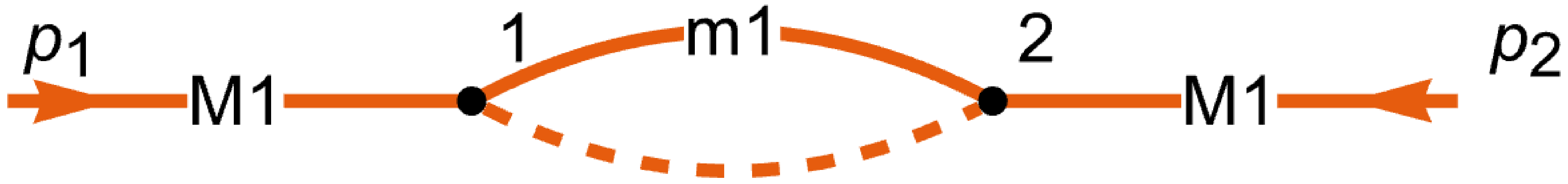
$$\omega \equiv \sum_{e=1}^{N} \nu_e - \frac{L\mathrm{D}}{2}$$

$\mathcal{U}\,,\mathcal{F} = \text{Symanzik polys.}$

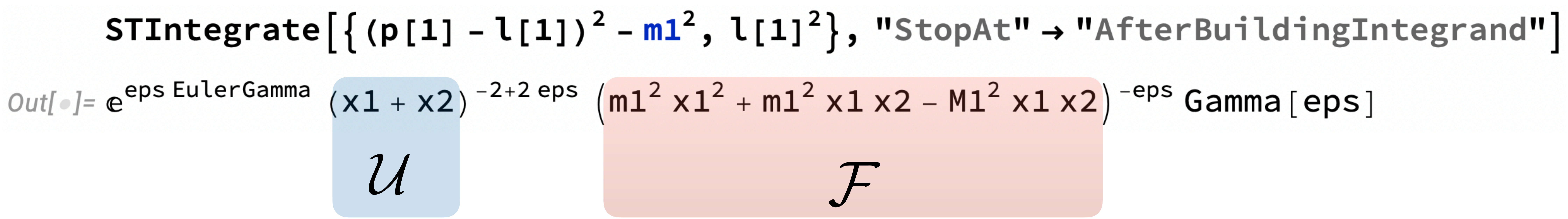
💭 Make sense of integrating non-integer dimensional vectors ?

`Shift + Enter` *in* `SubTropica`

$$p_1^2 = p_2^2 = M_1^2$$



Enter, e.g., a propagator basis here

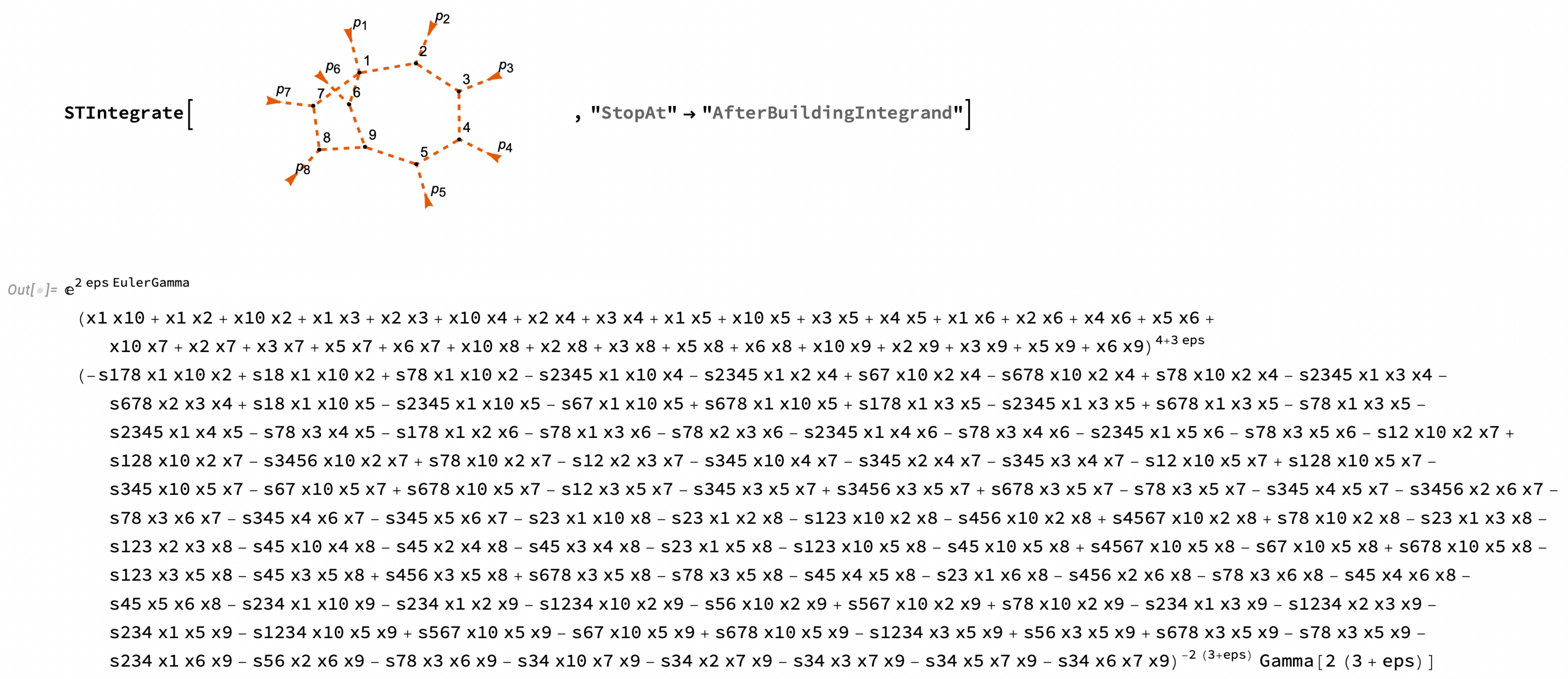


```
STIntegrate[{(p[1] - l[1])^2 - m1^2, l[1]^2}, "StopAt" → "AfterBuildingIntegrand"]
Out[ ]= e^(eps EulerGamma) (x1 + x2)^(-2+2 eps) (m1^2 x1^2 + m1^2 x1 x2 - M1^2 x1 x2)^(-eps) Gamma[eps]
```

$$\underbrace{(x_1 + x_2)}_{\mathcal{U}} \quad \underbrace{(m_1^2 x_1^2 + m_1^2 x_1 x_2 - M_1^2 x_1 x_2)}_{\mathcal{F}}$$

x's are to be integrated from $[0, \infty)$

Make sense of integrating non-integer dimensional vectors ?



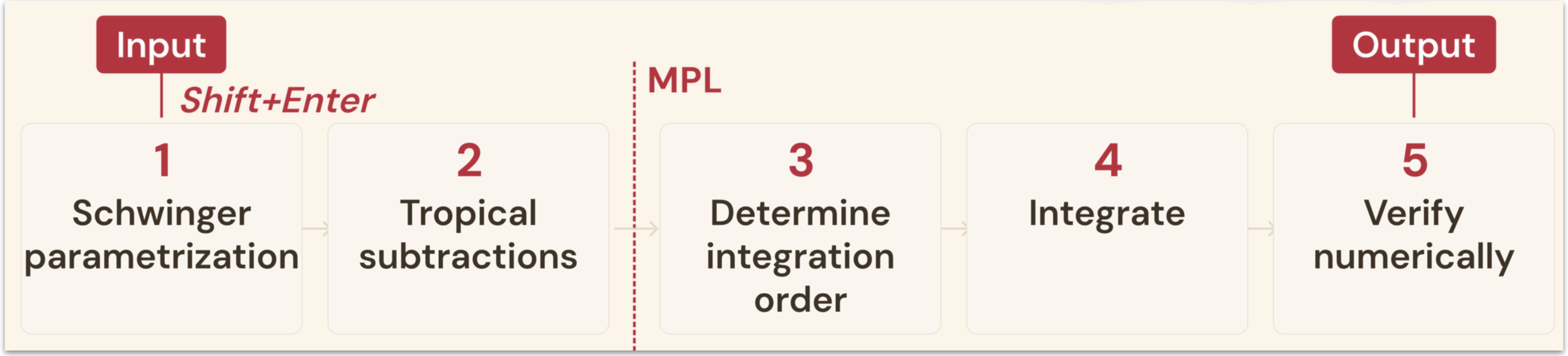
```
STIntegrate[ , "StopAt" → "AfterBuildingIntegrand"]

Out[•]= e 2 eps EulerGamma
 (x1 x10 + x1 x2 + x10 x2 + x1 x3 + x2 x3 + x10 x4 + x2 x4 + x3 x4 + x1 x5 + x10 x5 + x3 x5 + x4 x5 + x1 x6 + x2 x6 + x4 x6 + x5 x6 +
   x10 x7 + x2 x7 + x3 x7 + x5 x7 + x6 x7 + x10 x8 + x2 x8 + x3 x8 + x5 x8 + x6 x8 + x10 x9 + x2 x9 + x3 x9 + x5 x9 + x6 x9) 4+3 eps
 (-s178 x1 x10 x2 + s18 x1 x10 x2 + s78 x1 x10 x2 - s2345 x1 x10 x4 - s2345 x1 x2 x4 + s67 x10 x2 x4 - s678 x10 x2 x4 + s78 x10 x2 x4 - s2345 x1 x3 x4 -
   s678 x2 x3 x4 + s18 x1 x10 x5 - s2345 x1 x10 x5 - s67 x1 x10 x5 + s678 x1 x10 x5 + s178 x1 x3 x5 - s2345 x1 x3 x5 + s678 x1 x3 x5 - s78 x1 x3 x5 -
   s2345 x1 x4 x5 - s78 x3 x4 x5 - s178 x1 x2 x6 - s78 x1 x3 x6 - s78 x2 x3 x6 - s2345 x1 x4 x6 - s78 x3 x4 x6 - s2345 x1 x5 x6 - s78 x3 x5 x6 - s12 x10 x2 x7 +
   s128 x10 x2 x7 - s3456 x10 x2 x7 + s78 x10 x2 x7 - s12 x2 x3 x7 - s345 x10 x4 x7 - s345 x2 x4 x7 - s345 x3 x4 x7 - s12 x10 x5 x7 + s128 x10 x5 x7 -
   s345 x10 x5 x7 - s67 x10 x5 x7 + s678 x10 x5 x7 - s12 x3 x5 x7 - s345 x3 x5 x7 + s3456 x3 x5 x7 + s678 x3 x5 x7 - s78 x3 x5 x7 - s345 x4 x5 x7 - s3456 x2 x6 x7 -
   s78 x3 x6 x7 - s345 x4 x6 x7 - s345 x5 x6 x7 - s23 x1 x10 x8 - s23 x1 x2 x8 - s123 x10 x2 x8 - s456 x10 x2 x8 + s4567 x10 x2 x8 + s78 x10 x2 x8 - s23 x1 x3 x8 -
   s123 x2 x3 x8 - s45 x10 x4 x8 - s45 x2 x4 x8 - s45 x3 x4 x8 - s23 x1 x5 x8 - s123 x10 x5 x8 - s45 x10 x5 x8 + s4567 x10 x5 x8 - s67 x10 x5 x8 + s678 x10 x5 x8 -
   s123 x3 x5 x8 - s45 x3 x5 x8 + s456 x3 x5 x8 + s678 x3 x5 x8 - s78 x3 x5 x8 - s45 x4 x5 x8 - s23 x1 x6 x8 - s456 x2 x6 x8 - s78 x3 x6 x8 - s45 x4 x6 x8 -
   s45 x5 x6 x8 - s234 x1 x10 x9 - s234 x1 x2 x9 - s1234 x10 x2 x9 - s56 x10 x2 x9 + s567 x10 x2 x9 + s78 x10 x2 x9 - s234 x1 x3 x9 - s1234 x2 x3 x9 -
   s234 x1 x5 x9 - s1234 x10 x5 x9 + s567 x10 x5 x9 - s67 x10 x5 x9 + s678 x10 x5 x9 - s1234 x3 x5 x9 + s56 x3 x5 x9 + s678 x3 x5 x9 - s78 x3 x5 x9 -
   s234 x1 x6 x9 - s56 x2 x6 x9 - s78 x3 x6 x9 - s34 x10 x7 x9 - s34 x2 x7 x9 - s34 x3 x7 x9 - s34 x5 x7 x9 - s34 x6 x7 x9) -2 (3+eps) Gamma[2 (3 + eps)]
```

$$s_{i_1,\ldots,i_n} = (p_{i_1} + \ldots + p_{i_n})^2$$

💭 Make sense of integrating non-integer dimensional vectors ?

Schwinger representation

$$I_{\vec{\nu}}(\mathbf{s}) = \mathrm{e}^{\varepsilon L \gamma_{\mathrm{E}}} \Gamma(\omega) \left(\prod_{e=1}^{N} \frac{(-1)^{\nu_e}}{\Gamma(\nu_e)} \right) \int \frac{\mathrm{d}^N x}{\mathrm{GL}(1)} \prod_{e=1}^{N} x_e^{\nu_e - 1} \frac{\tilde{\mathcal{N}}(\mathbf{s})}{\mathcal{U}^{\mathrm{D}/2-\omega} [-\mathcal{F}(\mathbf{s})]^{\omega}}$$



Input

Shift+Enter

MPL

Output

1 Schwinger parametrization

2 Tropical subtractions

3 Determine integration order

4 Integrate

5 Verify numerically

💭 How do we deal with divergences as $\varepsilon = 0$?

Solution to this problem is more subtle and technical, so I will focus on illustrating the idea

Philosophy: we should always aim to integrate *after* expanding in ε (simpler!)

$$\tau_\varepsilon \int \mathcal{I}(s,\varepsilon) \overset{?}{=} \int \tau_\varepsilon \mathcal{I}(s,\varepsilon)$$

⚠ In general ***wrong***: RHS may diverge while LHS finite

$$\int_0^1 \frac{\mathrm{d}\alpha}{\alpha}\left(\alpha^{\varepsilon} - 2\alpha^{2\varepsilon}\right) \; = \; 0 \; \neq \; \int_0^1 \frac{\mathrm{d}\alpha}{\alpha}\left(-1 + \mathcal{O}(\varepsilon)\right) \; = \; \infty$$

Lesson. ε-expansion commutes with integration ⟺ integrand is locally finite

(i.e., does not develop a singularity on the integration contour)

💭 How do we deal with divergences as $\varepsilon = 0$?

A simple idea to organize divergences: rewrite

$$\mathcal{I} \;=\; \left[\mathcal{I} - \mathcal{I}^{\mathrm{ct}}\right] + \mathcal{I}^{\mathrm{ct}}$$

with counter-terms (c.t.) $\mathcal{I}^{\mathrm{ct}}$ chosen so that, schematically:

1. $\mathcal{I}^{\mathrm{ren}} \equiv \mathcal{I} - \mathcal{I}^{\mathrm{ct}}$ *is locally finite (i.e., c.t have same leading pole structure as* $\mathcal{I}$*)*

2. *but are simple enough* to be integrated* ***directly*** *and* ***exactly*** *in* ε

*Strategy is thus to first expose all divergences and subtract them one c.t. at a time***

---

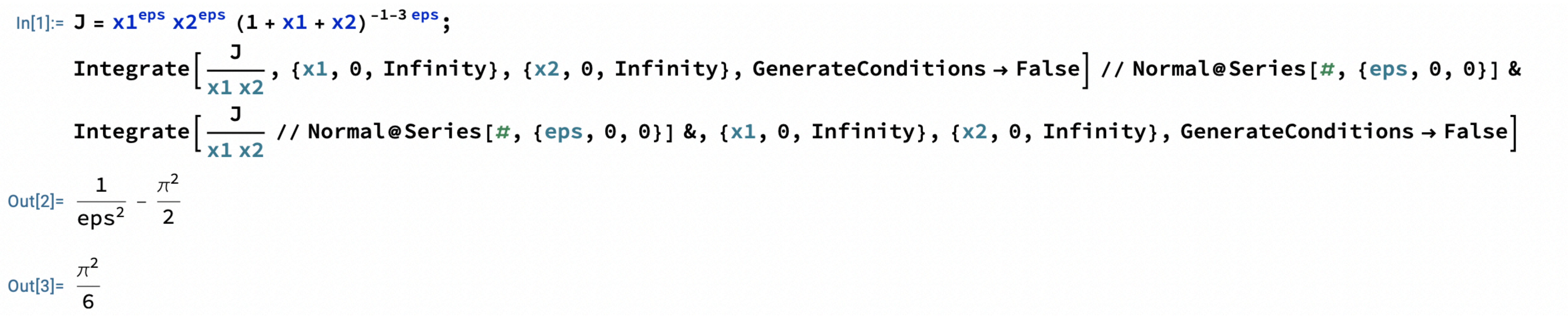
* *e.g., separable and depends on less integration variables* $\Longrightarrow$ *terminates when no variables are left over*

** *Ask me later or see* `[Salvatori, 2406.14606]` *for the theory behind it*

A simple example: constructing c.t. & integrating expansions

$$I(\varepsilon) \;=\; \int_{\mathbb{R}^2_{\geq 0}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1\,x_2}\; x_1^{\varepsilon}\, x_2^{\varepsilon}\,(1+x_1+x_2)^{-1-3\varepsilon}$$

Simple enough to be performed with Mathematica's `Integrate[]` *and can check operations do not commute*



```
In[1]:= J = x1^eps x2^eps (1 + x1 + x2)^(-1-3 eps);
        Integrate[J/(x1 x2), {x1, 0, Infinity}, {x2, 0, Infinity}, GenerateConditions → False] // Normal@Series[#, {eps, 0, 0}] &
        Integrate[J/(x1 x2) // Normal@Series[#, {eps, 0, 0}] &, {x1, 0, Infinity}, {x2, 0, Infinity}, GenerateConditions → False]
Out[2]= 1/eps^2 - π^2/2
Out[3]= π^2/6
```

A simple example: constructing c.t. & integrating expansions

$$I(\varepsilon) \;=\; \int_{\mathbb{R}^2_{\geq 0}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1\,x_2}\; x_1^{\varepsilon}\,x_2^{\varepsilon}\,(1+x_1+x_2)^{-1-3\varepsilon}$$

One can use *tropical geometry* to ***systematically*** construct counter-terms*

Counter-terms

$$\mathcal{I}^{\text{ren}} \;=\; \mathcal{I} \;-\; \bar{v}_{\rho_1}\mathcal{I}\Big|_{\rho_1} \;-\; \bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_2} \;+\; \bar{v}_{\rho_1}\bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_1\rho_2}$$

$$\mathcal{I}\Big|_{\rho_1} = x_1^{\varepsilon}x_2^{\varepsilon}(1+x_2)^{-1-3\varepsilon}, \qquad \mathcal{I}\Big|_{\rho_2} = x_1^{\varepsilon}x_2^{\varepsilon}(1+x_1)^{-1-3\varepsilon}, \qquad \mathcal{I}\Big|_{\rho_1\rho_2} = x_1^{\varepsilon}x_2^{\varepsilon} \qquad \bar{v}_{\rho_i} = \frac{1}{1+x_i}$$

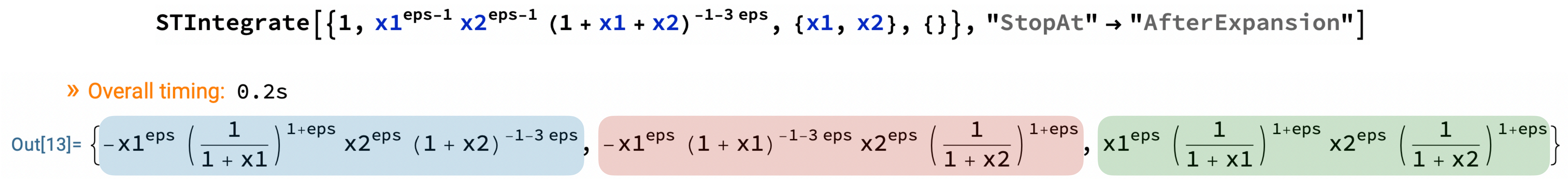
---

* *Ask me later or see* `[Salvatori, 2406.14606]` *for the theory behind it*

A simple example: constructing c.t. & integrating expansions

$$\mathcal{I}^{\text{ren}} = \mathcal{I} - \bar{v}_{\rho_1}\mathcal{I}\Big|_{\rho_1} - \bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_2} + \bar{v}_{\rho_1}\bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_1\rho_2}$$

Just a “`Shift + Enter`” *in* `SubTropica`



```
STIntegrate[{1, x1^(eps-1) x2^(eps-1) (1 + x1 + x2)^(-1-3 eps), {x1, x2}, {}}, "StopAt" → "AfterExpansion"]
```

» Overall timing: `0.2s`

Out[13]=

$$\left\{-x1^{\text{eps}}\left(\frac{1}{1+x1}\right)^{1+\text{eps}} x2^{\text{eps}}\,(1+x2)^{-1-3\,\text{eps}},\ -x1^{\text{eps}}\,(1+x1)^{-1-3\,\text{eps}}\,x2^{\text{eps}}\left(\frac{1}{1+x2}\right)^{1+\text{eps}},\ x1^{\text{eps}}\left(\frac{1}{1+x1}\right)^{1+\text{eps}} x2^{\text{eps}}\left(\frac{1}{1+x2}\right)^{1+\text{eps}}\right\}$$

A simple example: constructing c.t. & integrating expansions

Summary:

$$I(\varepsilon) = \int_{\mathbb{R}^2_{\geq 0}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1\,x_2}\Big[\mathcal{I} \;=\; \mathcal{I}^{\mathrm{ren}} \;+\; \bar{v}_{\rho_1}\mathcal{I}\Big|_{\rho_1} \;+\; \bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_2} \;-\; \bar{v}_{\rho_1}\bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_1\,\rho_2}\Big]$$

Now $\mathcal{I}^{\mathrm{ren}}$ can be expanded in ε before integration while the c.t. can be evaluate as exact function in ε

$$\int \bar{v}_{\rho_1}\mathcal{I}\Big|_{\rho_1} \frac{\mathrm{d}x_1\mathrm{d}x_2}{x_1x_2} \;=\; \underbrace{\int_0^\infty \frac{x_1^{\varepsilon-1}}{1+x_1}\,\mathrm{d}x_1}_{B(\varepsilon,\,1-\varepsilon)} \cdot \underbrace{\int_0^\infty x_2^{\varepsilon-1}(1+x_2)^{-1-3\varepsilon}\,\mathrm{d}x_2}_{B(\varepsilon,\,1+2\varepsilon)}$$

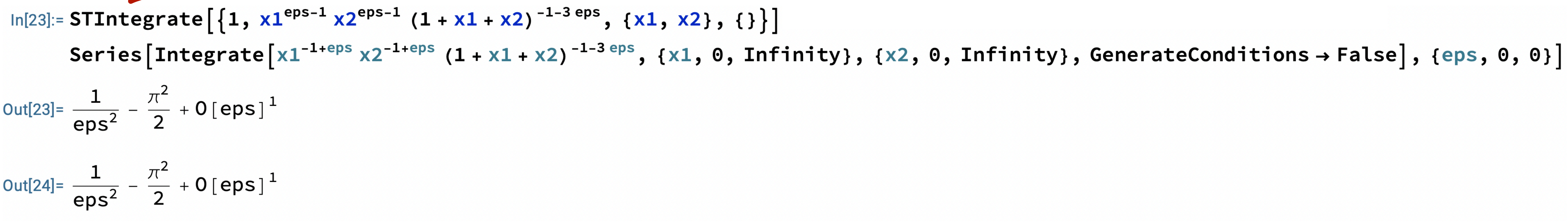
A simple example: constructing c.t. & integrating expansions

Sanity check

$$I(\varepsilon) \;=\; \int_{\mathbb{R}^2_{\geq 0}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1\,x_2}\; x_1^{\varepsilon}\, x_2^{\varepsilon}\,(1+x_1+x_2)^{-1-3\varepsilon}$$

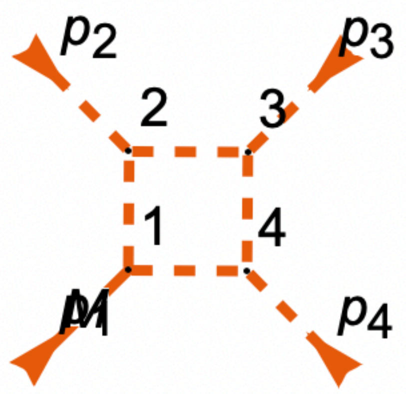
Again, just a “`Shift + Enter`” *in* `SubTropica`

*Series here is taken **before** integrating!*

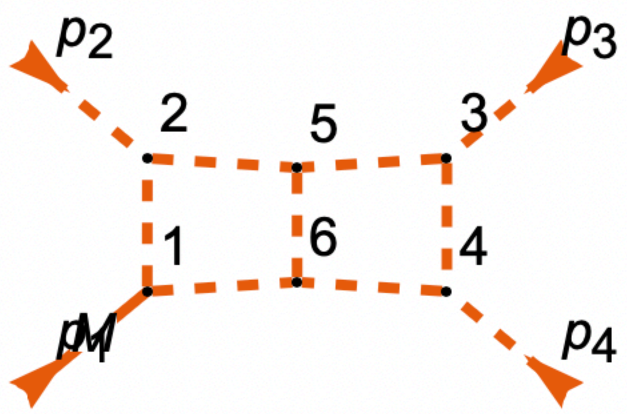


```
In[23]:= STIntegrate[{1, x1^(eps-1) x2^(eps-1) (1 + x1 + x2)^(-1-3 eps), {x1, x2}, {}}]
         Series[Integrate[x1^(-1+eps) x2^(-1+eps) (1 + x1 + x2)^(-1-3 eps), {x1, 0, Infinity}, {x2, 0, Infinity}, GenerateConditions → False], {eps, 0, 0}]

Out[23]= 1/eps^2 - π^2/2 + O[eps]^1

Out[24]= 1/eps^2 - π^2/2 + O[eps]^1
```

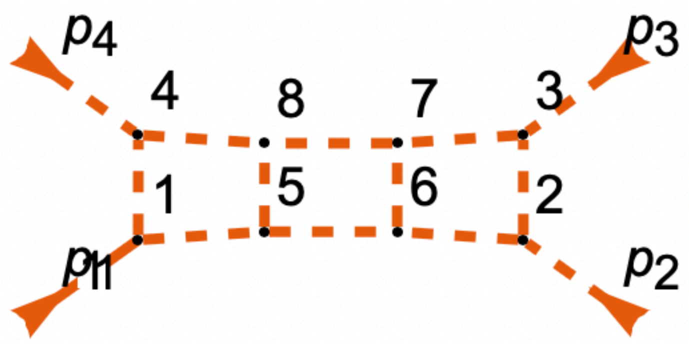
Number of counter terms grows fast

`STIntegrate[`  `, "StopAt" → "AfterExpansion"]["Result"]⟦1, 2, 1, 1⟧ // Length`

```
Out[•]= 6
```

`STIntegrate[`  `, "StopAt" → "AfterExpansion"]["Result"]⟦1, 2, 1, 1⟧ // Length`

```
Out[•]= 32
```

`STIntegrate[`  `, "StopAt" → "AfterExpansion"]["Result"]⟦1, 2, 1, 1⟧ // Length`

```
Out[•]= 96
```

💭 How do we deal with divergences as $\varepsilon = 0$?

So at the end of the day, `SubTropica` *uses a tropical subtraction scheme to write*

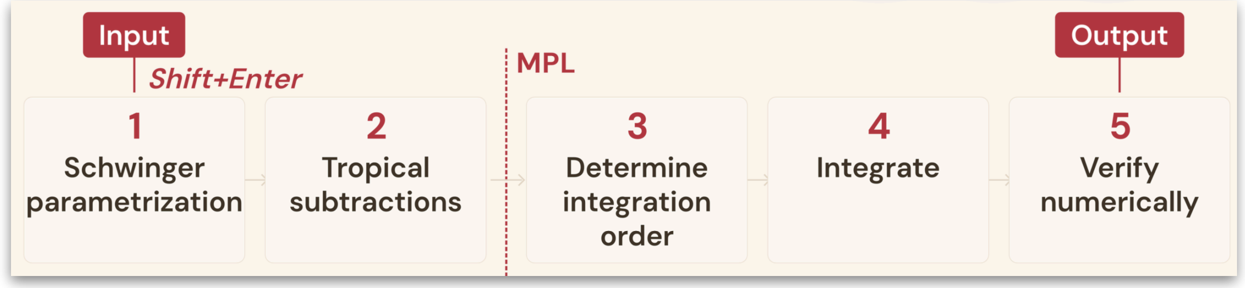
$$I_{\vec{\nu}}(\mathbf{s}) \equiv \frac{\mathrm{e}^{\varepsilon L \gamma_{\mathrm{E}}}}{(i\pi^{\mathrm{D}/2})^{L}} \int \frac{\mathcal{N}\, \mathrm{d}^{\mathrm{D}}\ell_1\, \mathrm{d}^{\mathrm{D}}\ell_2 \cdots \mathrm{d}^{\mathrm{D}}\ell_L}{(q_1^2 - m_1^2)^{\nu_1}(q_2^2 - m_2^2)^{\nu_2} \cdots (q_N^2 - m_N^2)^{\nu_N}}$$

$$= \sum_{i=-n} \varepsilon^{i}\, I_i^{\mathrm{finite}}(\varepsilon, \mathbf{s})$$

Polynomials

$$I^{\mathrm{finite}}(\varepsilon, \mathbf{s}) \;=\; \int_0^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_i P_i(x, \mathbf{s})^{a_i + b_i \varepsilon}$$

Integration commutes with expansion for those



Input

Shift+Enter

MPL

Output

1 Schwinger parametrization

2 Tropical subtractions

3 Determine integration order

4 Integrate

5 Verify numerically

What are the possible roadblocks to direct symbolic integration?

⚠️ Divergences require to work in D $= 4 - 2\varepsilon$, where $0 < |\varepsilon| \ll 1$

- 💭 Make sense of integrating non-integer dimensional vectors ? ✓
- 💭 How do we deal with divergences ? ✓

⚠️ A priori no control over the zoo of functions these evaluate to

- 💭 Which restriction(s) on the function space to impose?

The development of **`SubTropica`** *necessitated a (partial) answer to all these questions*

What are we left with after expanding the integrands in ε ?

Schematically an integral of the form

Polynomials

$$I^{\text{finite}}(\varepsilon, \mathbf{s}) \;=\; \int_0^\infty \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_i P_i(x, \mathbf{s})^{a_i + b_i \varepsilon}$$

$$= \int \mathrm{d}^n x \; \underbrace{\frac{\sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \left(\sum_i b_i \log P_i\right)^k}{\prod_{a_i<0} P_i^{|a_i|}} \prod_{a_i>0} P_i^{a_i}}_{\equiv\, g_0}$$

Question: is there any condition under which these integrals be evaluated systematically ?

Yes! Writing

$$I^{\text{finite}} = \underbrace{\int_0^\infty \mathrm{d}x_n \cdots \underbrace{\int_0^\infty \mathrm{d}x_2 \underbrace{\int_0^\infty \mathrm{d}x_1\; g_0(x_1,\ldots,x_n)}_{g_1(x_2,\ldots,x_n)}}_{g_2(x_3,\ldots,x_n)}}_{\substack{\vdots \\ g_{n-1}(x_n)}}$$

Just iterated integrals over dlog forms
(i.e., easy to evaluate numerically)

There is a algorithmic procedure to perform the integral in terms of *hyperlogarithms* if at every step k, all denominator factors of g_{k-1} are linear in x_k

This condition is called *linear reducibility* [Brown, 2009; Panzer, 2014]

What are we left with after expanding the integrands in ε ?

Schematically an integral of the form

Polynomials

$$I^{\text{finite}}(\varepsilon, \mathbf{s}) \;=\; \int_0^\infty \mathrm{d}x_1 \cdots \mathrm{d}x_n \prod_i P_i(x, \mathbf{s})^{a_i + b_i \varepsilon}$$

$$= \int \mathrm{d}^n x \; \frac{\sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \left(\sum_i b_i \log P_i\right)^k}{\prod_{a_i<0} P_i^{|a_i|}} \prod_{a_i>0} P_i^{a_i}$$

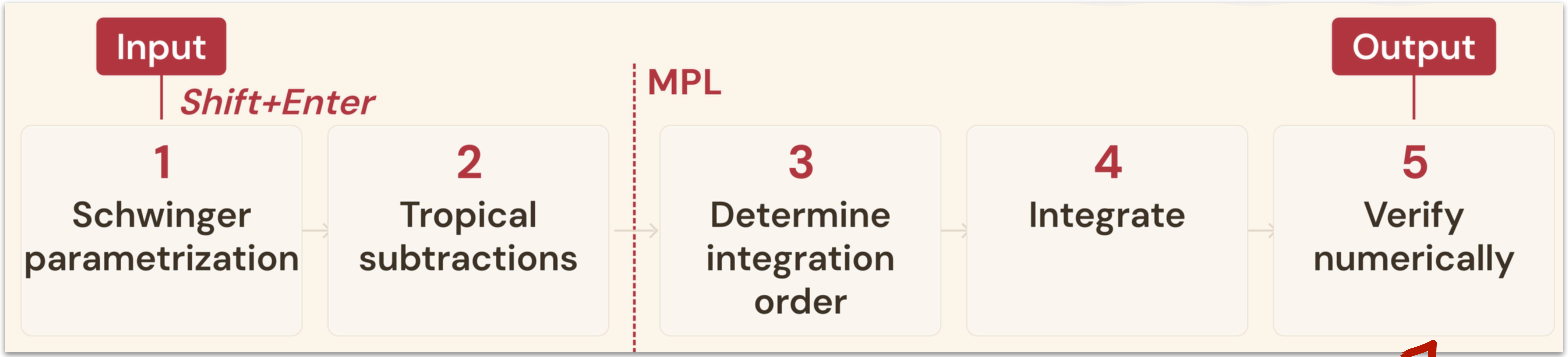
$$\equiv g_0$$

Given g_0, `SubTropica` *implements* an algorithmic procedure to check this LR condition* ***prior*** *integration*

Scans over all possible integration orders, if it finds a LR one, it proceeds automatically with integration

** Ask me later for more details*

Summary of the `SubTropica` *pipeline*



Input

Shift+Enter

MPL

Output

1 Schwinger parametrization

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3 Determine integration order

4 Integrate

5 Verify numerically

`SubTropica` *can optionally stress-test the symbolic output by comparing independently with* `PySecDec` *and* `FIESTA` *at a point:*
Call `STVerify[]` after a run

Extra slides II

A simple example: constructing c.t. & integrating expansions

$$I(\varepsilon) \;=\; \int_{\mathbb{R}^2_{\geq 0}} \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1\,x_2}\; x_1^{\varepsilon}\, x_2^{\varepsilon}\, (1+x_1+x_2)^{-1-3\varepsilon}$$

Step #1. Find all *leading* divergent scalings $x_i \to x_i \lambda^{-\rho_i}$ $(\lambda \to 0^+)$ such that integrand $\sim \lambda^{-|m|+n\varepsilon}$

$\vec{\rho} = (\rho_1, \rho_2)$

No $\vec{\rho} = (a,b) \in \mathbb{Z}^2$: integrand $\sim \lambda^{-|m|+n\varepsilon} \Rightarrow$ no power divergent scaling

$\vec{\rho}_1 = (-1,0),\; \vec{\rho}_2 = (0,-1)$: integrand $\sim \lambda^{\varepsilon} \Rightarrow$ marginal/log-divergent at $\varepsilon = 0$

$\vec{\rho}_4 = (-1,-1)$: integrand $\sim \lambda^{2\varepsilon} \Rightarrow$ marginal but less (not leading) at $\varepsilon = 0$

$\vec{\rho}_3 = (1,1)$: integrand $\sim \lambda^{1+\varepsilon} \Rightarrow$ convergent at $\varepsilon = 0$

$\vdots$

Only those matter!

* *For the experts: this "power counting" is what the "Newton polytope" in tropical geometry gives you and is actually how these are obtained in* `SubTropica`

A simple example: constructing c.t. & integrating expansions

Step #2. Next we want to subtract to $I(\varepsilon)$ itself restricted on the leading divergent rays $\vec{\rho} = (\rho_1, \rho_2)$

A naive guess for the "renormalized integrand" would be

$$\mathcal{I}^{\text{ren}} \overset{?}{=} \mathcal{I} - \mathcal{I}\Big|_{\rho_1} - \mathcal{I}\Big|_{\rho_2}$$

$$\mathcal{I}\Big|_{\rho_1} = x_1^{\varepsilon} x_2^{\varepsilon} (1 + x_2)^{-1-3\varepsilon}, \qquad \mathcal{I}\Big|_{\rho_2} = x_1^{\varepsilon} x_2^{\varepsilon} (1 + x_1)^{-1-3\varepsilon}$$

However, we cannot stop there because while this remove the divergence along $\vec{\rho}_i$

it introduces a "UV" divergence at the opposite boundary $x_i \to \infty$

A simple example: constructing c.t. & integrating expansions

Step #2. Next we want to subtract to $I(\varepsilon)$ itself restricted on the leading divergent rays $\vec{\rho} = (\rho_1, \rho_2)$

One obvious option to smooth this out is to allow coefficients with the right support properties

$$\mathcal{I}^{\text{ren}} \overset{?}{=} \mathcal{I} - \bar{v}_{\rho_1}\mathcal{I}\Big|_{\rho_1} - \bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_2}$$

where $\bar{v}_{\rho_i}$ *goes to 1 on the ray* $\vec{\rho}_i$ *and vanishes as* $x_i \to \infty$. *The simplest choice is*

$$\bar{v}_{\rho_i} = \frac{1}{1 + x_i}$$

This way we kill the growth far from the ray without changing the boundary cancellation

A simple example: constructing c.t. & integrating expansions

Step #2. Next we want to subtract to $I(\varepsilon)$ itself restricted on the leading divergent rays $\vec{\rho} = (\rho_1, \rho_2)$

However, this is still not enough because $\mathcal{I}^{\text{ren}}$ should be finite (e.g., vanish) along the dangerous scalings, e.g.,

$$\mathcal{I}^{\text{ren}}|_{\rho_i} \overset{?}{=} \mathcal{I}|_{\rho_i} - \underbrace{\bar{v}_{\rho_i}|_{\rho_i}}_{=1}\mathcal{I}\Big|_{\rho_i} - \Big(\bar{v}_{\rho_j}\mathcal{I}\Big|_{\rho_j}\Big)\Big|_{\rho_i} = -\Big(\bar{v}_{\rho_i}\bar{v}_{\rho_j}\mathcal{I}\Big|_{\rho_j}\Big)\Big|_{\rho_i} = -\Big(\bar{v}_{\rho_i}\bar{v}_{\rho_j}\mathcal{I}\Big|_{\rho_j,\rho_i}\Big)\Big|_{\rho_i}$$

and so $\mathcal{I}^{\text{ren}}$ is only finite after we, e.g., add the "contact term"

Predictable = implementable
Special case of Möbius inversion formula

$$\mathcal{I}^{\text{ren}} = \mathcal{I} - \bar{v}_{\rho_1}\mathcal{I}\Big|_{\rho_1} - \bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_2} + \bar{v}_{\rho_1}\bar{v}_{\rho_2}\mathcal{I}\Big|_{\rho_1\rho_2}$$

$$\mathcal{I}\Big|_{\rho_1} = x_1^{\varepsilon}x_2^{\varepsilon}(1+x_2)^{-1-3\varepsilon}, \qquad \mathcal{I}\Big|_{\rho_2} = x_1^{\varepsilon}x_2^{\varepsilon}(1+x_1)^{-1-3\varepsilon}, \qquad \mathcal{I}\Big|_{\rho_1\rho_2} = x_1^{\varepsilon}x_2^{\varepsilon}$$

Hyperlogarithms in a nutshell

$$\boxed{\mathrm{H}(z;\sigma_1,\ldots,\sigma_w) \;\equiv\; \int_0^z \frac{\mathrm{d}t_1}{t_1-\sigma_1}\int_0^{t_1}\frac{\mathrm{d}t_2}{t_2-\sigma_2}\cdots\int_0^{t_{w-1}}\frac{\mathrm{d}t_w}{t_w-\sigma_w}, \quad \mathrm{H}(z;\emptyset)=1}$$

$$\mathrm{H}(z;0)=\log z, \qquad \mathrm{H}(z;\sigma)=\log(z-\sigma)-\log(-\sigma) \;\;(\sigma\neq 0)$$

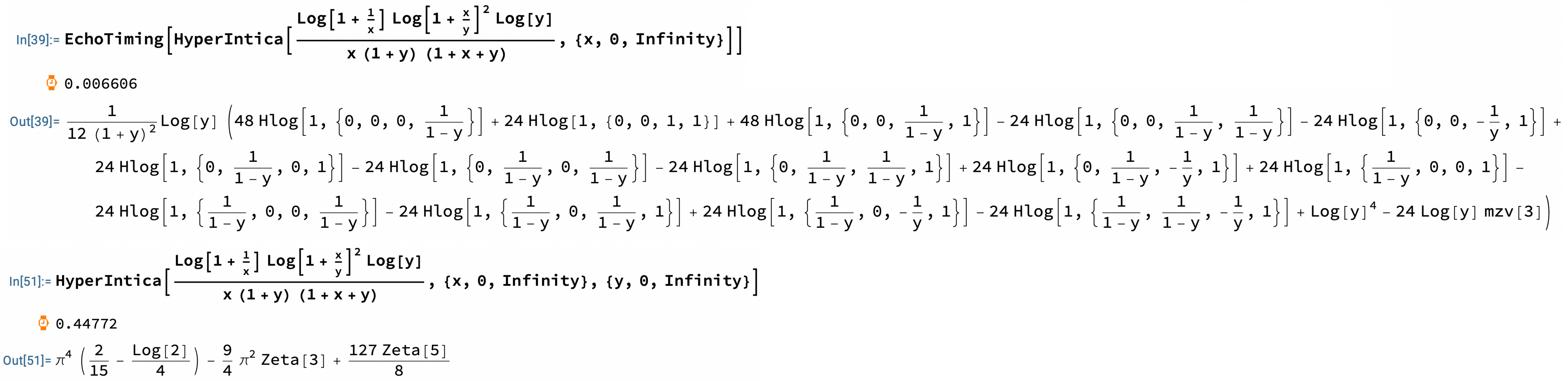
$$\log(1-z)=\mathrm{H}(z;1), \quad \mathrm{Li}_n(z)=-\mathrm{H}(z;\underbrace{0,\ldots,0}_{n-1},1), \quad \mathrm{Li}_2(z)=-\mathrm{H}(z;0,1)$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{H}(z;\sigma_1,\sigma_2,\ldots,\sigma_w) \;=\; \frac{1}{z-\sigma_1}\,\mathrm{H}(z;\sigma_2,\ldots,\sigma_w)}$$

What are we left with after expanding the integrands in ε ?

$$I^{\text{finite}}(\varepsilon,\mathbf{s}) = \int \mathrm{d}^n x \, \frac{\sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!}\left(\sum_i b_i \log P_i\right)^k}{\prod_{a_i<0} P_i^{|a_i|}} \prod_{a_i>0} P_i^{a_i}$$

The procedure to express this kind of LR integral is implemented through the command* **`HyperIntica[ ]`**



```
In[39]:= EchoTiming[HyperIntica[(Log[1 + 1/x] Log[1 + x/y]^2 Log[y])/(x (1 + y) (1 + x + y)), {x, 0, Infinity}]]

  0.006606

Out[39]= 1/(12 (1 + y)^2) Log[y] (48 Hlog[1, {0, 0, 0, 1/(1 - y)}] + 24 Hlog[1, {0, 0, 1, 1}] + 48 Hlog[1, {0, 0, 1/(1 - y), 1}] - 24 Hlog[1, {0, 0, 1/(1 - y), 1/(1 - y)}] - 24 Hlog[1, {0, 0, -1/y, 1}] +
      24 Hlog[1, {0, 1/(1 - y), 0, 1}] - 24 Hlog[1, {0, 1/(1 - y), 0, 1/(1 - y)}] - 24 Hlog[1, {0, 1/(1 - y), 1/(1 - y), 1}] + 24 Hlog[1, {0, 1/(1 - y), -1/y, 1}] + 24 Hlog[1, {1/(1 - y), 0, 0, 1}] -
      24 Hlog[1, {1/(1 - y), 0, 0, 1/(1 - y)}] - 24 Hlog[1, {1/(1 - y), 0, 1/(1 - y), 1}] + 24 Hlog[1, {1/(1 - y), 0, -1/y, 1}] - 24 Hlog[1, {1/(1 - y), 1/(1 - y), -1/y, 1}] + Log[y]^4 - 24 Log[y] mzv[3])

In[51]:= HyperIntica[(Log[1 + 1/x] Log[1 + x/y]^2 Log[y])/(x (1 + y) (1 + x + y)), {x, 0, Infinity}, {y, 0, Infinity}]

  0.44772

Out[51]= π^4 (2/15 - Log[2]/4) - 9/4 π^2 Zeta[3] + (127 Zeta[5])/8
```

`HyperIntica[ ]` is (one of) the integration engine in **`SubTropica`**

* *Ask me later for more details*

Hyperlogarithms in a nutshell

$$\boxed{\mathrm{H}(z;u)\cdot\mathrm{H}(z;v) \;=\; \sum_{w\,\in\,u\,\sqcup\!\sqcup\,v} \mathrm{H}(z;w)}$$

where $u \sqcup\!\sqcup v =$ all mergings of u, v preserving internal order

$$\log^2 z \;=\; \mathrm{H}(z;0)^2 \;=\; \mathrm{H}\big(z;[0]\sqcup\!\sqcup[0]\big) \;=\; 2\,\mathrm{H}(z;0,0)$$

$$\log z\,\log(1-z) \;=\; \mathrm{H}(z;0)\cdot\mathrm{H}(z;1) \;=\; \mathrm{H}(z;0,1)+\mathrm{H}(z;1,0)$$

$$\mathrm{H}(z;0,1) = -\mathrm{Li}_2(z), \qquad \mathrm{H}(z;1,0) = \log(z)\log(1-z)+\mathrm{Li}_2(z)$$

Integration algorithm in a nutshell

Step 1 — standard form via partial fractions & shuffle:

$$f_{n-1} \;=\; \sum_{\vec{\sigma},\tau,k} \frac{\mathrm{H}(x;\vec{\sigma})}{(x-\tau)^k}\,\lambda_{\vec{\sigma},\tau,k}$$

Step 2 — antiderivative F, $\partial_x F = f_{n-1}$. Simple pole $(k=1)$:

$$\int \frac{\mathrm{H}(x;\vec{\sigma})}{x-\tau}\,\mathrm{d}x \;=\; \mathrm{H}(x;\tau,\vec{\sigma}) \qquad (\tau \to \text{new first letter, weight} + 1)$$

Step 3 — evaluate at endpoints:

$$f_{\text{integrated}} \;=\; \lim_{x\to\infty} F(x) \;-\; \lim_{x\to 0^+} F(x)$$

Integration algorithm in a nutshell

Trailing zeros $(z \to 0)$:

$$\mathrm{H}(z;\sigma,0) \ = \ \mathrm{H}(z;\sigma)\ \log z - \mathrm{H}(z;0,\sigma) \quad \Longrightarrow \quad \mathrm{Reg}_0\, \mathrm{H}(z;\sigma,0) \ = \ -\mathrm{H}(z;0,\sigma)$$

Trailing $(-\lambda)$'s $(z \to \infty,\ \lambda > 0)$:

$$\mathrm{H}(z;-\lambda) \ = \ \log(z+\lambda) - \log\lambda \quad \Longrightarrow \quad \mathrm{Reg}_\infty\, \mathrm{H}(z;-\lambda) \ = \ 0$$