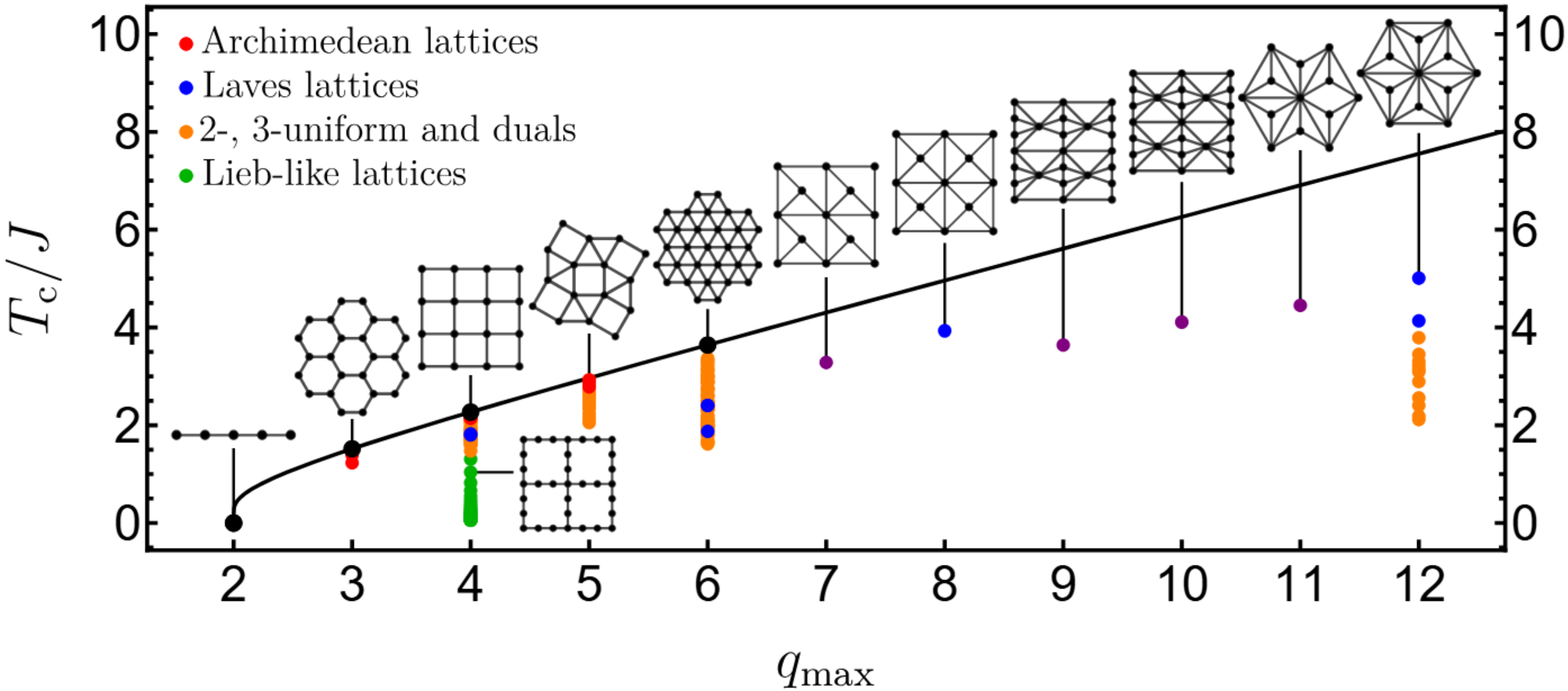
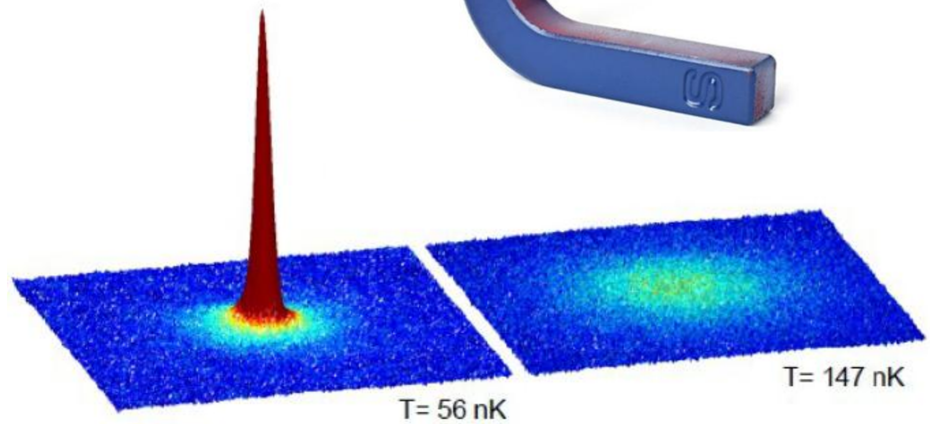
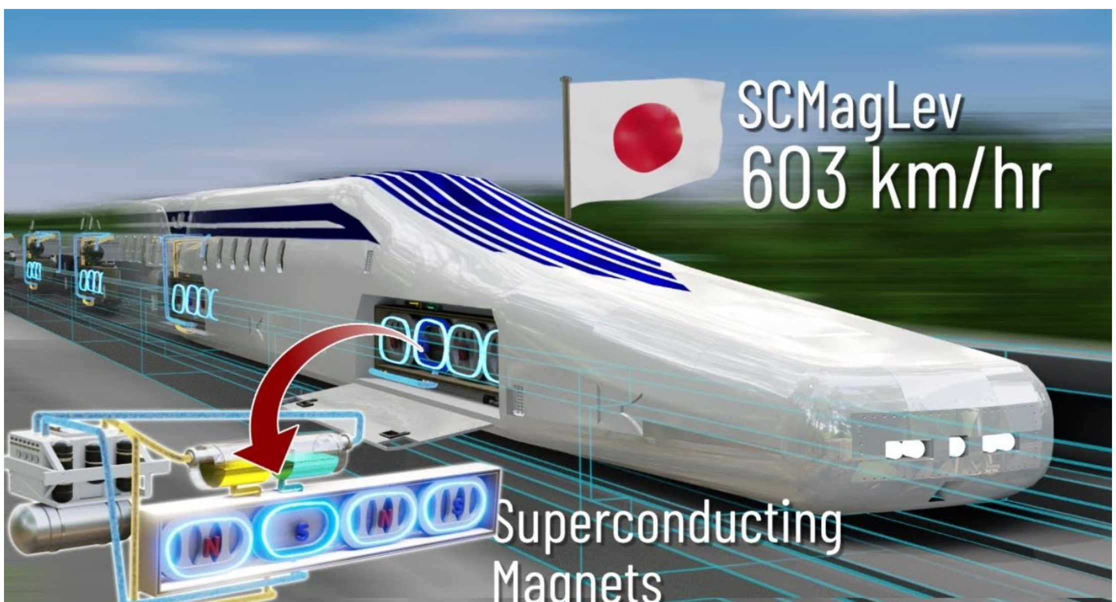
Exact critical temperature bounds for two-dimensional Ising models



Igor Boettcher, Davidson Joseph, Connor Walsh
University of Alberta, Quantum Horizons Alberta

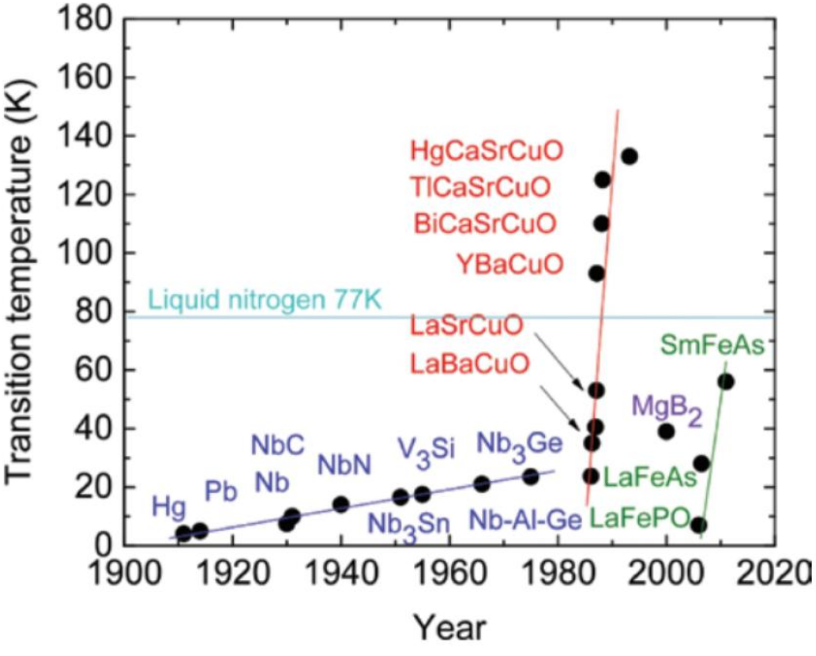
Phase Transitions



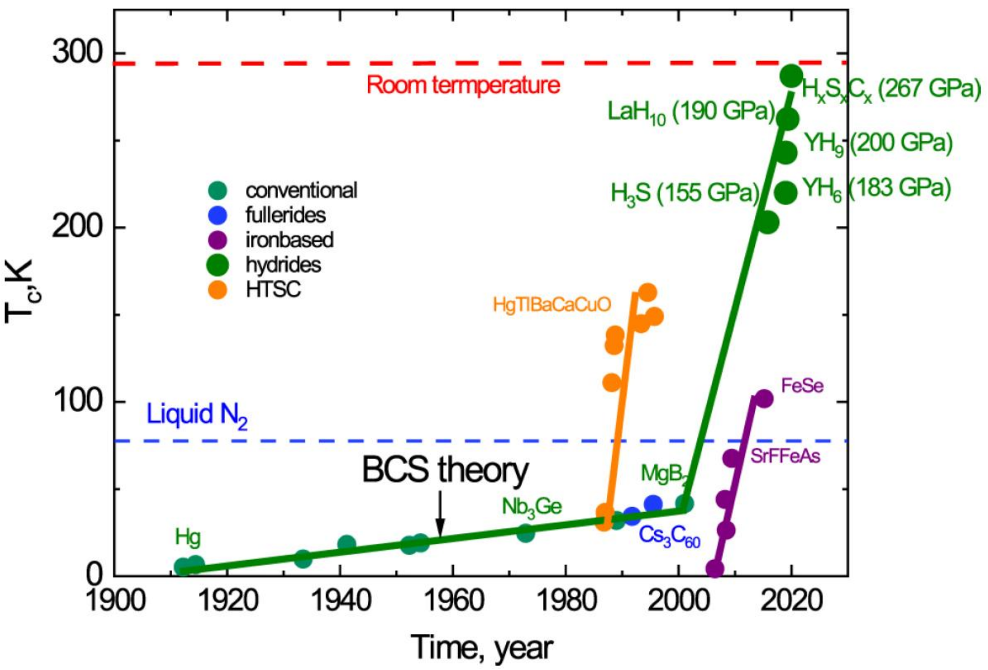


Tc of superconducting materials

before 2015

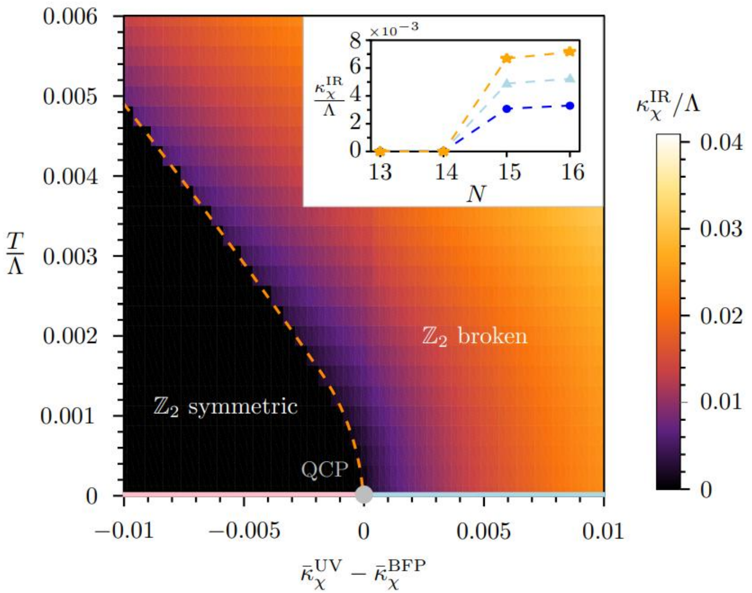


after 2015

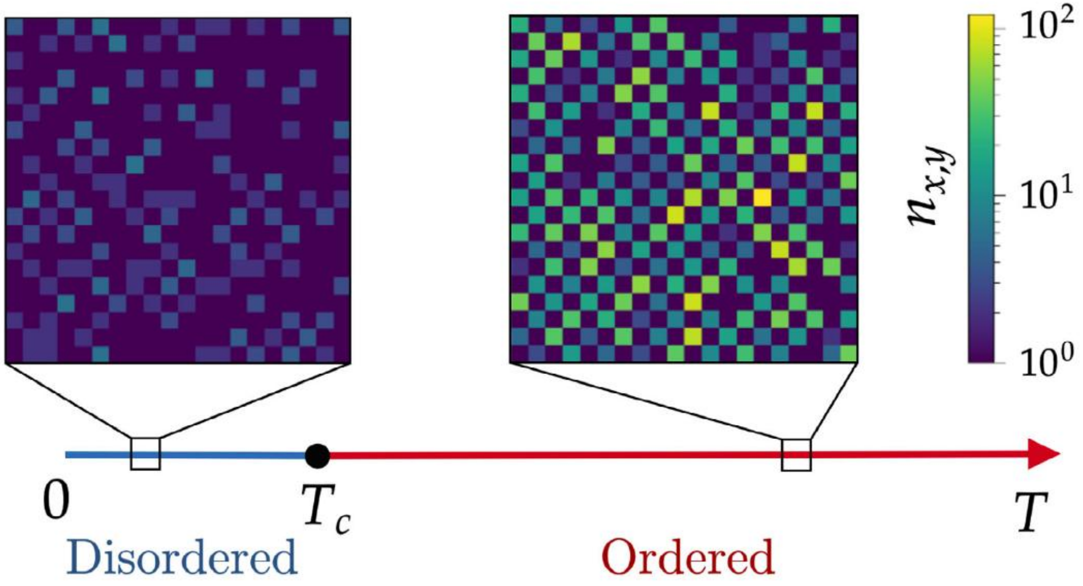


Is there an upper limit to T_c?

Pomeranchuk Effect / Entropic Order:
Tc can be infinite for certain systems!



Hawashin, Rong, Scherer,
Phys. Rev. Lett. 134, 041602 (2025)



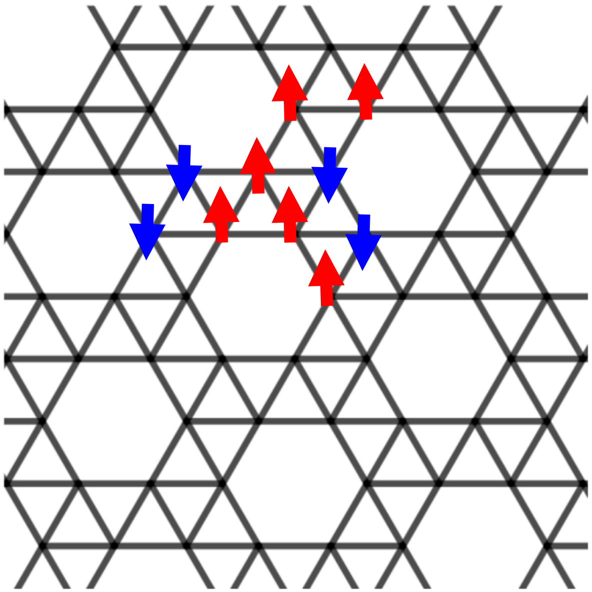
Han, Huang, Komargodski, Lucas, Popov,
Nat. Commun. 17, 87 (2026)

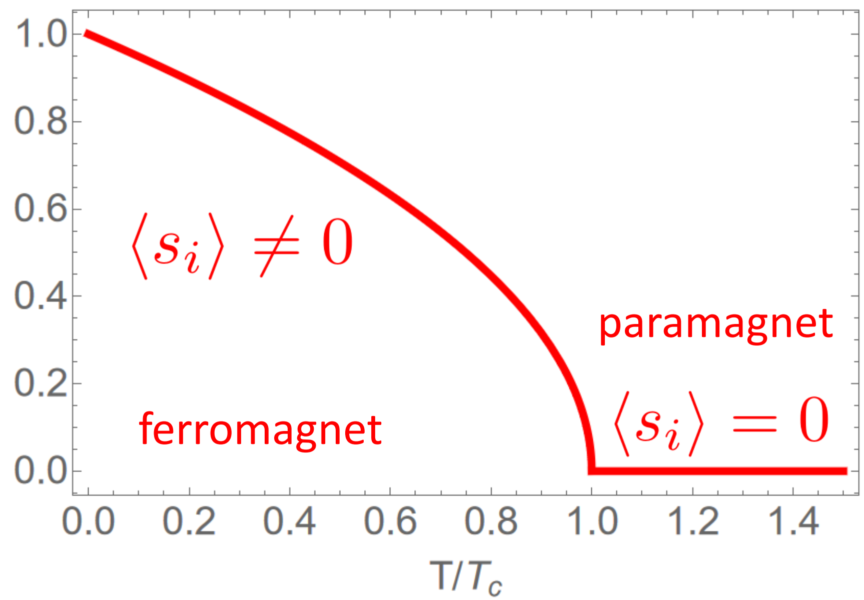
Is this true for all systems?

classical Ising model

$$H(\vec{s}) = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i$$

$$\vec{s} = (s_1, \ldots, s_N), \;\; s_i = \pm 1$$





classical Ising model

$$H(\vec{s}) = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i$$

$$\vec{s} = (s_1, \ldots, s_N), \; s_i = \pm 1$$

$$Z(T,h) = \sum_{\vec{s}} e^{-\beta H(\vec{s})}$$ ⟵ 2^N terms

$$f(T,h) = \frac{F}{N} = -\frac{1}{\beta N} \log Z$$

classical Ising model

$$H(\vec{s}) = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i$$

	1D	2D	2D bilayer	3D
$f(T)$	Ising 1925	Onsager 1944	x	x
$f(T,h)$	Ising 1925	x	x	x

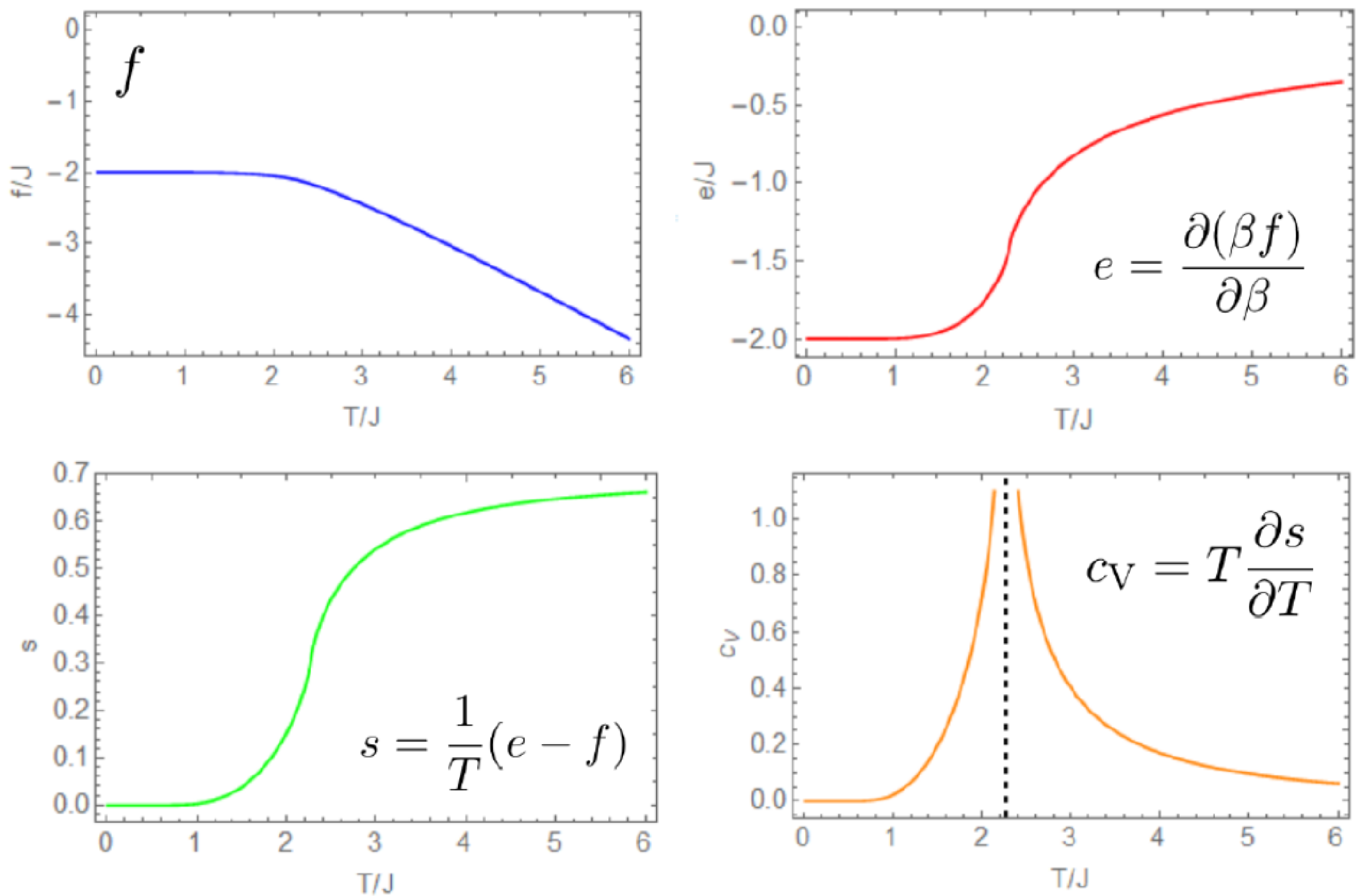
frontier

integrability / complexity / computability

	1D	2D	2D bilayer	3D
$f(T)$	Ising 1925	Onsager 1944	x	x
$f(T,h)$	Ising 1925	x	x	x

Onsager's solution (square lattice)

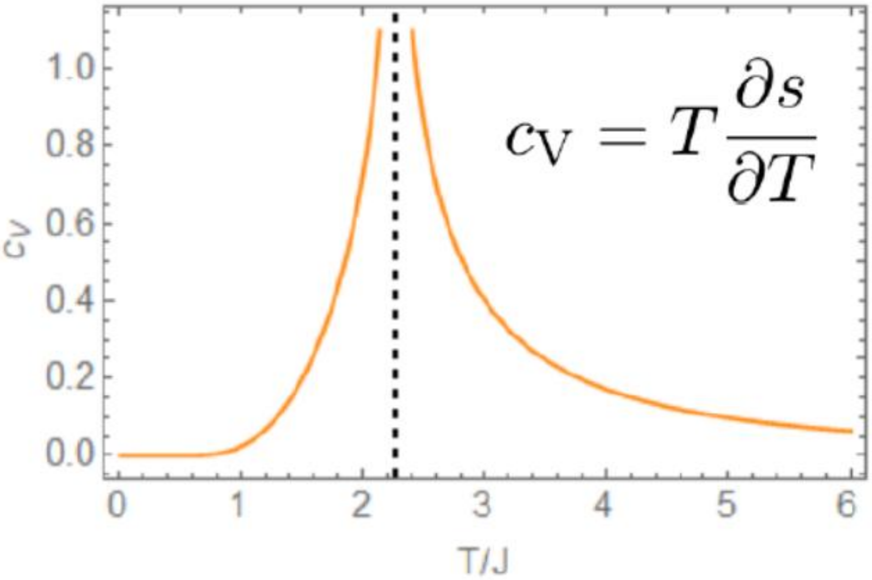
$$\beta f_{\mathrm{2D}} = -\log 2 - \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\log\Big[\cosh(2\beta J)^2 - \sinh(2\beta J)(\cos k_1 + \cos k_2)\Big]$$



Onsager's solution (square lattice)

$$\beta f_{2\mathrm{D}} = -\log 2 - \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\log\Big[\underbrace{\cosh(2\beta J)^2 - \sinh(2\beta J)(\cos k_1 + \cos k_2)}_{\text{vanishes for } \vec{k}=0 \text{ at } T_\mathrm{c}}\Big]$$

$$\begin{aligned}0 &= \cosh(2\beta_\mathrm{c}J)^2 - 2\sinh(2\beta_\mathrm{c}J)\\ &= 1 + \sinh(2\beta_\mathrm{c}J)^2 - 2\sinh(2\beta_\mathrm{c}J)\\ &= \Big(1 - \sinh(2\beta_\mathrm{c}J)\Big)^2\end{aligned}$$



$$\sinh(2\beta_\mathrm{c}J) = 1 \;\Rightarrow\; T_\mathrm{c}/J = \frac{2}{\log(1+\sqrt{2})} = 2.26919$$

Onsager's solution (square lattice)

$$\beta f_{2\mathrm{D}} = -\log 2 - \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\log\Big[\cosh(2\beta J)^2 - \sinh(2\beta J)(\cos k_1 + \cos k_2)\Big]$$
$$= -\log 2 - \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\log\Big[\cosh(2\beta J)^2 + \frac{1}{2}\sinh(2\beta J)\varepsilon_\square(\mathbf{k})\Big]$$

integral over 2D Brillouin zone

square lattice dispersion $\varepsilon_\square(\mathbf{k}) = -2(\cos k_1 + \cos k_2)$ shows up

1950/51: Triangular and Honeycomb lattice
(Houtappel; Temperley; Newell; Wannier; Husimi, Syozi)

$$\beta f_{\mathrm{T}} = -\log 2 - \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\log\left[\cosh(2\beta J)^3 - \sinh(2\beta J)^3 + \frac{1}{2}\sinh(2\beta J)\varepsilon_\Delta(\mathbf{k})\right)\right]$$

$$T_{\mathrm{c,T}}/J = \frac{2}{\log\sqrt{3}} = 3.64096$$

$$\beta f_{\mathrm{HC}} = -\log 2 - \frac{1}{4}\int_0^{2\pi}\frac{\mathrm{d}k_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}k_2}{2\pi}\log\left[\frac{1}{2}\left([\cosh(2\beta J)]^3 + 1 + \frac{1}{2}[\sinh(2\beta J)]^2\varepsilon_\Delta(\mathbf{k})\right)\right]$$

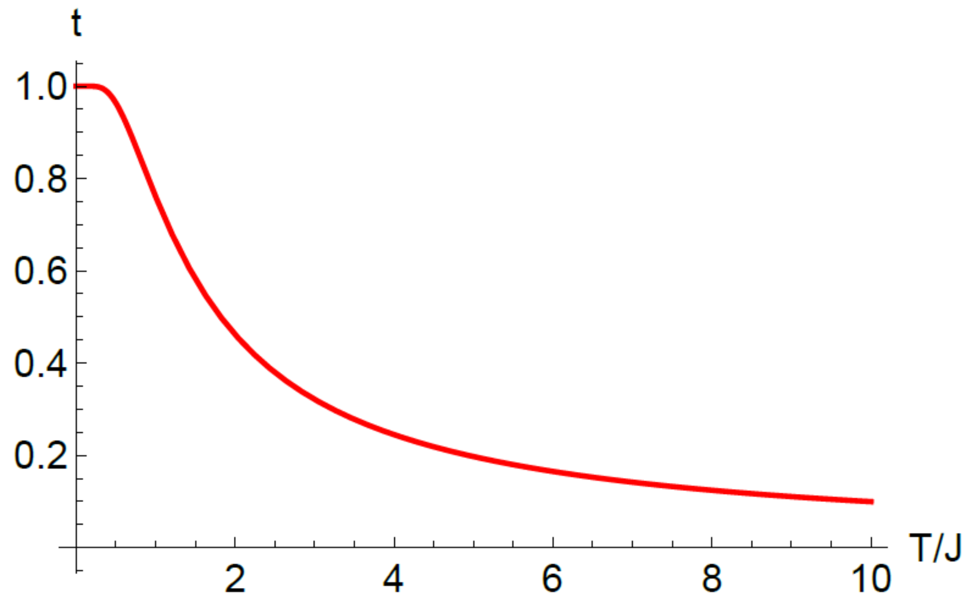
$$T_{\mathrm{c,HC}}/J = \frac{2}{\log(2+\sqrt{3})} = 1.51865$$

Kagome lattice (q=4): Tc does not only depend on q
(Kano, Naya, 1954)

$$\beta f(t) = -\log 2 + \frac{\bar{q}}{4}\log(1-t^2) - \frac{1}{2N_{\mathrm{u}}}\int_{\mathbf{k}} \log\det(\mathbb{1} - tW(\mathbf{k}))$$

Kac-Ward matrix

$$t = \tanh(\beta J)$$



$$\beta f(t) = -\log 2 + \frac{\bar{q}}{4}\log(1-t^2) - \frac{1}{2N_\mathrm{u}}\int_{\mathbf{k}} \log\det(\mathbb{1} - tW(\mathbf{k}))$$

Square lattice

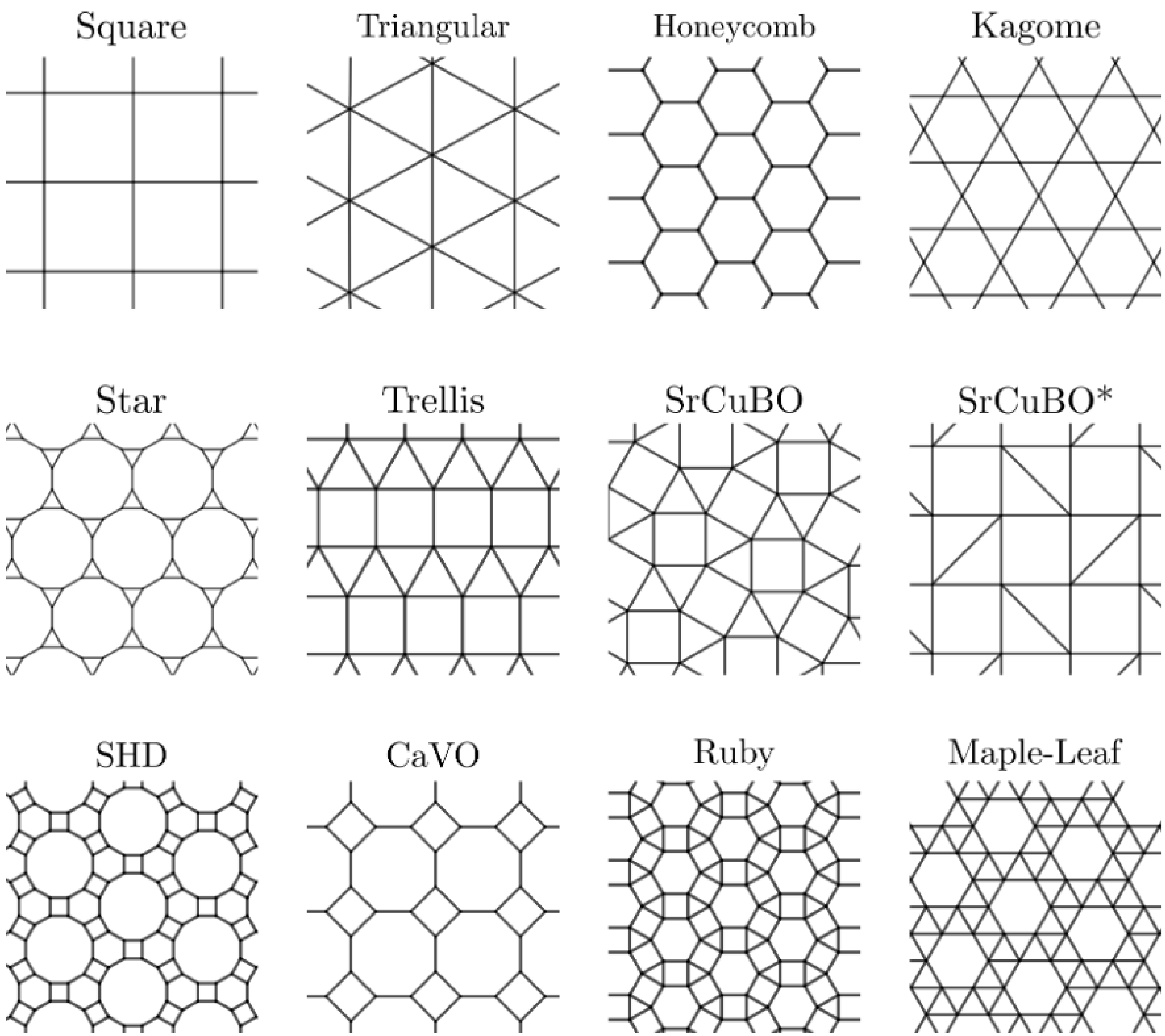
$$W(\mathbf{k}) = B_1(\mathbf{k})\Phi_{11} = \begin{pmatrix} e^{-\mathrm{i}k_1} & 0 & 0 & 0 \\ 0 & e^{-\mathrm{i}k_2} & 0 & \\ 0 & 0 & e^{\mathrm{i}k_1} & 0 \\ 0 & 0 & 0 & e^{\mathrm{i}k_2} \end{pmatrix} \begin{pmatrix} 1 & e^{\mathrm{i}\pi/4} & 0 & e^{-\mathrm{i}\pi/4} \\ e^{-\mathrm{i}\pi/4} & 1 & e^{\mathrm{i}\pi/4} & 0 \\ 0 & e^{-\mathrm{i}\pi/4} & 1 & e^{\mathrm{i}\pi/4} \\ e^{\mathrm{i}\pi/4} & 0 & e^{-\mathrm{i}\pi/4} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\mathrm{i}k_1} & e^{-\mathrm{i}k_1+\mathrm{i}\pi/4} & 0 & e^{-\mathrm{i}k_1-\mathrm{i}\pi/4} \\ e^{-\mathrm{i}k_2-\mathrm{i}\pi/4} & e^{-\mathrm{i}k_2} & e^{-\mathrm{i}k_2+\mathrm{i}\pi/4} & 0 \\ 0 & e^{\mathrm{i}k_1-\mathrm{i}\pi/4} & e^{\mathrm{i}k_1} & e^{\mathrm{i}k_1+\mathrm{i}\pi/4} \\ e^{\mathrm{i}k_2+\mathrm{i}\pi/4} & 0 & e^{\mathrm{i}k_2-\mathrm{i}\pi/4} & e^{\mathrm{i}k_2} \end{pmatrix}$$

$$\det(\mathbb{1} - tW(\mathbf{k})) = (1+t^2)^2 + t(1-t^2)\varepsilon_\square(\mathbf{k})$$

11 Archimedean Lattices

- constant coordination number q



11 Archimedean Lattices

- constant coordination number q

Archimedean Lattices				
Lattice	q_{max}	$\bar{q}$	t_c	T_c/J
Triangular	6	6	0.26795	3.64095
SrCuBO	5	5	0.32902	2.9263
Trellis	5	5	1/3	2.88542
Maple-Leaf	5	5	0.34430	2.7858
Square	4	4	0.41421	2.26921
Kagome	4	4	0.43542	2.14332
Ruby	4	4	0.43542	2.14332
Honeycomb	3	3	0.57735	1.51865
CaVO	3	3	0.60123	1.4387
SHD	3	3	0.61661	1.38982
Star	3	3	0.67070	1.23151

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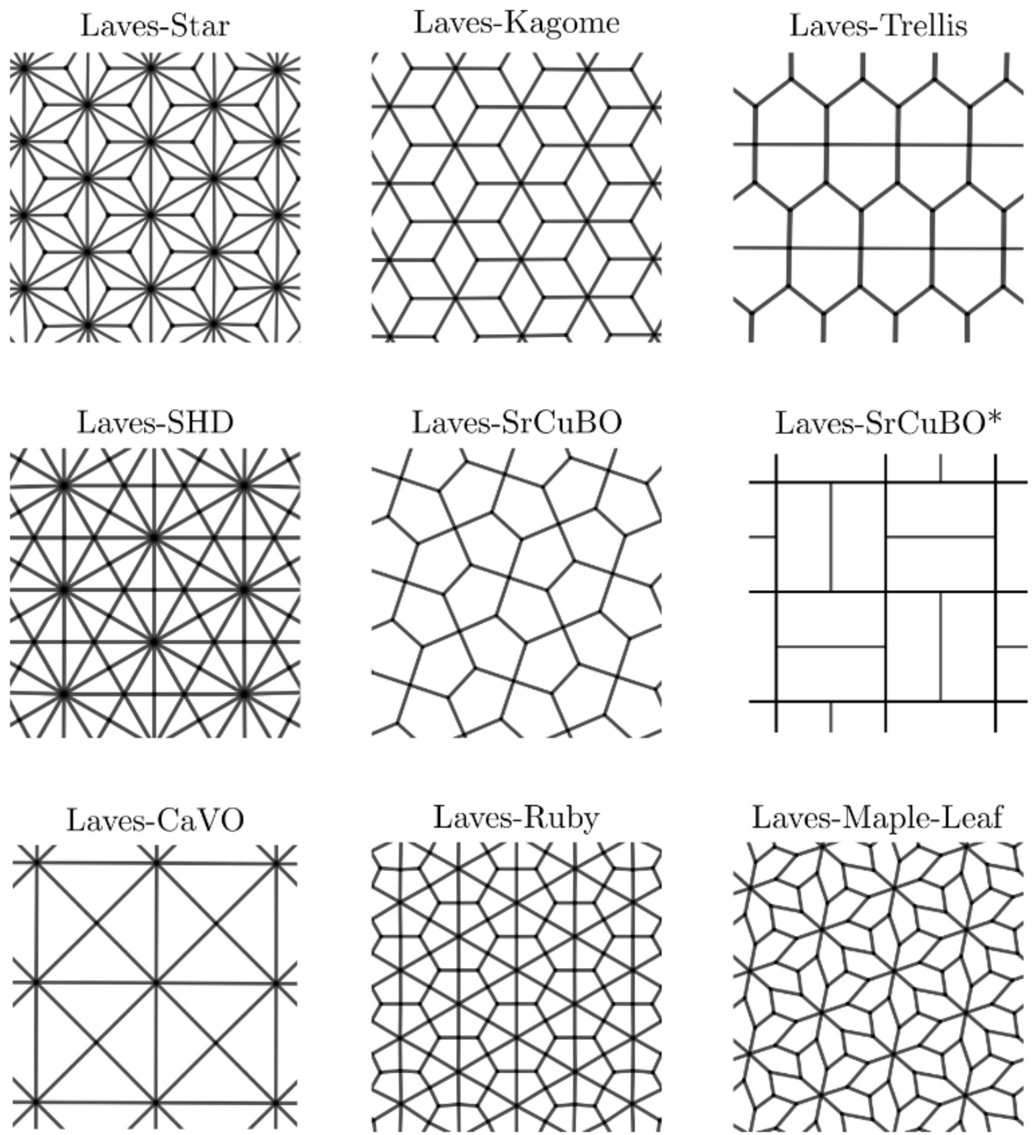
increasing Tc

Tc and q clearly related!

Laves Lattices = Dual Archimedean

- **non-constant** coordination number q



Laves Lattices = Dual Archimedean

- **non-constant** coordination number q

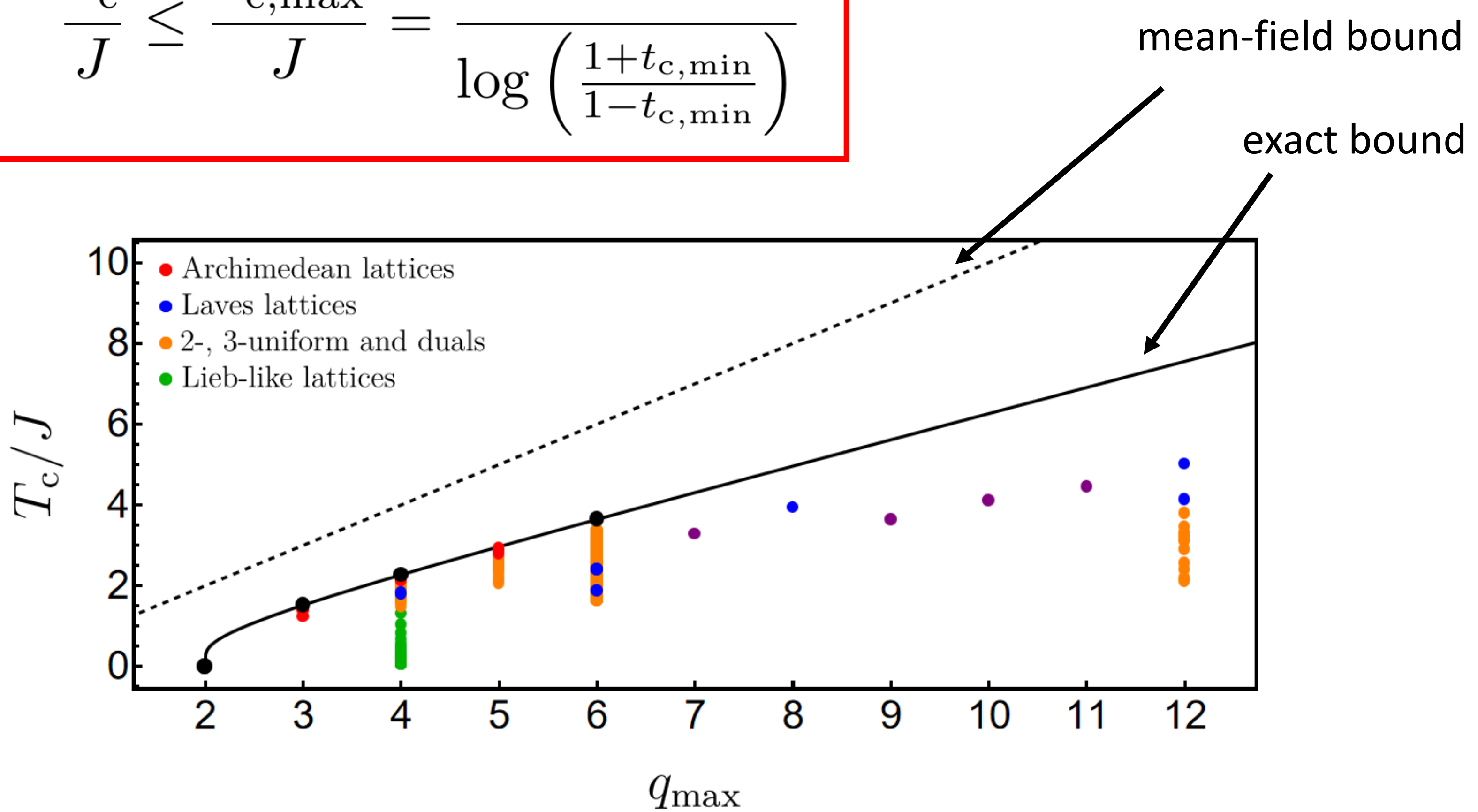
Laves Lattices				
Lattice	$q_{\max}$	$\bar{q}$	t_c	T_c/J
Laves-Star	12	6	0.19711	5.00704
Laves-SHD	12	6	0.23716	4.13629
Laves-CaVO	8	6	0.24904	3.93102
Laves-Kagome	6	4	0.39332	2.40546
Laves-Ruby	6	4	0.39332	2.40546
Laves-Maple-Leaf	6	3.33	0.48777	1.87572
Laves-Trellis	4	3.33	1/2	1.82048
Laves-SrCuBO	4	3.33	0.50486	1.79917

Tc and qmax are related

For the zero-field, ferromagnetic Ising model on any periodic tessellation of the plane with maximal coordination number qmax we prove that:

$$t_{\mathrm{c}} \geq t_{\mathrm{c,min}} = \tan\left(\frac{\pi}{2q_{\mathrm{max}}}\right)$$

$$\frac{T_{\mathrm{c}}}{J} \leq \frac{T_{\mathrm{c,max}}}{J} = \frac{2}{\log\left(\frac{1+t_{\mathrm{c,min}}}{1-t_{\mathrm{c,min}}}\right)}$$



$$\beta f(t) = -\log 2 + \frac{\bar{q}}{4}\log(1-t^2) - \frac{1}{2N_{\mathrm{u}}}\int_{\mathbf{k}} \log\det(\mathbb{1} - tW(\mathbf{k}))$$

$$0 = \det(\mathbb{1} - t_{\mathrm{c}} W_0), \;\; W_0 := W(\mathbf{0})$$

$$\Rightarrow \;\; t_{\mathrm{c}}^{-1} = \text{largest eigenvalue of } W_0$$

non-universal, lattice-dependent

$$\beta f(t) = -\log 2 + \frac{\bar{q}}{4}\log(1-t^2) - \frac{1}{2N_\mathrm{u}}\int_{\mathbf{k}} \log\det(\mathbb{1} - tW(\mathbf{k}))$$

$$0 = \det(\mathbb{1} - t_\mathrm{c}W_0),\ \ W_0 := W(\mathbf{0})$$

$$\Rightarrow\ t_\mathrm{c}^{-1} = \text{largest eigenvalue of } W_0$$

$$z = re^{\mathrm{i}\phi}$$

$$r = \sqrt{z^* z} \geq 0$$

polar decomposition

$$M = RU$$

$$R = \sqrt{M^\dagger M} \geq 0$$

$$\beta f(t) = -\log 2 + \frac{\bar{q}}{4}\log(1-t^2) - \frac{1}{2N_{\mathrm{u}}}\int_{\mathbf{k}} \log\det(\mathbb{1} - tW(\mathbf{k}))$$

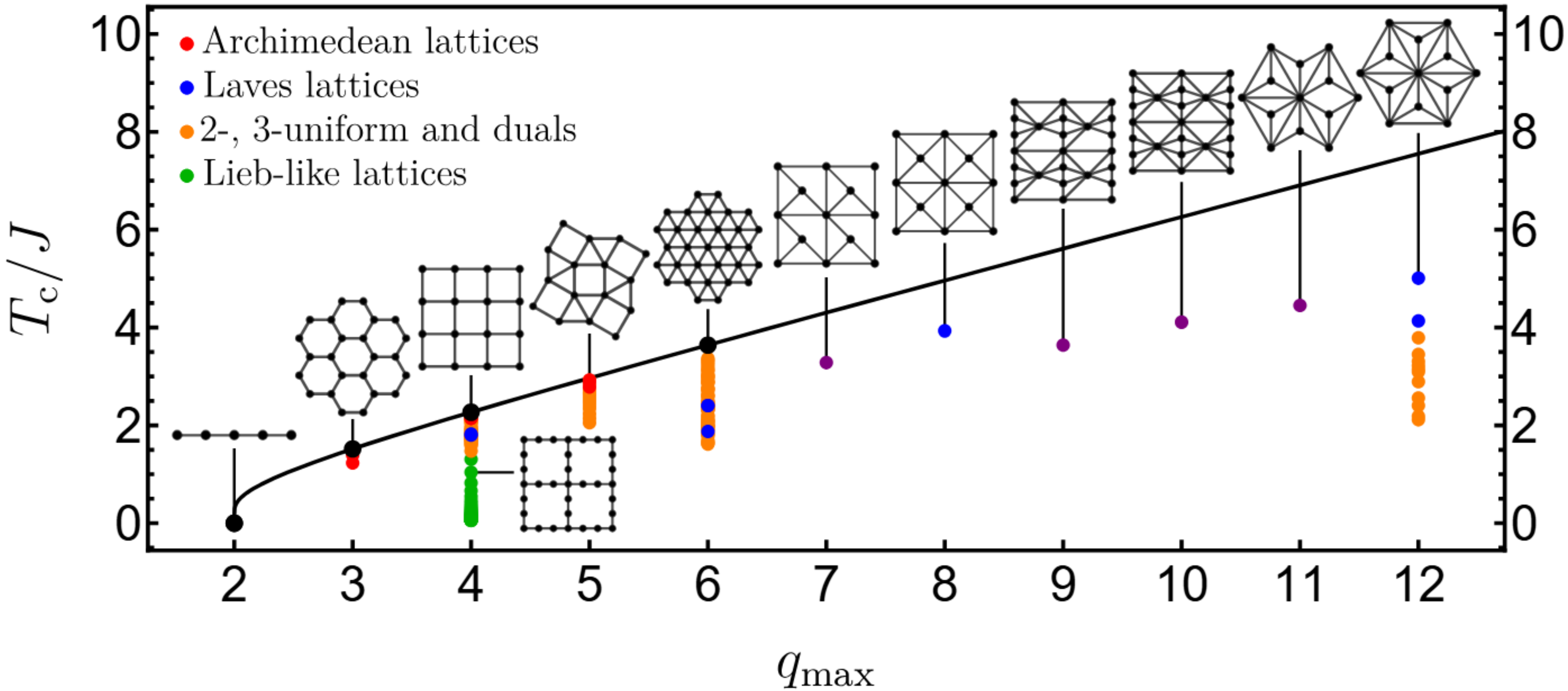
$$0 = \det(\mathbb{1} - t_{\mathrm{c}} W_0), \;\; W_0 := W(\mathbf{0})$$

$$\Rightarrow \;\; t_{\mathrm{c}}^{-1} = \text{largest eigenvalue of } W_0$$

$$\Rightarrow \;\; t_{\mathrm{c}}^{-1} \leq \sqrt{\mu_{\max}} = \sqrt{\text{largest eigenvalue of } W_0^{\dagger} W_0}$$

only depends on qmax

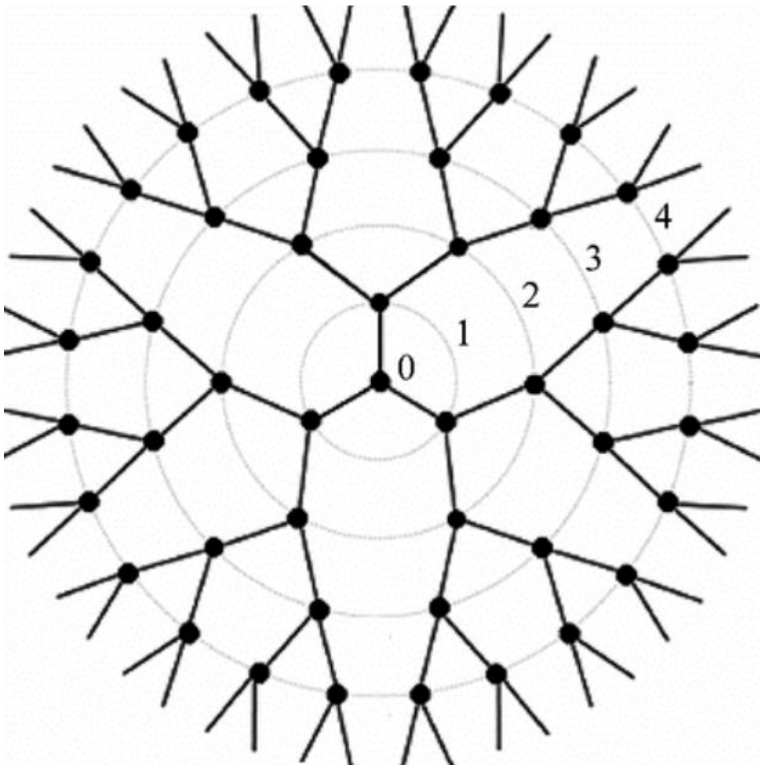
DN Joseph, **IB**, Phys. Rev. E 113, 064113 (2026)



data from 236 lattices

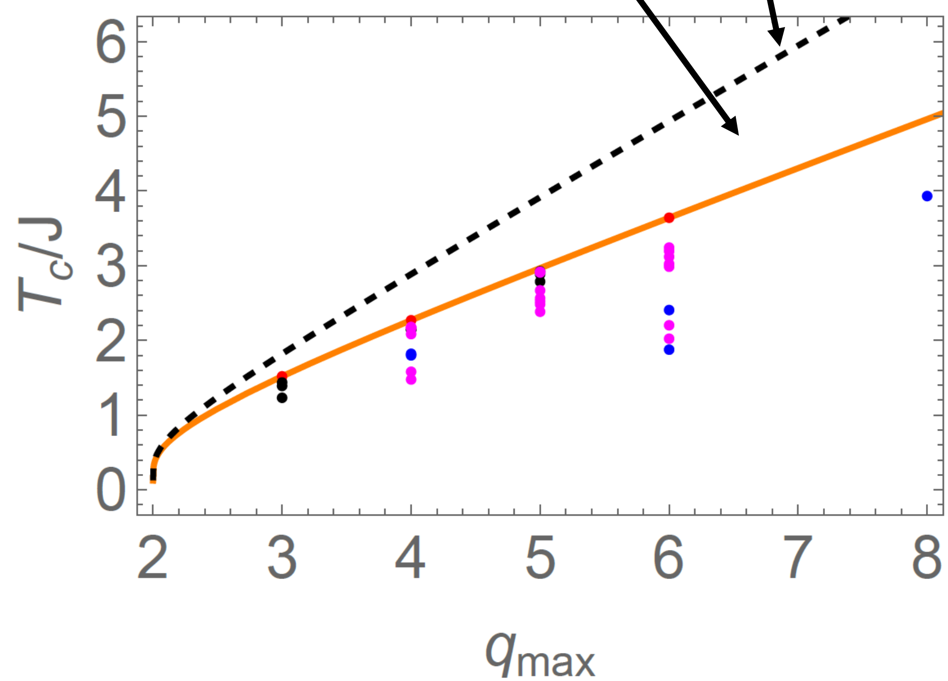
DN Joseph, **IB**, Phys. Rev. E 113, 064113 (2026)

Non-Euclidean tessellations of the plane?

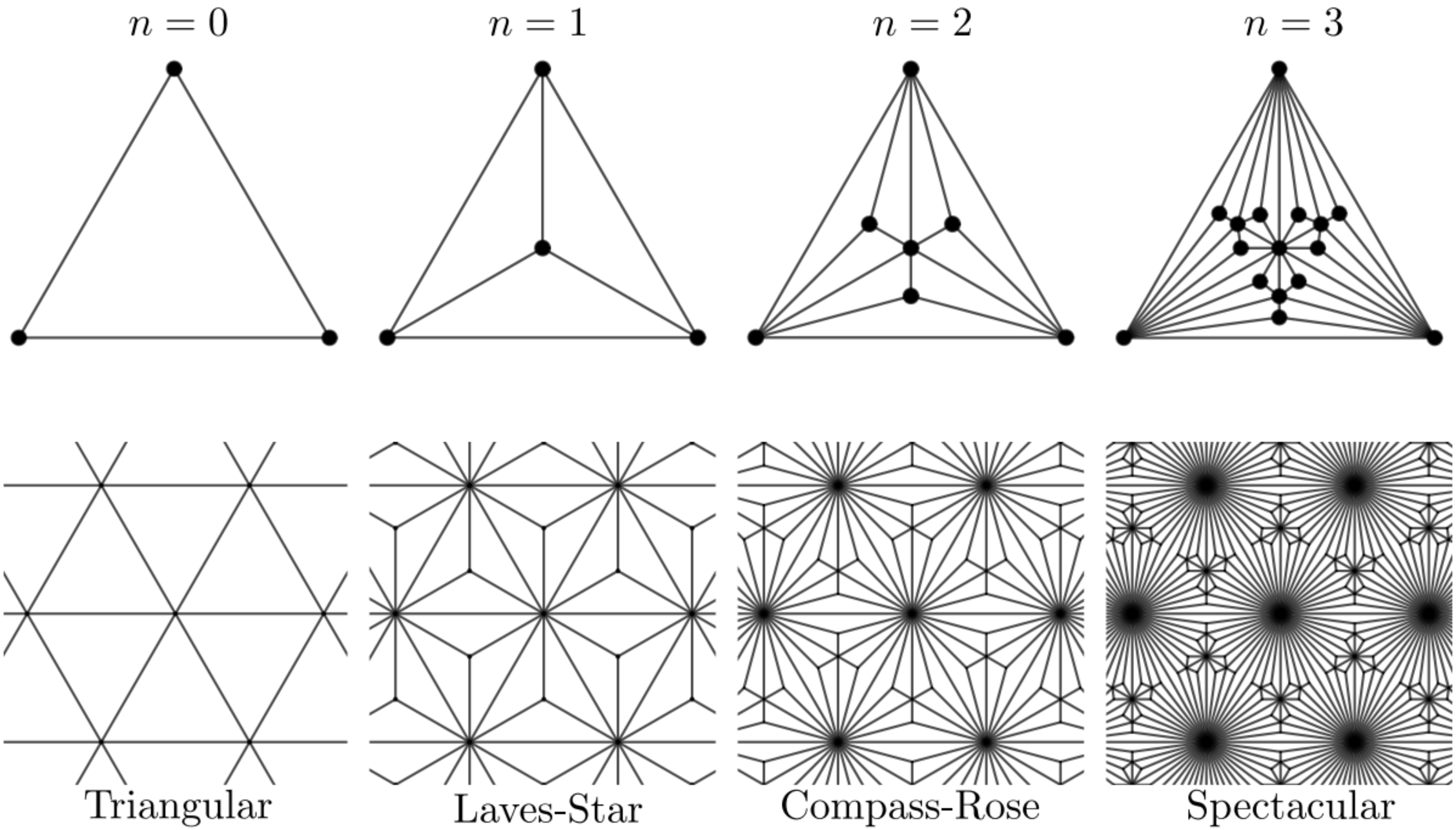


Bethe Lattice: $\frac{T_c}{J} = \frac{2}{\log \frac{q}{q-2}}$

Hyperbolic Lattices



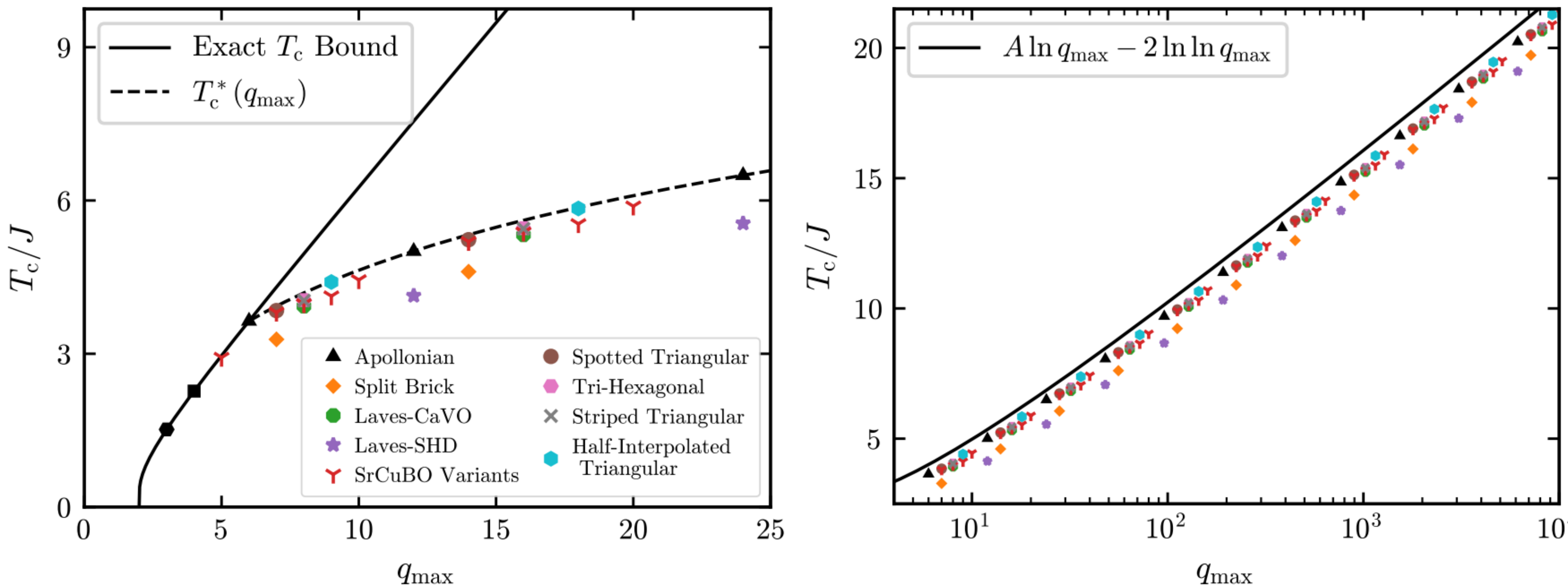
Tc Engineering: Apollonian Lattices



Triangular	Laves-Star	Compass-Rose	Spectacular
$q_{\max} = 6$	$q_{\max} = 12$	$q_{\max} = 24$	$q_{\max} = 48$
$T_c/J = 3.64$	$T_c/J = 5.01$	$T_c/J = 6.49$	$T_c/J = 8.06$

DN Joseph, CM Walsh, **IB**, arXiv:2605.10017

Tc Engineering: Apollonian Lattices



$$\frac{T_c^*}{J} \sim A \log q_{max}, \quad A = \frac{2}{\log 2} = 2.89$$

DN Joseph, CM Walsh, **IB**, arXiv:2605.10017

Igor Boettcher

Davidson Joseph

Connor Walsh

DN Joseph, **IB**, Phys. Rev. E 113, 064113 (2026)

DN Joseph, CM Walsh, **IB**, arXiv:2605.10017

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