The Analytic Functional in the Conformal Bootstrap

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based on arXiv:2307.11144 with Kausik Ghosh & work in progress

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Conformal field theory: literally central

Not a corner of theoretical physics — almost literally its *center*.

- Organize all QFTs by renormalization-group flow: every flow ends on a fixed point, and those scale-invariant fixed points *are* CFTs — the anchors of the whole landscape.
- The same CFTs then surface everywhere: critical magnets & the liquid–vapor point, the string worldsheet, quantum-critical matter.
- Via AdS/CFT, a CFT *is* quantum gravity one dimension up.
- In $d > 2$ these CFTs are typically strongly coupled, with no Lagrangian — the conformal bootstrap is the most mature way to study them.

And it works — the 3d Ising CFT pinned to record precision

$$\Delta_\sigma = 0.518148806(24) \qquad \Delta_\epsilon = 1.41262528(29)$$

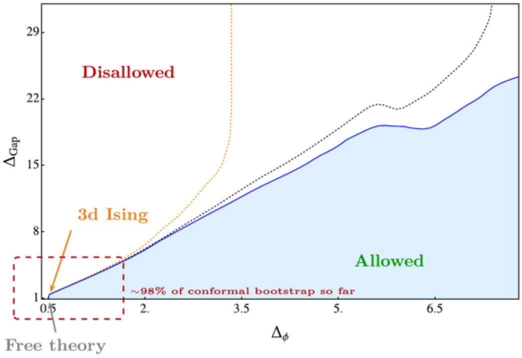
[Chang *et al.*, arXiv:2411.15300]

The conformal bootstrap landscape

- The bootstrap carves the space of CFTs into allowed / disallowed regions.
- The **3d Ising** CFT sits at a sharp kink on the boundary.

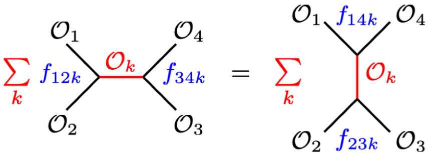
The catch

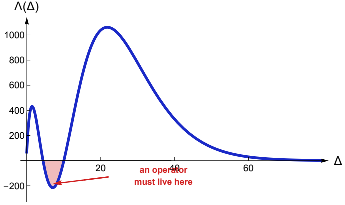
Standard numerics use a *derivative basis*—powerful, but the cost grows fast. Can a smarter, analytic basis do better?



Δ_{gap} vs. external dimension Δ_ϕ in 3d

Conformal bootstrap: a crash course





Sum rule forces operators where $\Lambda(\Delta) = \omega(F_\Delta) < 0$ (shaded).

Crossing = conformal-block sum in two channels:

$$\sum_{\mathcal{O}} f_{\mathcal{O}}^2 \, F_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(u,v) = 0,$$

$$F_{\Delta,\ell} = v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u).$$

Act with a linear functional ω:

$$\omega(F_{\mathbb{1}}) + \sum_{\mathcal{O}} f_{\mathcal{O}}^2 \, \omega(F_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}) = 0.$$

If $\omega(F_{\mathbb{1}}) > 0$ and $\omega(F_{\Delta,\ell}) \geq 0 \ \forall \, \Delta \geq \Delta_{\text{gap}}(\ell)$: the sum cannot vanish $\Rightarrow$ no such CFT exists.

The optimal functional — and the basis problem

- The extremal functional saturating a bound has double zeros on the physical spectrum.
- Standard numerics search ω in the *derivative basis*

$$\omega = \sum_{m+n \leq \Lambda} \lambda_{mn} \, \partial_z^m \partial_{\bar{z}}^n \big|_{z=\bar{z}=\frac{1}{2}} .$$

- Effective, but Λ derivatives is a generic basis — no knowledge of the answer built in.

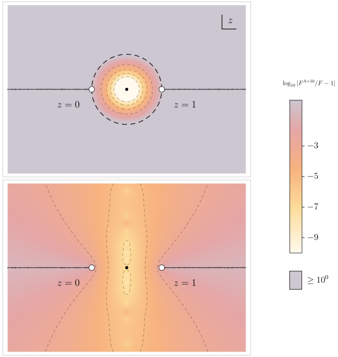
1d analytic functionals [Mazac, Paulos, . . .]

Two expansions of the *same* crossing vector $F_\Delta(z|\Delta_\phi)$:

power expansion at $z = \frac{1}{2}$: $$F_\Delta(z|\Delta_\phi) = \sum_n \frac{(z-\frac{1}{2})^n}{n!}\,\partial_z^n F_\Delta(\tfrac{1}{2}|\Delta_\phi)$$

analytic-functional basis: $$F_\Delta(z|\Delta_\phi) = \sum_n \left[\alpha_n^-(\Delta)\,F_{\Delta_n}(z|\Delta_\phi) + \beta_n^-(\Delta)\,\partial F_{\Delta_n}(z|\Delta_\phi)\right]$$

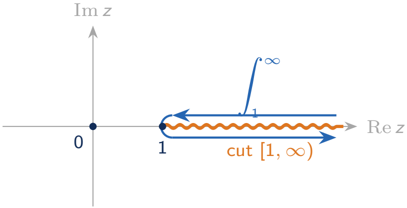
- The coefficients α_n^-, β_n^- *are* the analytic functionals (dual basis, $\alpha_n(F_{\Delta_m}) = \delta_{nm}$), with double zeros on the GFF spectrum Δ_n.
- Pays off as a huge domain of convergence: accurate almost everywhere on the z-plane (bottom), whereas the power-law basis works only inside $|z-\frac{1}{2}| < \frac{1}{2}$ (top).



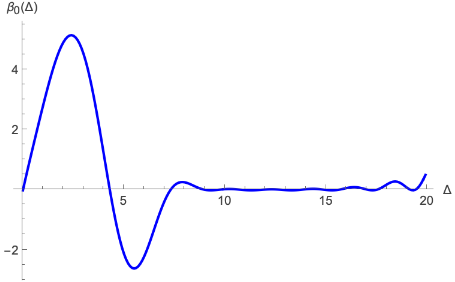
The 1d analytic functional, explicitly

Each functional is a contour integral of the crossing vector against a kernel $h_\pm$, weighting its discontinuity $\mathcal{I}_z$ across the cut $[1, \infty)$:

$$\omega_\pm(\Delta \,|\, \Delta_\phi) = \int_1^\infty \frac{\mathrm{d}z}{\pi}\, h_\pm(z)\, \mathcal{I}_z\, F_{\pm,\Delta}(z \,|\, \Delta_\phi), \qquad \mathcal{I}_z F(z) = \lim_{\epsilon \to 0^+} \frac{F(z+i\epsilon) - F(z-i\epsilon)}{2i}.$$



$\mathcal{I}_z$ = discontinuity picked up along the cut.



A functional profile $\beta_0(\Delta)$ — double zeros on the GFF spectrum Δ_n.

Higher d: identifying the right basis

Proposals for analytic functionals in higher d are numerous [Mazáč, Rastelli, Zhou, Paulos, Penedones, Silva, Zhiboedov, Caron-Huot, Simmons-Duffin, Gopakumar, Sinha, Zahed, ...] — but seldom in a form *useful for numerics*. We therefore first identify a basis that meets every criterion:

	Derivative	ω^F, ω^B (1d)	$\nu_{i,j}$ & $\mu_{i,j}$	**Product func.**
Finiteness	✓	✓	✓	✓
Swapping	✓	✓	✓	✓
Completeness	✓	✓	✓	✓
Positivity	✓	✓	✗	✓
Computability	✓	✓	✓	✓

The winning product functional $\omega^- \otimes \omega^+$ acts as

$$\left(\omega^- \otimes \omega^+\right)\left(F_{\Delta,\ell}(z,\bar{z})\right) := 2\int_{++} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi^2}\, h_-(z)\, h_+(\bar{z}) \left[\mathcal{I}_z\mathcal{I}_{\bar{z}}\, F_{\Delta,\ell}(z,\bar{z}) + \mathcal{I}_z\mathcal{I}_{\bar{z}}\, F_{\Delta,\ell}(z,1-\bar{z})\right].$$

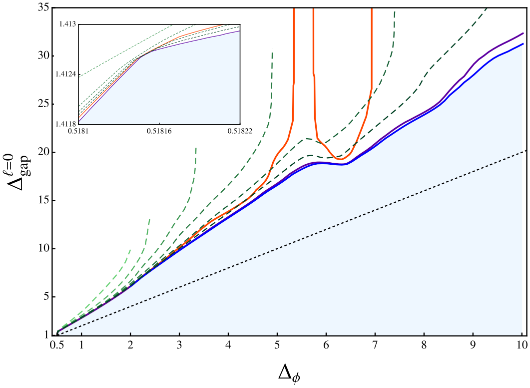
Proof of concept: the 2d Ising spectrum

Scalar-channel gap maximization with 50 functional-basis functionals at $\Delta_\phi = 0.125$, compared to the exact $2d$ critical Ising data:

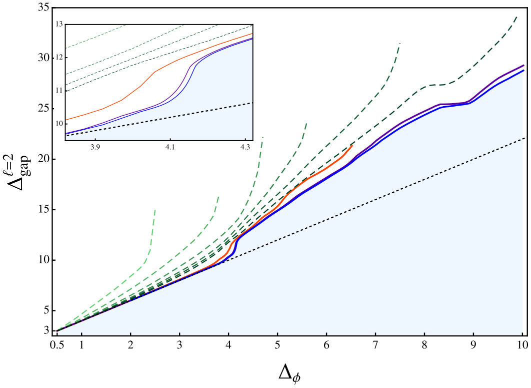
Spin	Δ	OPE^2	Δ_{num}	$\mathrm{Err}_\Delta(\%)$	$\mathsf{OPE}^2_{\mathsf{num}}$	$\mathrm{Err}_{\mathrm{OPE}^2}(\%)$
0	1	2.50000000e-1	1.00000009	9.0e-6	2.49999985e-1	6.0e-6
	4	2.44140625e-4	4.00001322	3.3e-4	2.44137861e-4	1.1e-3
	8	4.82812729e-8	7.99525353	5.9e-2	4.78906432e-8	8.1e-1
2	2	3.12500000e-2	2.00000000	0	3.12499999e-2	3.2e-7
	6	6.86644375e-6	6.00080943	1.3e-2	6.80179598e-6	9.4e-1
4	4	4.39451562e-4	4.00000000	0	4.39455331e-4	8.6e-4
	5	3.05175590e-5	5.00004158	8.3e-4	3.05149040e-5	8.7e-3
	8	2.12871504e-7	8.00476230	6.0e-2	2.11291737e-7	7.4e-1
6	6	1.36239239e-5	6.00022916	3.8e-3	1.36305441e-5	4.9e-2
	7	1.52586728e-6	7.00259564	3.7e-2	1.51943252e-6	4.2e-1
8	8	5.39324266e-7	8.00012269	1.5e-3	5.39297105e-7	5.0e-3

All operators with $\Delta \leq 8$ shown (numerical artifacts discarded). Both *dimensions* and *OPE coefficients* are recovered to high accuracy.

3d results: gap bounds



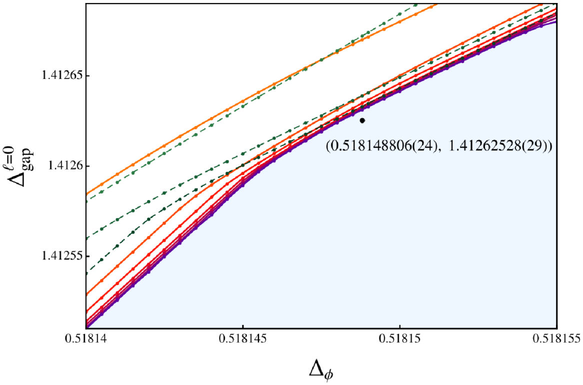
Scalar gap $\Delta_{\rm gap}^{\ell=0}$ vs. Δ_ϕ.



Spin-2 gap $\Delta_{\rm gap}^{\ell=2}$ vs. Δ_ϕ.

Colored curves: analytic functionals (increasing order) **vs.** dashed: standard derivative bootstrap. The **3d Ising kink** is reproduced.

Zooming in on the 3d Ising point



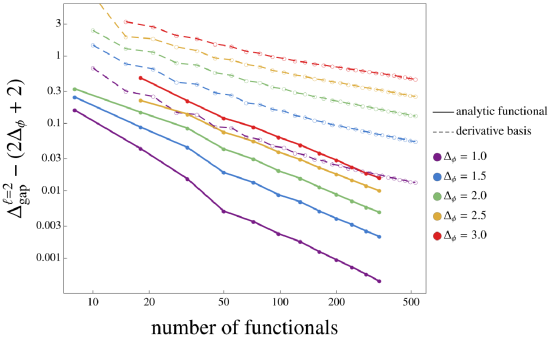
- The method localizes the 3d Ising CFT to high precision:

At the kink

$$\Delta_\phi = 0.518148806(24)$$
$$\Delta_{\mathrm{gap}}^{\ell=0} = 1.41262528(29)$$

- Consistent with the world's best determinations of $(\Delta_\sigma, \Delta_\epsilon)$.

Convergence & performance



Error in the gap bound vs. number of functionals.

- Analytic basis (solid) converges much faster than the derivative basis (dashed) at fixed cost.
- Controlled *functional by functional*, with analytic asymptotics.

Same precision, a fraction of the cost

$\sim$1 day $\longrightarrow$ $\sim$30 s

$\approx$ 3000$\times$ faster — $\sim$3 orders of magnitude

One nuclear supercarrier's engines $\approx$ the power of 3000 family cars.

Summary & outlook

On the horizon

- Global symmetry
- Mixed correlators
- $d > 3$, especially even dimensions

Based on arXiv:2307.11144 (w/ K. Ghosh) + work in progress.

A note on AI usage

- The numerical pipeline was developed with the assistance of Claude Code and Codex.
- These slides were prepared entirely with Claude Code.