4d quantum gravity and higher symmetries

Dynamic Suprematism 1915
K. Malevich
https://www.tate.org.uk/art/artists/kazimir-malevich-1561

Florian Girelli
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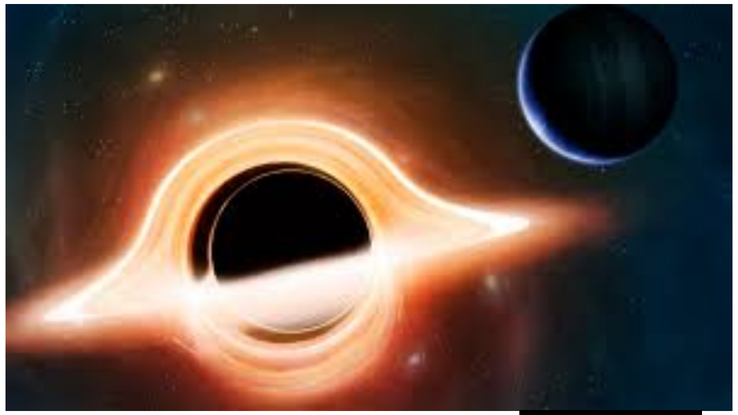
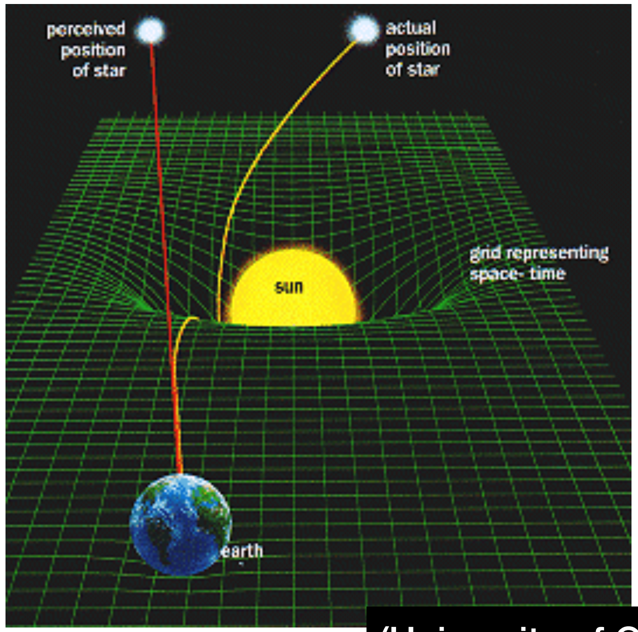
In collaboration with H. Chen, M. Dupuis, O. Hrytseniak, T. Oliveira Ferreira, C. Pollack, A. Riello, P. Tsimiklis.

Gravity=spacetime geometry



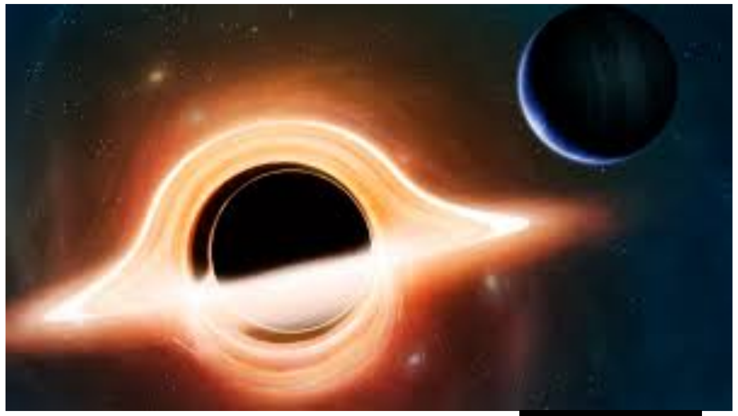
(Science)



(University of Oregon)

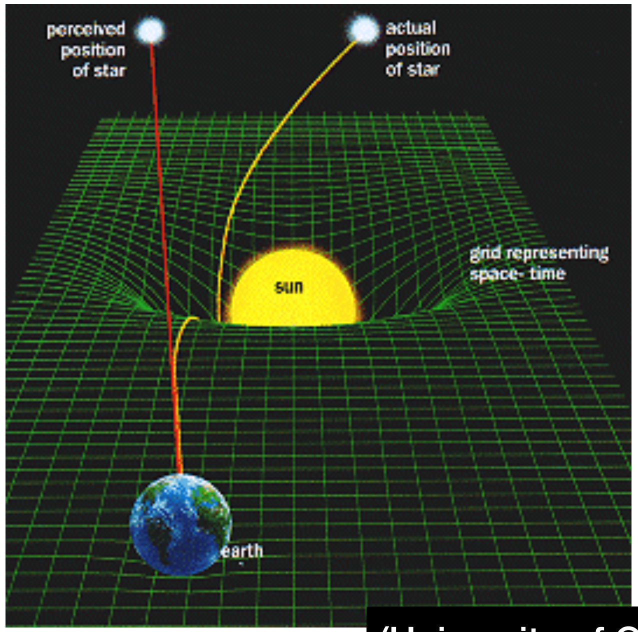
Usual formulation in terms of metric $g_{\mu\nu}$

Gravity=spacetime geometry



(Science)

Choice of variables matters



(University of Oregon)

Alternative characterization of gravity: Cartan formalism using (co)frame field e and spin connection A

$$g_{\mu\nu} = e^{A}{}_{\mu}\,\eta_{AB}\,e^{B}{}_{\nu}$$

$$A_{\mu}{}^{ab} = e^{a}{}_{\nu}\,\nabla_{\mu}e^{b\nu} = e^{a}{}_{\nu}\left(\partial_{\mu}e^{b\nu} + \Gamma^{\nu}_{\mu\rho}e^{b\rho}\right)$$

Proper variables to include spin

Quantum gravity goal: make sense of the partition function

$$\int [Dg]\, e^{iS_{GR}[g]}$$

Or

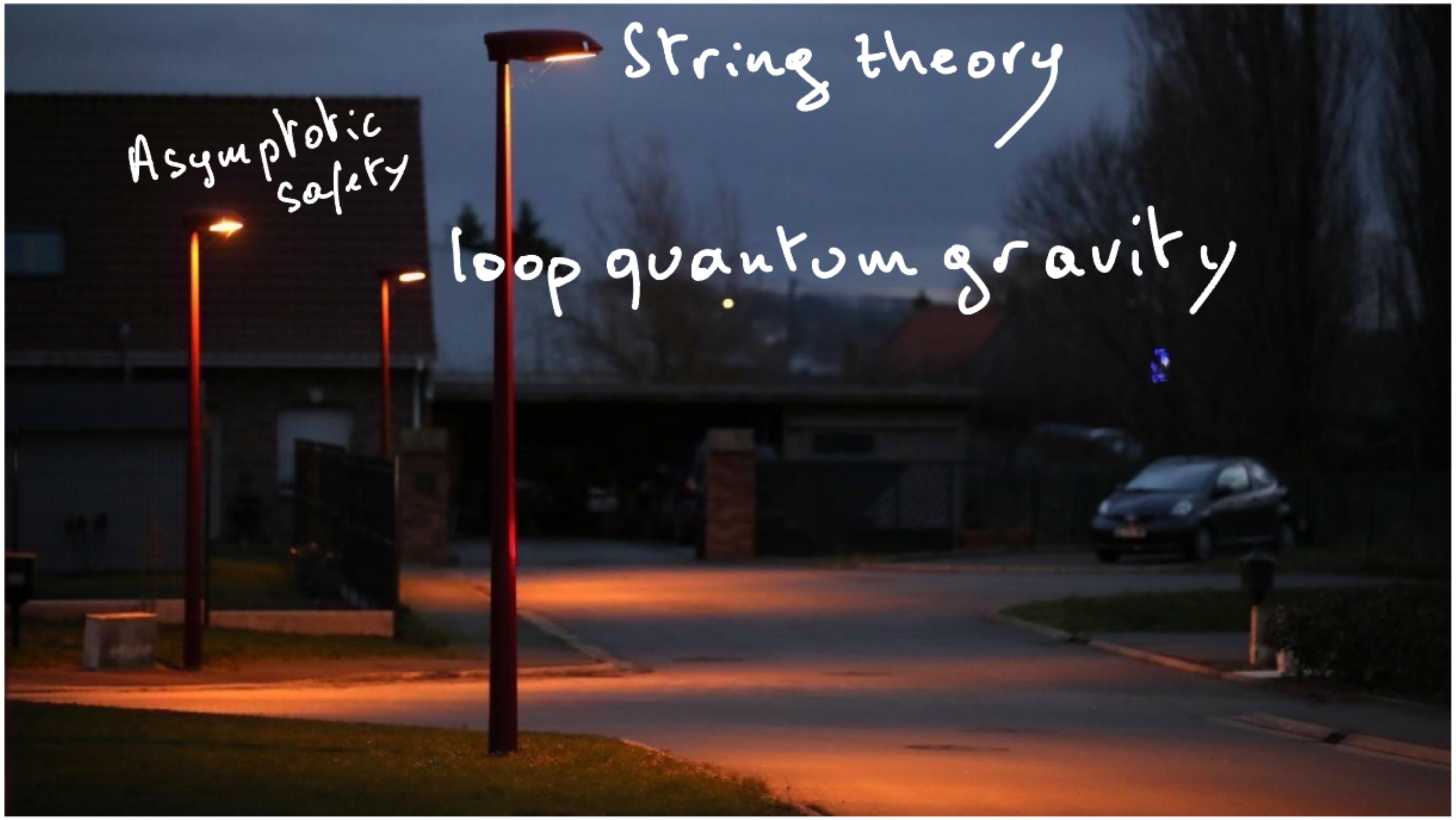
$$\int [De][DA]\, e^{iS_{GR}[e,A]}$$

Choice of variables matters?

What about the action?

© lavoix du nord

Looking for your keys?



Looking for the Quantum Gravity keys?

ultimately we hope experiments will somehow indicates where the quantum gravity regime keys are.


the 2-group lamppost
Is this where the QG theory lies?
I don't know but at least here I can see better!
topological models
Cool math
2-groups

(Quantum) topological models can be seen as backbone of a (quantum) gravity theory

Gravity

"4d BF theory"

$$\int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2} T(\mathscr{B}) \wedge \mathscr{B}$$

$$\mathscr{B} \in \Lambda^2 M \otimes \mathfrak{g}^*$$

$$\mathscr{A} \in \Lambda^1 M \otimes \mathfrak{g}$$

$$\mathscr{F}(\mathscr{A}) \in \Lambda^2 M \otimes \mathfrak{g}$$

$$T : \mathfrak{g}^* \to \mathfrak{g}$$

EOM:

$T\mathscr{B} = \mathscr{F}$ "Fake flatness"

$d_{\mathscr{A}}\mathscr{B} = 0$ "2-curvature"

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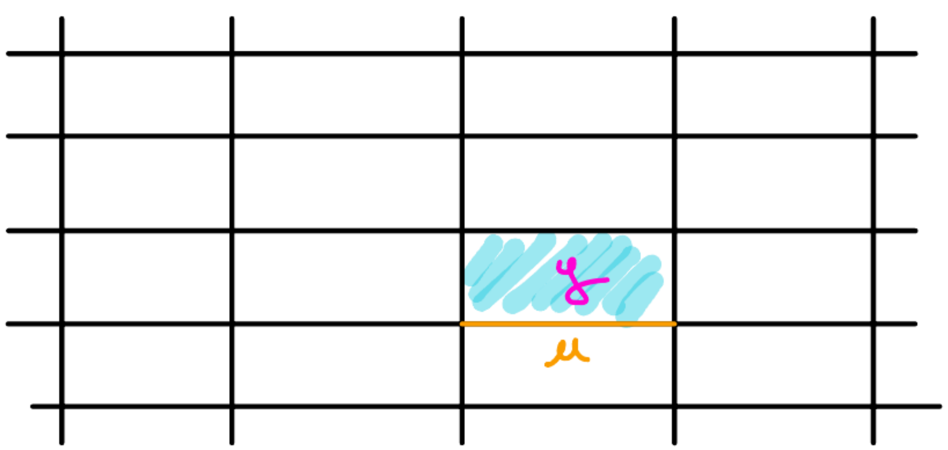
"4d BF theory"

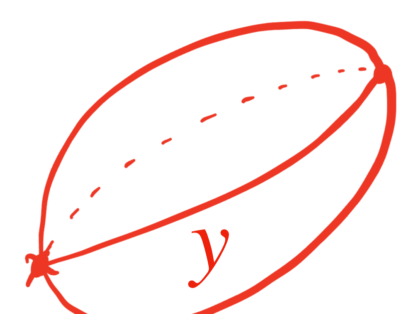
$$\int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2} T(\mathscr{B}) \wedge \mathscr{B}$$

Lattice with edge and face decorations



$T\mathscr{B} = \mathscr{F}$ $\qquad$ $t(y)u_s = u_\tau$ "Fake flatness"

2-holonomy $\quad d_{\mathscr{A}}\mathscr{B} = 0$



$y = 1$

"2-curvature"

A strict Lie 2-algebra is crossed module of Lie algebras

$$(\mathfrak{h} \xrightarrow{T} \mathfrak{g}, \triangleright) \qquad J_i \in \mathfrak{g} \qquad X_i \in \mathfrak{h}$$

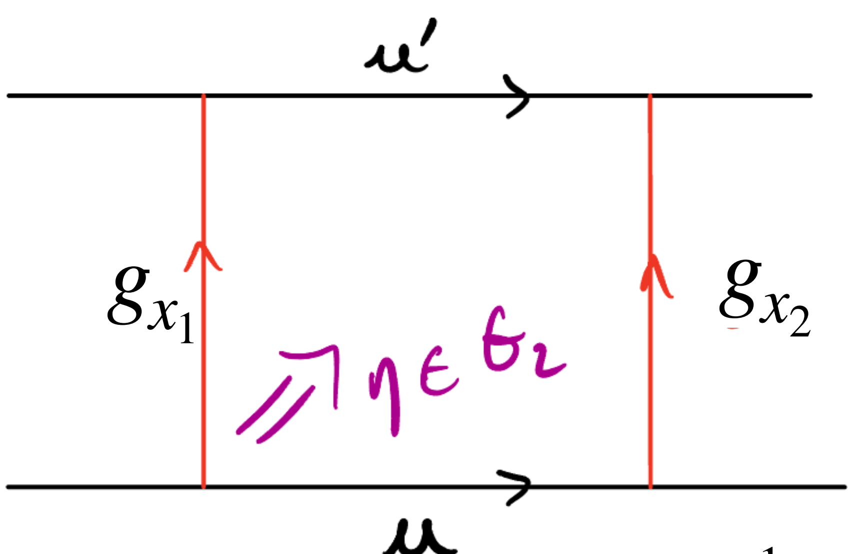
$$T([X_1, X_2]) = [T(X_1), T(X_2)]$$

$$T(X_1) \triangleright X_2 = [X_1, X_2]$$

$$T(J \triangleright X) = [J, T(X)]$$

Strict Lie 2-algebra are
exponentiated to strict **Lie 2-group**

Baez-Crans



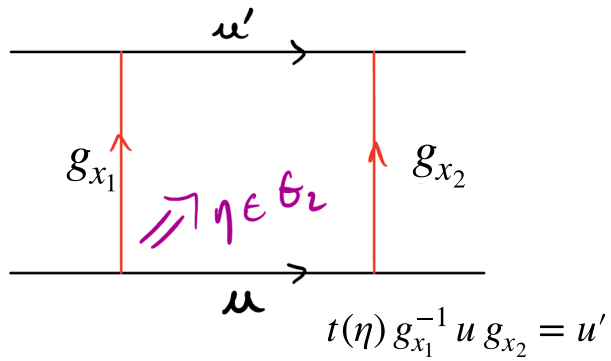
$$t(\eta)\, g_{x_1}^{-1}\, u\, g_{x_2} = u'$$

2-principal bundle

Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Higher gauge thy for Lie 2-algebra $(\mathfrak{h} \xrightarrow{T} \mathfrak{g}, \rhd)$

Baez
Girelli, Pfeiffer
Jurco, Raspollini, Saemann, Wolf



Connection $\mathfrak{A} = (A, \Sigma) \in (\mathfrak{g} \otimes \Lambda^1 M) \oplus (\mathfrak{h} \otimes \Lambda^2 M)$

Curvature data, $\mathfrak{F} = (F - T\Sigma, \quad d_A\Sigma = d\Sigma + A \rhd \Sigma)\,.$

1- and 2-gauge transf parameterized by $(\alpha, \phi) \in \mathfrak{g} \otimes \Lambda^0 M \oplus \mathfrak{h} \otimes \Lambda^1 M$,

$$\delta_{(\alpha,\phi)}\mathfrak{A} = d(\alpha, \phi) + [\mathscr{A}, (\alpha, \phi)]_2 = \left(d\alpha + [A, \alpha] + T(\phi), \alpha \rhd \Sigma + d_A\phi + \frac{1}{2}[\phi, \phi]\right)$$

$$\delta_{(\alpha,\phi)}\mathfrak{F} = [\mathscr{F}, (\alpha, \phi)]_2 = \left([F - T\Sigma, \alpha], \alpha \rhd d_A\Sigma + (F - T\Sigma) \rhd \phi\right)$$

2-adjoint action!

Generalized Maurer Cartan forms: $\mathfrak{A} = (A, \Sigma) \in (\mathfrak{g} \otimes \Lambda^1 M) \oplus (\mathfrak{h} \otimes \Lambda^2 M)$

$$dA + \frac{1}{2}[A \wedge A] = t\Sigma, \quad d_A\Sigma = 0$$

solution: $A = u^{-1}du + t\phi \qquad B = d_A\phi + \frac{1}{2}[\phi \wedge \phi]$

"Pure 2-gauge"

For Lie (quantum) 2-groups, we do **not** have the representation theory for most relevant groups, nor the harmonic analysis (Peter-Weyl theorem)…

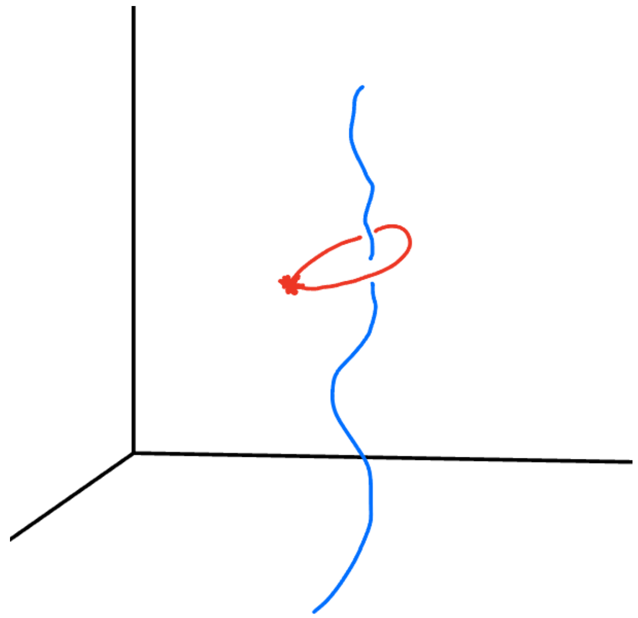
See however Baez-Baratin-Freidel-Wise

Some interesting math to do!

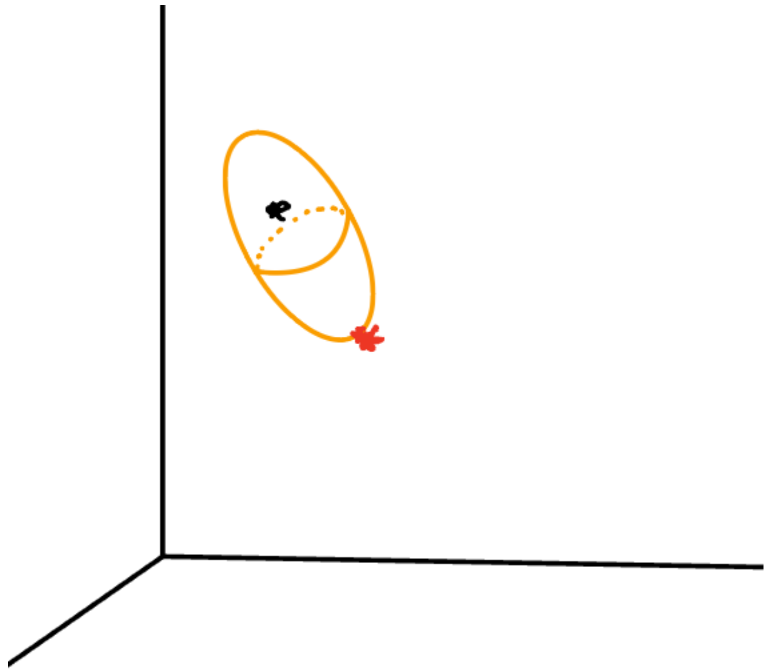
Defects

Curvature defects can be introduced:

String-like: curvature $\mathscr{F}(\mathscr{A}) - T\mathscr{B} = S\delta^2(x)$



Point-like: 2-curvature $d_{\mathscr{A}}\mathscr{B} = p\delta^3(x)$



Exactly the structure to construct
Kitaev model in 3+1
(typically for finite 2-groups)

(Quantum) topological models can be seen as backbone of a (quantum) gravity theory

Plebanski $\mathscr{A} \in \mathfrak{so}(3,1) \otimes \Lambda^1 M$

$$\mathcal{S}_{Pl} = \int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{\Lambda}{2} \mathscr{B} \wedge \mathscr{B} - \frac{1}{2} \phi(\mathscr{B}) \wedge \mathscr{B}$$

Lagrange multiplier

$$T_{PL}(\mathscr{B}) = (\Lambda \mathrm{id} + \phi)\mathscr{B} = \mathscr{F}$$

$$d_{\mathscr{A}} \mathscr{B} = 0$$

Cf review by Freidel-Speziale

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Freidel-Starodubtsev
(MacDowell-Mansouri) $\mathscr{A} = (\omega, e) \in \mathfrak{so}(4,1) \otimes \Lambda^1 M$

Gives Immirzi

$\alpha \propto G\lambda \sim 10^{-120}$

$$\int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2}\beta\mathscr{B} \wedge \mathscr{B} - \frac{\alpha}{4}\epsilon^{4IJKL}\mathscr{B}_{IJ} \wedge \mathscr{B}_{KL}$$

$$T_{MM}(\mathscr{B}) = (\beta \mathrm{id} + \frac{\alpha}{2}\epsilon)\mathscr{B} = \mathscr{F}$$

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Herffray-Krasnov $\mathscr{A} \in \mathfrak{so}(3,\mathbb{C}) \otimes \Lambda^1 M$

$$S_{HK} = \int \mathscr{B} \wedge \mathscr{F} - \frac{\Lambda}{2}\mathscr{B} \wedge \mathscr{B} + \frac{\alpha}{2}\left(Tr(\sqrt{\mathscr{B}^I \wedge \mathscr{B}^J})\right)^2$$

$$T^I_{HK}(\mathscr{B}) = \left(\Lambda\delta^{IJ} - \alpha Tr(\sqrt{X})\,(X^{-1})^{IJ}\right)\mathscr{B}_J = \mathscr{F}^I, \quad X^{IJ} = \mathscr{B}^I \wedge \mathscr{B}^J$$

$$d_{\mathscr{A}}\mathscr{B} = 0$$

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“4d BF theory”

$$\int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2} T(\mathscr{B}) \wedge \mathscr{B}$$

$$Z_{QG} \sim \int [D\mathscr{B}][D\mathscr{A}] e^{i \int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2} T(\mathscr{B}) \wedge \mathscr{B} + \text{Non topo}}$$

Doable

Non topo: hard but proposals (Eg EPRL-FK model)

Topological theory from matter

Conversely, consider **matter fields** Φ with *no spin* on a 4d *flat* spacetime, which background geometry is given by **frame field** e and **spin connection** A.

$$\mathscr{L}_m(e_{bckgd}, \Phi)$$

We assume constant curvature and no torsion: $F(A) = 0, \quad T(e, A) = d_A e = 0$

$$\mathscr{A} = (A, e), \quad \mathfrak{F} = (F(A), T(e, A)) \in \mathfrak{iso}(3,1) \otimes \Lambda^2 M,$$

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Use Lagrange multipliers $\mathfrak{B} = (B, \Sigma) \in \mathfrak{iso}^*(3,1) \otimes \Lambda^2 M$ to implement these constraints

$$\mathscr{L}_m(e_{bckgd}, \Phi) \to \mathscr{L}_m(e, \Phi) + \mathfrak{B} \wedge \mathfrak{F}$$

Topological sector appears!

We have enlarged the space of possible backgrounds: degenerate frame field/metrics are also allowed.

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Analogue of Einstein equation can be seen as a 2-curvature data

$$\frac{\delta \mathscr{L}}{\delta e}(e, \Phi) = d_A \Sigma$$

Matter can be seen as 2-curvature excitation.

Gauge symmetry on frame field sector = stress energy tensor conservation

Dupuis, Girelli, Hyrtseniak, wip

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Topological sector appears!

Similar result can be obtained from Feynman diagrams of scalar field theory

Feynman diagrams (scalar field in Minkowski) =
particle excitations in a topological 2-state-sum

Baratin-Freidel

$$\int [d\phi] e^{i\int S[\phi,e]} \to \int [d\phi][d\mathscr{A}][d\mathscr{B}] e^{i\int S[\phi,e] + \mathscr{B}\wedge\mathscr{F}}$$

Two possible improvements of standard approach

Plebanski/ Spinfoam/ LQG

$\mathscr{A} \in \mathfrak{so}(3,1) \otimes \Lambda^1 M$

$$\mathcal{S}_{Pl} = \int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2}\phi(\mathscr{B}) \wedge \mathscr{B}$$

Lagrange multiplier

$T_{PL}(\mathscr{B}) = \phi\mathscr{B} = \mathscr{F}$

Frame field not explicitly present in this formulation
Makes things complicated to couple to matter!
Is it really the best formulation?

$$Z_{QG} \sim \int [D\mathscr{B}][D\mathscr{A}] e^{i\int \mathscr{B}\wedge\mathscr{F}(\mathscr{A}) - \frac{1}{2}T(\mathscr{B})\wedge\mathscr{B} + \text{Non topo}}$$

Using standard group representations, not 2-groups!

Shouldn't we use 2-group representations?

A better formulation?

Spinorial BF action $\quad \phi, \psi \in \Lambda^1 M \otimes \mathbb{C}^2, \quad {}^{\phi}\pi, {}^{\psi}\pi \in \Lambda^2 M \otimes \mathbb{C}^2$

$$\int {}^{\psi}\pi^A \wedge d\psi_A + {}^{\phi}\pi^A \wedge d\phi_A + {}^{\psi}\pi^A \wedge {}^{\phi}\pi_A$$

Global $SL(2,\mathbb{C})$ symmetry

2-group gauge symmetry $\mathbb{C}^2 \to \mathbb{C}^2$

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2-group gauge symmetry $\mathbb{C}^2 \to \mathbb{C}^2$

Make $SL(2,\mathbb{C})$ local: introduce constraint using Lagrange multiplier ${A_A}^B$

Dupuis, Girelli, Hyrtseniak, Wieland

$$\int {}^{\psi}\pi^A \wedge d\psi_A + {}^{\phi}\pi^A \wedge d\phi_A + {}^{\psi}\pi^A \wedge {}^{\phi}\pi_A + {A_A}^B \wedge ({}^{\psi}\pi^A \wedge \psi_B + {}^{\phi}\pi^A \wedge \phi_B)$$

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$$= \int {}^{\psi}\pi^A \wedge \nabla\psi_A + {}^{\phi}\pi^A \wedge \nabla\phi_A + {}^{\psi}\pi^A \wedge {}^{\phi}\pi_A$$

= self dual gravity (up to boundary term)

Robinson

$$\psi_A = e_{A0'}, \quad \phi_A = ie_{A1'}$$

A better formulation?

Make $SL(2,\mathbb{C})$ local: introduce constraint using Lagrange multiplier $A_A{}^B$ **Dupuis, Girelli, Hyrtseniak, Wieland**

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= self dual gravity (up to boundary term) **Robinson**

Gravity appears from topological theory (2-symmetries)
by imposing local gauge symmetry
(which breaks some of these 2-symmetries)

Particle can be introduced at the topological level

Dupuis, Girelli, Hyrtseniak, Wieland

Can construct a Kodama state for zero cosmological constant!

Summary

4d BF theory have symmetries encoded by Lie 2-groups.

4d BF theory can be used to define gravity action in many different ways

4d BF theory naturally appears from matter Lagrangians

We proposed yet another formulation of gravity using 4d BF theory

— > Gravity appears by making a global $SL(2,\mathbb{C})$ symmetry local.

— > Frame field is present from the get-go.

—> This allows to include particles as topological defects

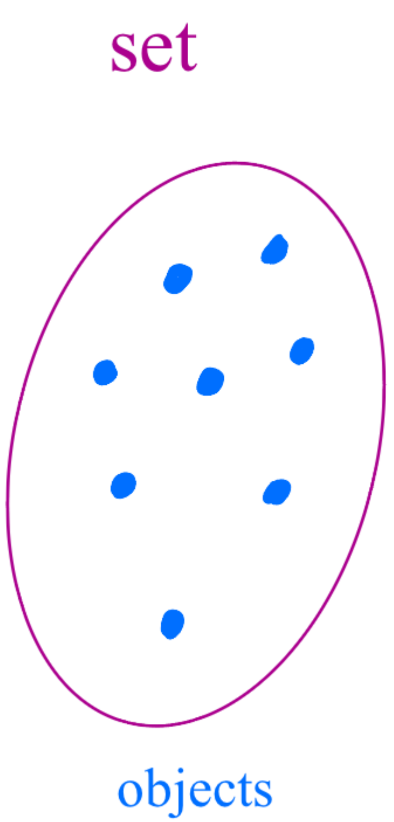
Need to study Lie (quantum) 2-group representations to define 4d BF partition functions

To tweak them to define QG models

$$Z_{QG} \sim \int [D\mathscr{B}][D\mathscr{A}] e^{i\int \mathscr{B}\wedge\mathscr{F}(\mathscr{A}) - \frac{1}{2}T(\mathscr{B})\wedge\mathscr{B} + \text{Non topo}}$$

Back up slides

- Many formulations of 4d gravity start from a topological theory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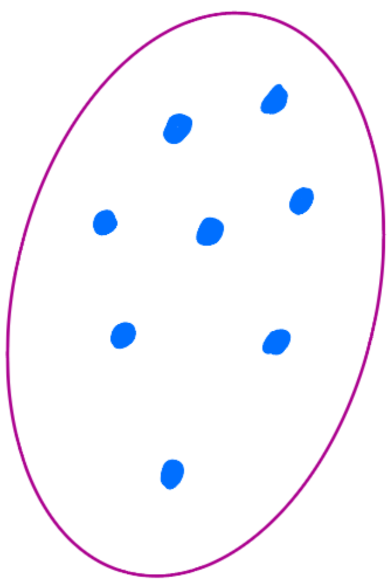

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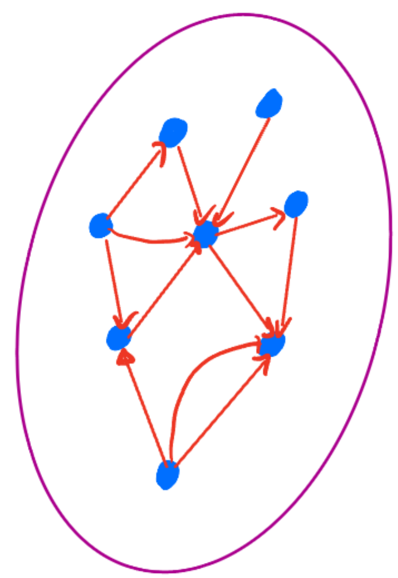

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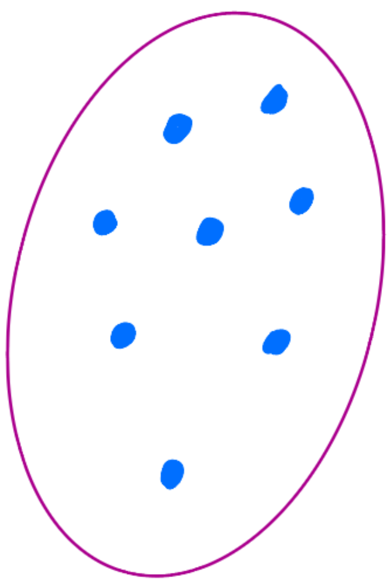

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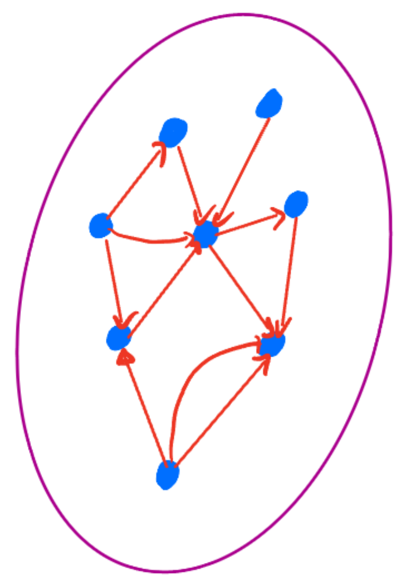

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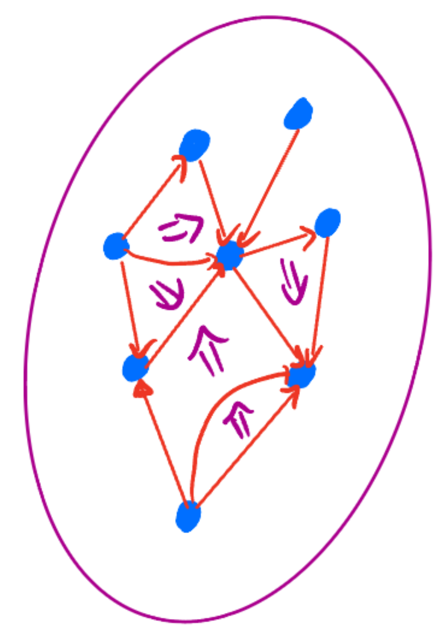

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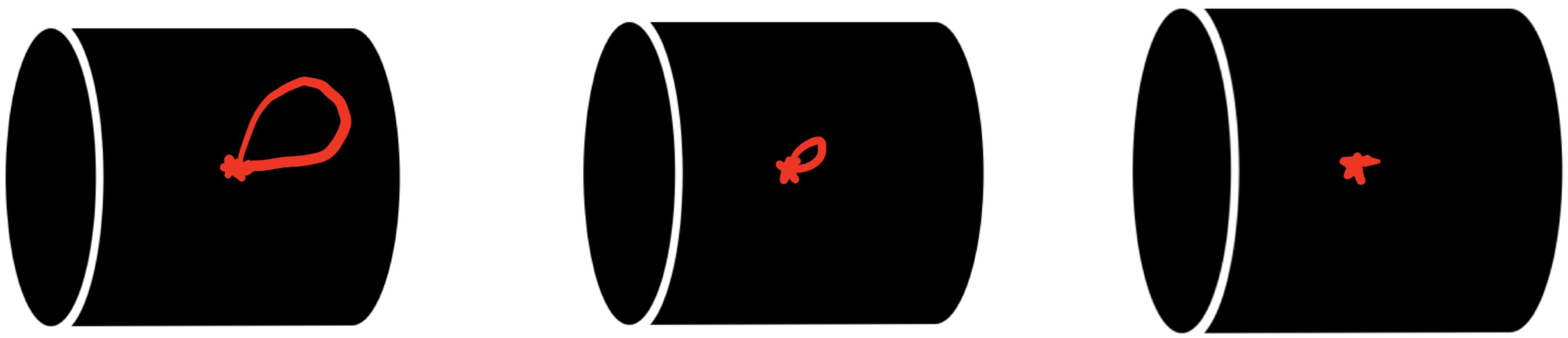
Probing topology



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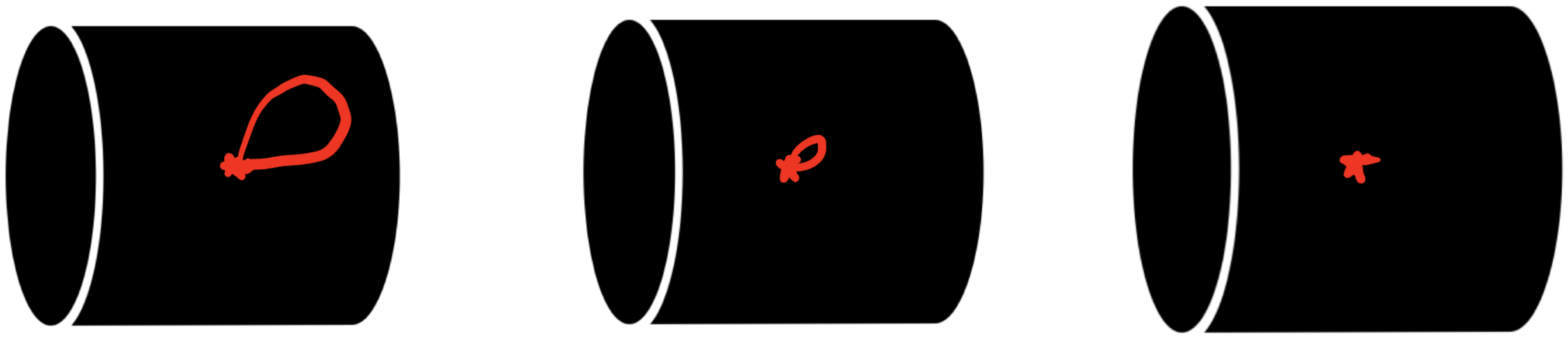
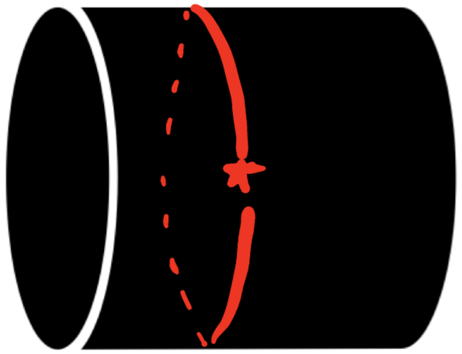
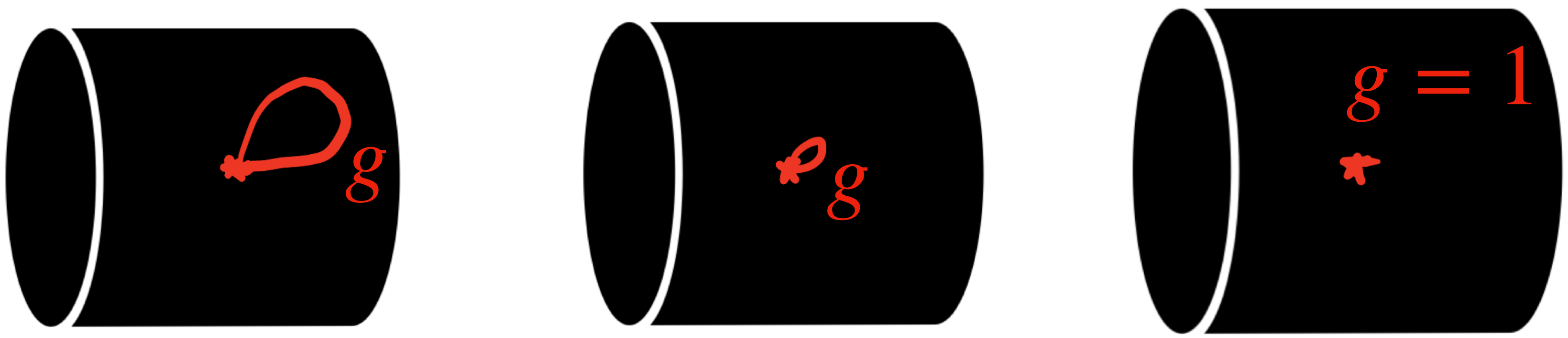


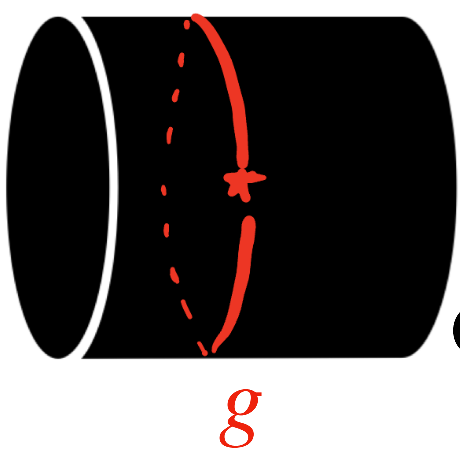


Cylinder: circle cannot be shrunk

Probing the fundamental homotopy group $\pi_1(M)$

Probing topology



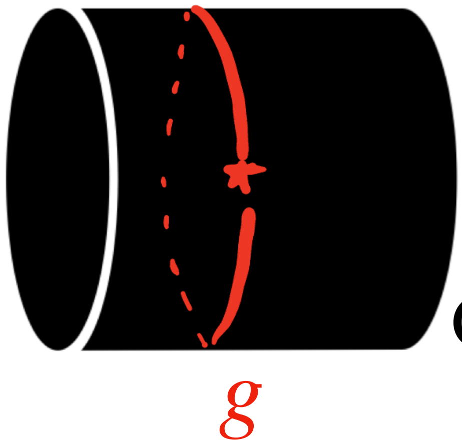


Cylinder: circle cannot be shrunk

Can be decorated by some group element keeping track of topology.

Probing the fundamental homotopy group $\pi_1(M)$

Probing topology



Cylinder: circle cannot be shrunk

Can be decorated by some group element keeping track of topology.

Probing the fundamental homotopy group $\pi_1(M)$

Essential 2-sphere: 2-sphere cannot be shrunk

Need a surface to probe such object

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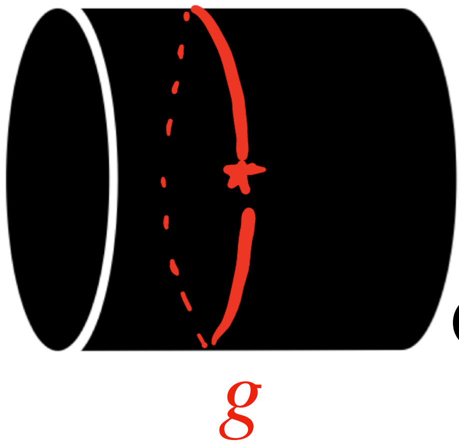

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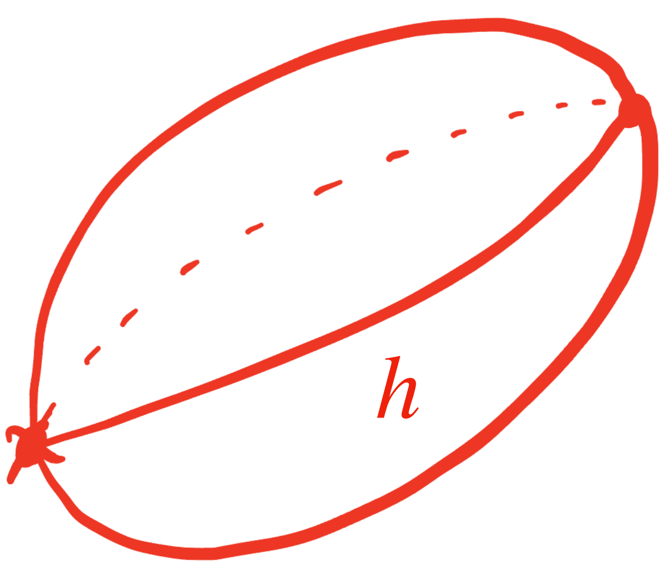

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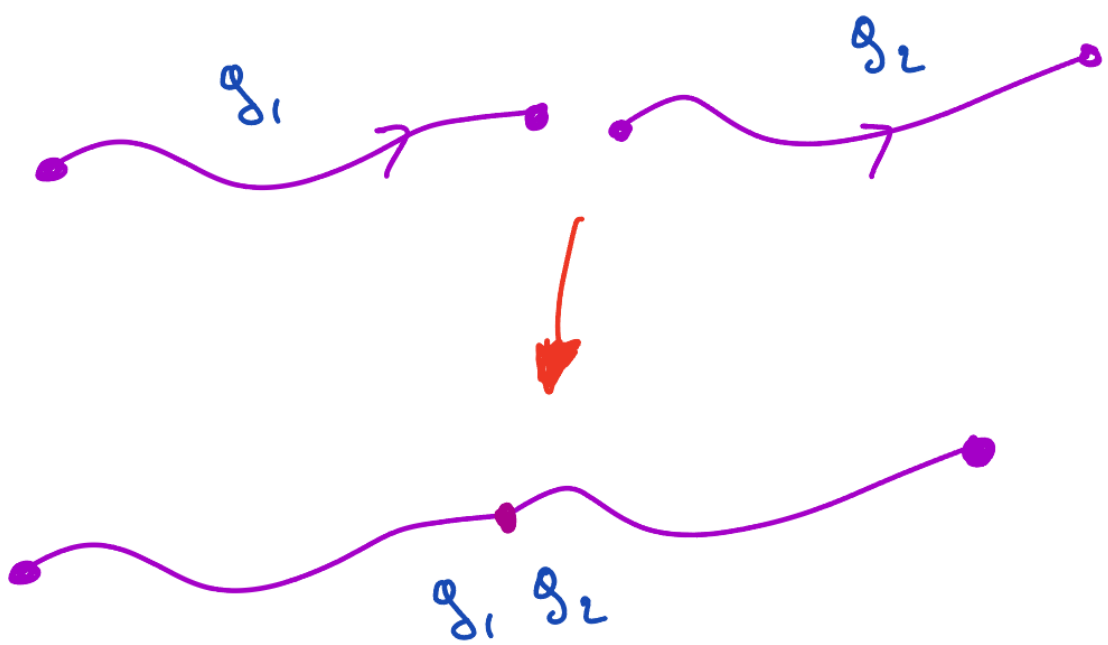
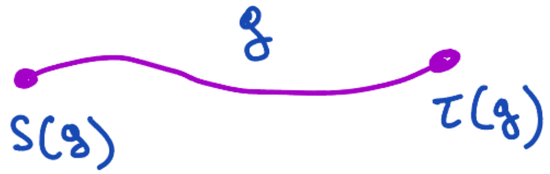
Can be decorated by some group element keeping track of topology.

Decorated paths composition



$g_i \in G$.
group

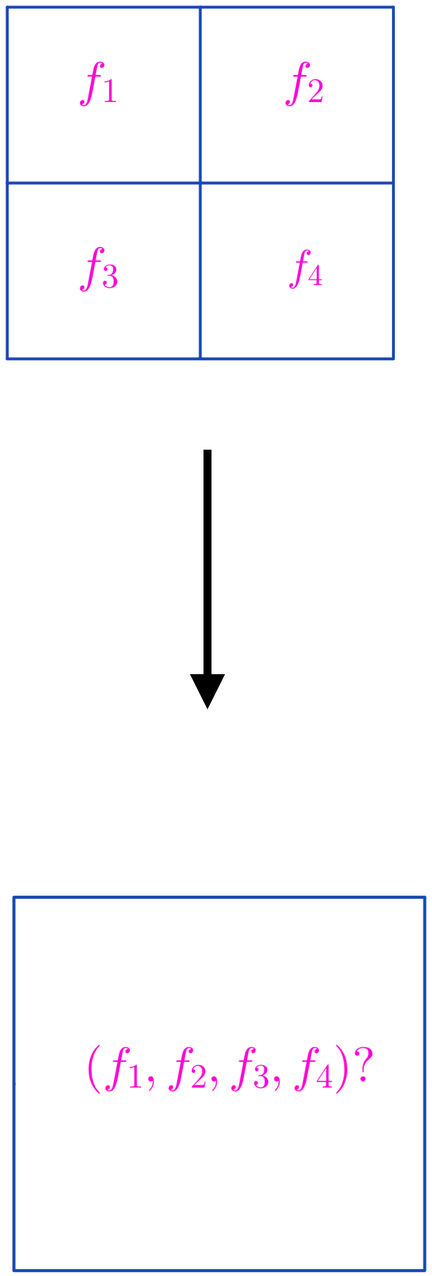
s: source $\qquad$ τ : target



Product : $g_1 g_2$ if $\tau(g_1) = s(g_2)$.

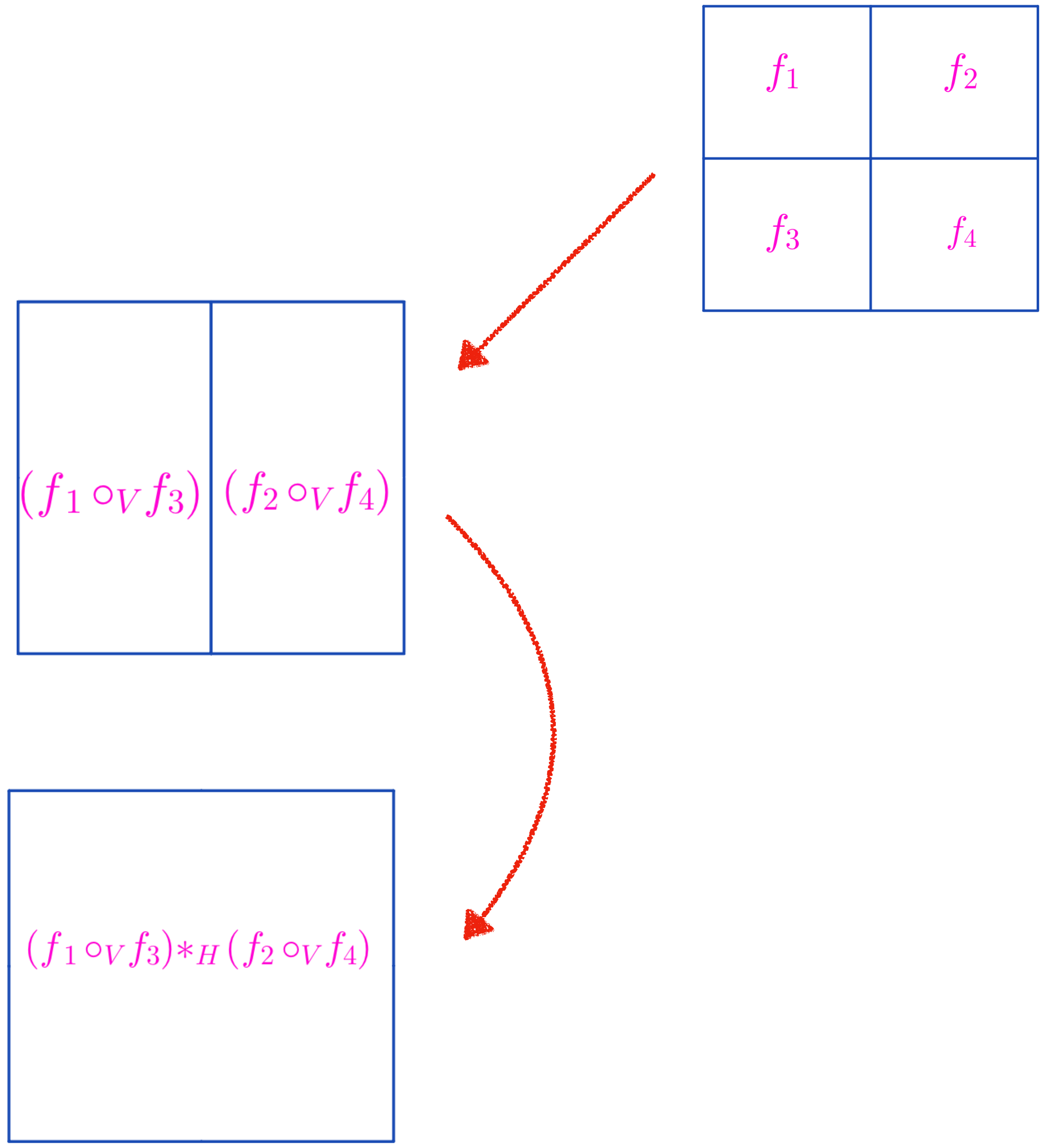
Groupoid structure. $\qquad G \underset{\tau}{\overset{s}{\rightrightarrows}} M$

Composing surfaces?



Composing surfaces: geometric compatibility

$f_i \in G$



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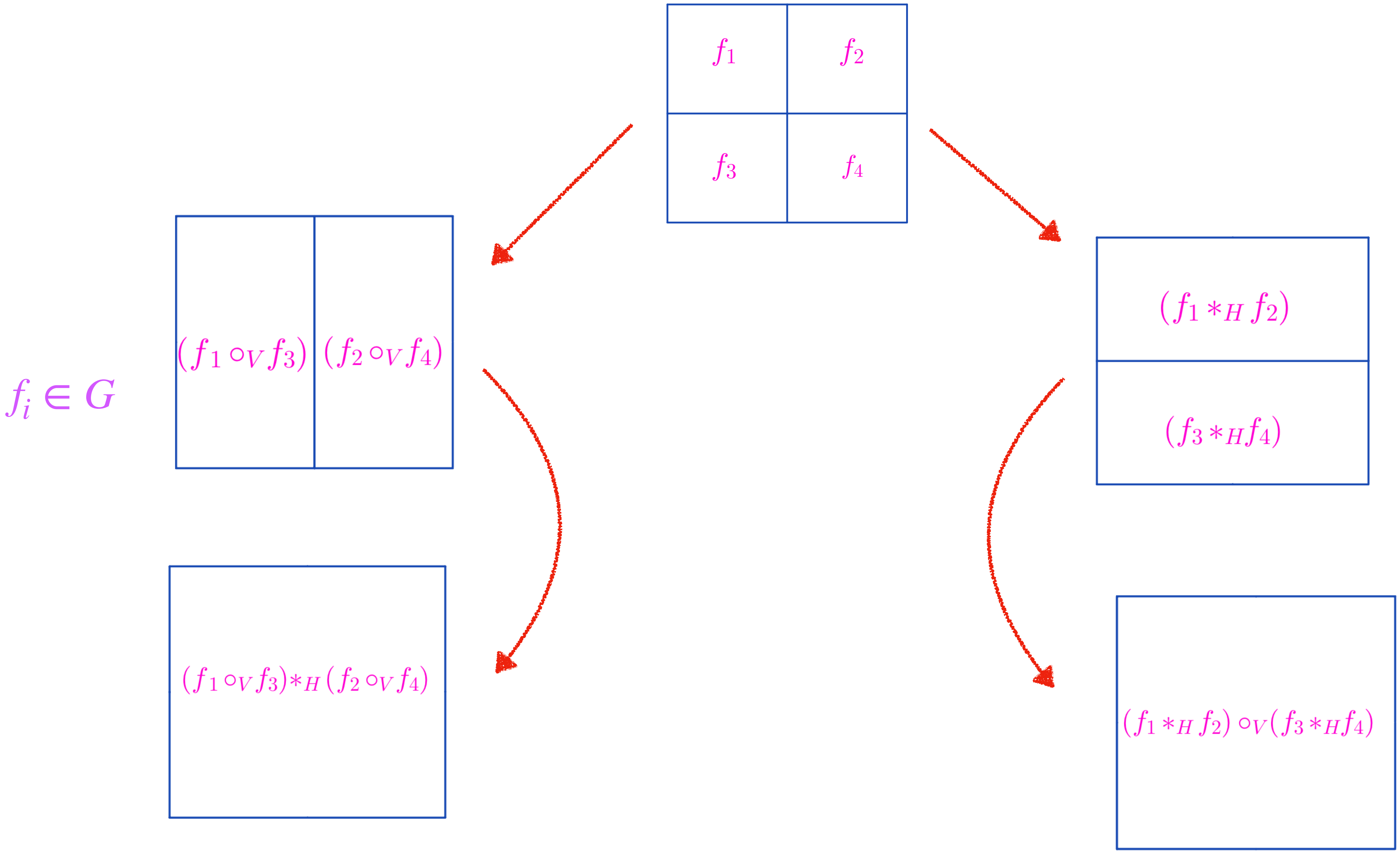
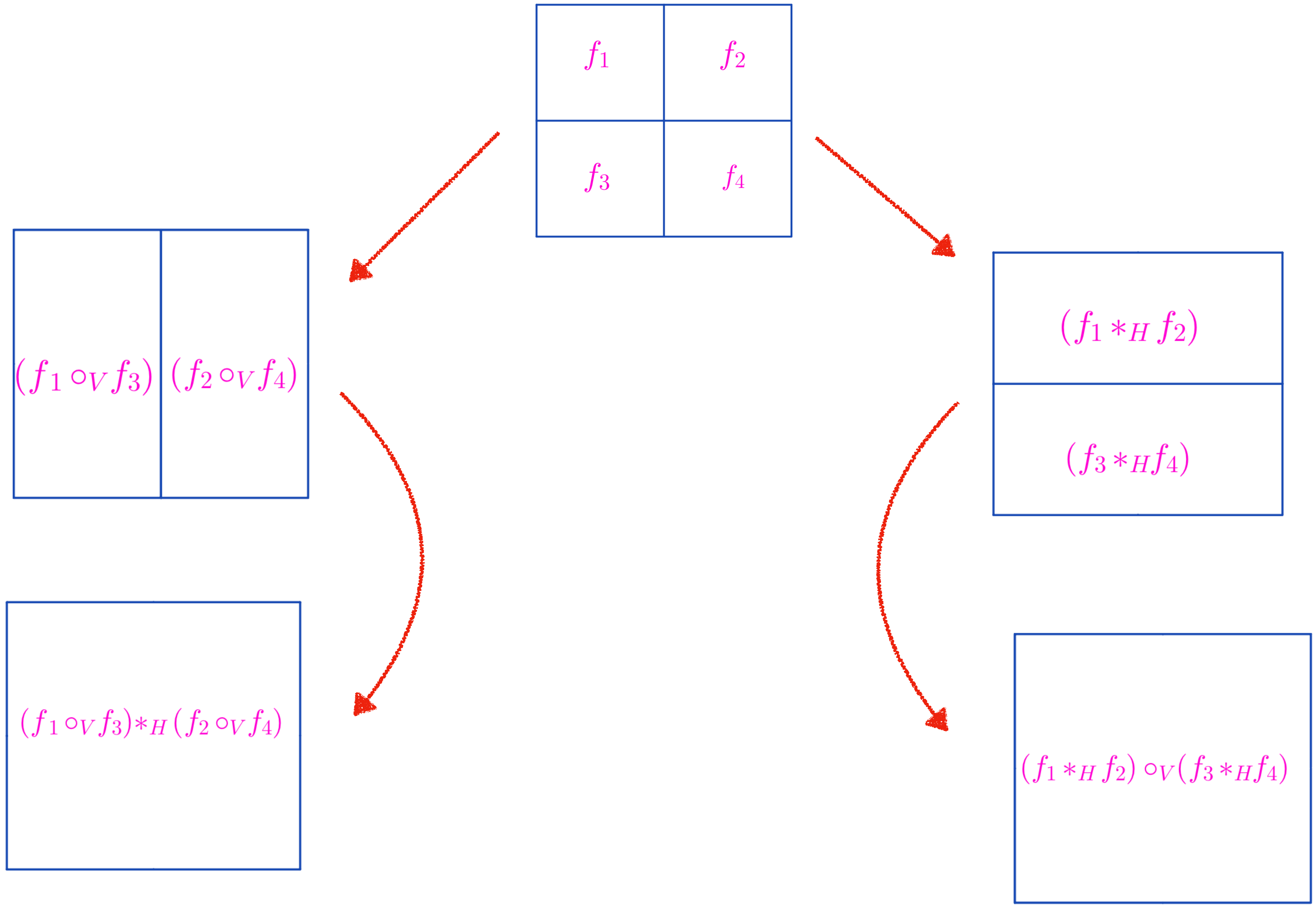


$$(f_1 \circ_V f_3) *_H (f_2 \circ_V f_4) = (f_1 *_H f_2) \circ_V (f_3 *_H f_4)$$

Composing surfaces: geometric compatibility

$f_i \in G$

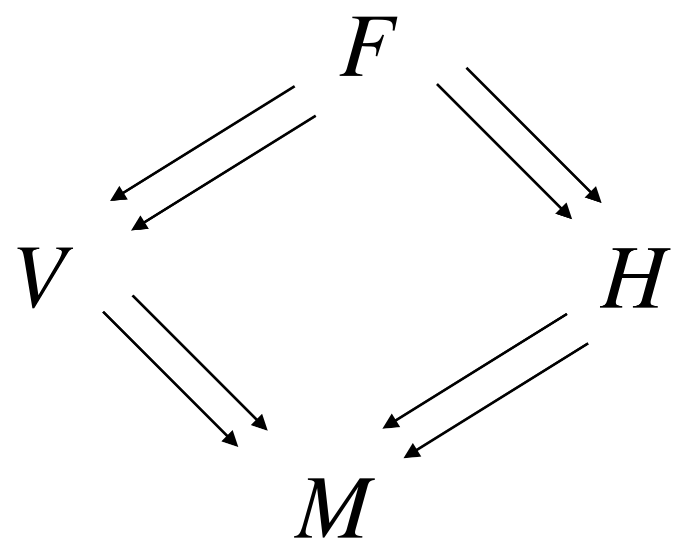


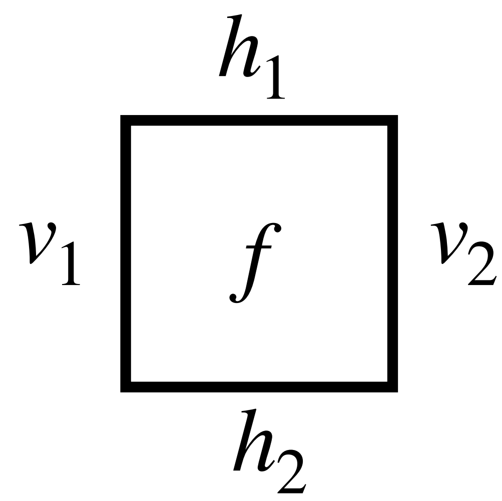
Group product: $(f_1 * f_3) * (f_2 * f_4) = (f_1 * f_2) * (f_3 * f_4)$

Eckmann Hilton argument $f_3 * f_2 = f_2 * f_3$

Composing surfaces: geometric compatibility

Solution: double groupoid





\+ Compatibility rules

Brown, Spencer

Composing surfaces: geometric compatibility

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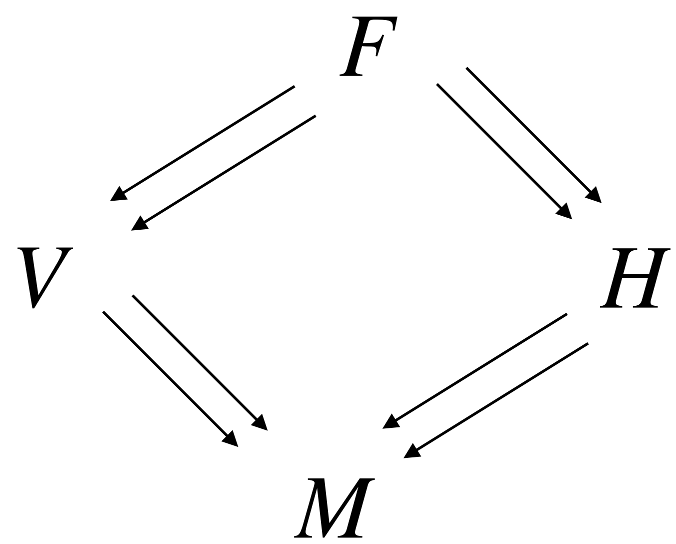

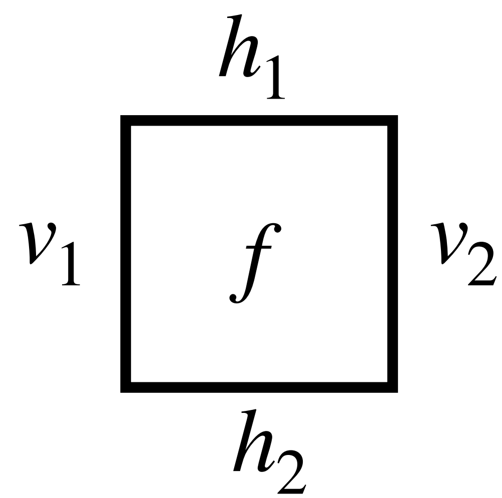

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Brown, Spencer

Exemple: **crossed module = strict 2-group**

$$H = G_1, \quad V = 1$$

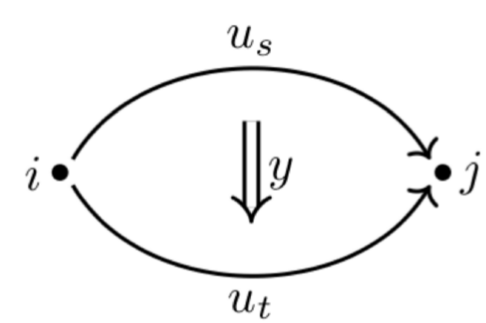
$$F = G_1 \ltimes G_2 \ni f = (u, y)$$

$t : G_2 \longrightarrow G_1$ “Boundary map”

$$G_1 \ni t(y) \equiv \tau(f) s^{-1}(f)$$

$$t(u \triangleright y) = u\, t(y) u^{-1}$$

$$t(y) \triangleright y_1 = y\, y_1\, y^{-1}$$



$$t(y) u_s = u_t$$

Composing surfaces: geometric compatibility

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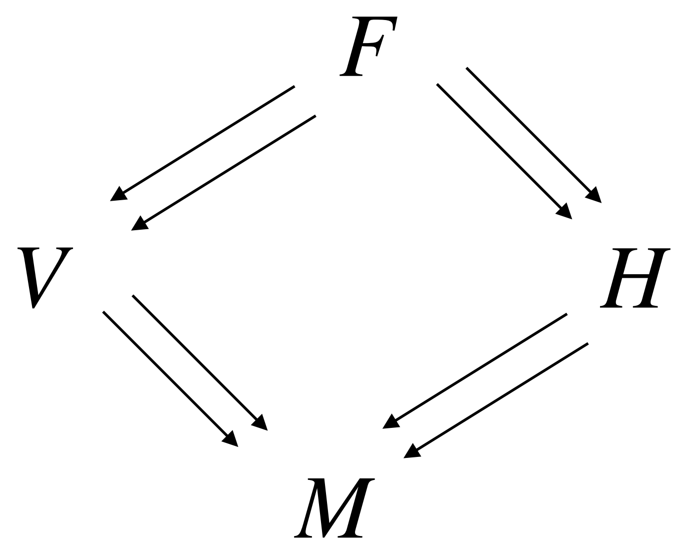

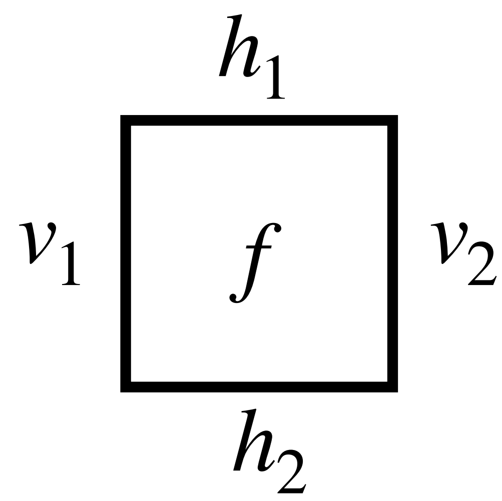

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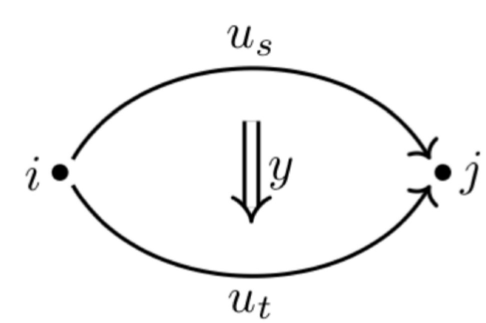

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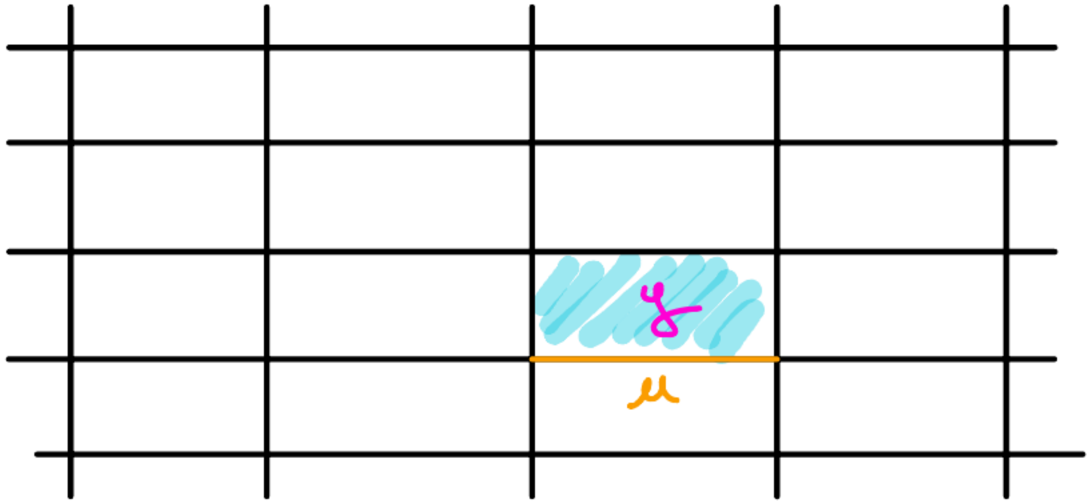
Identity 2-group: $t = id$

face is totally characterized by boundary

Skeletal 2-group: $t = 1$

face is totally independent from boundary

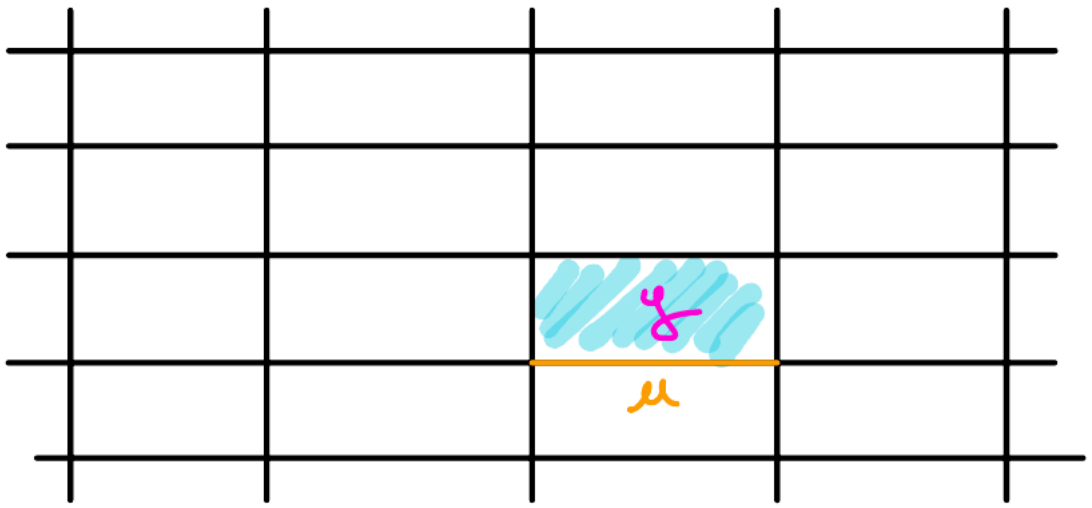
Generalization of lattice gauge theory



$$t(y)u_s = u_t$$

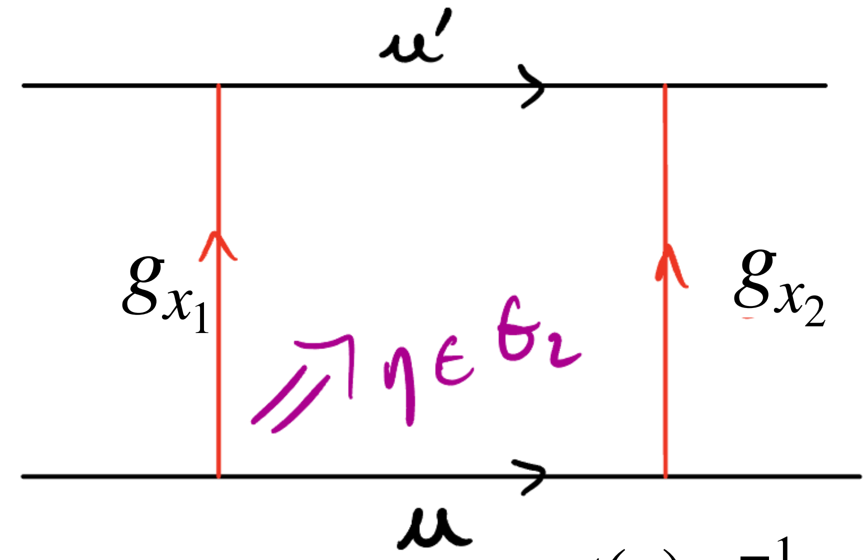
Lattice with edge and face decorations

Generalization of lattice gauge theory



$$t(y)u_s = u_\tau$$

Lattice with edge and face decorations



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2-principal bundle

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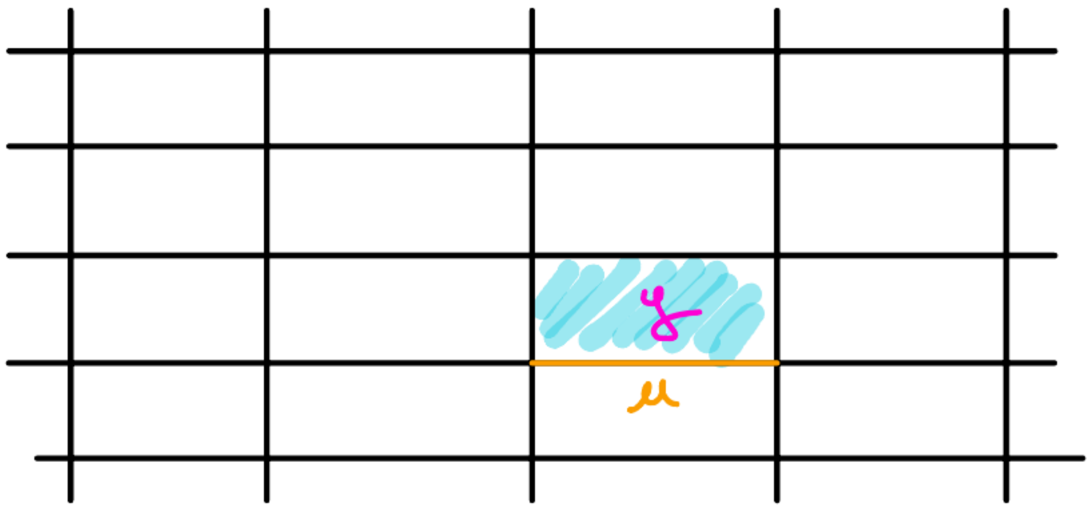

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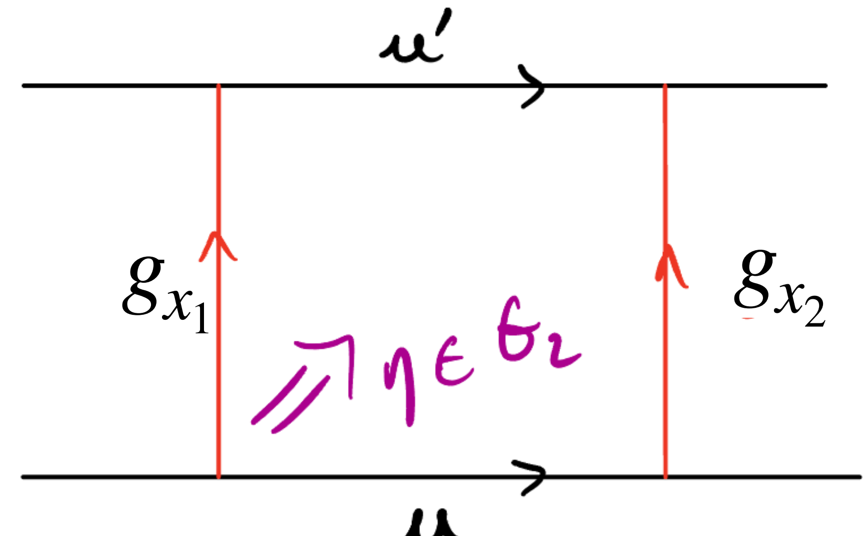

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2-principal bundle

Interpret these group elements as holonomies: $u = Pexp\left(\int_\gamma A\right), \quad y = Sexp\left(\int\int \Sigma\right)$

need to have connection 1-form and 2-form, need to perform infinitesimal limit.

Baez, Schreiber

Lie 2-algebra??

Need 2-vector space definition!

Attention: *there are 2 different notions of 2-vector spaces.*

- Angulo's representation theory
- 2-Hopf algebras (Chen-Girelli)
- Crossed modules

- Condensed matter models (topological order): mainly inspired by finite groups and category theory
- Douglas Reuter state sum.
- Baez-Baratin-Freidel-Wise 2-representation theory (building up on Crane,Yetter…..)

"Natural" from an algebra perspective 2-adj action!

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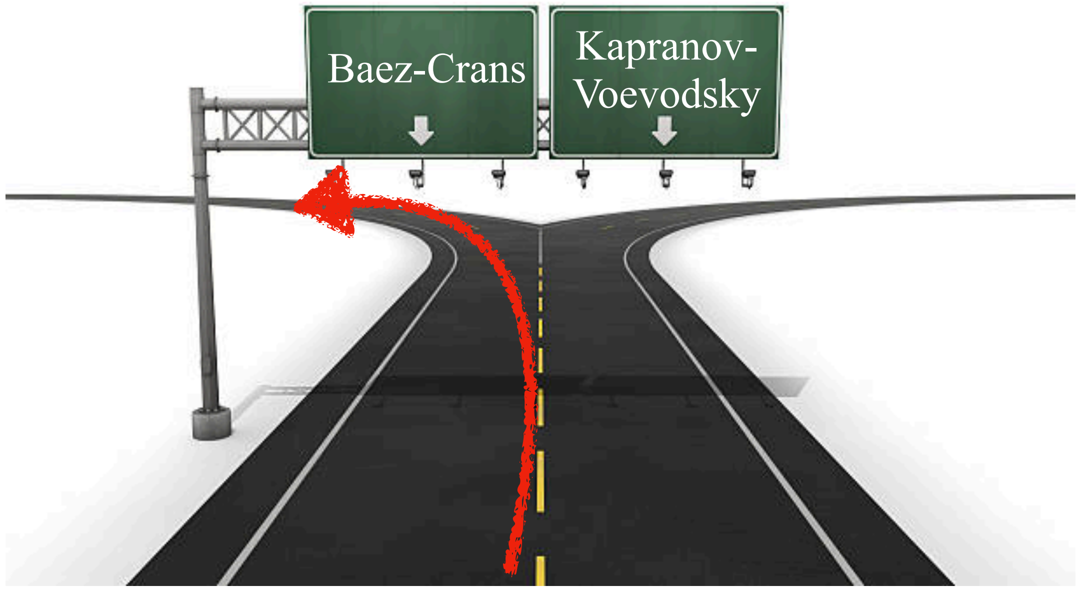

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"Natural" from an algebra perspective
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Baez-Crans 2-vector space

Baez-Crans 2-vector space is given in terms of a pair of vector spaces, with a linear map $t : W \to V$

W and t encode the transformations on V $\quad v \to v + tw$

2-vector space can be made a *Lie 2-algebra*, with a 2-bracket.

Replace the vector spaces by Lie algebras, with an action of $\mathfrak{g}$ on $\mathfrak{h}$

$$V \rightsquigarrow \mathfrak{g}, \quad W \rightsquigarrow \mathfrak{h}, \quad V \oplus W \rightsquigarrow \mathfrak{g} \ltimes \mathfrak{h}$$

Lie 2-algebra structure $\quad \mathfrak{g} \ltimes \mathfrak{h} \rightrightarrows \mathfrak{g}$

Lie 2-bracket:
(Satisfies 2-Jacobi)

$$[(J_1, X_1); (J_2, X_2)]_2 = ([J_1; J_2], \tau(J_1, X_1) \rhd X_2 - s(J_2, X_2) \rhd X_1)$$
$$= ([J_1; J_2], [X_1; X_2] + J_1 \rhd X_2 - J_2 \rhd X_1)$$

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Equivalently a strict Lie 2-algebra is crossed module of Lie algebras

$$(\mathfrak{h} \xrightarrow{t} \mathfrak{g}, \rhd\ )$$

$$t([X_1, X_2]) = [t(X_1), t(X_2)]$$
$$t(X_1) \rhd X_2 = [X_1, X_2]$$
$$t(J \rhd X) = [J, t(X)]$$

Strict Lie 2-algebra are
exponentiated to strict Lie 2-group

Baez-Crans

Elements of 2-representations

Consider a vector space V, a representation of a Lie algebra/Lie group on V is given in terms of a **matrix**

What is a "matrix" on a Baez-Crans 2-vector space $\mathbb{V} = (W \xrightarrow{\partial} V)$?

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It is given in terms of a pair of matrices (M, m) and a map A

$$(M, m) \in GL(W) \oplus GL(V) = GL_0(\mathbb{V}) : \text{objects} \qquad \partial M = m\partial$$

$$A \in Hom(V, W) = GL_1(\mathbb{V}) : \text{maps between objects} \qquad \Delta A = (id + A\partial, id + \partial A)$$

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$GL(\mathbb{V})$ is itself a Lie 2-group.

2-Adjoint action is a 2-representation of this kind!

General representation theory of strict Lie 2-groups is not generally known

If W=0, this collapses to standard representations/matrices

Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Higher gauge thy for Lie 2-algebra $\mathfrak{g} \ltimes \mathfrak{h} \rightrightarrows \mathfrak{g}$

Baez
Girelli, Pfeiffer
Jurco, Raspollini, Saemann, Wolf

Connection $\mathscr{A} = (A, \Sigma) \in (\mathfrak{g} \otimes \Lambda^1 M) \oplus (\mathfrak{h} \otimes \Lambda^2 M)$

Curvature data, $\mathscr{F} = (F - t\Sigma, \quad d_A\Sigma = d\Sigma + A \triangleright \Sigma)\,.$

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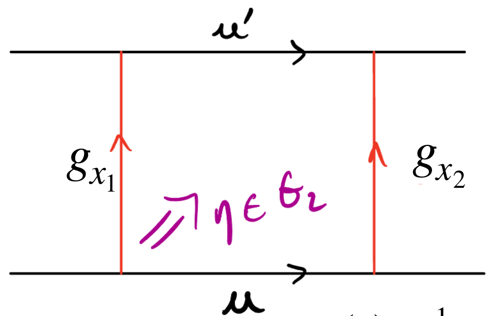

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1- and 2-gauge transf parameterized by $(\alpha, \phi) \in \mathfrak{g} \otimes \Lambda^0 M \oplus \mathfrak{h} \otimes \Lambda^1 M,$

$$\delta_{(\alpha,\phi)}\mathscr{A} = d(\alpha, \phi) + [\mathscr{A}, (\alpha, \phi)]_2 = (d\alpha + [A, \alpha] + t(\phi), \alpha \rhd \Sigma + d_A\phi + \frac{1}{2}[\phi, \phi])$$

$$\delta_{(\alpha,\phi)}\mathscr{F} = [\mathscr{F}, (\alpha, \phi)]_2 = ([F - t\Sigma, \alpha], \alpha \rhd d_A\Sigma + (F - t\Sigma) \rhd \phi]$$

2-adjoint action!

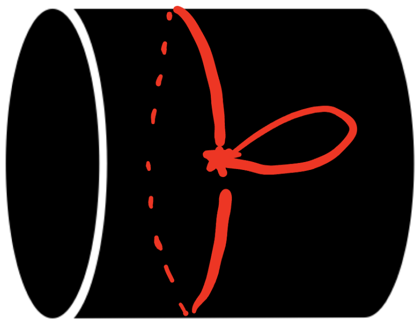
Generalized Maurer Cartan forms: $\mathscr{A} = (A, \Sigma) \in (\mathfrak{g} \otimes \Lambda^1 M) \oplus (\mathfrak{h} \otimes \Lambda^2 M)$

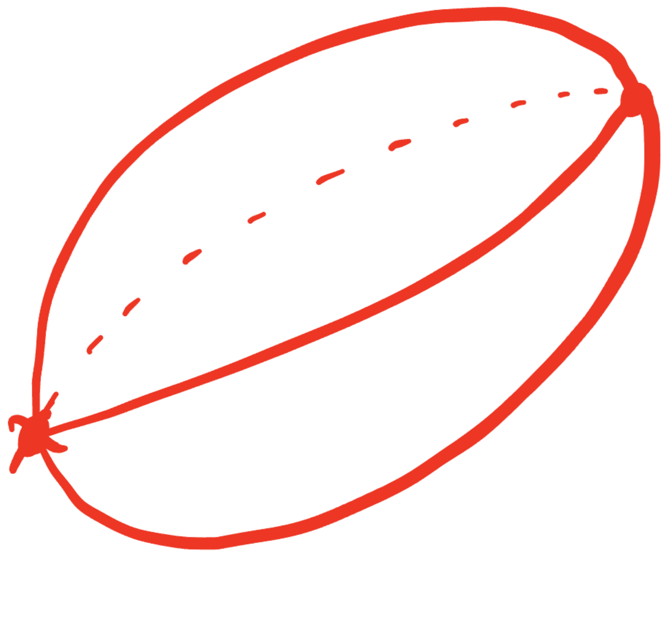
$$dA + \frac{1}{2}[A \wedge A] = t\Sigma, \quad d_A\Sigma = 0$$

solution: $A = u^{-1}du + t\phi \qquad B = d_A\phi + \frac{1}{2}[\phi \wedge \phi]$

"Pure 2-gauge"

Higher gauge theory natural object to discuss topological theory in 4d





Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Construct a 4d action to implement flat 2-curvature $\mathscr{F} = 0$, a "2-BF" theory

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$$\mathcal{S}_{2BF} = \langle B \wedge (F - t\Sigma) \rangle + \langle C \wedge d_A \Sigma \rangle \, .$$

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Canonical variables

$\theta = B \wedge \delta A + C \wedge \delta\Sigma$ $\qquad (B, C) \leftrightarrow (A, \Sigma)$ $\qquad$ 1-2-connections dual to each other.

There is a dual 2-symmetry, for the dual sector, due to 2-Bianchi identity.

Natural symmetry given by a matched pair of Lie 2-algebras, $Lie\mathbb{G}^* \bowtie Lie\mathbb{G}$

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Generalization of bi-algebra used for quantum group, integrable systems…

(Used by Hank Chen to construct higher dimensional integrable systems, generalization of Fock-Rosly construction to 4d.)

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This 2-BF theory is a BF theory with symmetry $\big((\mathfrak{h}^* \ltimes \mathfrak{g}) \ltimes (\mathfrak{g}^* \ltimes \mathfrak{h}) \rightrightarrows \mathfrak{h}^* \ltimes \mathfrak{g}\big)$ **up to a boundary term**

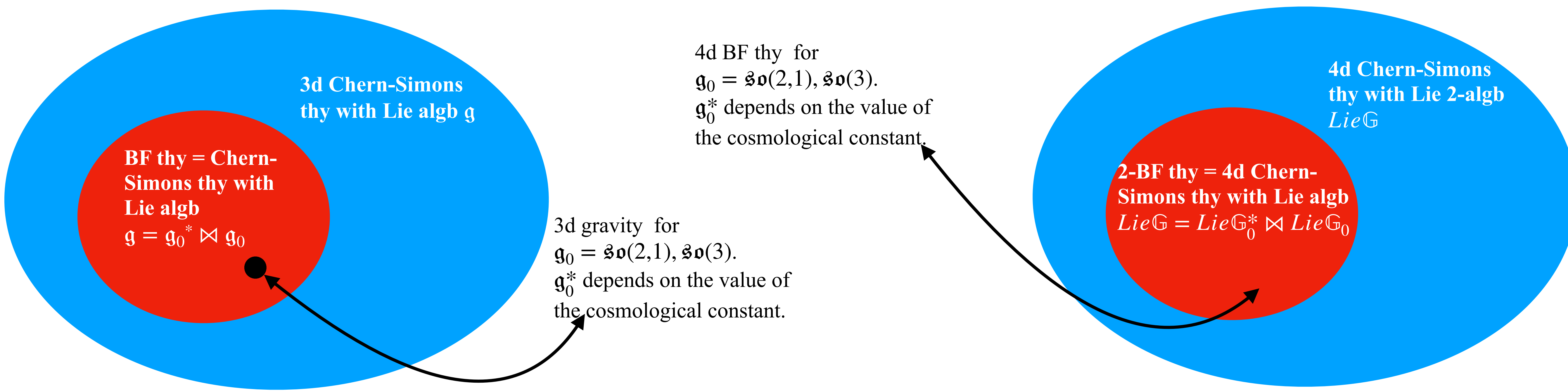
$$\mathcal{S}_{2BF} + d\langle C \wedge \Sigma\rangle = \langle B \wedge (F - t\Sigma)\rangle + \langle \Sigma \wedge d_A C\rangle = \langle \mathscr{B} \wedge \mathscr{F}\rangle - \frac{1}{2}\langle T(\mathscr{B}) \wedge \mathscr{B}\rangle\,.$$

Canonical variables

$$\theta' = B \wedge \delta A + \Sigma \wedge \delta C = \mathscr{B} \wedge \delta \mathscr{A} \qquad \mathscr{B} = (B, \Sigma) \leftrightarrow (A, C) = \mathscr{A}$$

Symmetries of 4d BF theory, aka 4d Chern-Simons thy

Not all BF theories are double of 2-symmetries. This is the analogue of Chern-Simons vs BF theory in 3d



Phase space of discrete geometries from 2BF

2-BF thy = 4d Chern-Simons thy with Lie algb
$Lie\mathbb{G} = Lie\mathbb{G}_0^* \bowtie Lie\mathbb{G}_0$

No loss of information by discretizing the manifold and smearing the variables (lattice picture)

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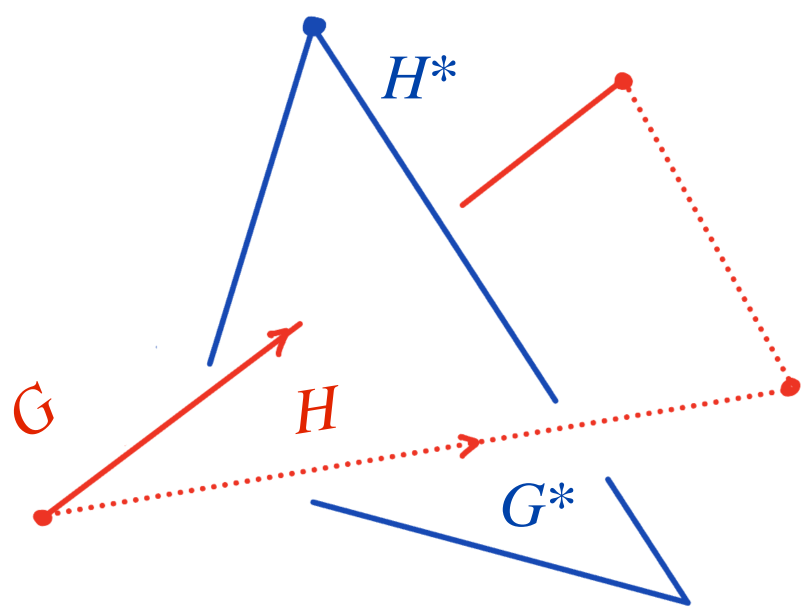
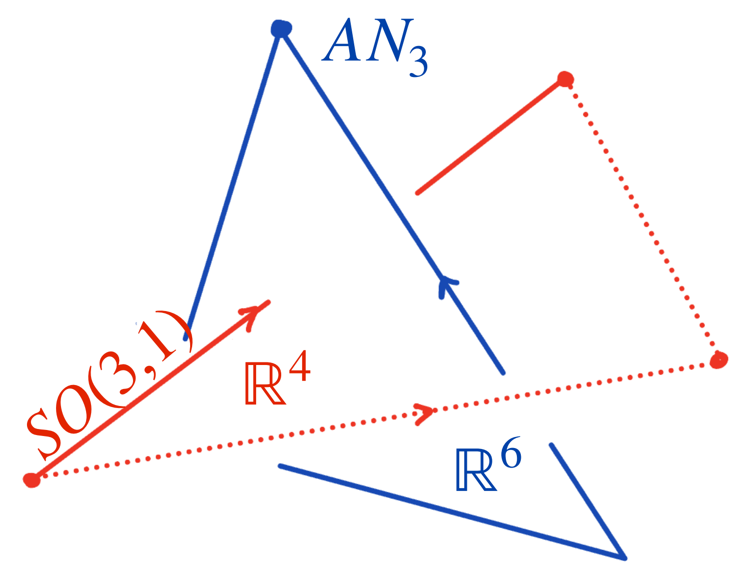
Phase space constructed in terms of decorated triangulation and dual triangulation.





In particular we can have edge information: frame field

Can define homogeneously curved 3d discrete geometry
Use non abelian decorations on the triangulation

Girelli Tsimiklis
Ferreira Oliveira, Girelli, Riello, wip

Quantum theory

For **some** choices of Lie 2-group, we know how to discretise the partition function

$$\int [dA][dC][d\Sigma][dB] e^{i\int_M \langle \mathscr{B} \wedge \mathscr{F} \rangle} \sim \int [dg][dh] \prod_{face} \delta(t(h) \prod_{links} g) \, \delta(\prod_{polyhedra} h)$$

This partition function is a topological invariant for the manifold: invariant under Pachner moves.

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$$\int [dA][dC][d\Sigma][dB] e^{i\int_M \langle \mathscr{B} \wedge (\mathscr{F} - \frac{1}{2} T(\mathscr{B})) \rangle} \sim ??$$

Standard picture up to now for $\mathscr{A} \in \mathfrak{so}(4)$

$$\int d\mathscr{B} d\mathscr{A} e^{i\int_M \langle \mathscr{B} \wedge \mathscr{F} \rangle} \sim \{15j\}_{SO(4)}$$

Does not use any 2-representations: missing data?
Lacking edge information, not easy to introduce matter.

Quantum theory

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For Lie 2-groups, we do **not** have the representation theory for most relevant groups, nor the harmonic analysis (Peter-Weyl theorem)…

Some interesting math to do!

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Note that:

- state-sums have been constructed using 2-categories associated with finite 2-groups

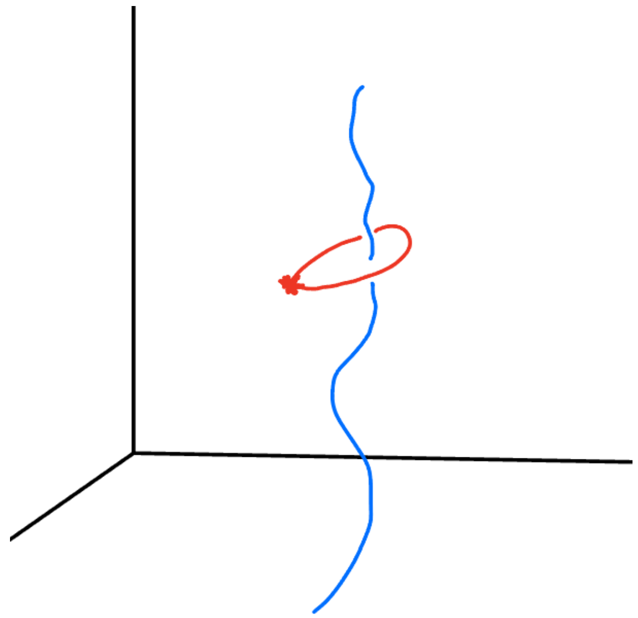
Cui
Douglas-Reutter

- For the Poincaré 2-group, we were able to connect the BF partition with a 2-state-sum built independently by Korepanov and Baratin-Freidel.

Asante-Dittrich-Girelli-Riello-Tsimiklis

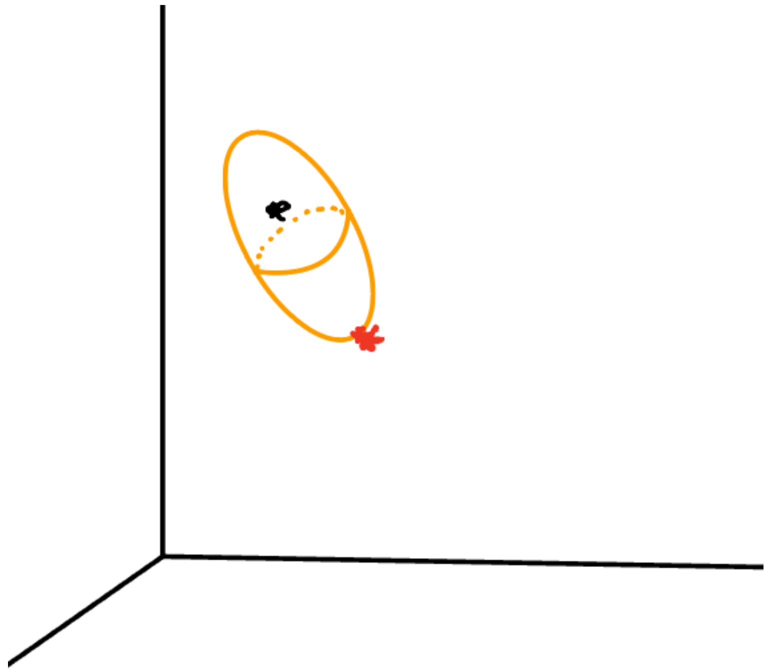
Defects

Curvature defects can be introduced:



String-like: curvature $F[A] - t\Sigma = S\delta^2(x)$

Point-like: 2-curvature $d_A\Sigma = p\delta^3(x)$



Exactly the structure to construct
Kitaev model in 3+1
(typically for finite 2-groups)

Matter and topological theory

Consider topological theory

$$\mathscr{L} = \mathfrak{B} \wedge \mathfrak{F}$$

$$\mathscr{A} = (A, e) \in \mathfrak{so}(4,1) \otimes \Lambda^1 M, \quad \mathscr{F} = (F(A) + \Lambda e \wedge e, T(e, A)) \in \mathfrak{so}(4,1) \otimes \Lambda^2 M,$$

$$\mathfrak{B} = (B, \Sigma) \in \mathfrak{so}^*(4,1) \otimes \Lambda^2 M$$

Implement point like topological defects : 2-curvature defects $\mathscr{L}' = \mathscr{B} \wedge \mathscr{F} - \mathscr{A} \cdot \pi$

$$\pi = (\sigma\delta^2(x), p\delta^3(x))$$

$$d_{\mathscr{A}}\mathscr{B} = \pi$$

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Introduce fields to re-establish gauge invariance in the action (same as particle in 3d gravity)

We recover the usual spinning particle moving on a constantly curved space

$$\nabla_\tau s^{ab} + e_\tau{}^a p^b - e_\tau{}^b p^a = 0$$

Freidel-Kowalski-Glikman-Starodubtsev

$$(\nabla_\tau p_a) e_\mu{}^a = \frac{1}{2} s_{ab}\, R_{\mu\nu}{}^{ab}\, \dot{z}^\nu + p_a T_{\mu\nu}{}^a\, \dot{z}^\nu.$$

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Analogue of Einstein equation can be seen as a 2-curvature data $d_{\mathscr{A}} \mathscr{B} = \pi$

Matter can be seen as 2-curvature excitation.

Standard particle is already present at the topological level

Matter and topological theory

Conversely, consider **matter fields** Φ with *no spin* on a 4d *flat* spacetime, which geometry is given by **frame field** e and **spin connection** A.

$$\mathscr{L}_m(e, \Phi)$$

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$$\mathscr{A}=(A,e), \quad \mathfrak{F}=(F(A),T(e,A))\in \mathfrak{iso}(3,1)\otimes\Lambda^2 M,$$

Use Lagrange multipliers $\mathfrak{B}=(B,\Sigma)\in \mathfrak{iso}^*(3,1)\otimes\Lambda^2 M$ to implement these constraints

$$\mathscr{L}_m(e,\Phi)\approx \mathscr{L}_m(e,\Phi)+\mathfrak{B}\wedge\mathfrak{F}$$

Topological sector appears!

We have enlarged the space of possible backgrounds: degenerate frame field/metric is also allowed.

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Analogue of Einstein equation can be seen as a 2-curvature data

$$\frac{\delta \mathscr{L}}{\delta e}(e, \Phi) = d_A \Sigma$$

Matter can be seen as 2-curvature excitation.

Gauge symmetry on frame field sector = stress energy tensor conservation

Dupuis, Girelli, Hyrtseniak, wip

Matter and topological theory

Feynman diagrams (scalar field in Minkowski) = particle excitations in a topological 2-state-sum

Baratin-Freidel

$$I_\Gamma = \int_{\mathbb{R}^4} \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_n \, \mathcal{O}_\Gamma(|\vec{x}_i - \vec{x}_j|), \qquad \mathcal{O}_\Gamma = \prod_{(ij)\in\Gamma} G^F(\vec{x}_i - \vec{x}_j)$$

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⟷

This 2-state-sum is associated with Poincaré 2-group

Baratin-Freidel

↕

Such 2-state-sum is associated with BF theory with the Poincaré 2-group

Asante, Dittrich, Girelli, Riello, Tsimiklis

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Asante, Dittrich, Girelli, Riello, Tsimiklis

Particle as defects in BF theory with the Poincaré 2-group is the same as Feynman diagram contribution?

Dupuis, Girelli, Hyrtseniak, wip

4d gravity from topological theory

Many formulations of gravity from topological gravity

4d gravity is not topological, it breaks higher symmetries. Just like Yang-Mills theory can be built from a BF theory, we can do something similar for gravity.

Freidel-Starodubtsev (MacDowell-Mansouri) $\mathscr{A} = (\omega, e) \in \mathfrak{so}(4,1) \otimes \Lambda^1 M$

Gives Immirzi

$\alpha \propto G\lambda \sim 10^{-120}$

$$\int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \frac{1}{2}\beta \mathscr{B} \wedge \mathscr{B} - \frac{\alpha}{4}\epsilon^{4IJKL}\mathscr{B}_{IJ} \wedge \mathscr{B}_{KL}$$

$$d_{\mathscr{A}}\mathscr{B} = 0$$

$$(\beta \mathrm{id} + \frac{\alpha}{2}\epsilon)\mathscr{B} = \mathscr{F}$$

Many other formulations

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$$T_{MM}(\mathscr{B}) = (\beta \mathrm{id} + \frac{\alpha}{2}\epsilon)\mathscr{B} = \mathscr{F}$$

Plebanski $\mathscr{A} \in \mathfrak{so}(3,1) \otimes \Lambda^1 M$

$$\mathscr{S}_{Pl} = \int \mathscr{B} \wedge \mathscr{F}(\mathscr{A}) - \phi(\mathscr{B}) \wedge \mathscr{B}$$

Lagrange multiplier

$$T_{PL}(\mathscr{B}) = (\lambda \mathrm{id} + \phi)\mathscr{B} = \mathscr{F}$$

Herffray-Krasnov $\mathscr{A} \in \mathfrak{so}(3,\mathbb{C}) \otimes \Lambda^1 M$

$$S_{HK} = \int \mathscr{B} \wedge \mathscr{F} - \frac{\lambda}{2}\mathscr{B} \wedge \mathscr{B} + \frac{\alpha}{2}\left(Tr(\sqrt{\mathscr{B}^I \wedge \mathscr{B}^J})\right)^2$$

$$T^I_{HK}(\mathscr{B}) = \left(\lambda\delta^{IJ} - \alpha Tr(\sqrt{X})\,(X^{-1})^{IJ}\right)\mathscr{B}_J = \mathscr{F}^I, \quad X^{IJ} = \mathscr{B}^I \wedge \mathscr{B}^J$$

$$d_{\mathscr{A}}\mathscr{B} = 0$$

Cf review by Freidel-Speziale

Many other formulations

Spinorial BF action $\phi, \psi \in \mathbb{C}^2 \otimes \Lambda^1 M, \quad {}^{\phi}\pi, {}^{\psi}\pi \in \mathbb{C}^2 \otimes \Lambda^2 M$

$$\int {}^{\psi}\pi^A \wedge d\psi_A + {}^{\phi}\pi^A \wedge d\phi_A + {}^{\psi}\pi^A \wedge {}^{\phi}\pi_A$$

Global $SL(2,\mathbb{C})$ symmetry

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Make it local: introduce constraint using Lagrange multiplier ${A_A}^B$ **Dupuis, Girelli, Hyrtseniak, Wieland wip**

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$$= \int {}^{\psi}\pi^A \wedge \nabla\psi_A + {}^{\phi}\pi^A \wedge \nabla\phi_A + {}^{\psi}\pi^A \wedge {}^{\phi}\pi_A$$

= self dual gravity (up to boundary term) **Robinson**

$$\psi_A = e_{A0'}, \quad \phi_A = ie_{A1'}$$

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= self dual gravity (up to boundary term) **Robinson**

Particle can be introduced at the topological level **Dupuis, Girelli, Hyrtseniak, Wieland wip**

4d quantum gravity from topological theory?

Quantum gravity

Plebanski approach: $\mathcal{S}_{Pl} = \int \mathcal{B} \wedge \mathcal{F}(\mathcal{A}) - \phi(\mathcal{B}) \wedge \mathcal{B}$

$$\int d\mathcal{B} d\mathcal{A} e^{iS_{PL}} \sim \{15j\}_{SO(4)} \text{ with } \mathcal{B} \wedge \mathcal{B} = 0$$

EPRL model

$$\int d\mathcal{B} d\mathcal{A} e^{iS_{BF}}$$

No 2-reps, no matter included, no frame field: needs 2-improvement

Quantum gravity

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$$\int d\mathscr{B} d\mathscr{A} e^{iS_{MM}} \sim \int d\mathscr{B} d\mathscr{A} \left(1 + \alpha\epsilon\mathscr{B}\mathscr{B} + \dots\right) e^{iS_{2BF}}$$

Goal

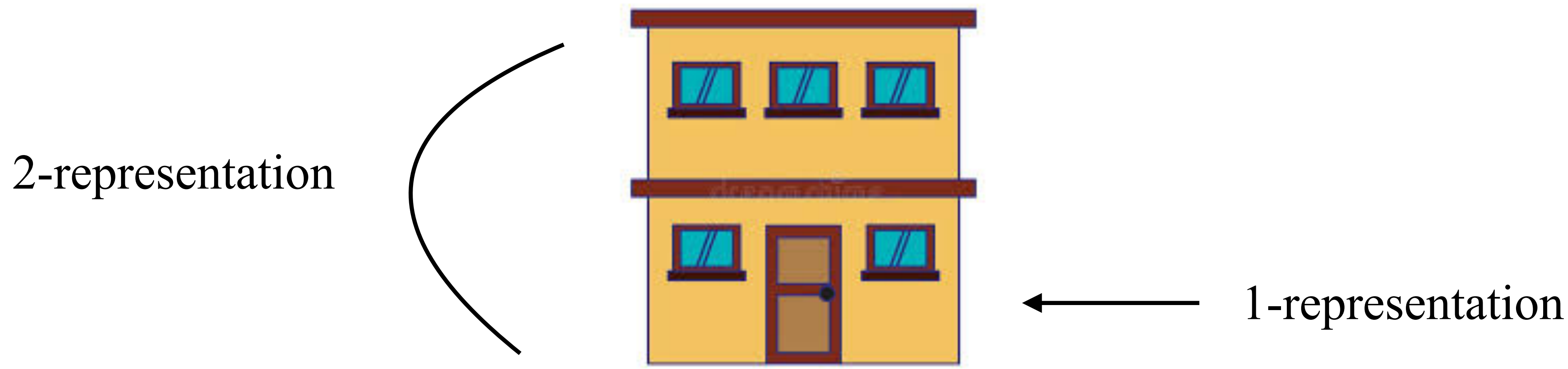
Build state sum on 2-representations, introduce gravity in a perturbative manner.

Important to focus on 2-representations, and not only 1-representation.

Introduce matter at the topological level and it will percolate to the gravity case!!

Outlook

2-symmetries are naturally present in 4d, but have not been leveraged



Still some work to do…

4d topological models are based on 2-symmetries/representations

Strict Lie/quantum 2-groups 2-representations have to be studied.

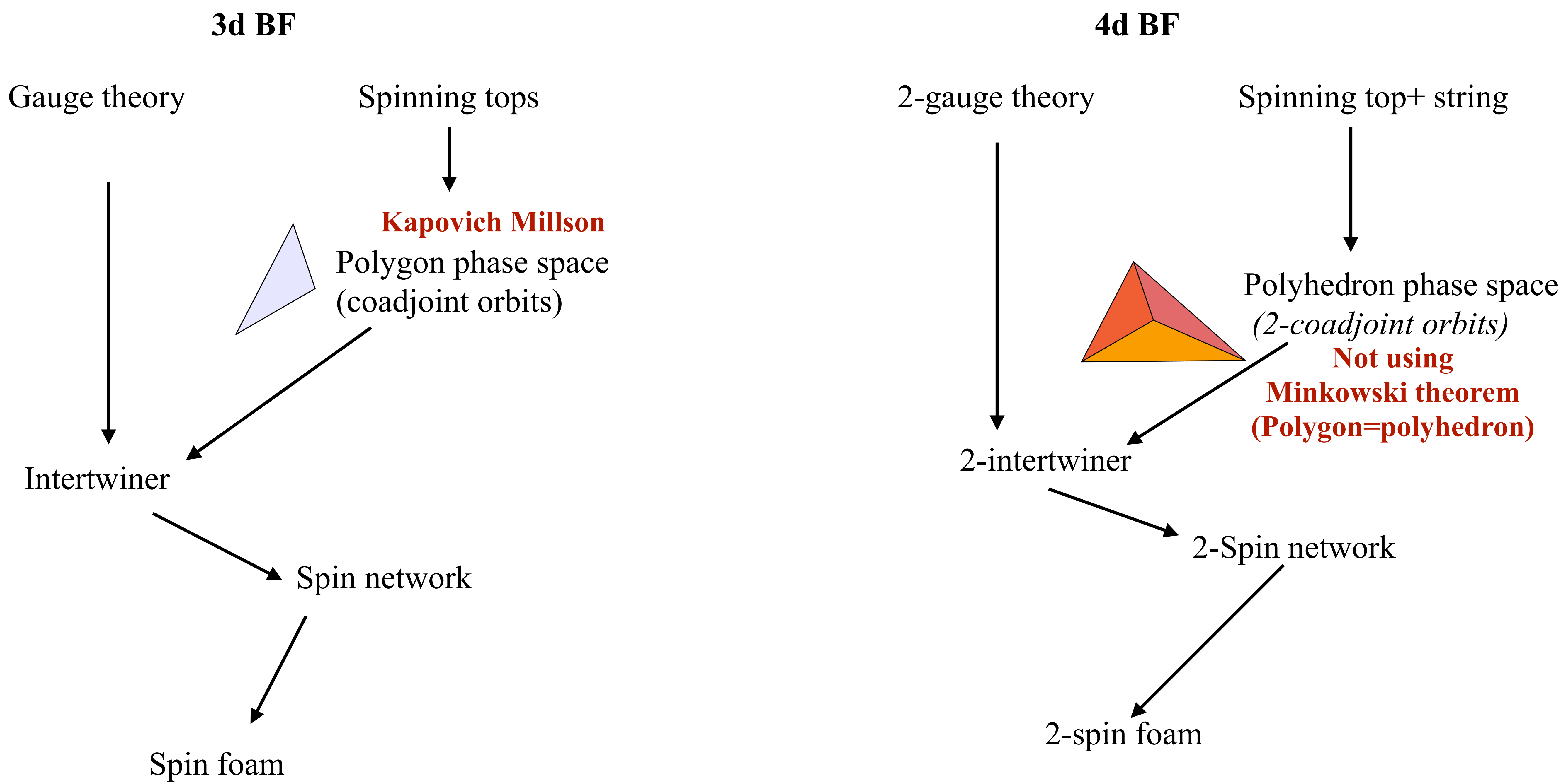
State-sum from strict Lie/quantum 2-groups 2-representations have to be derived.

Highlight: Have the frame field present (already at the topological level) due to higher gauge.

Can construct homogeneously curved discrete 3d geometries

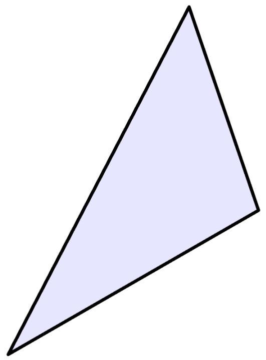
Can introduce matter at the topological level

Road map to LQG 2.0
Work in progress



From polygon to polyhedron

Polygon phase space (with fixed edge length) Kapovich-Millson



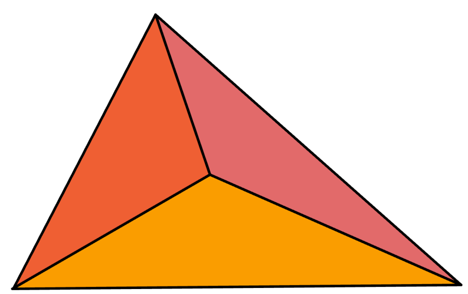
$$\mathscr{P}_{\triangle} = \left(S^2_\ell \times S^2_\ell \times S^2_\ell\right)//C \leftarrow C = \sum_a X^a = 0$$

$SU(2)$ co-adjoint orbit

$$\left.\begin{array}{l} X \in S^2_\ell \subset \mathfrak{su}^*(2) \sim \mathbb{R}^3 \\ X^2 = \ell^2 \end{array}\right) + \text{Kirillov symplectic form} \quad \{X_i, X_j\} = \epsilon_{ij}^k X_k$$

Polyhedron phase space?

WIP
Girelli-Oliveira-Riello



$$C_i = X_i - \sum_j J_j = 0$$

$$C = \sum X_i = 0$$

$$\mathbb{G} = (SU(2) \ltimes SU(2) \rightrightarrows SU(2)) \text{ 2-coadjoint orbits}$$

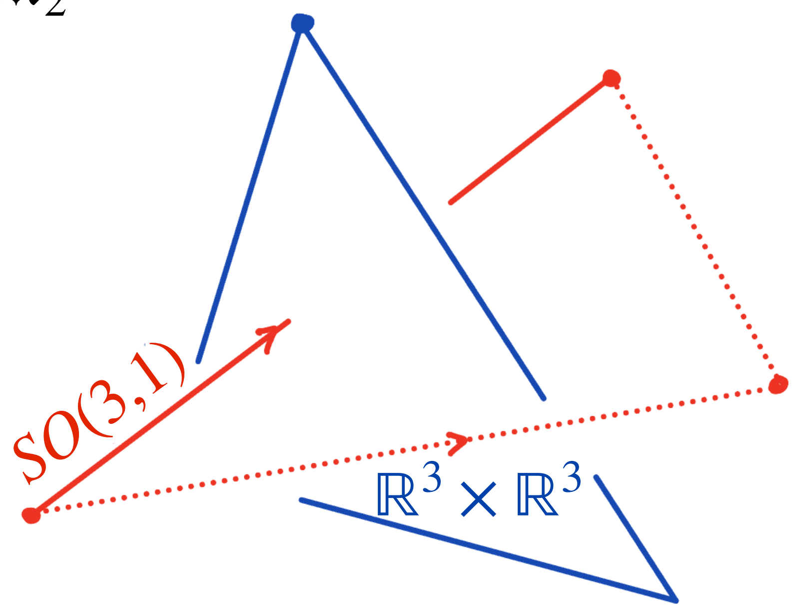
$$\left.\begin{array}{l} (X, J) \in \mathfrak{su}^*(2) \oplus \mathfrak{su}^*(2) \\ X \cdot J = s, \quad J^2 = \ell^2 \end{array}\right) + \text{2-Kirillov symplectic form}$$

$$\{X_i, X_j\} = \epsilon_{ij}^k X_k$$
$$\{J_i, X_j\} = \epsilon_{ij}^k X_k$$
$$\{J_i, J_j\} = \epsilon_{ij}^k J_k$$

Standard $\mathfrak{so}(3,1)$ BF theory (no BB interaction) can be viewed either as a double of trivial 2-symmetries, or as a double of (deformed) 2-symmetries. This is the same as the change of polarization from before.

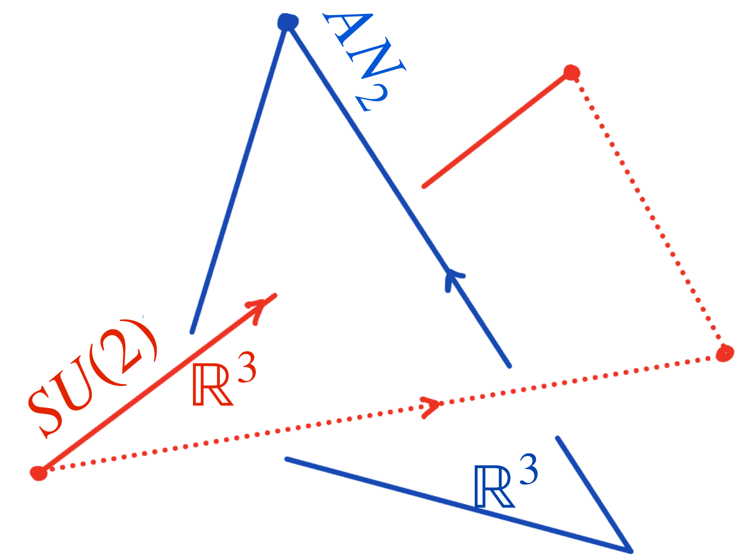
$\mathfrak{so}(3,1) \approx \mathfrak{su}(2) \bowtie \mathfrak{an}_2$

$\mathfrak{so}(3,1)^* \approx \mathbb{R}^3 \times \mathbb{R}^3$



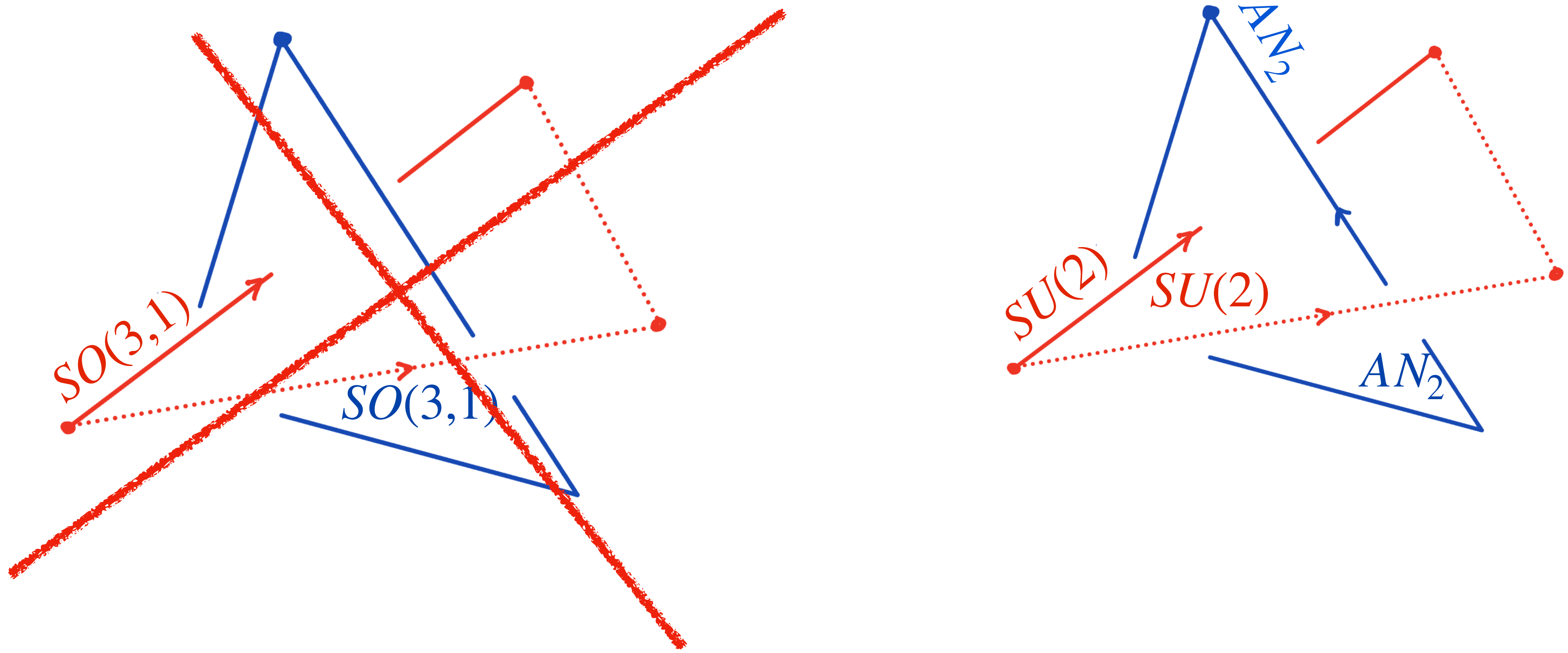
Partition function as state sum in terms of 1-category of $\mathfrak{so}(3,1)$ representations.

Girelli-Tsimiklis
Girelli-Laudonio-Tsimiklis



Partition function as state sum in terms of 2-category of Euclidian 2-group $\mathfrak{su}(2) \ltimes \mathbb{R}^3$ 2-representations.

Symmetries of 4d BF theory, aka 4d Chern-Simons thy



Standard Lorentz BF theory **with BB interaction** cannot be viewed as a *standard gauge theory. This is why quantum 2-group structure should arise.*

$$Lie\mathbb{G} = \left(\mathfrak{so}(3,1) \ltimes \mathfrak{so}(3,1) \rightrightarrows \mathfrak{so}(3,1)\right) \approx Lie\mathbb{G}^*$$

$$\mathfrak{so}(3,1)^* \approx \mathfrak{so}(3,1)$$

$$\mathfrak{so}(3,1) \approx \mathfrak{su}(2) \bowtie \mathfrak{an}_2$$