MAGNETIC LONG-RANGE ORDER AT FINITE TEMPERATURE IN TWO-DIMENSIONAL HYPERBOLIC LATTICES

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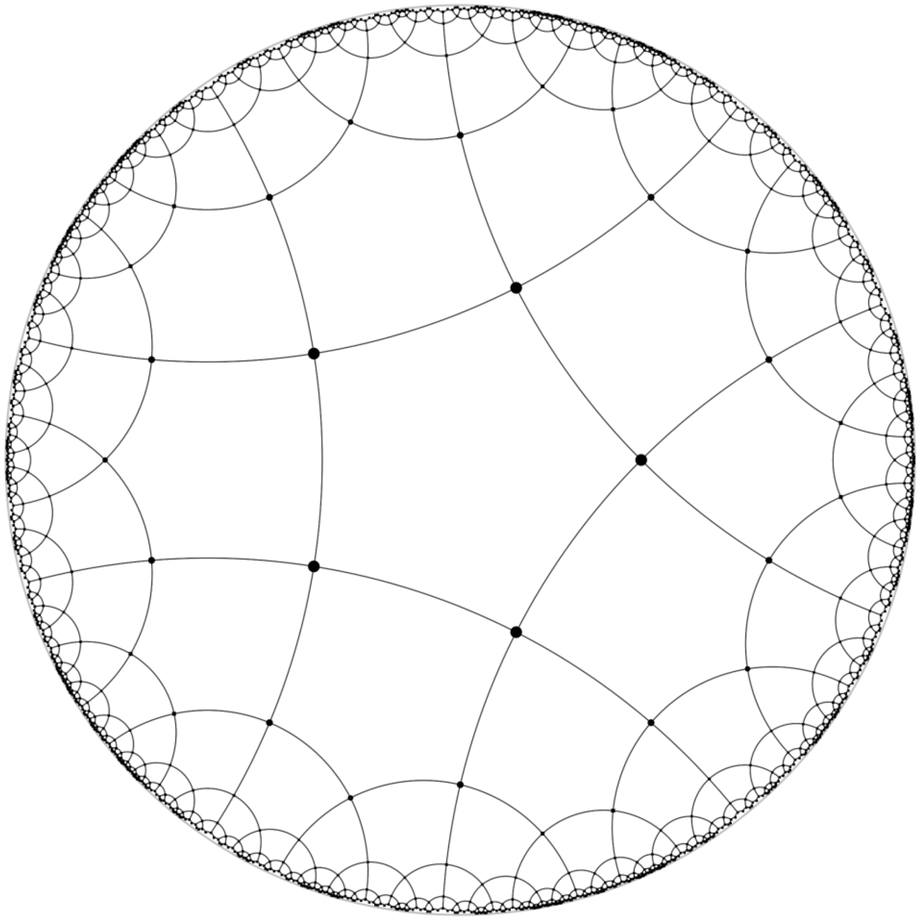
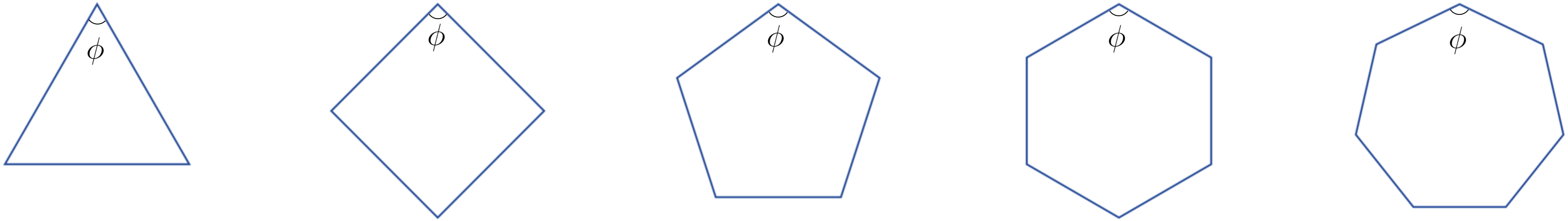
QUANTUM™ HORIZONS ALBERTA



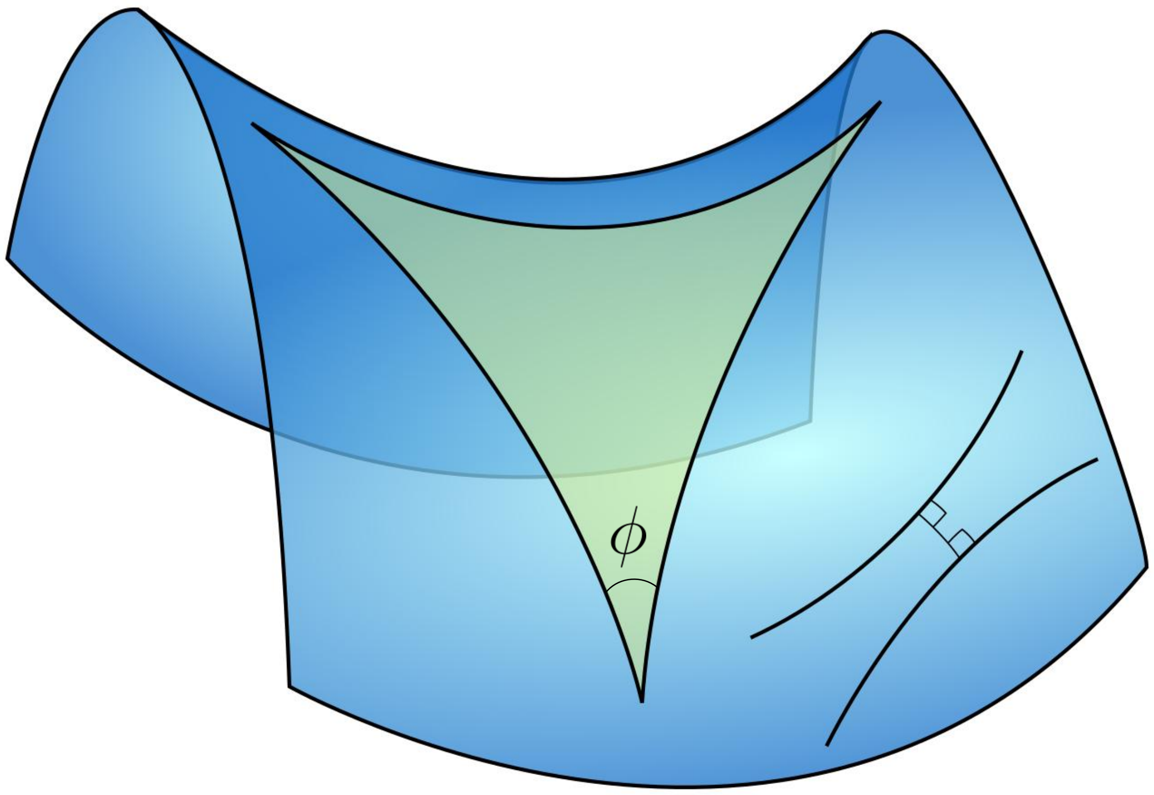
Regular polygons



$p = 3$ $\quad$ $p = 4$ $\quad$ $p = 5$ $\quad$ $p = 6$ $\quad$ $p = 7$

$$\sum_{i=1}^{p} \phi = p\phi = (p-2)\pi$$

Polygons in negative curvature



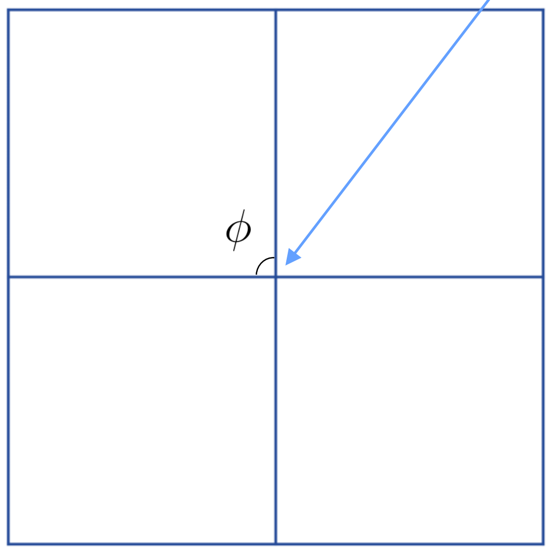
$$p\phi < (p-2)\pi$$

$$p\phi - (p-2)\pi = \kappa A$$

Gaussian curvature

Fitting polygons together

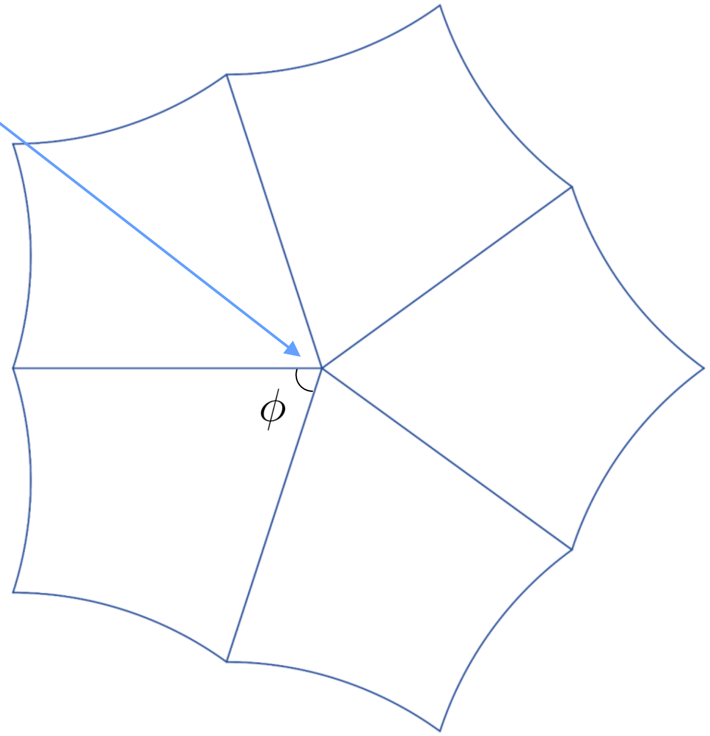
coordination number q



$\kappa = 0$

$$q\phi = 2\pi$$

$$p\phi - (p-2)\pi = \kappa A$$



$\kappa < 0$

Euclidean tessellations

Schläfli symbol $\{p, q\}$

Regular p-gons → p; Coordination q → q

$$(p-2)(q-2) = 4$$

Euclidean tessellations

$$\text{Schläfli symbol } \{p, q\}$$

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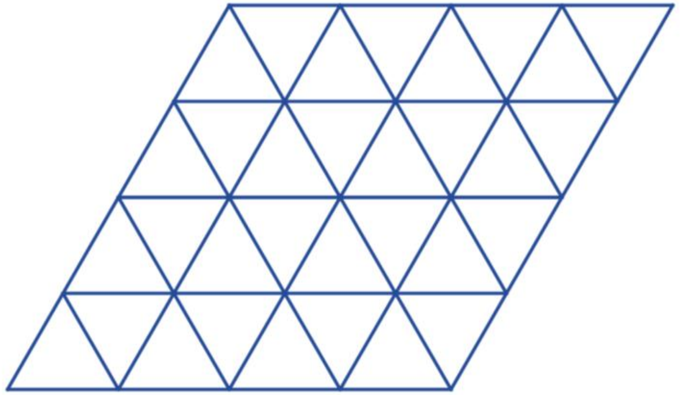
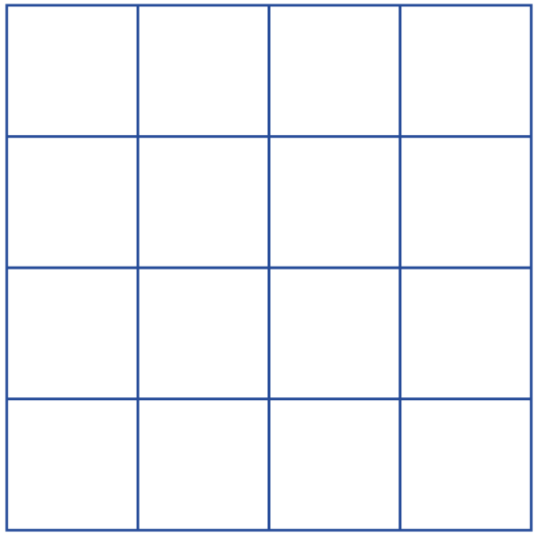
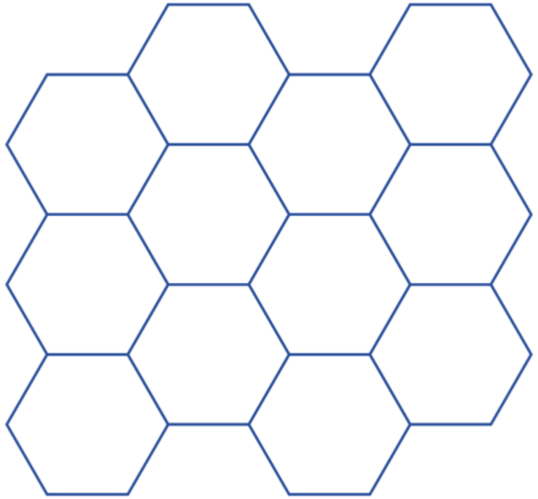


$\{3, 6\}$



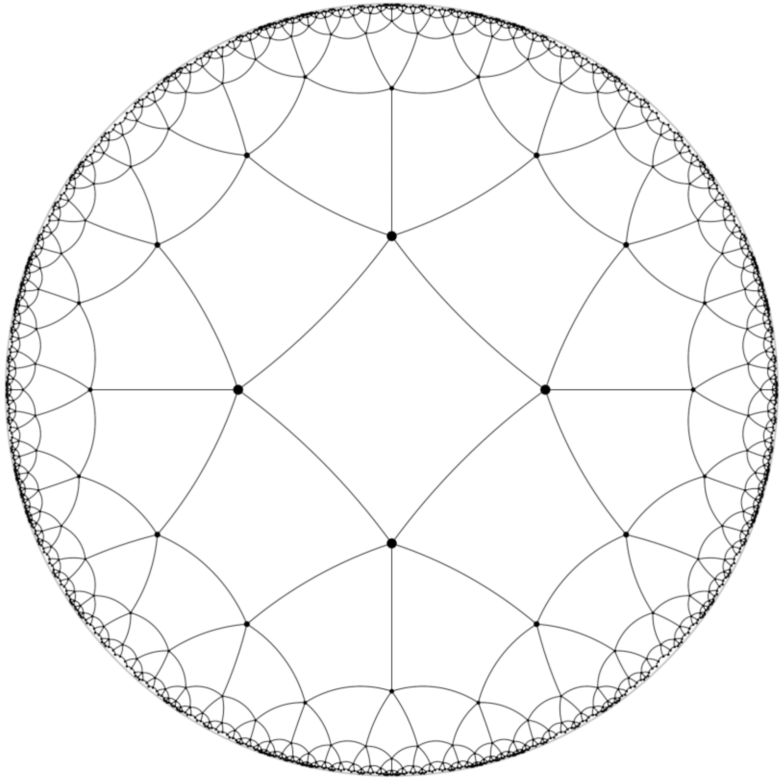
$\{4, 4\}$



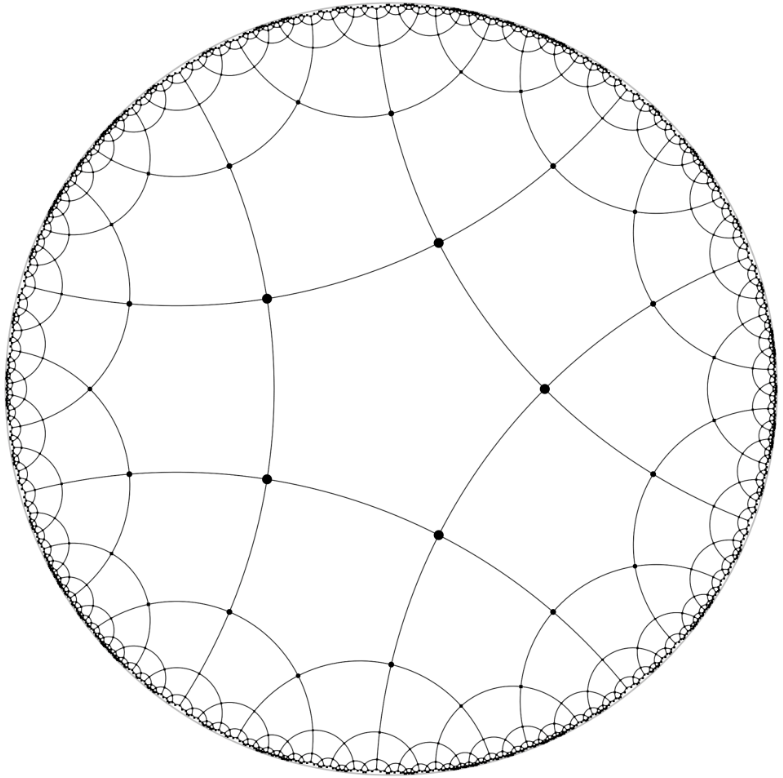
$\{6, 3\}$

Hyperbolic tessellations

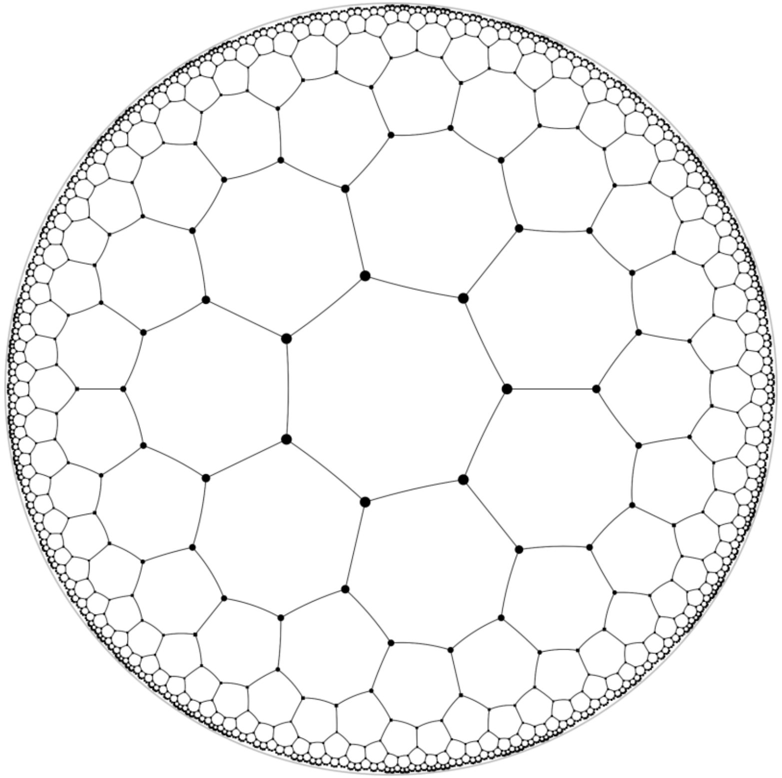
$$\{p, q\} \text{ with } (p-2)(q-2) > 4$$



$\{4, 5\}$



$\{5, 4\}$



$\{7, 3\}$

Poincaré disk representation

Plane of constant negative curvature:

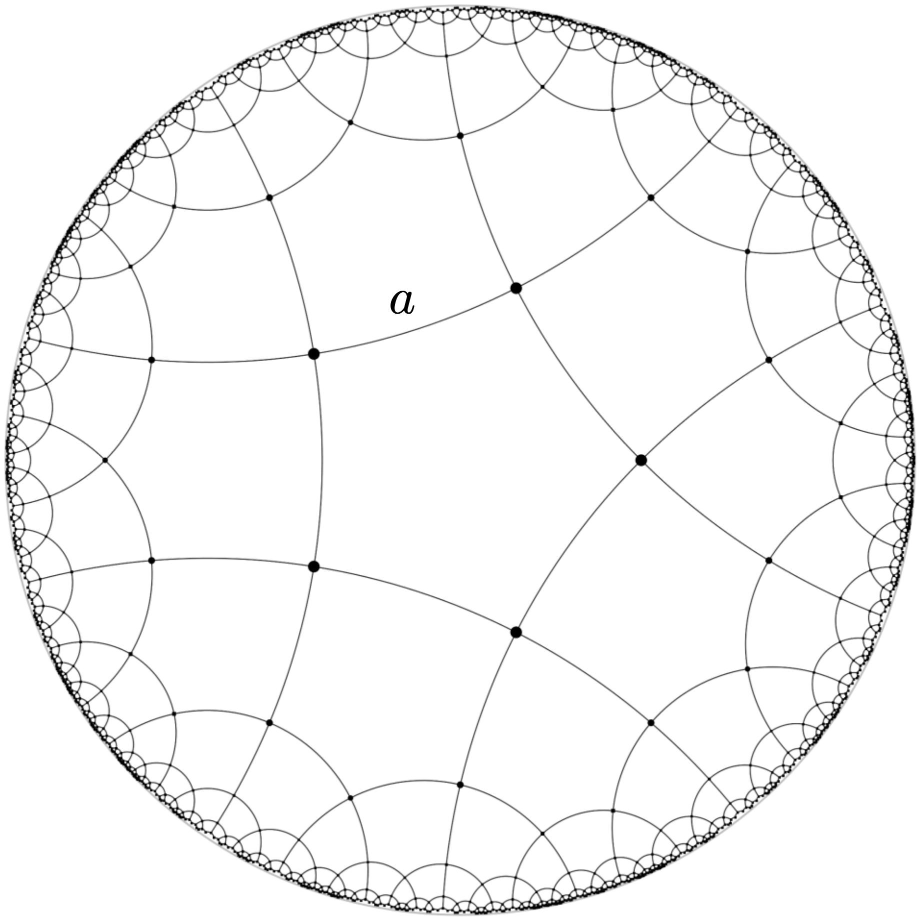
$$\mathbb{D} = \left\{x, y \in \mathbb{R} : x^2 + y^2 < 1\right\},$$

with metric

$$ds^2 = \frac{4}{|\kappa|} \frac{dx^2 + dy^2}{\left(1 - x^2 - y^2\right)^2} \;.$$

Hyperbolic geometry implies:

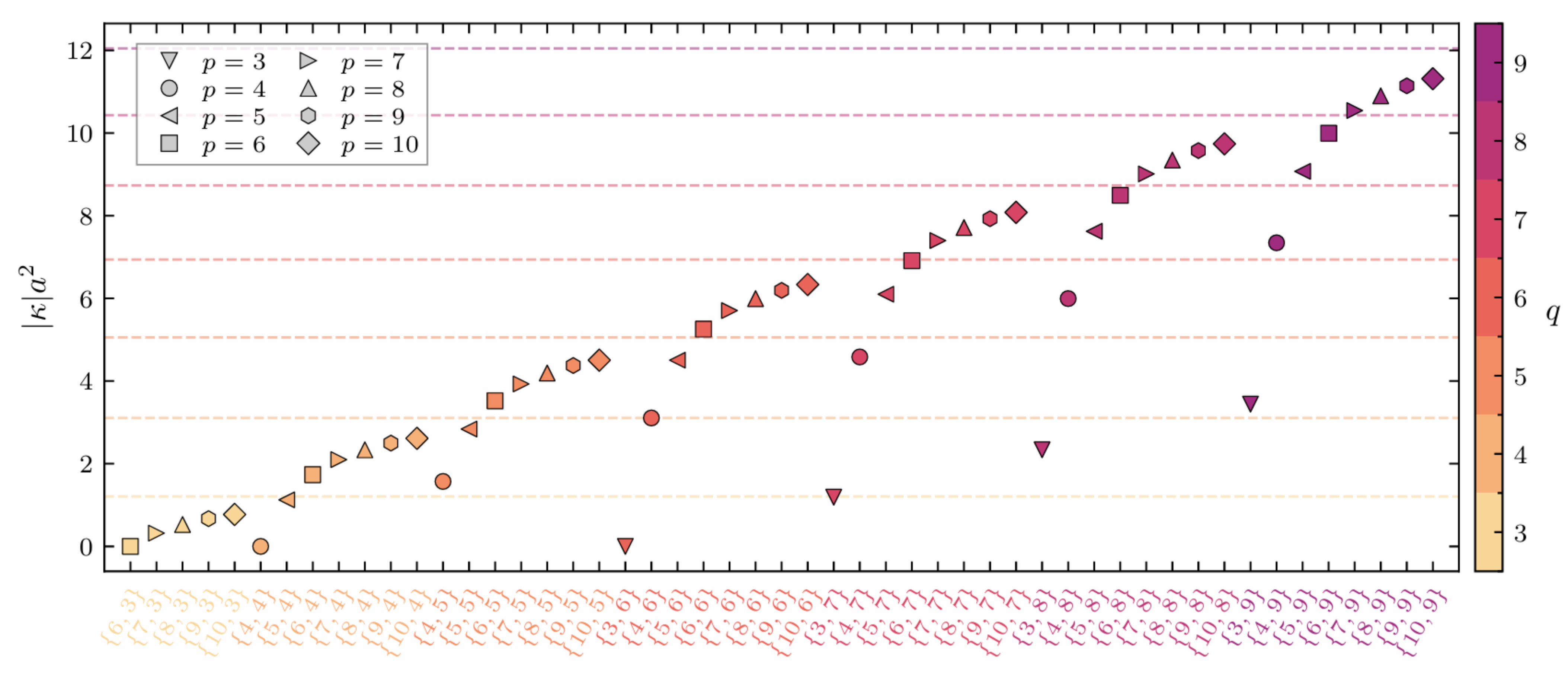
$$\cosh\left(\frac{1}{2}\sqrt{|\kappa| a^2}\right) = \frac{\cos\left(\pi/p\right)}{\sin\left(\pi/q\right)} \;.$$



$$\{p, q\} = \{5, 4\}$$

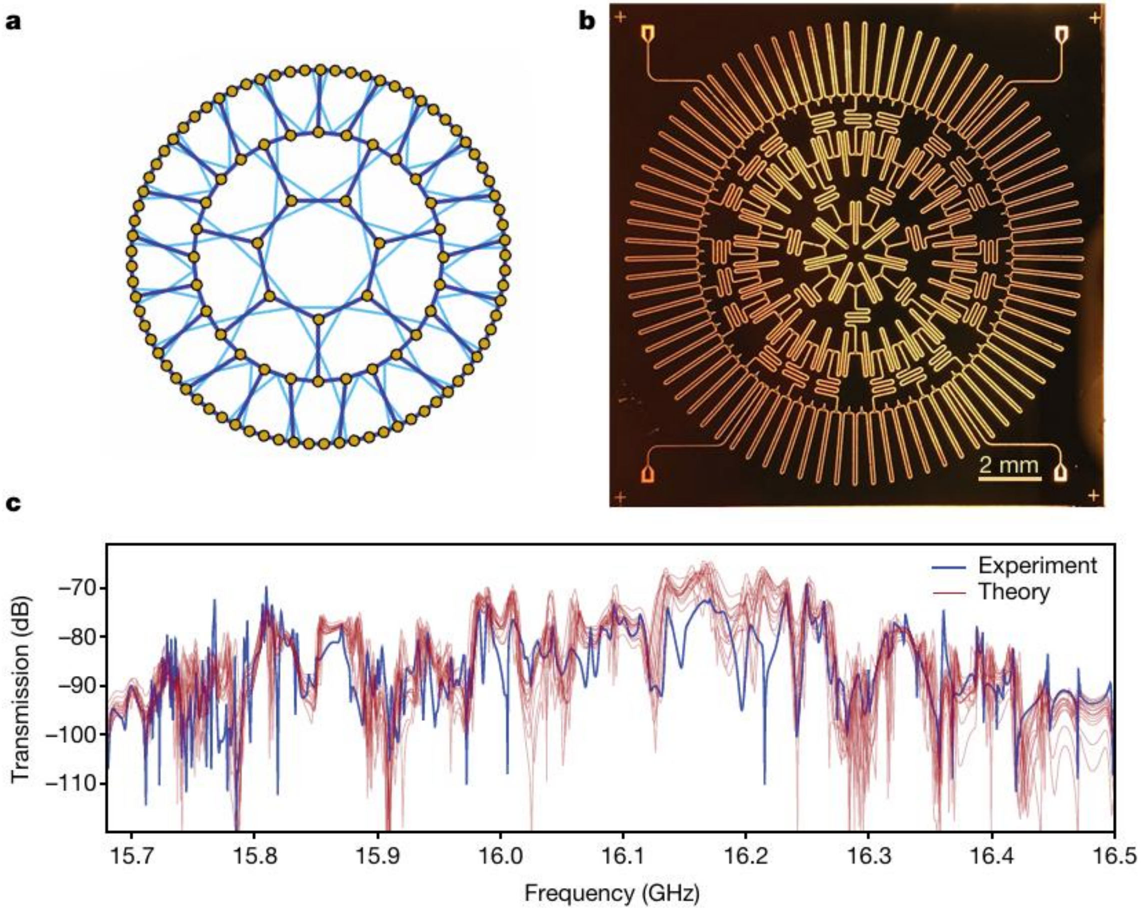
“Intrinsic” curvature scale

$$\cosh\left(\frac{1}{2}\sqrt{|\kappa|a^2}\right) = \frac{\cos(\pi/p)}{\sin(\pi/q)}$$



Hyperbolic matter

- Geometry beyond Euclidean crystals: curvature becomes an additional degree of freedom.
- Lattice ≠ atomic crystal: circuit QED, topolectric, photonic, and programmable architectures can realize graph connectivity directly.
- Hyperbolic lattices appear naturally in AdS/CFT tensor-network and quantum-error-correction constructions.
- Literature on single-particle systems is relatively mature, strongly correlated phenomena such as magnetism, superconductivity, Hubbard physics, and spin liquids are still developing.



A. J. Kollár *et al.*, Nature **571**, 45-50 (2019)

Hyperbolic Heisenberg model

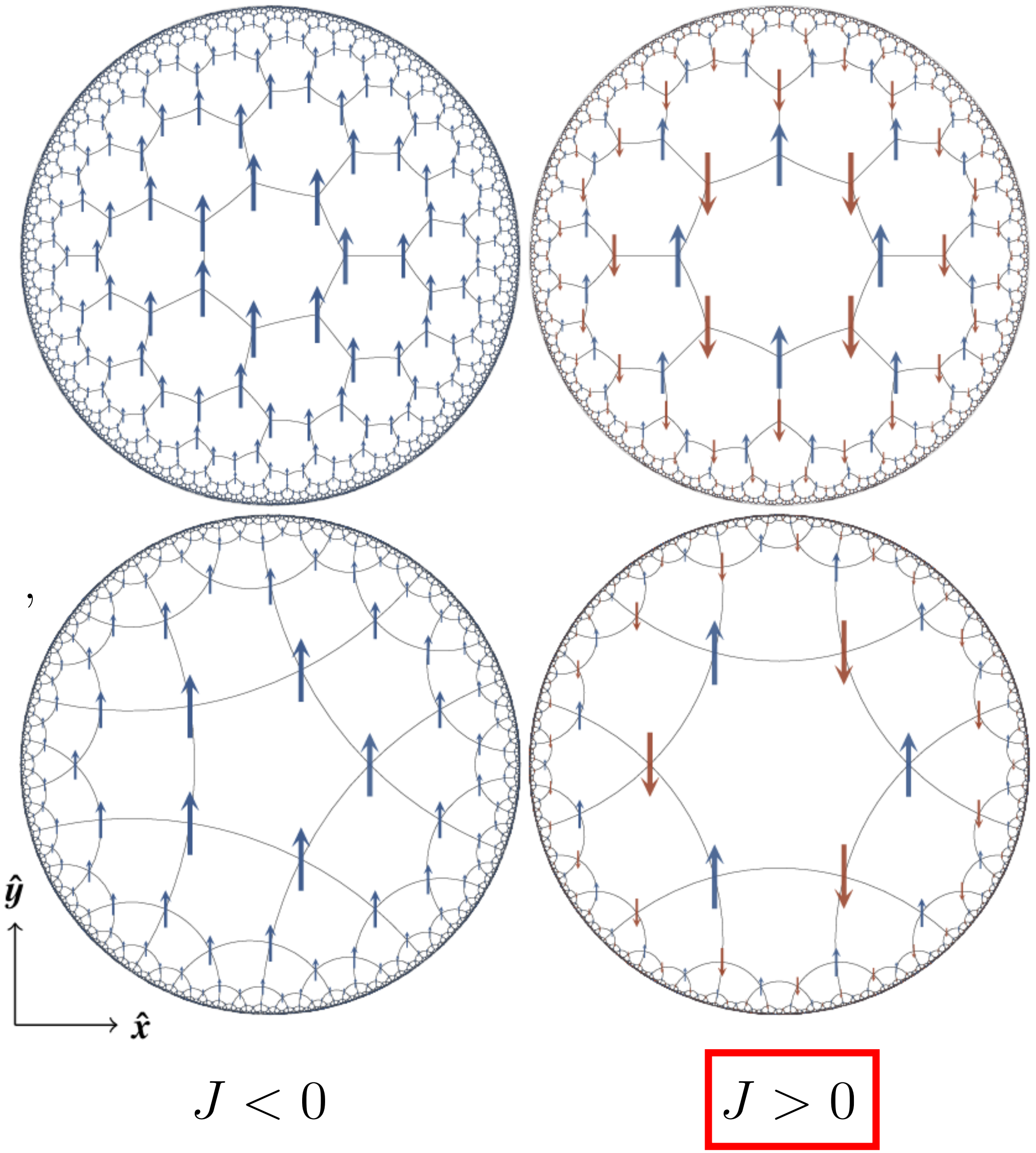
$$H = J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

Holstein-Primakoff transformation

$$\boldsymbol{S}_i = \left(S - a_i^\dagger a_i\right) \hat{\boldsymbol{e}}_{i,0} + \sqrt{S} \left[\left(1 - \frac{a_i^\dagger a_i}{2S}\right)^{1/2} a_i \hat{\boldsymbol{e}}_{i,-} + \mathrm{H.c.} \right],$$

$$H = \sum_{n=0}^{\infty} S^{2-n/2} H_n$$

We study the low energy, bulk properties of the hyperbolic Heisenberg antiferromagnet.

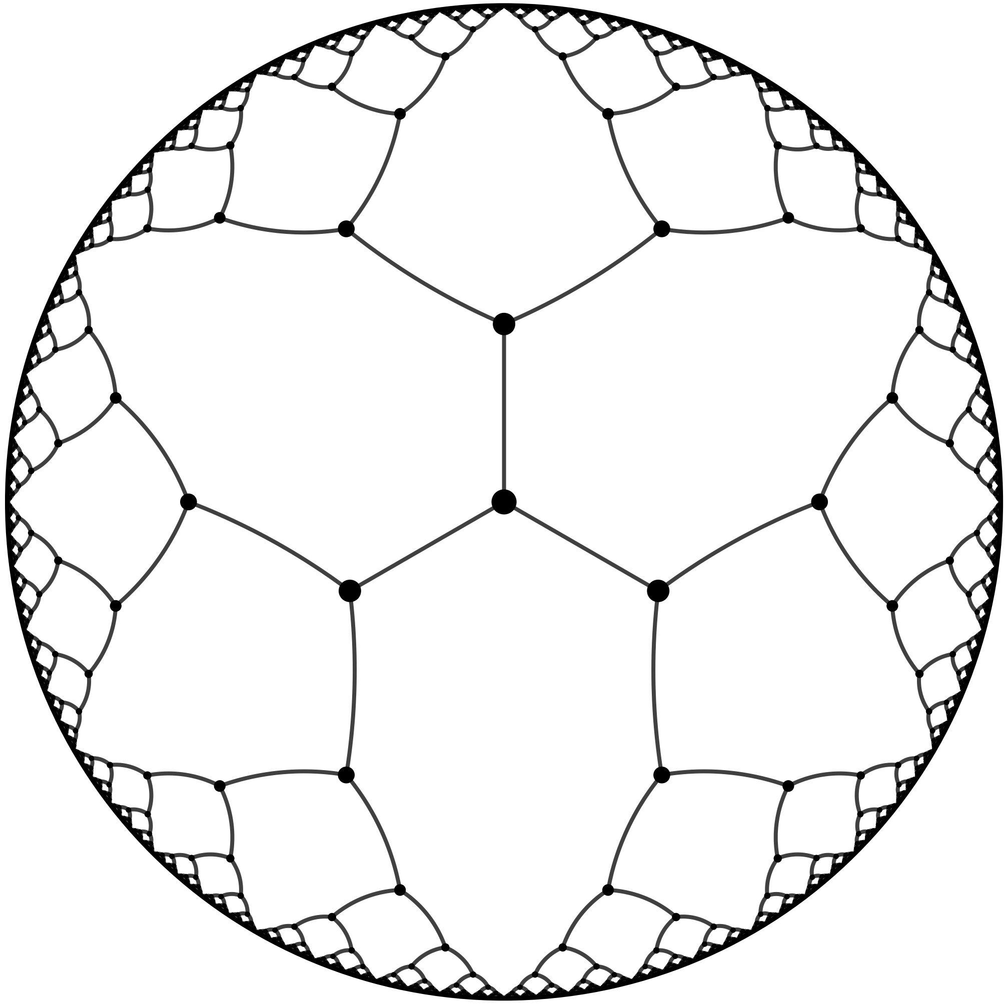


Bethe lattice limit

The limit $\{p \to \infty, q\}$ is the infinite regular tree graph – the Bethe lattice.

Resolvent of the adjacency matrix:

$$\left[(z - \boldsymbol{A})^{-1}\right]_{ij} = \frac{\alpha(z)^{d(i,j)}}{z - q\alpha(z)}$$



$q = 3$

Continuum limit

O(3) Nonlinear sigma model:

$$S_{\mathrm{AFM}}[\boldsymbol{n}] = \frac{\rho_s}{2} \int \mathrm{d}\tau \mathrm{d}^2 x \sqrt{|g|} \left[\frac{1}{c^2} \left(\partial_\tau \boldsymbol{n}\right)^2 + g^{ab} \partial_a \boldsymbol{n} \cdot \partial_b \boldsymbol{n} \right],$$

$$\Delta_{\mathrm{AFM}} = \frac{JSq}{2\sqrt{2}} \sqrt{|\kappa| a^2}.$$

$$\langle S_r^+ S_0^- + S_r^- S_0^+ \rangle \sim \frac{e^{-\sqrt{|\kappa|}\, r}}{\sqrt{r}}.$$

Euclidean overview

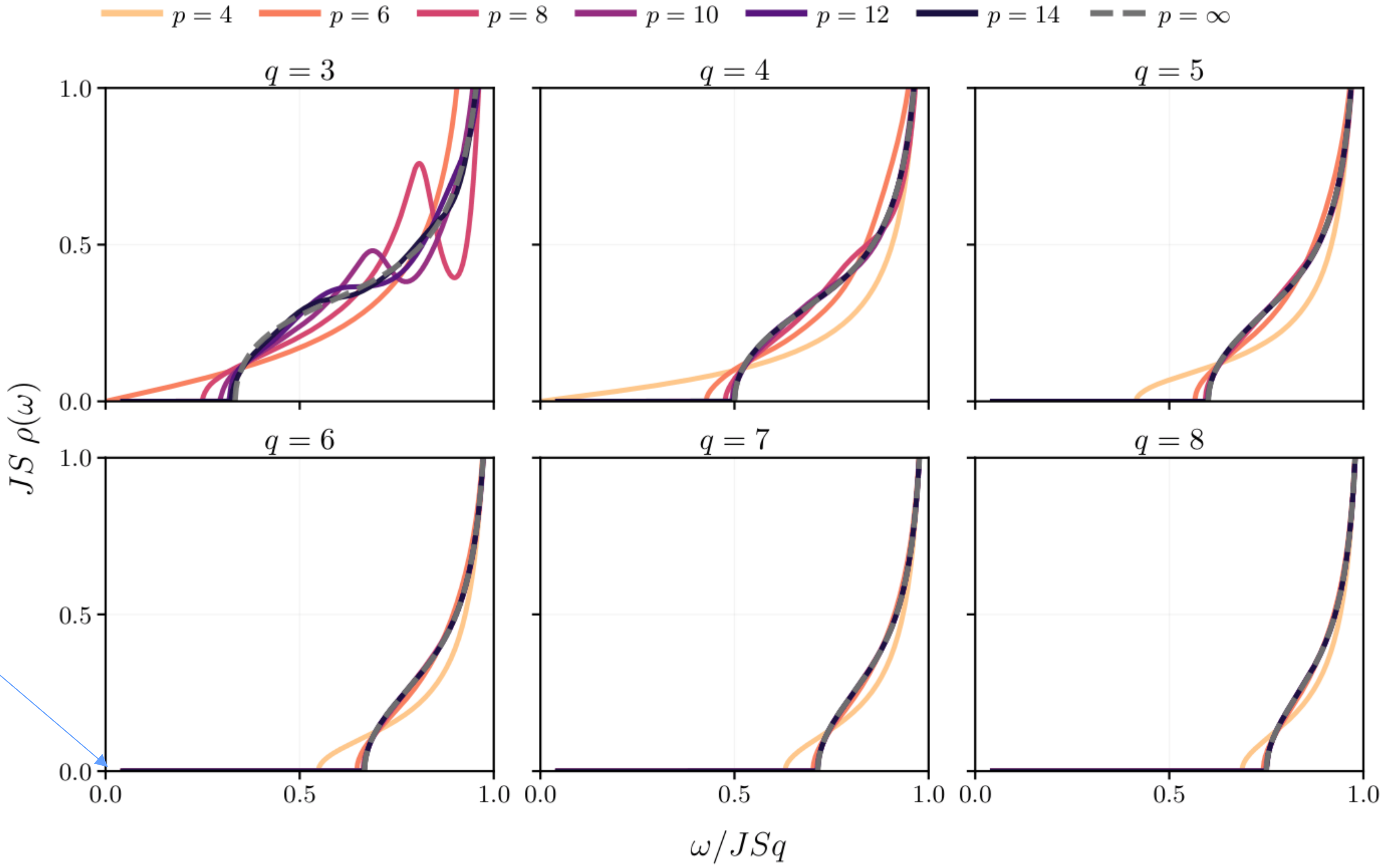
- Ground state (T=0) is Néel ordered, low energy subspace is Goldstone mode $\omega_{\boldsymbol{k}} \sim c|\boldsymbol{k}|$.
- Correlations (T=0) are long-range (massless):

$$C_\perp(r_{ij}) = \frac{1}{2S}\langle S_i^+ S_j^- + S_i^- S_j^+\rangle \sim \frac{1}{r_{ij}} \quad .$$

- Finite temperature fluctuations destroy magnetic order (Mermin-Wagner theorem):

$$\langle \boldsymbol{S}_i \rangle \sim S - \delta m(0) - \frac{\mathcal{C}_0 T}{S}\int_0^{\omega_*} \frac{\mathrm{d}\omega}{\omega}.$$

Bulk density of states



Global symmetry (Goldstone) mode contributes O(1/N) at zero energy.

Spectral gap

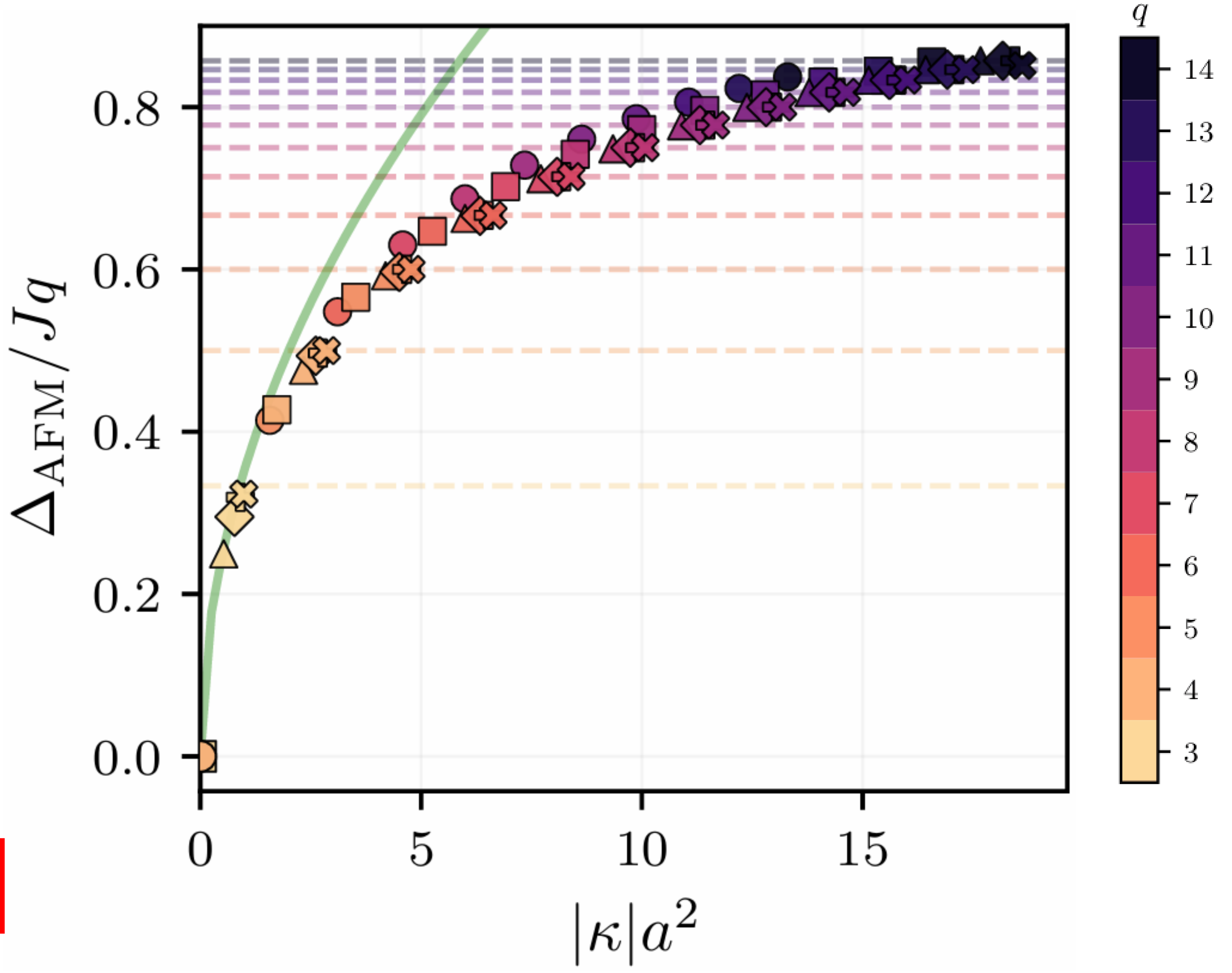
q-regular Bethe lattice:

$$\Delta_{\mathrm{AFM}} = J(q-2).$$

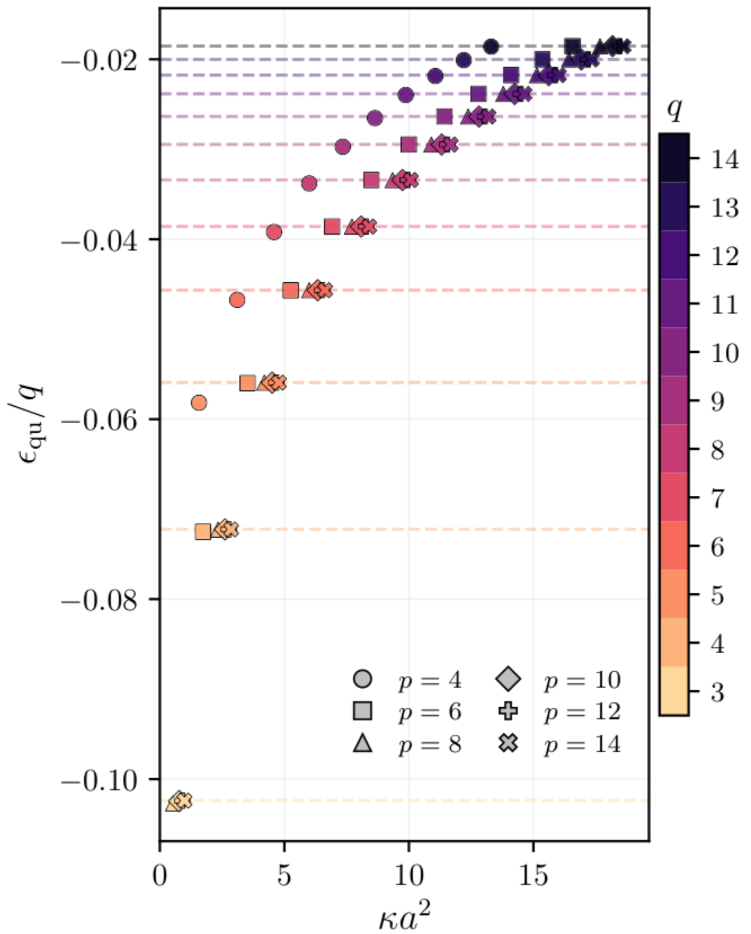
Continuum:

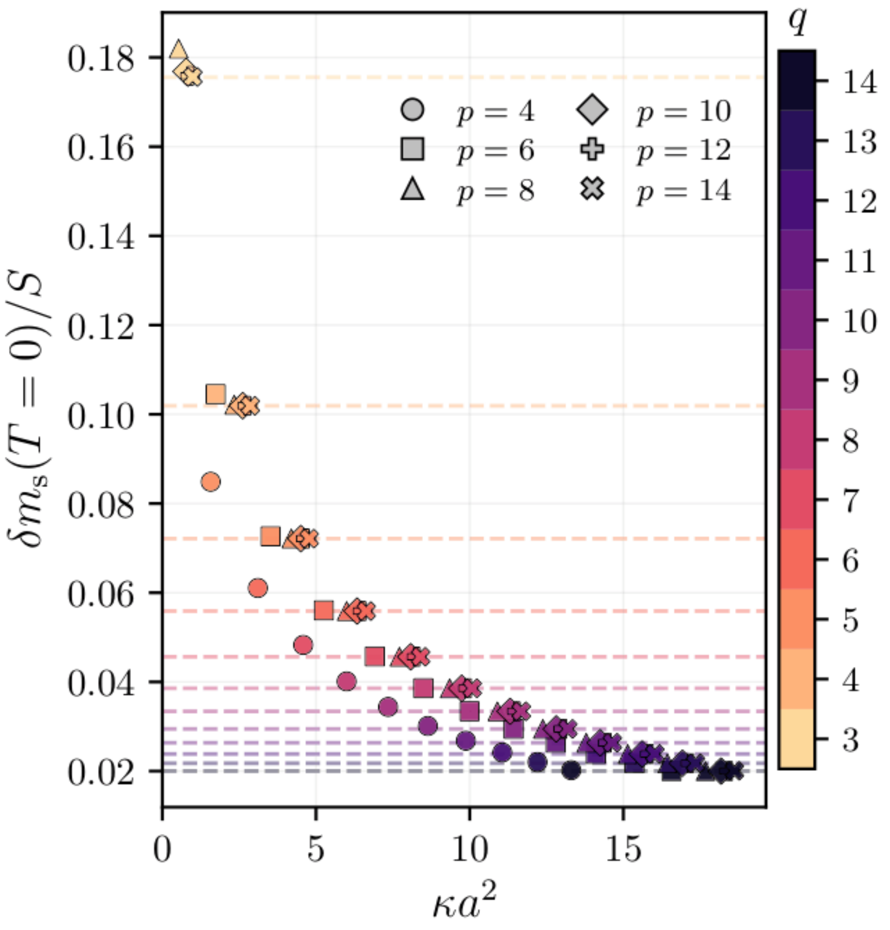
$$\Delta_{\mathrm{AFM}} = \frac{Jq}{2\sqrt{2}}\sqrt{|\kappa|a^2}.$$

Curvature induces magnon gap



Zero-point fluctuations





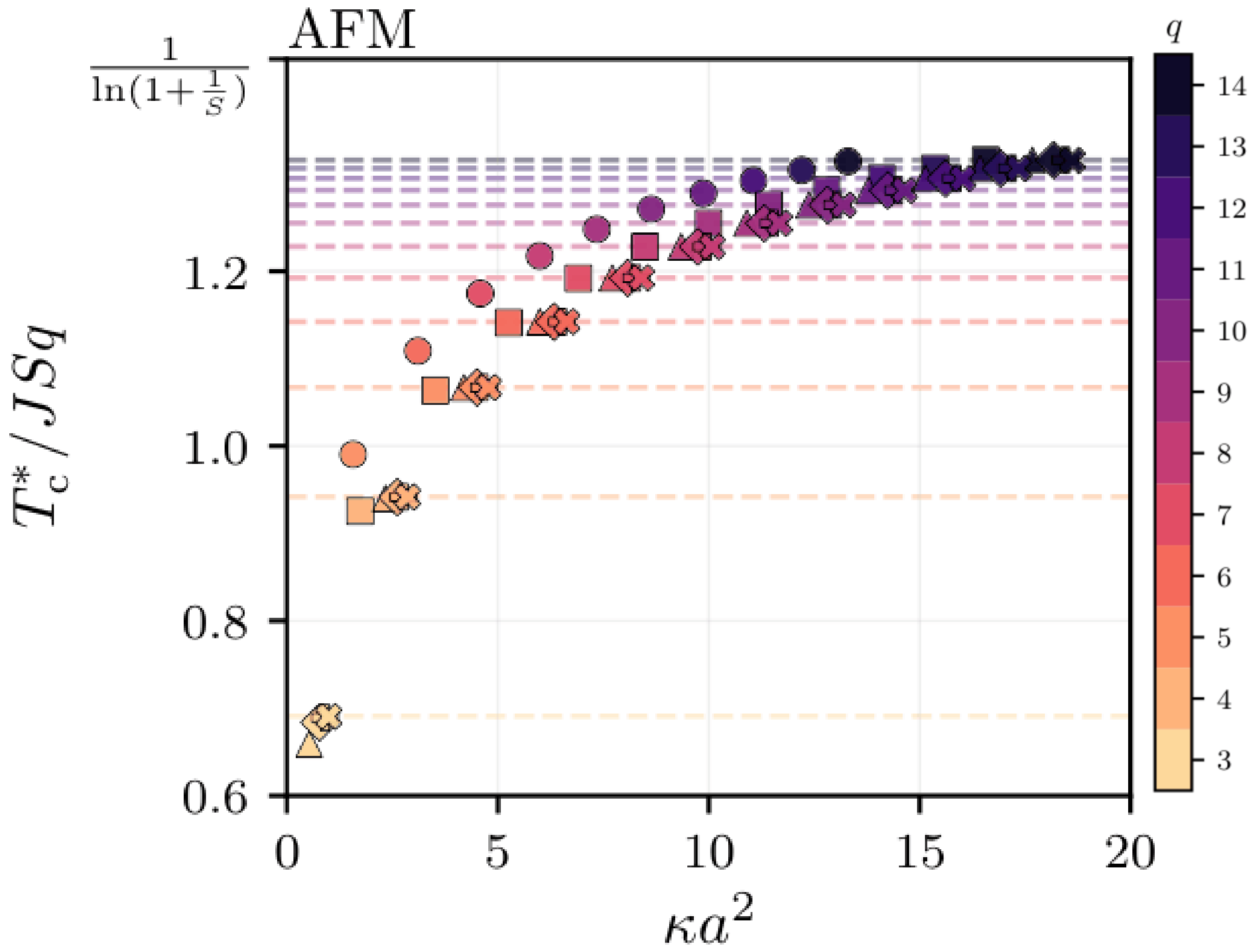
Curvature reduces quantum fluctuations

Critical temperature

Solve for $T = T_\mathrm{c}^*$, such that $\langle \boldsymbol{S}_i \rangle = 0$.

Infrared fluctuations are no longer divergent:

$$\langle \boldsymbol{S}_i \rangle \sim S - \delta m(0) - \mathcal{C}\,\frac{\sqrt{\pi}}{2}\left(\frac{T}{S}\right)^{3/2} e^{-S\Delta/T}.$$



Transverse correlations

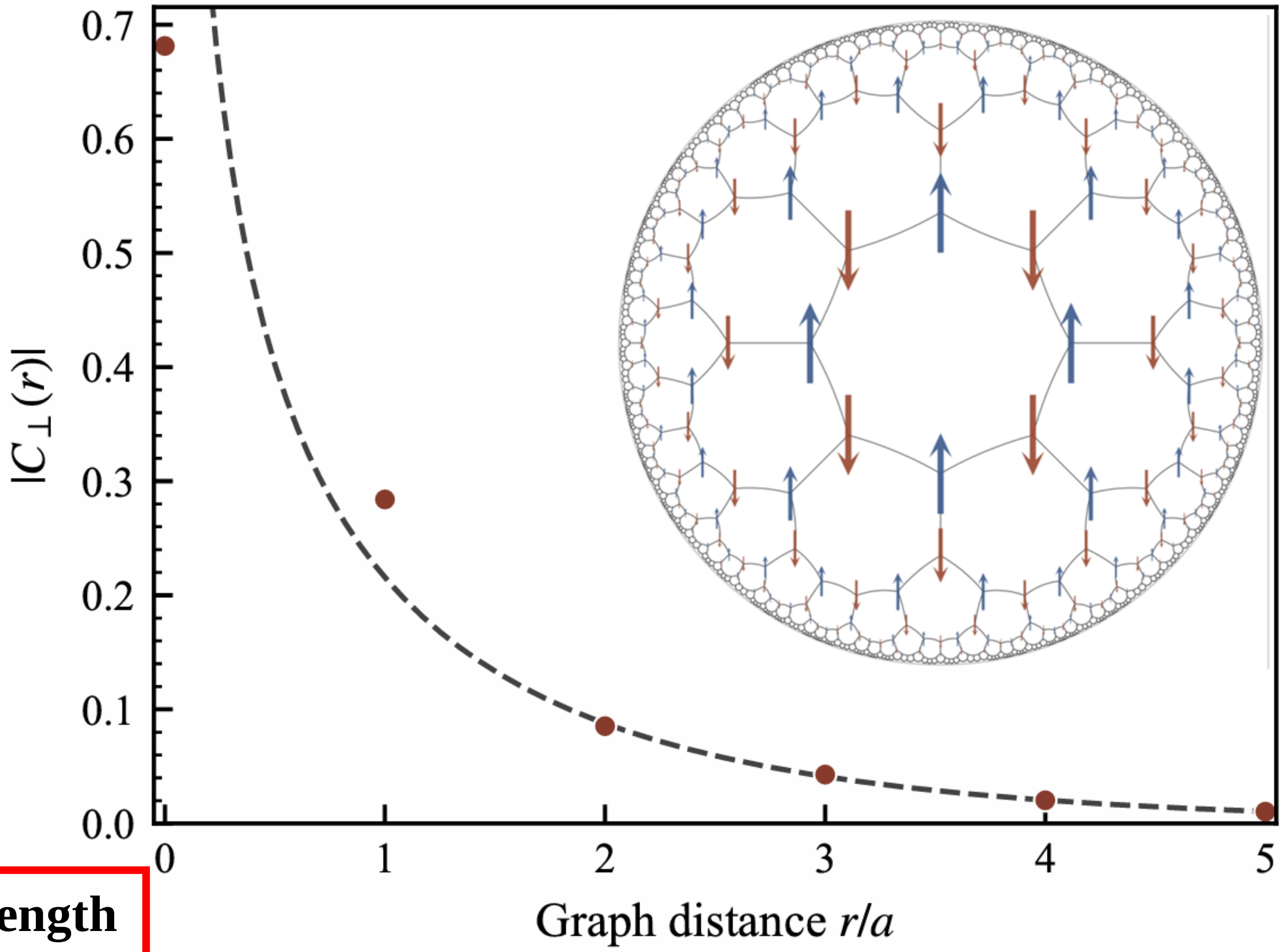
$$\langle S_r^+ S_0^- + S_r^- S_0^+ \rangle \sim \frac{e^{-r/\xi_\perp}}{\sqrt{r}}.$$

$\{8,3\}$: $\xi_\perp / a \approx 1.8$

Bethe lattice: $$\xi_\perp / a = \frac{1}{\ln(q-1)}$$

Continuum: $$\xi_\perp / a = \frac{1}{\sqrt{\kappa a^2}}$$

$\{8,3\}$

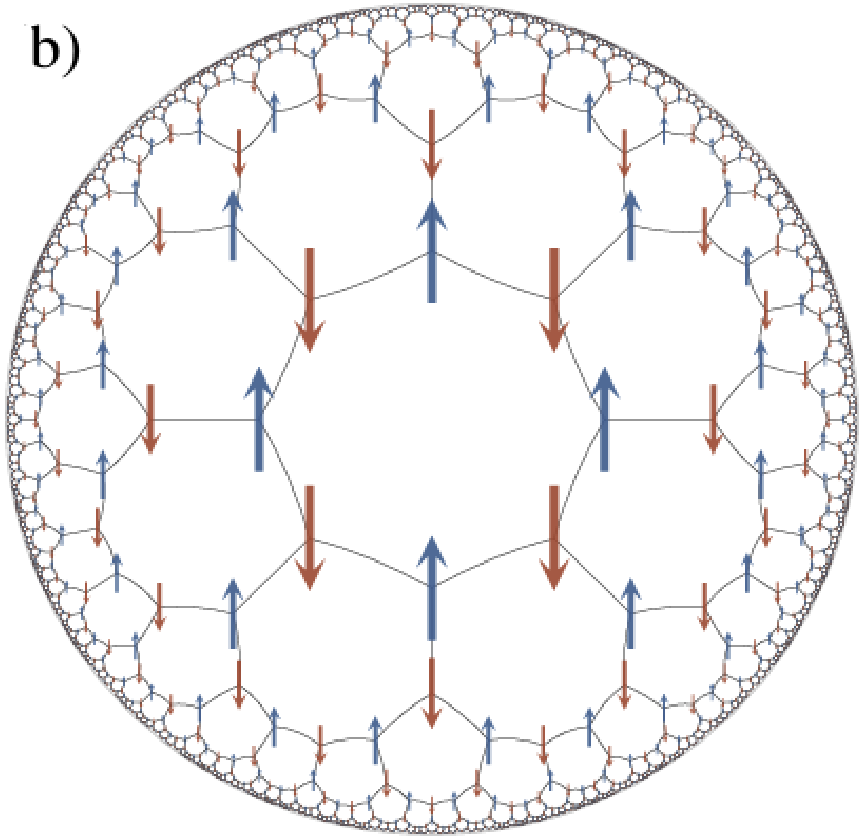


Transverse correlations have finite correlation length

Transverse correlation length

$$\mathcal{N}(r) \sim \lambda^r = e^{r \ln \lambda}$$

$$C_\perp(r) \sim \frac{1}{\sqrt{r}} e^{-r/\xi_\perp}$$



$$\mathcal{S}_{\text{bulk}}^{\perp} = \sum_r \mathcal{N}(r)\, C_\perp(r) \sim \sum_r \frac{1}{\sqrt{r}} \exp\left[r\left(\ln\lambda - \frac{1}{\xi_\perp}\right)\right]$$

Diverges when:

$$\xi_\perp \geq \frac{1}{\ln\lambda}$$

Conclusion

- Hyperbolic geometry changes the infrared phase space of spin waves, allowing finite-temperature magnetic order with SU(2) symmetry.
- Negative curvature induces a bulk spectral gap, short-ranged transverse correlations, and reduced quantum fluctuations.