

Bulk-Boundary Correspondence and Exceptional Points of Bloch Hamiltonians

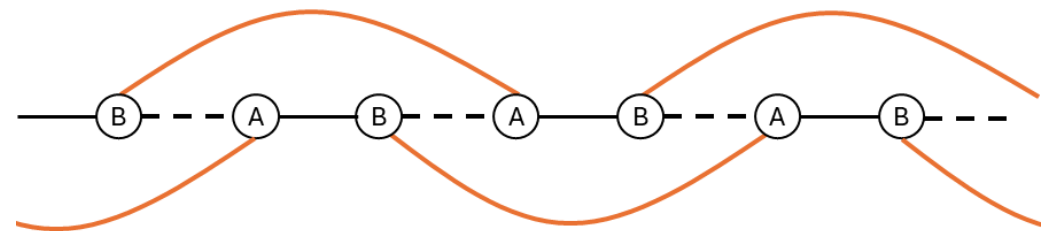
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Plan

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4. Exceptional Points and Adiabatic Protection
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6. Exceptional points and winding numbers

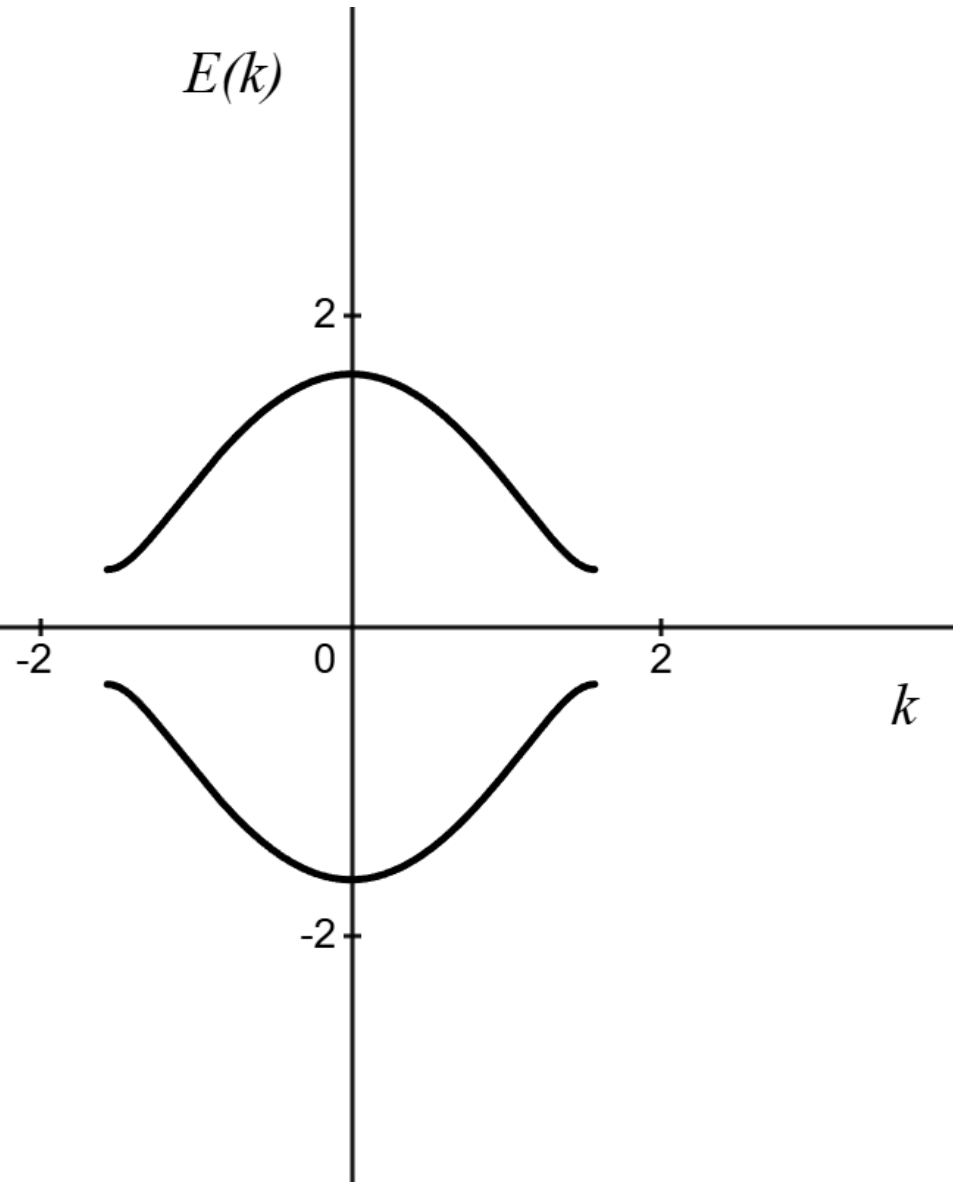
1. What is Bulk-Boundary Correspondence in 1D

- Bulk = Bloch Hamiltonian of a chain
- In 1D only possible in the presence of chiral symmetry
- $h(k) = \begin{pmatrix} 0 & q(k) \\ q^\dagger(k) & 0 \end{pmatrix}$, an $N \times N$ matrix (N even)
- The spectrum of $h(k)$ is symmetric $E(k) \rightarrow -E(k)$



1. What is Bulk-Boundary Correspondence in 1D

- If spectrum is gapped, $E(k) \neq 0 \Rightarrow \det(q(k)) \neq 0$ and $q(k) \in GL(N/2, \mathbb{C})$
- In 1D, $k \in S^1$
- $q(k): S^1 \rightarrow GL(N/2, \mathbb{C})$
- Gapped phases can be classified by their topological invariant.



1. What is Bulk-Boundary Correspondence in 1D: Edge States

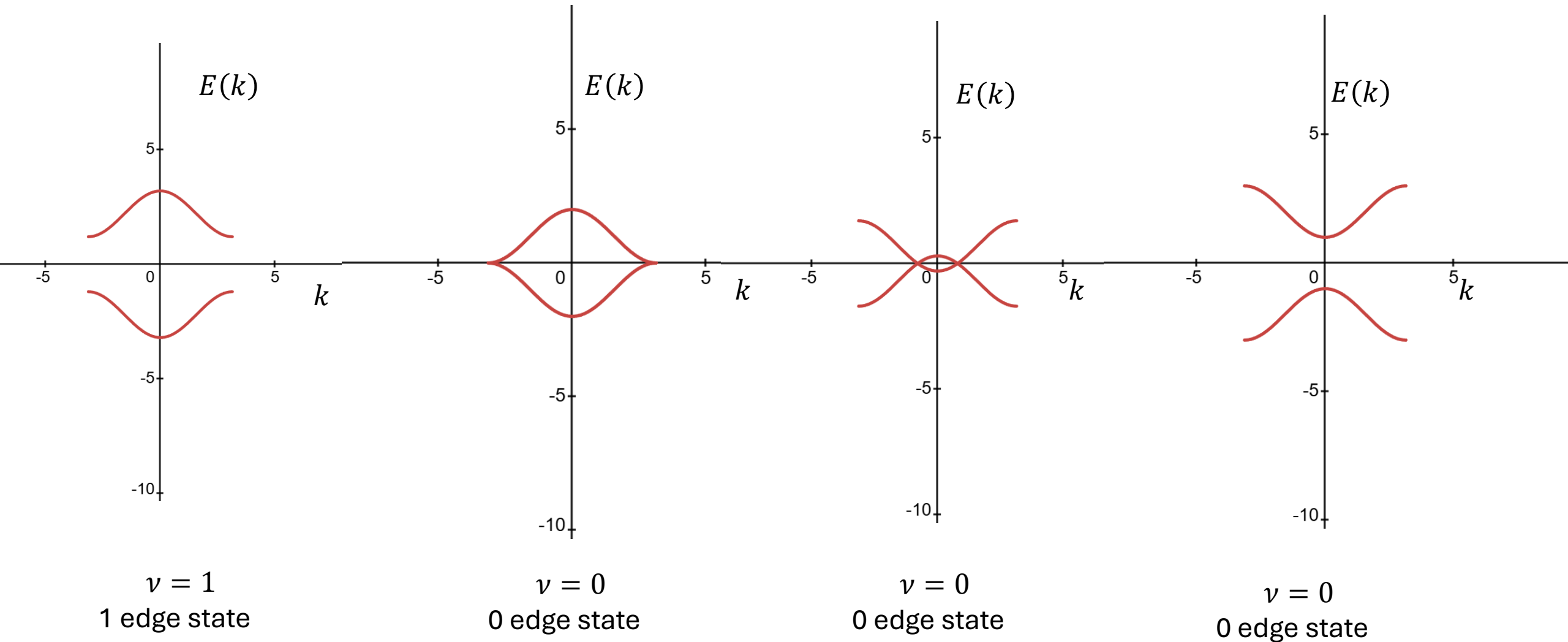
- In a semi-infinite chain, there are adiabatically protected states exponentially localized at the edge called topological edge states.
- They can only be destroyed or created if the corresponding bulk Hamiltonian becomes a conducting

Bulk-Boundary Correspondence:

edge states \sim topological invariant

What follows is an alternate derivation of this principle.

Adiabatic protection



2. Analytical Continuation of Bloch Hamiltonians

- Finite hopping Bloch Hamiltonian can be written

$$h(e^{ika}) = e^{-ikMa} h_{-M} + \dots + e^{-ika} h_{-1} + h_0 + e^{ika} h_1 + e^{ikMa} h_M$$

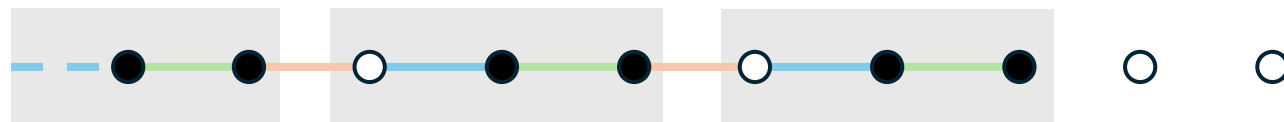
- To analytically continue, $e^{ika} \in S^1 \rightarrow \lambda \in \mathbb{C}$

$$h(\lambda) = \lambda^{-M} h_{-M} + \dots + \lambda^{-1} h_{-1} + h_0 + \lambda h_1 + \lambda^M h_M$$

- Analytically continued Hamiltonian is non-Hermitian but has property $h^\dagger(1/\lambda^*) = h(\lambda)$ called para-Hermiticity.
- $\langle m(1/\lambda^*) | n(\lambda) \rangle = \delta_{mn}$ with C_m being some constant.

3. Boundary Condition and Wavefunction Zeros

- To find the edge states apply a boundary condition to the eigenstates of the bulk Hamiltonian.
- The simplest possible boundary condition is $\langle \psi | n \rangle = 0$ where n is a site index.
- Simplest scenario is Bloch states that satisfy this condition
- An eigenstate of $h(\lambda)$ will have a zero sublattice

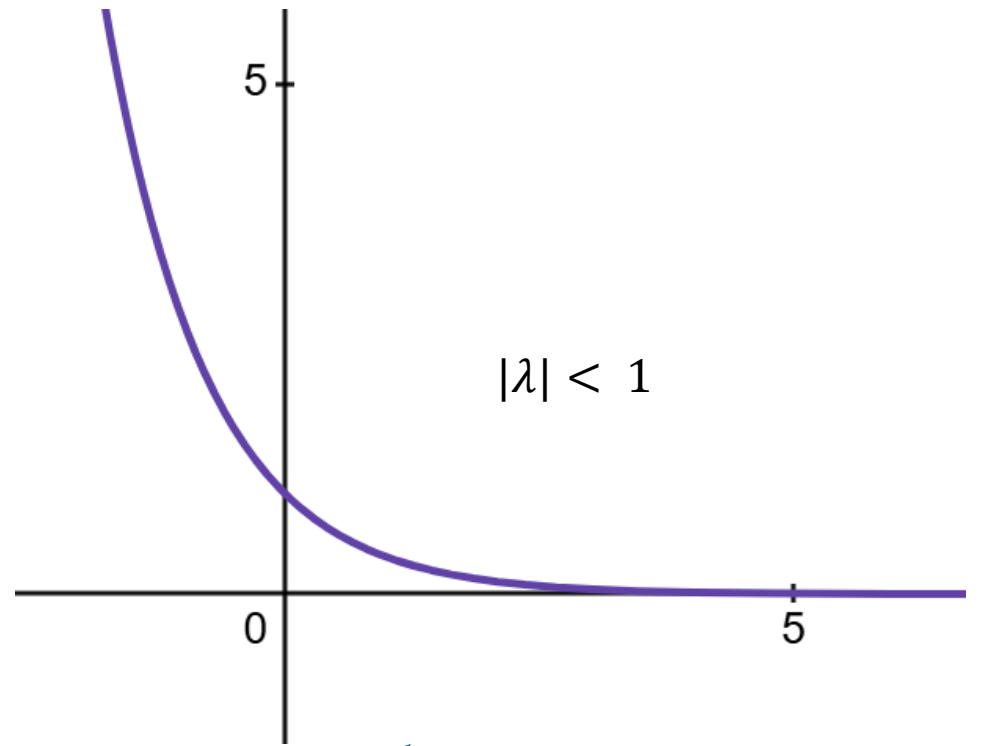
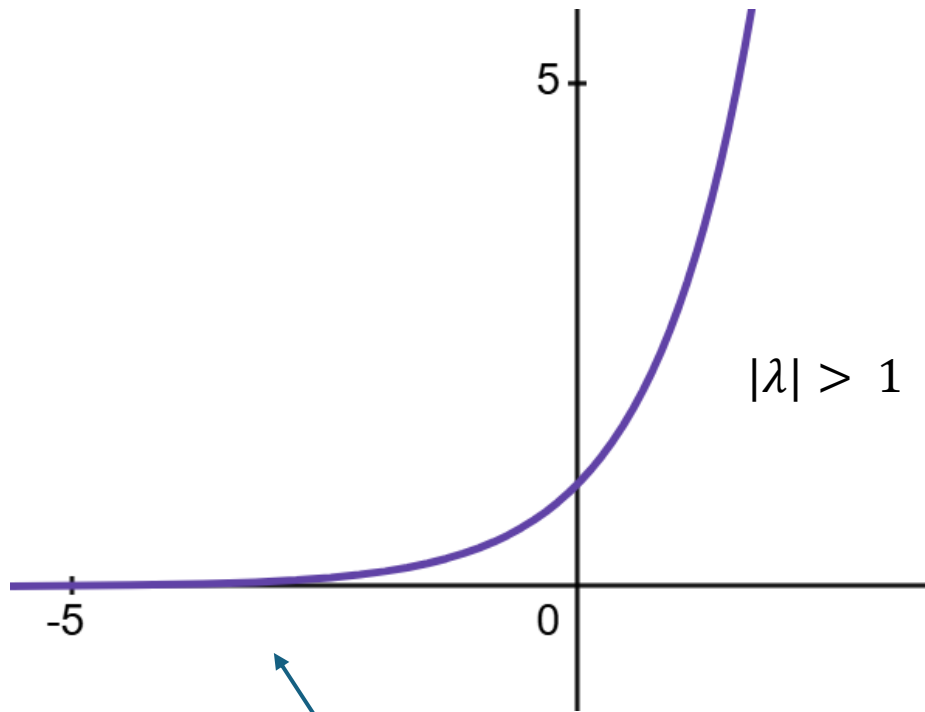


3. Boundary Condition and Wavefunction Zeros

- Let's suppose that sublattice zero at $|\lambda| \neq 1$ corresponds to an edge state. States with $|\lambda| > 1$ grow exponentially whereas states with $|\lambda| < 1$ exponentially decay. In a semi-infinite chain, only one of the two is allowed.

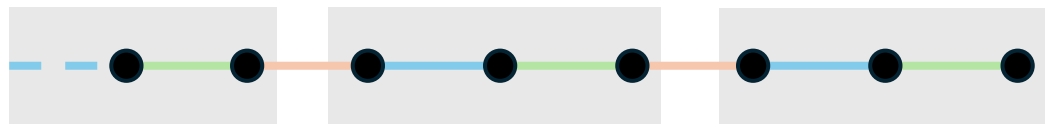
$$|\alpha(\lambda)\rangle = (\alpha_1 \alpha_2 \dots 0 \dots \alpha_N)^T$$
$$|\psi(\lambda)\rangle = \sum_{m=-\infty}^{\infty} T^m \lambda^m |\alpha(\lambda)\rangle$$

where T is the translation by one unit cell operator

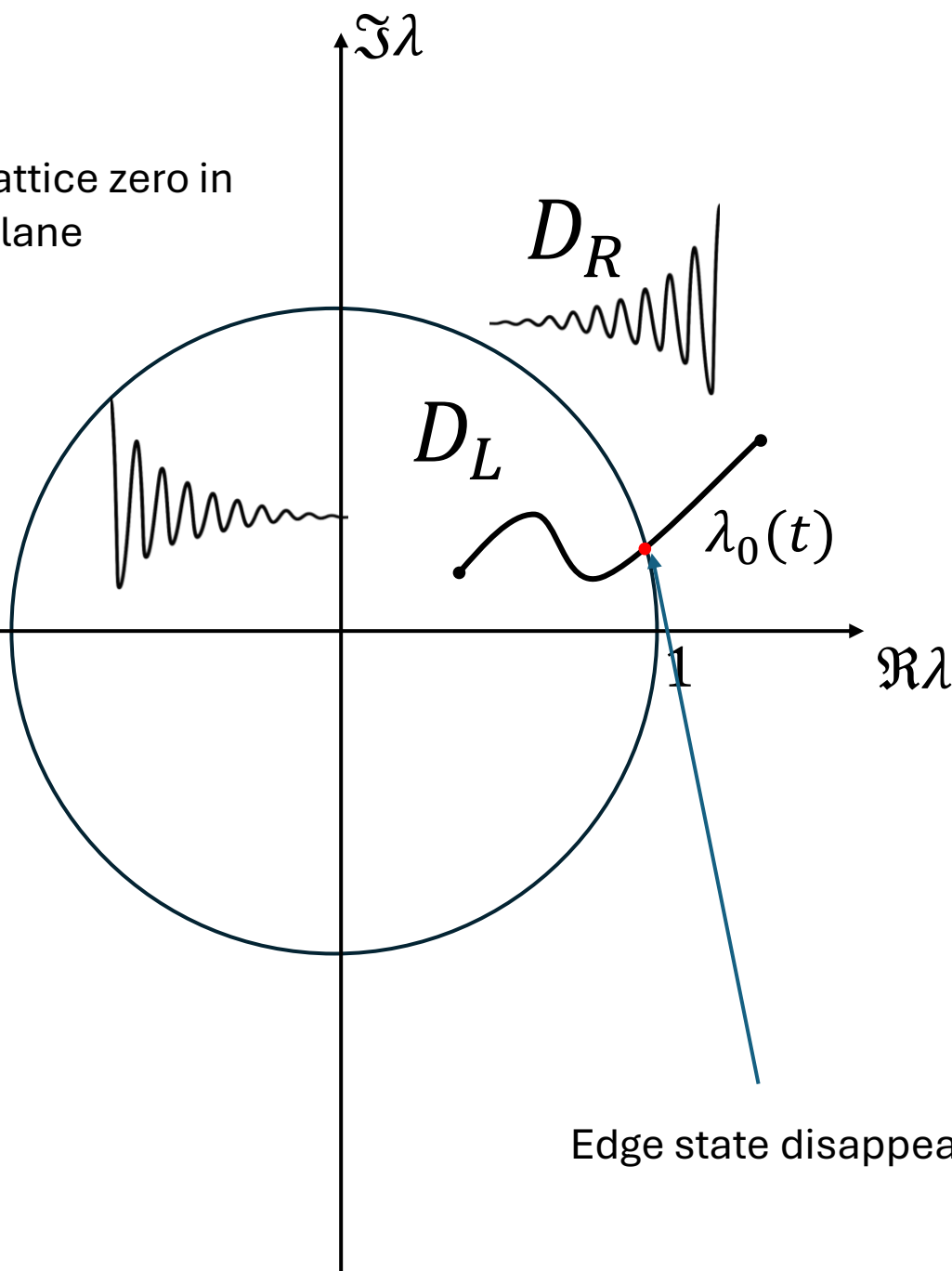


Allows this

Does not allow this



Position of sublattice zero in the complex λ plane



4. Exceptional Points and Adiabatic Protection

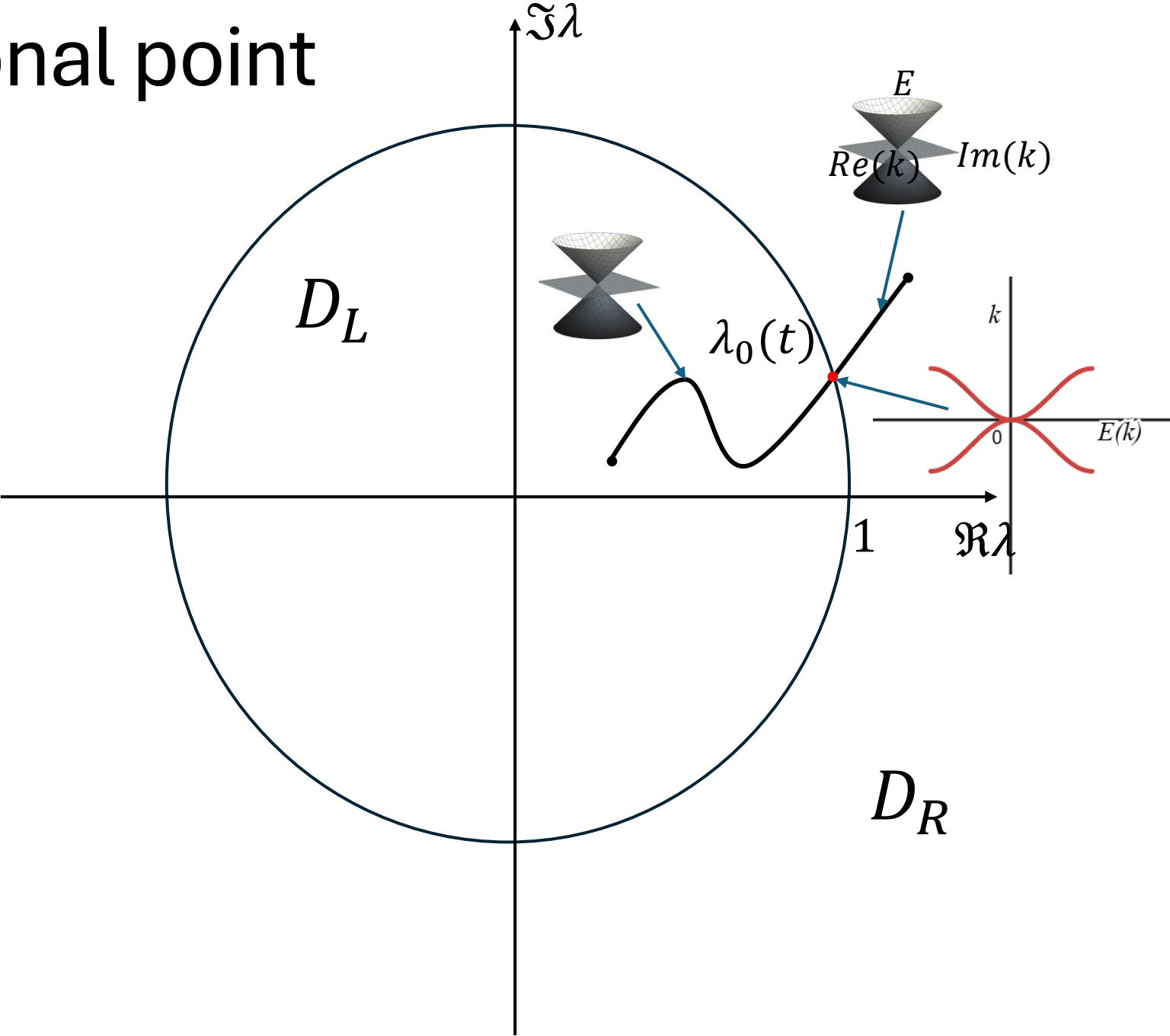
Under what condition does a zero crossing from one region to the other must result in a gap closing?

If the zero sublattice state belongs to two bands simultaneously, the state is protected by a gap closing

Points of non-Hermitian Hamiltonians where two eigenstates coalesce are called exceptional points.

Edge state disappears here

If exceptional point



5. Chiral Symmetry and Exceptional Points

Chiral symmetric Hamiltonians are of the form

$$h(\lambda) = \begin{pmatrix} 0 & q(\lambda) \\ q^\dagger(1/\lambda^*) & 0 \end{pmatrix}$$

with eigenstates of the form $|\alpha(\lambda)\rangle = (\psi_A \pm \psi_B)^T$ and energy $\pm E$, where ψ_A and ψ_B are $N/2$ component vectors.

States with $\psi_A = 0$ or $\psi_B = 0$ simultaneously have a sublattice zero (satisfy a boundary condition) and arise at exceptional points.

6. Exceptional points and winding numbers

- We can assign a winding number to whether $\psi_A = 0$ or $\psi_B = 0$.
- Let $U(\lambda) = \begin{pmatrix} U_A & U_A \\ U_B & -U_B \end{pmatrix}$ be the matrix diagonalizing the Hamiltonian. When two states coalesce, $\det U(\lambda) = 0$ and either $\det U_A = 0$ or $\det U_B = 0$. Those zeros can be counted with

$$\nu_{A/B} = \frac{1}{2\pi i} \oint d\lambda \frac{d}{d\lambda} \ln \det U_{A/B} = \frac{1}{2\pi i} \oint d\lambda \operatorname{Tr} U_{A/B}^{-1} \frac{d}{d\lambda} U_{A/B}$$

- The usual topological invariant is

$$\nu = \frac{1}{2\pi i} \oint dk \operatorname{Tr} q^{-1}(k) \frac{d}{dk} q(k) = \nu_B - \nu_A$$