Bootstrapping Gravity with Crossing Symmetric Dispersion Relations

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Set-up

Goal: Derive bounds on the couplings in gravitational EFTs that have a consistent UV completion.

We will use the S-matrix bootstrap to study scattering amplitudes:

$$S = \mathbb{1} + iT\,, \qquad SS^\dagger = \mathbb{1}$$

- Assumptions: unitarity, causality, analyticity and Lorentz invariance
- Restrict to EFT at weak coupling, ignoring any low-energy loop effects
- Use smeared amplitudes:

$$\mathcal{M}_f(s) = \int_0^{p_{\max}} dp f(p) \mathcal{M}\left(s, -p^2\right)\,, \qquad \lim_{|s|\to\infty} \frac{\mathcal{M}_f(s,u)}{|s|^2} = 0 \quad \text{for fixed } u < 0.$$

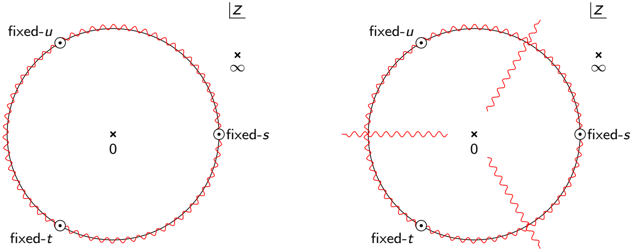
with wavefunctions $f(p)$ that have compact support in momentum space and decay rapidly at large impact parameter b [Caron-Huot et al., 2021, Häring and Zhiboedov, 2024].

Crossing symmetric variables

Mandelstam variables are defined in terms of z and an auxiliary momentum variable p as [Sinha and Zahed, 2021]:

$$s_n = -p^2 + \frac{p^2 (z - z_n)^3}{z^3 - 1}, \quad n = 1, 2, 3, \qquad \frac{1}{p^2} = -\frac{1}{s} - \frac{1}{t} - \frac{1}{u}$$

where z_n are cubic roots of unity: $z_1 = 1, z_2 = e^{2\pi i/3}, z_3 = e^{4\pi i/3}$.



The UV branch cuts shown in red, $M^2 = 1$. Left: $p_{\max} = M/\sqrt{3}$, Right: $p_{\max} = M$.

Crossing symmetric variables

Solving z in terms of s, the Mandelstam invariants in (s, p^2) are:

$$t = \frac{1}{2}s\left(-1 - \frac{\sqrt{s-3p^2}}{\sqrt{s+p^2}}\right), \quad u = \frac{1}{2}s\left(-1 + \frac{\sqrt{s-3p^2}}{\sqrt{s+p^2}}\right).$$

Note: $u = -(p_1 + p_3)^2$ and in the large s limit, $u \to -p^2$.

Taking $s = m^2$, the scattering angle is

$$\cos\theta = 1 + \frac{2u}{s} = \frac{\sqrt{m^2 - 3p^2}}{\sqrt{m^2 + p^2}}.$$

Crossing symmetric dispersion relations

The crossing symmetric kernel is

$$\mathcal{K}_k(s,p^2) = \frac{2s+3p^2}{s(s+p^2)}\left(\frac{s+p^2}{s^3}\right)^{\frac{k}{2}}. \qquad k = 2,4\ldots$$

Deforming the contour around the high-energy branch cuts and the low-energy poles, we get dispersion relations:

$$\mathcal{C}_k(p^2) \equiv \underset{z=0}{\text{Res}}\left[\mathcal{K}_k(z,p^2)\mathcal{M}_{\text{low}}\right] = \left\langle \mathcal{K}_k(m^2,p^2)\,\pi_J\left(\frac{\sqrt{m^2-3p^2}}{\sqrt{m^2+p^2}}\right)\right\rangle$$

where $s = m^2$ and π_J are partial waves and the high-energy heavy averages are:

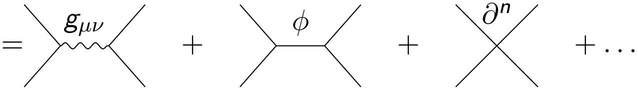
$$\langle(\cdots)\rangle \equiv \frac{1}{\pi}\sum_J n_J^{(D)}\int_{M^2}^{\infty}\frac{dm^2}{m^2}\rho_J\left(m^2\right)(\cdots)\,.$$

where unitarity implies $0 \leq \rho_J(s) \leq 2$.

Scalar theory with gravity

$2 \to 2$ scalar scattering:

$$\mathcal{M}_{\text{low}}(s,u) = 8\pi G \left(\frac{st}{u} + \frac{su}{t} + \frac{tu}{s}\right) - g^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$$
$$- g_0 + g_2 \left(s^2 + t^2 + u^2\right) + g_3(stu) + g_4 \left(s^2 + t^2 + u^2\right)^2 + \dots .$$



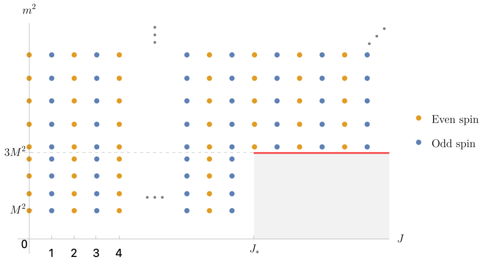
Crossing symmetric sum rules **naturally** isolate a finite subset of Wilson coefficients at each k:

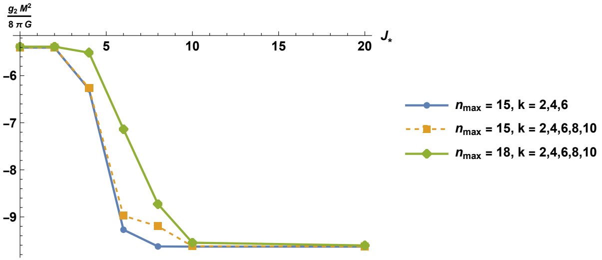
$$k = 2: \quad \frac{8\pi G}{p^2} + 2g_2 + p^2 g_3 = \left\langle \frac{2m^2 + 3p^2}{m^6} \mathcal{P}_J \left(\frac{\sqrt{m^2 - 3p^2}}{\sqrt{m^2 + p^2}} \right) \right\rangle$$

Spectral Assumptions

For a select number of heavy states, we allow for imaginary values in $\cos\theta = \frac{\sqrt{m^2 - 3p^2}}{\sqrt{m^2+p^2}}$.

We choose a spin J_* and include all $m \geq M$ for $J \leq J_*$, and $m \geq \sqrt{3}M$ for $J > J_*$.





The bounds converge for a given number of functionals if a large enough J_* is used.

Semi-definite programming

We use semi-definite programming, SDPB [Simmons-Duffin, 2015], to compute the coupling bounds using the following optimization problem:

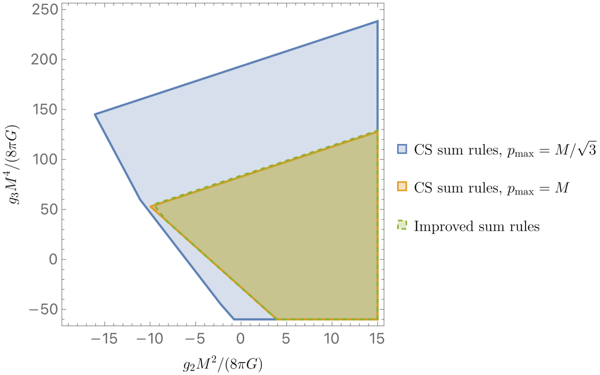
$$\text{maximise} \quad \sum_i \int_0^{p_{max}} dp\, f_i(p)\, B_i^{low}(p^2)$$

$$\text{subject to} \quad \sum_i \int_0^{p_{max}} dp\, f_i(p)\, B_i^{high}\left[p^2, m^2, J\right] \succeq 0 \quad \forall\, m \geq M, \forall\, J.$$

Label i enumerates the independent dispersive sum rules.

Scalar theory with gravity (D=6)

By smearing up to $p_{\max} = M = 1$ we reproduce bounds for g_2, g_3 found in [Caron-Huot et al., 2021]:



Taking $p_{\max} = \frac{M}{\sqrt{3}}$:
We reproduce suboptimal bounds found in [Chang and Parra-Martinez, 2025, Beadle et al., 2025].

$2 \to 2$ Graviton scattering (D=4)

Consider the maximal-helicity violating (MHV) amplitude:

$$\mathcal{M}\left(1^+2^-3^-4^+\right) = \langle 23\rangle^4 [14]^4 f(s,u)$$

where $f(s,u)$ is only crossing symmetric in $s \leftrightarrow u$,

$$f_{\text{low}}(s,u) = \frac{8\pi G}{stu} + \underbrace{\frac{2\pi G s u}{t}|\widehat{g}_3|^2}_{\propto \text{Riem}^3} + \underbrace{g_4 + g_5 t + g_6 t^2 - g_6' s u}_{\propto \text{Riem}^4} + \underbrace{\ldots}_{\ldots}$$

Goal:

Bound couplings of higher derivative operators that are modifications to GR.

2 →2 Graviton scattering

We take the following combinations of $f(s,u)$ to ensure crossing symmetry in s, t, u [Mahoux et al., 1974]:

$$\mathcal{M}^{(1)} = \bar{f}(s,t) + \bar{f}(t,u) + \bar{f}(s,u)$$

$$\mathcal{M}^{(2)} = \frac{\bar{f}(s,t) - \bar{f}(s,u)}{t-u} + \text{cyc perm}$$

$$\mathcal{M}^{(3)} = \left(\frac{\bar{f}(s,t) - \bar{f}(s,u)}{t-u} - \frac{\bar{f}(s,t) - \bar{f}(t,u)}{s-u} \right) \frac{1}{s-t} + \text{cyc perm}$$

where we define

$$\bar{f}(s,t) = u^4 f(s,t), \qquad \bar{f}(t,u) = s^4 f(t,u), \qquad \bar{f}(s,u) = t^4 f(s,u)\,.$$

to ensure the combinations in $\mathcal{M}^{(\mathrm{n})}$ all have the same Regge limit since:

$$\lim_{|s|\to\infty} f\,(s,u) \leq Cs^{-3}, \qquad \lim_{|s|\to\infty} f\,(s,t) \leq Cs$$

2 →2 Graviton scattering

For the high-energy side, we use the partial wave expansion for spinning particles

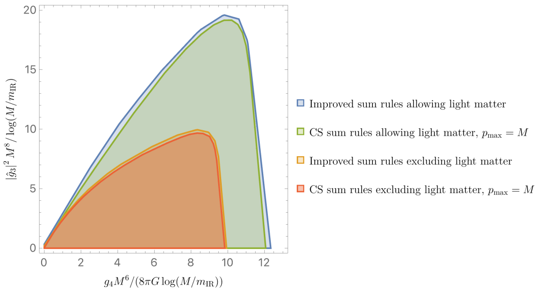
$$\mathcal{M}\left(1^{h_1}2^{h_2}3^{h_3}4^{h_4}\right) = 16\pi \sum_J (2J+1) a_J^{\{h\}}(s) d^J_{h_{12},h_{34}}\left(-\frac{\sqrt{m^2-3p^2}}{\sqrt{m^2+p^2}}\right).$$

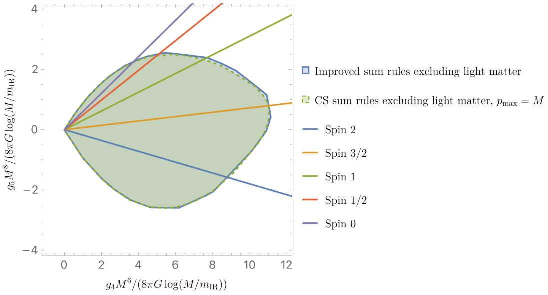
The first few sum rules are:

$$\begin{aligned} B_2^{(1)}: \quad \frac{2(8\pi G)}{p^2} &= \left\langle \left(\frac{2m^2+3p^2}{m^6}\right)\left(\bar{f}(s,u)+\bar{f}(s,t)+\bar{f}(t,u)\right)\right\rangle \\ B_3^{(2)}: \quad -3p^2 g_4 &= \left\langle \left(\frac{2m^2+3p^2}{m^6}\right)\left(\frac{\bar{f}(s,t)-\bar{f}(s,u)}{t-u}+\text{cyc.}\right)\right\rangle . \end{aligned}$$

2 →2 Graviton scattering: Numerical bounds

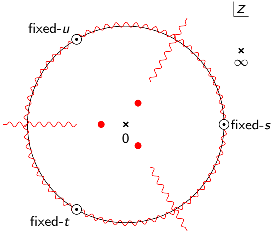
Comparing bounds using improved sum rules and CS sum rules for $|\hat{g}_3|$ and g_4 (left), g_5 and g_4 (right) for which we allow heavy matter loops of various spin:





Bounds with higher spin spectral assumptions

Assume an additional massive state with mass m_X is integrated in below the new EFT cut-off M' where $M' \geq m_X$.



For the MHV amplitude we have:

$$f(s,u) = \frac{(g^{+-}_{GG4})^2 \tilde{d}^4_{4,4}\left(1+\frac{2u}{s}\right)}{m^2_{J=4}-s} + \frac{(g^{++}_{GG4})^2 \tilde{d}^4_{0,0}\left(1+\frac{2u}{t}\right)}{m^2_{J=4}-t} + \frac{(g^{+-}_{GG4})^2 \tilde{d}^4_{4,4}\left(1+\frac{2s}{u}\right)}{m^2_{J=4}-u} + \text{analytic}\,.$$

Spin-4 exchange

The upper bound on the coupling of gravitons with the massive spin-4 state are shown as a function of the mass ratio of the spin-4 mass to the EFT cut-off M.

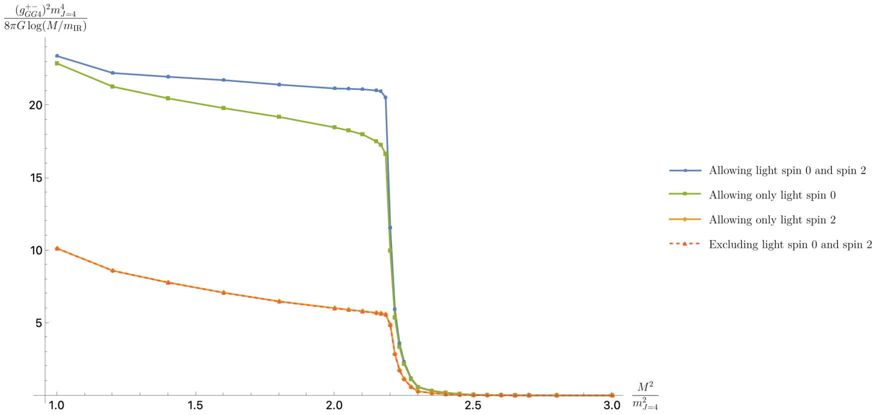


Figure: Upper bound on the coupling between gravitons and a massive spin-4 state including different combinations of heavy spin-0 and spin-2 states below the cut-off M.

Outlook

Summary:

- Crossing symmetric (CS) dispersion relations naturally isolate EFT Wilson coefficients
- CS bounds exactly match previous work for amplitudes with s, t, u-symmetry
- CS sum rules can be applied to amplitudes that do not have crossing symmetry
- New bounds on the coupling of gravitons with massive spin-4 exchange computed

Future directions:

- apply CS relations to amplitudes with even less crossing symmetry
- bootstrapping massive spin-2 KK couplings
  (WIP with Simon, Junsei, Julia - presented yesterday)

Thank you!

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