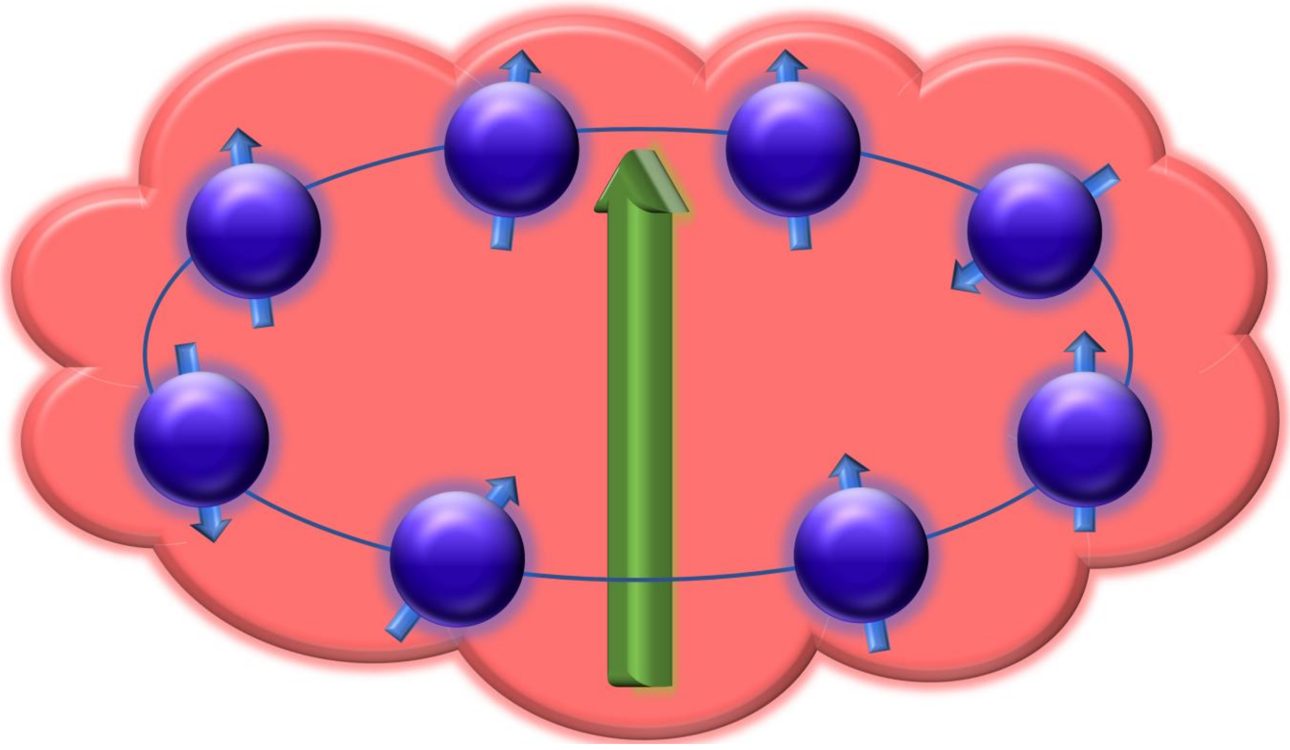
Finite-Size Effects in Quantum Metrology at Strong Coupling: Microscopic vs Phenomenological Approaches



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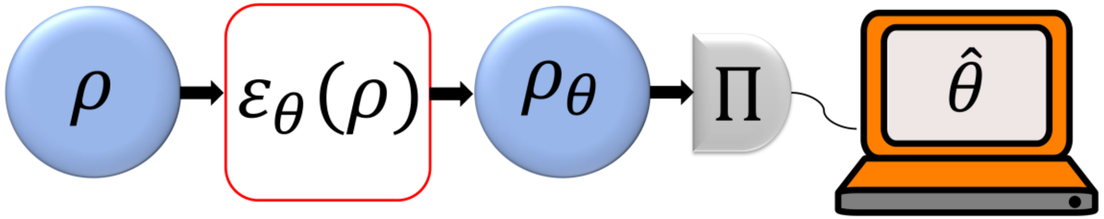


- In quantum metrology, instead of employing classical probes, quantum ones can be utilized to leverage quantum effects and achieve greater precision.

- For a parameterized state, the lower bound for the variance of any unbiased estimator of the parameter, is given by the Quantum Cramer Rao Bound

$$(\Delta\hat{\theta})^2 \geq \frac{1}{N\mathcal{F}(\theta)}$$

- For a pure state quantum Fisher information can be calculated as

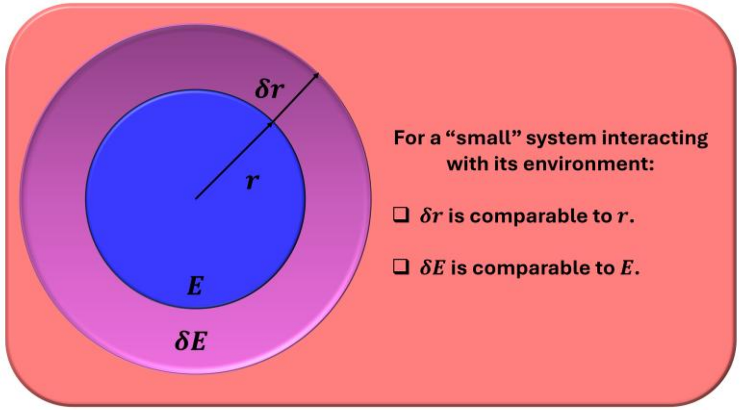
$$\mathcal{F}(\theta) = 4[\langle\partial_\theta\psi|\partial_\theta\psi\rangle - |\langle\partial_\theta\psi|\psi\rangle|^2]$$

- For a mixed state we can write

$$\mathcal{F}(\theta) = \mathrm{Tr}\left[\hat{\rho}\hat{L}_\theta^2\right]$$

- SLD is implicitly defined as $\partial_\theta\hat{\rho} = \frac{\hat{L}_\theta\hat{\rho}+\hat{\rho}\hat{L}_\theta}{2}$ $\Rightarrow$ $\mathcal{F}(\theta) = 2\sum_{k,l}\frac{|\langle k|\partial_\theta\hat{\rho}|l\rangle|^2}{\lambda_k+\lambda_l}$

Hamiltonian of Mean Force & Effective Hamiltonians



- The total Hamiltonian is written as:

$$\hat{H}_{\text{tot}} = \hat{H}_S + \hat{H}_B + \hat{H}_I$$

- Away from the weak coupling limit the equilibrium state is given by the MFG.

$\hat{\rho}^* = \frac{e^{-\beta \hat{H}_S^*}}{\mathcal{Z}_S^*}$ in which $\hat{H}_S^*$ is the Hamiltonian of mean force (HMF) and $\mathcal{Z}_S^*$

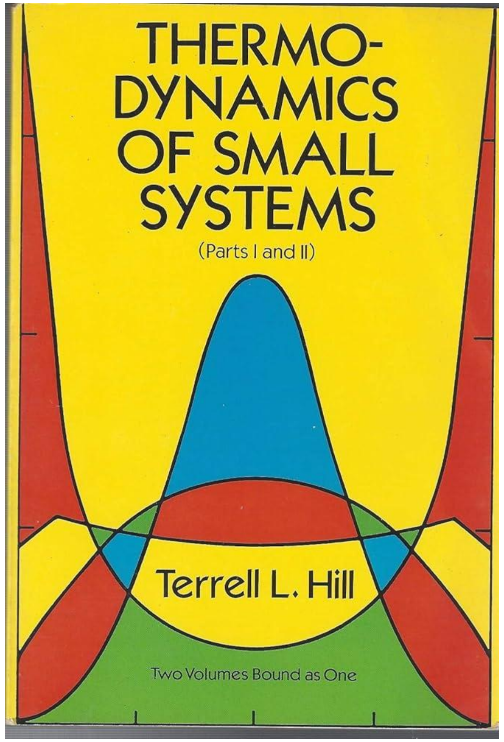
is the partition function associated with the HMF. HMF is:

$$\hat{H}_S^* = -\frac{1}{\beta}\ln\left(\frac{\text{Tr}_B\left[e^{-\beta \hat{H}_{\text{tot}}}\right]}{\text{Tr}_B\left[e^{-\beta \hat{H}_B}\right]}\right) = -\frac{1}{\beta}\ln\left(\frac{\text{Tr}_B\left[e^{-\beta \hat{H}_{\text{tot}}}\right]}{\mathcal{Z}_B}\right)$$ (difficult to calculate so it is convenient to find another approach)

Polaron Transformation: The polaron transformation allows to unitarily map the system to the polaron frame where the system Hamiltonian is dressed by the system-bath coupling, giving an effective Hamiltonian.

$$\hat{H}_{\text{eff}} = e^{\hat{S}} \hat{H} e^{-\hat{S}}$$

Hill’s Nanothernmodynamics & TDEL



- The core idea of Hill's nanothermodynamics or thermodynamics of small systems is that a large ensemble of small systems should yield the same thermodynamic behavior as a large system.

- For an ensemble of nanosystems, the Euler equation is

$$E_t = TS_t - pV_t + \mu N_t + \mathcal{E}\mathcal{N}$$

- Dividing by the number of nanosystems, we get

$$E = TS - pV + \mu N + \mathcal{E} = U + \mathcal{E}$$

In the equation above $E = E_t/\mathcal{N}$, $V = V_t/\mathcal{N}$, $N = N_t/\mathcal{N}$ and U is the internal energy without considering FS effect.

The subdivision potential $\mathcal{E}$ is temperature-dependent and can be related to an effective Hamiltonian as

$$\mathcal{E}(\beta) = -\beta \left\langle \frac{\partial \hat{H}}{\partial \beta} \right\rangle = -\beta \sum_n p_n \frac{\partial E_n}{\partial \beta}$$

If the microscopic effective Hamiltonian is not known, we can extract the subdivision potential via a linear ansatz

$$F(\beta, N) = N F_b(\beta) + \mathcal{E}(\beta)$$

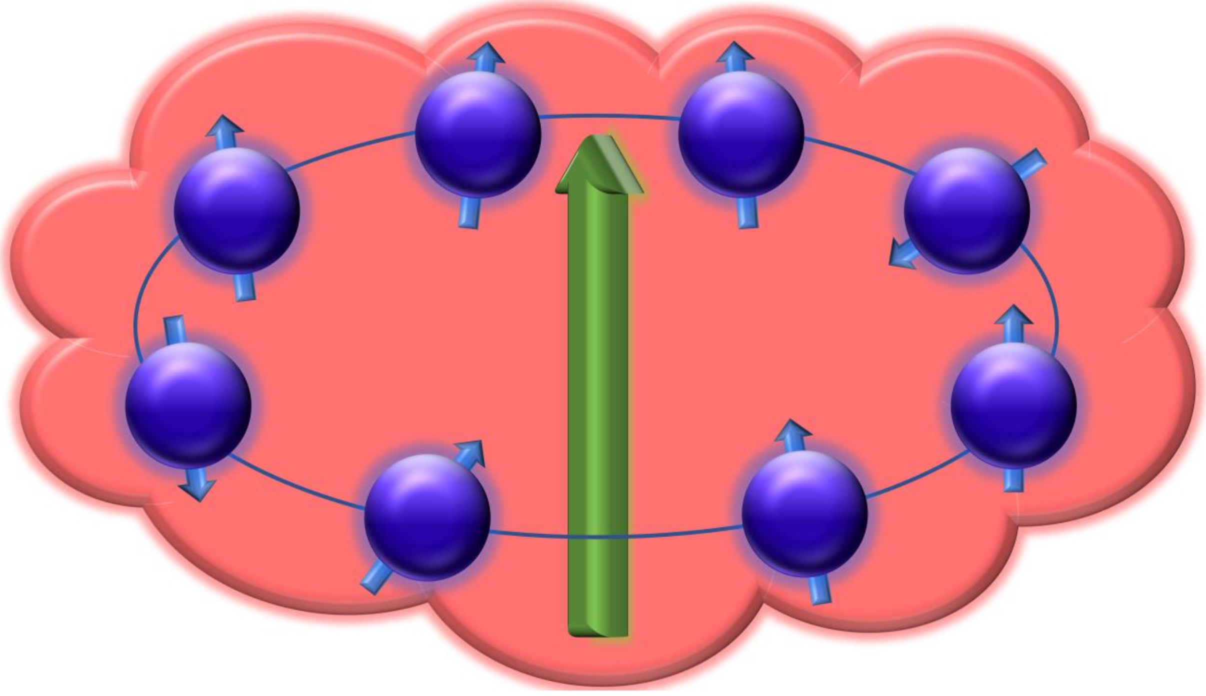
The Model

- The bare system, interaction and bath Hamiltonians are

$$\hat{H}_S = -\frac{J}{2}\sum_{n=1}^{N}\left[(1+\gamma)\hat{\sigma}_n^x\hat{\sigma}_{n+1}^x + (1-\gamma)\hat{\sigma}_n^y\hat{\sigma}_{n+1}^y\right] - h\sum_{n=1}^{N}\hat{\sigma}_n^z$$

$$\hat{H}_I = \sum_{n=1}^{N}\sum_{k} t_{n,k}\,\hat{\sigma}_n^x\left(\hat{b}^\dagger{}_{n,k} + \hat{b}_{n,k}\right)$$

$$\hat{H}_B = \sum_{n=1}^{N}\sum_{k} \nu_{n,k}\,\hat{b}^\dagger{}_{n,k}\hat{b}_{n,k}$$



- After performing a full polaron transformation the effective system Hamiltonian becomes

$$\hat{H}_S^\flat = -\frac{J^\flat}{2}\sum_{n=1}^{N}\left[\left(1+\gamma^\flat\right)\hat{\sigma}_n^x\hat{\sigma}_{n+1}^x + \left(1-\gamma^\flat\right)\hat{\sigma}_n^y\hat{\sigma}_{n+1}^y\right] - h^\flat\sum_{n=1}^{N}\hat{\sigma}_n^z$$

with

$$h^\flat = \langle\hat{\mathcal{C}}\rangle h,\quad \gamma^\flat = \frac{(1+\gamma)-\langle\hat{\mathcal{C}}\rangle^2(1-\gamma)}{(1+\gamma)+\langle\hat{\mathcal{C}}\rangle^2(1-\gamma)},\quad J^\flat = \frac{J}{2}\left((1+\gamma)+\langle\hat{\mathcal{C}}\rangle^2(1-\gamma)\right),\ \langle\hat{\mathcal{C}}\rangle = \exp\left(2g - \frac{4g}{(\beta\omega_c)^2}\psi^{(1)}\left(\frac{1}{\beta\omega_c}\right)\right)$$

Finite-size Partition Function

- We can diagonalize the bare and dressed Hamiltonians by applying Jordan-Wigner, Fourier and Bogoliubov-Valatin transformations. In the diagonal basis the quasiparticle field operators become:

$\hat{\eta}_k = \cos\left(\frac{\theta_k}{2}\right)\hat{c}_k - i\sin\left(\frac{\theta_k}{2}\right)\hat{c}^\dagger_{-k}$, $\hat{\eta}^\dagger_{-k} = \cos\left(\frac{\theta_k}{2}\right)\hat{c}^\dagger_{-k} - i\sin\left(\frac{\theta_k}{2}\right)\hat{c}_k$ with Bogoliubov angle $\theta_k = \arctan\left(\frac{J\gamma\sin(k)}{h-J\cos(k)}\right)$.

- The diagonalized Hamiltonian can be written as a direct sum of its positive and negative parity sector contributions.

$\hat{\mathcal{H}}^+ = \sum_{k\in\mathbf{K}^+}\epsilon_k\left(\hat{\eta}^\dagger_k\hat{\eta}_k - \frac{1}{2}\right) - Nh\hat{\Pi}^+$, $\quad \hat{\mathcal{H}}^- = \sum_{k\in\mathbf{K}^-\backslash\{0,\pi\}}\epsilon_k\left(\hat{\eta}^\dagger_k\hat{\eta}_k - \frac{1}{2}\right) + \epsilon_0\left(\hat{\eta}^\dagger_0\hat{\eta}_0 - \frac{1}{2}\right) + \epsilon_\pi\left(\hat{\eta}^\dagger_\pi\hat{\eta}_\pi - \frac{1}{2}\right) - Nh\hat{\Pi}^-$

- The fermionic dispersion relation becomes $\epsilon_k = 2\sqrt{(h - J\cos k)^2 + J^2\gamma^2\sin^2 k}$.

- The finite-size partition function is written as

$$\mathcal{Z} = \frac{1}{2}e^{Nh\beta}\left[\prod_{k\in\mathrm{K}^+} 2\cosh\left(\frac{\beta\epsilon_k}{2}\right) + \prod_{k\in\mathrm{K}^+} 2\sinh\left(\frac{\beta\epsilon_k}{2}\right) + \prod_{k\in\mathrm{K}^-} 2\cosh\left(\frac{\beta\epsilon_k}{2}\right) - \prod_{k\in\mathrm{K}^-} 2\sinh\left(\frac{\beta\epsilon_k}{2}\right)\right]$$

- In the asymptotic limit, PPA becomes accurate and partition function is written as a product.

$$\mathcal{Z}_{PPA} = e^{Nh\beta}\prod_{k\in K^+} 2\cosh\left(\frac{\beta\epsilon_k}{2}\right)$$

QFI Calculations

- The total QFI can be written as a sum of classical and quantum contributions.

$$\boldsymbol{\mathcal{F}(\alpha)} = \sum_{n} \frac{(\partial_\alpha p_n)^2}{p_n} + 2\sum_{n,m} \frac{(p_n - p_m)^2}{p_n + p_m} |\langle m|\dot{n}\rangle|^2$$

- Applying this to our system we get

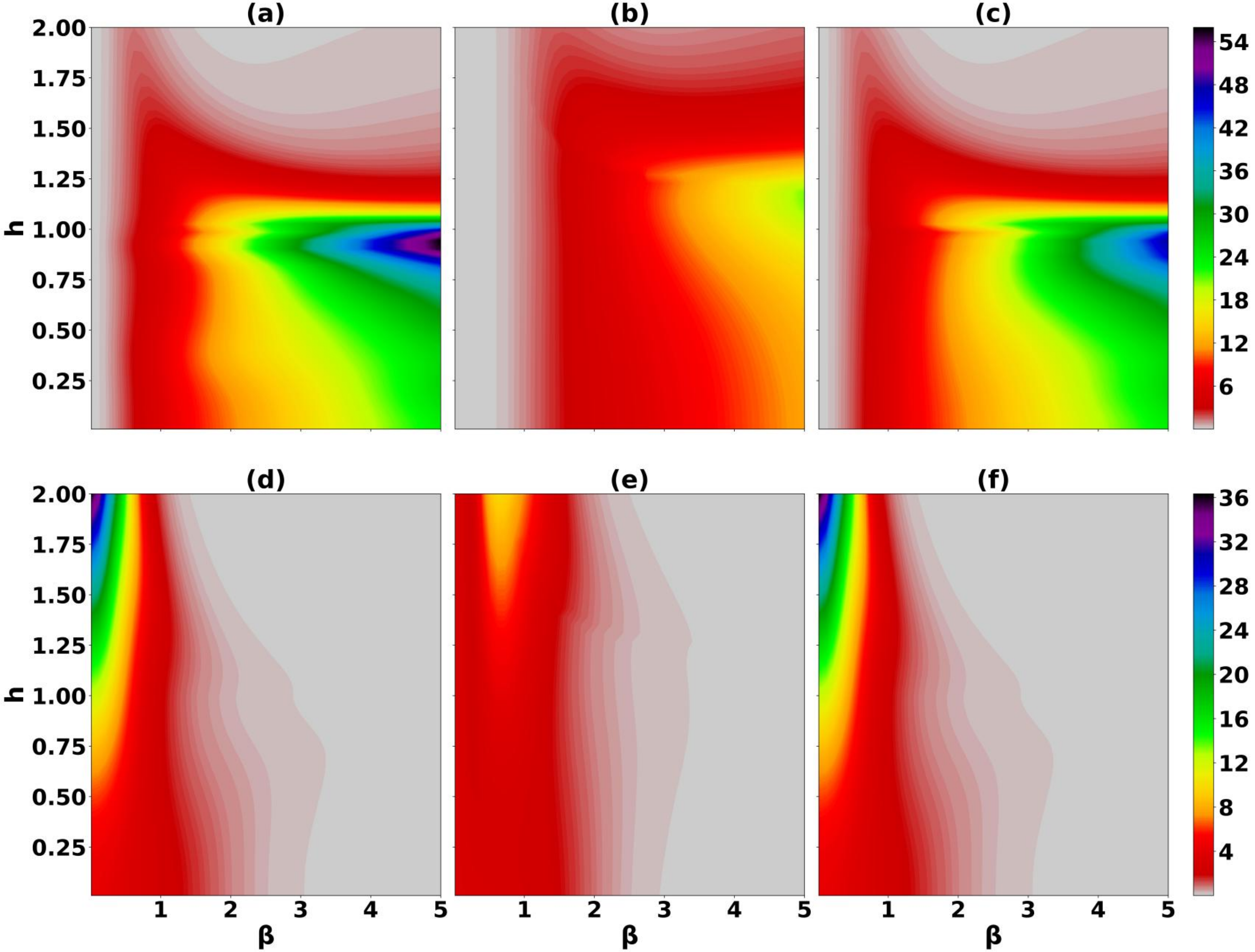
$$\mathcal{F}(\alpha) = \frac{9}{4\mathcal{Z}}\left[\sum_{k\in\mathbf{K}^+} \frac{e^{-\beta\epsilon_k}}{2}\mathbb{Z}_k^+ \frac{\left(e^{-2\beta\epsilon_k} - 1\right)^2}{e^{-2\beta\epsilon_k} + 1}\left(\frac{\partial\theta_k}{\partial\alpha}\right)^2 + \sum_{k\in\mathbf{K}^-\setminus\{0,\pi\}} \frac{e^{-\beta\epsilon_k}}{2}\mathbb{Z}_k^- \frac{\left(e^{-2\beta\epsilon_k} - 1\right)^2}{e^{-2\beta\epsilon_k} + 1}\left(\frac{\partial\theta_k}{\partial\alpha}\right)^2\right] + \frac{\partial^2\psi}{\partial\alpha^2} + \tilde{\mathcal{F}}^c$$

- The free entropy and CFI are related via $\frac{\partial^2\psi}{\partial\alpha^2} = \left\langle\left(\frac{\partial(\beta\hat{H})}{\partial\alpha}\right)^2\right\rangle - \left\langle\frac{\partial(\beta\hat{H})}{\partial\alpha}\right\rangle^2 - \left\langle\frac{\partial^2(\beta\hat{H})}{\partial\alpha^2}\right\rangle = \mathcal{F}^c(\alpha) - \left\langle\frac{\partial^2(\beta\hat{H})}{\partial\alpha^2}\right\rangle.$

- Using the phenomenological approach, we can also define an expression for QFI as

$$\boldsymbol{\mathcal{F}'(\alpha)} = \frac{\partial^2\psi'}{\partial\alpha^2} + \frac{\partial^2(\beta\mathcal{E})}{\partial\alpha^2}$$

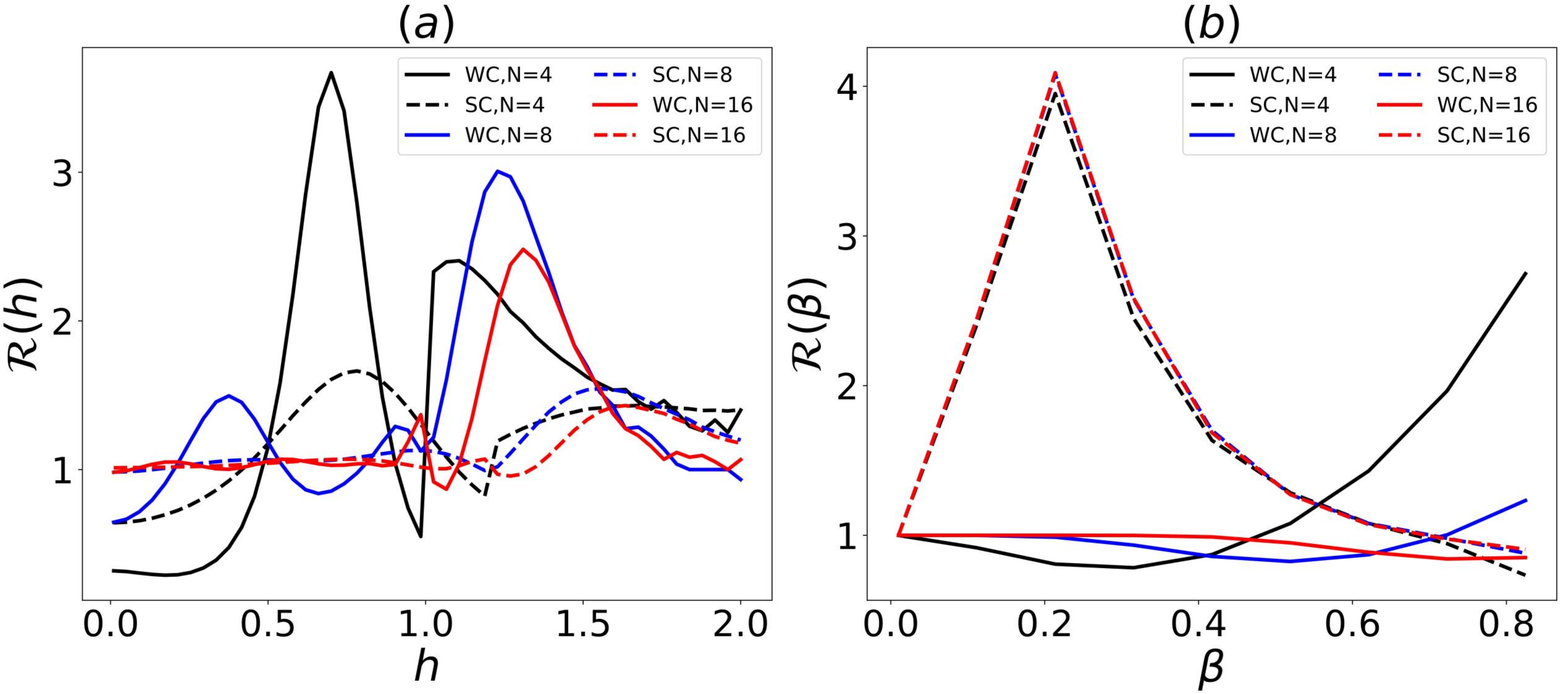
Results: Comparing QFI



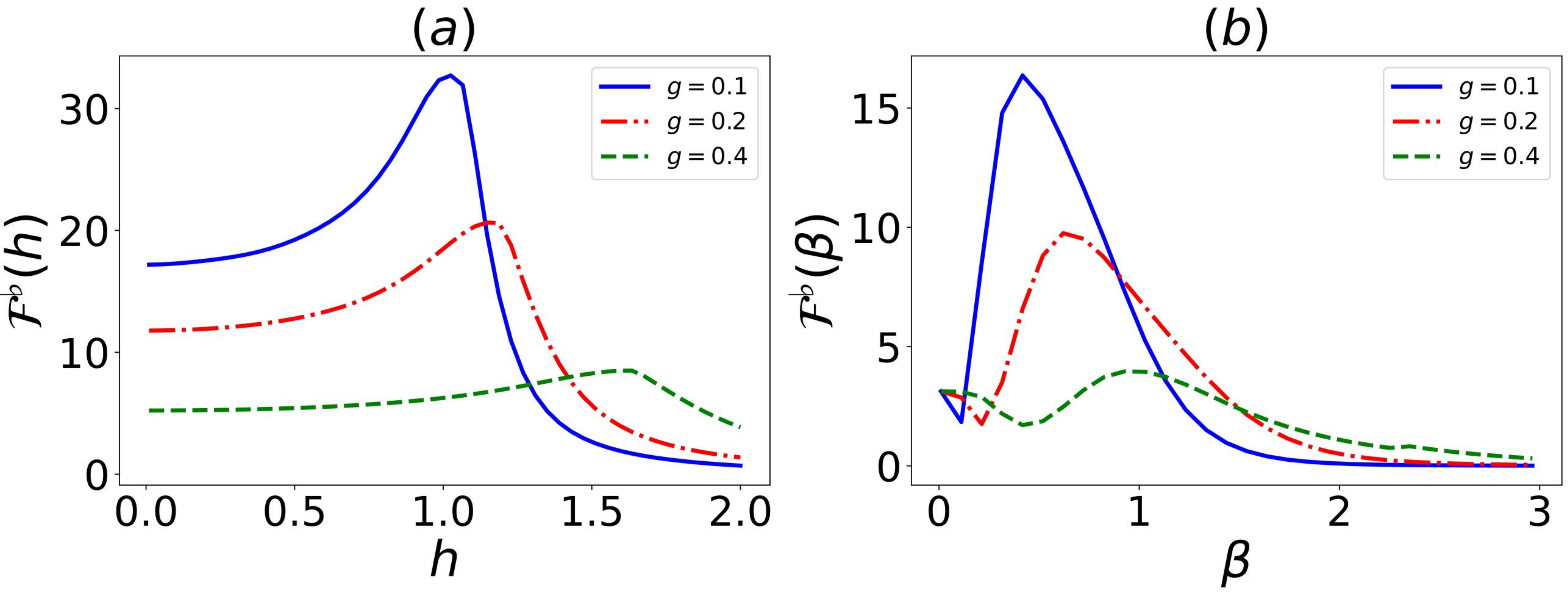
Comparison between QFI calculated for 3 cases. Top panel: QFI calculated for h (a) at WC, (b) at SC, (c) using the phenomenological approach. Bottom panel: QFI calculated for β (d) at WC (e) at SC, (f) using the phenomenological approach. The parameters are $N = 8, J = 1, \gamma = 0.25, g = 0.2$

Results: Effect of Neglecting FS Effects

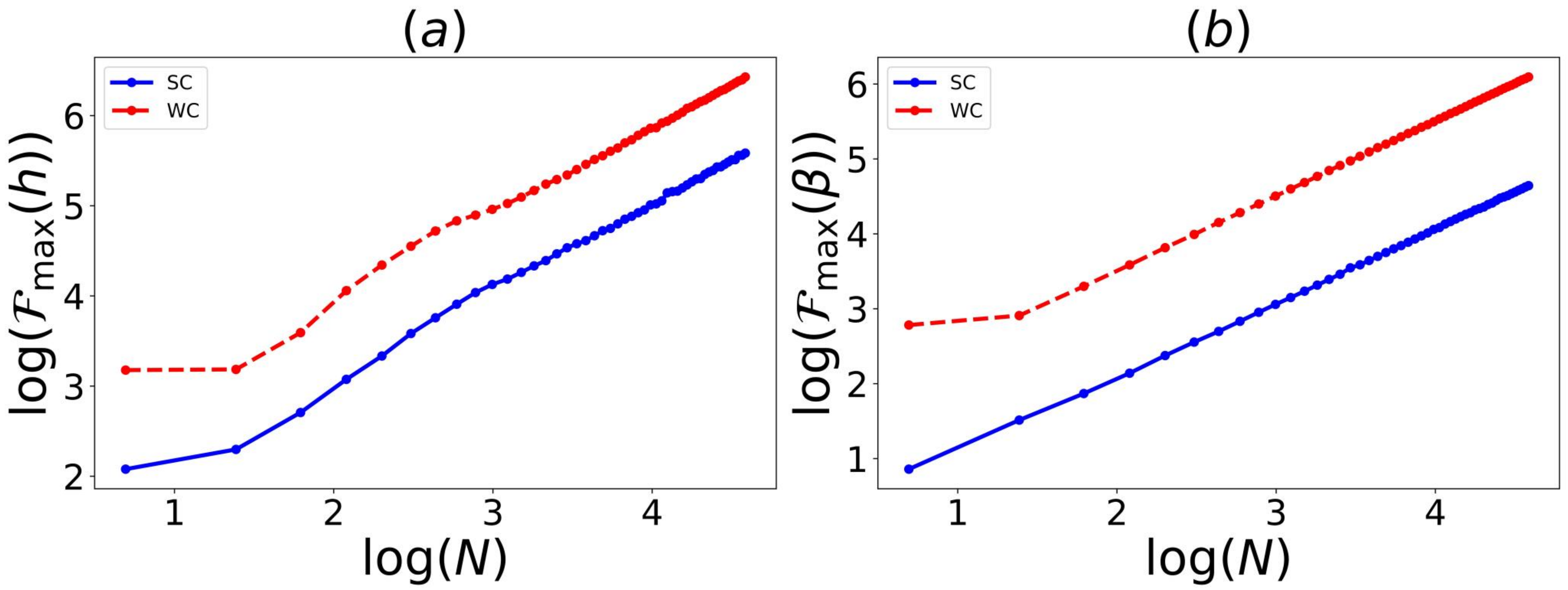
The ratio of the QFI with and without assuming PPA is given by $\mathcal{R}(\alpha) = \mathcal{F}_{\mathrm{PPA}}(\alpha)/\mathcal{F}(\alpha)$.



Results: Effect of System-Bath Coupling Strength



Results: Scaling



The ansatz is derived using finite-size scaling theory as $\mathcal{F}_{\max}(\alpha) = aN^{\xi} + \frac{b}{N} + c$.

	ξ	a	b	c
$\mathcal{F}_{\max}(h)$	0.999	6.093	-27.930	18.290
$\mathcal{F}_{\max}(\beta)$	0.989	4.794	18.671	-3.514
$\mathcal{F}^{\flat}_{\max}(h)$	0.973	3.055	-5.178	2.192
$\mathcal{F}^{\flat}_{\max}(\beta)$	0.995	1.255	0.310	-0.123

Conclusion

- FS introduce significant and systematic corrections to QFI that persist well beyond very small system sizes. Approximations based on thermodynamic-limit arguments, such as PPA, can therefore lead to substantial misestimation of achievable precision.

- Strong system–bath coupling qualitatively modifies metrological performance. While it generally suppresses sensitivity, it can enhance thermometric precision in low-temperature regimes and shift optimal operating points. Utility of SC in low-temperature thermometry is also shown.

- Phenomenological thermodynamic approaches, including those based on subdivision potentials and temperature-dependent energy levels, fail to reliably capture these effects, highlighting the necessity of a fully microscopic description.

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