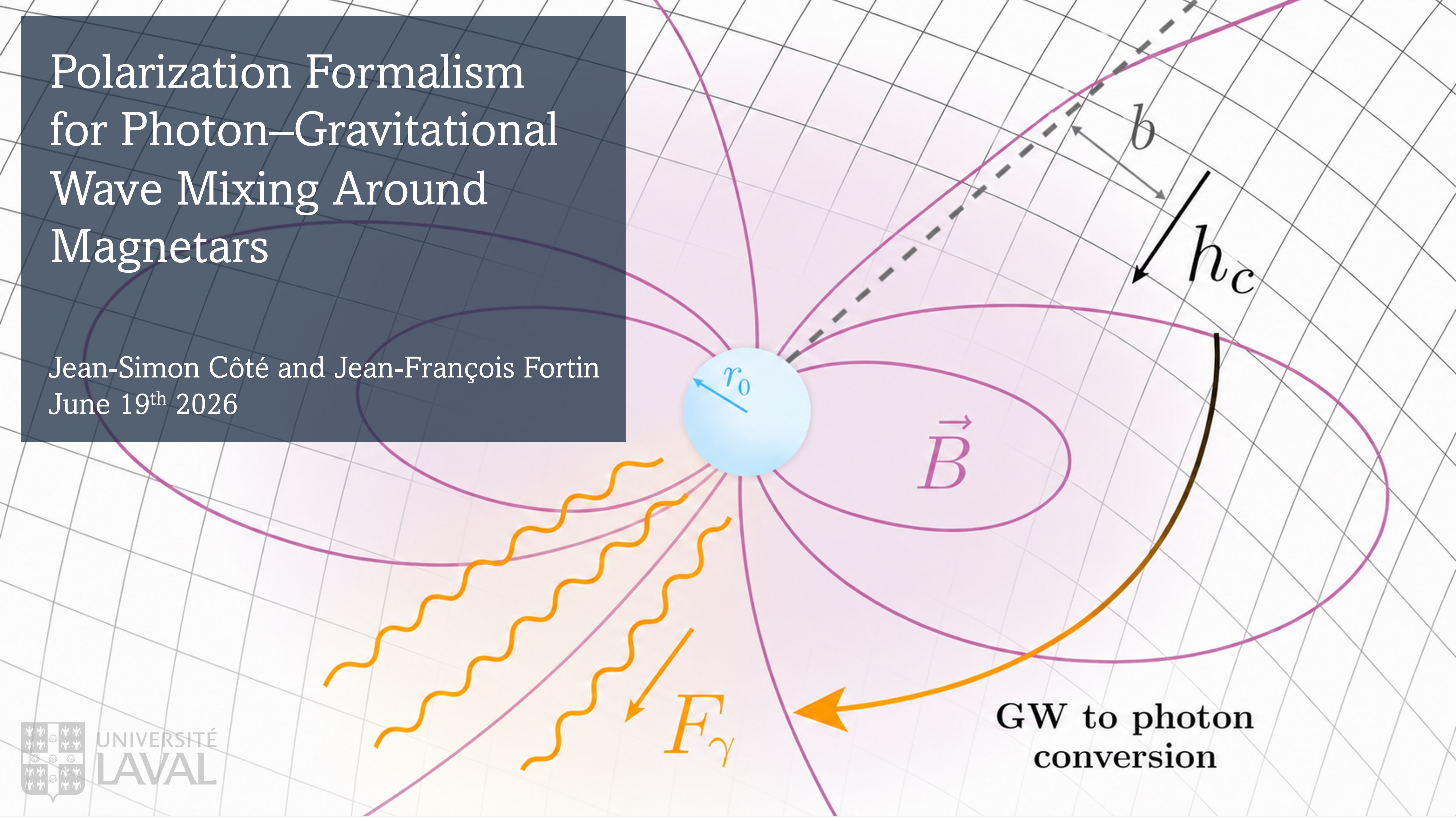
Polarization Formalism for Photon–Gravitational Wave Mixing Around Magnetars

Jean-Simon Côté and Jean-François Fortin
June 19th 2026



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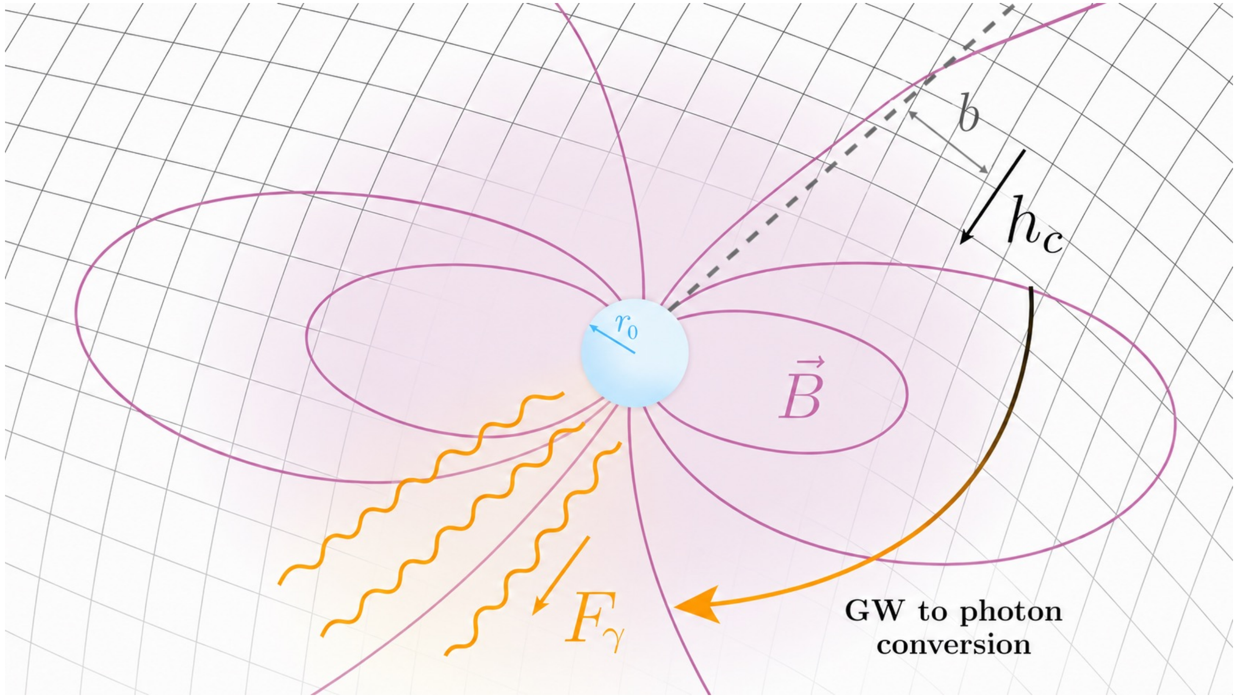
Introduction



V. Dandoy et al., (2024), JCAP, arXiv:2305.01832

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^\mu h^{\rho\sigma})(\partial_\mu h_{\rho\sigma}) + \frac{1}{m_P}h_{\mu\nu}F^{\mu\rho}\,F^\nu{}_\rho + \frac{\alpha_{EM}^2}{90m_e^4}\left[\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \frac{7}{4}\left(F^{\mu\nu}\tilde{F}_{\mu\nu}\right)^2\right]$$

Objective

Searching for a stochastic gravitational wave background SGWB

- Probe of early-Universe/BSM physics
- Analogue of the CMB in EM astrophysics

- Magnetars ($B_0 \sim 10^{14}$G)
- Gertsenshtein effect
  ⇒ photons ↔ gravitational waves (GW)
- Constraints on the SGWB amplitude from magnetar X-ray spectra

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arXiv : 2606.12546

Introduction

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\partial^{\mu}h^{\rho\sigma}\right)\left(\partial_{\mu}h_{\rho\sigma}\right)}_{\text{Kinetic Terms}} + \underbrace{\frac{1}{m_P}h_{\mu\nu}F^{\mu\rho}\,F^{\nu}{}_{\rho}} + \underbrace{\frac{\alpha_{EM}^{2}}{90m_e^{4}}\left[\left(F^{\mu\nu}F_{\mu\nu}\right)^{2} + \frac{7}{4}\left(F^{\mu\nu}\widetilde{F}_{\mu\nu}\right)^{2}\right]}$$

Gertsenshtein effect

- Consequence of metric expansion $g^{\mu\nu} = \eta^{\mu\nu} - \frac{2}{m_P}h^{\mu\nu}$
- Weak coupling between EM and GW fields ($\propto 1/m_P$)

Euler–Heisenberg Lagrangian

- Nonlinear dynamics of QED at high energies
- Photon–photon scattering

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Introduction

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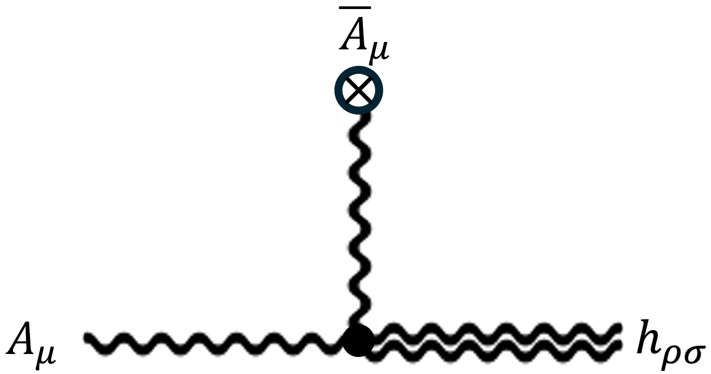

Euler–Heisenberg Lagrangian

- Nonlinear dynamics of QED at high energies
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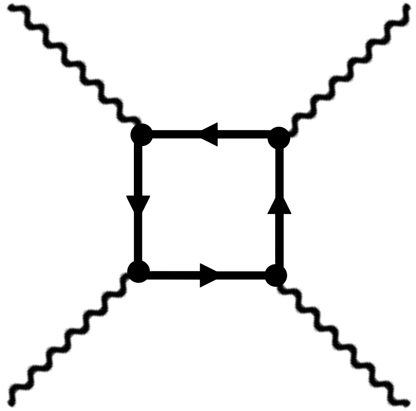

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Equations of Motion

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^{\mu}h^{\rho\sigma})(\partial_{\mu}h_{\rho\sigma}) + \frac{1}{m_P}h_{\mu\nu}F^{\mu\rho}\ F^{\nu}{}_{\rho} + \frac{\alpha_{EM}^2}{90m_e^4}\left[(F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}(F^{\mu\nu}\tilde{F}_{\mu\nu})^2\right]$$

Gauge fixing

- Transverse + traceless for GW
- Transverse + temporal for EM

Euler-Lagrange Equations

$$\partial_{\lambda}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\lambda}f_i)}\right] - \frac{\partial\mathcal{L}}{\partial f_i} = 0$$

Electromagnetic Field Linearisation

- Photon + background

$$\mathcal{A}_{\mu} = \overline{A}_{\mu} + A_{\mu}$$

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Equations of Motion

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^{\mu}h^{\rho\sigma})(\partial_{\mu}h_{\rho\sigma}) + \frac{1}{m_P}h_{\mu\nu}F^{\mu\rho}\,F^{\nu}{}_{\rho} + \frac{\alpha_{EM}^2}{90m_e^4}\left[\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \frac{7}{4}\left(F^{\mu\nu}\tilde{F}_{\mu\nu}\right)^2\right]$$

Gauge fixing Euler-Lagrange Equations Electromagnetic Field Linearisation

Magnetic Background

- ○ Propagation in z-direction
- ○ Purely magnetic background

Plane Wave

- ○ Approximate the fields as plane waves $A_\mu, h_{\mu\nu} \propto e^{i\omega t - ikz}$

WKB Approximation

- ○ Slow amplitude cf. fast frequency

$$\partial^2 \simeq -2\omega(\omega - i\partial_z)$$

Change of bases

- ○ Background magnetic field in spherical coordinates

$$\overline{\mathbf{B}} = B(\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$$

$$\begin{pmatrix} A_\perp \\ A_\parallel \\ h_\times \\ h_+ \end{pmatrix} = \begin{pmatrix} \sin\Phi & -\cos\Phi & 0 & 0 \\ \cos\Phi & \sin\Phi & 0 & 0 \\ 0 & 0 & \sin 2\Phi & \cos 2\Phi \\ 0 & 0 & -\cos 2\Phi & \sin 2\Phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ h_{xx} \\ h_{xy} \end{pmatrix}$$

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Equations of Motion

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^{\mu}h^{\rho\sigma})(\partial_{\mu}h_{\rho\sigma}) + \frac{1}{m_P}h_{\mu\nu}F^{\mu\rho}\,F^{\nu}{}_{\rho} + \frac{\alpha_{EM}^2}{90m_e^4}\left[\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \frac{7}{4}\left(F^{\mu\nu}\tilde{F}_{\mu\nu}\right)^2\right]$$

Gauge fixing Euler-Lagrange Equations Electromagnetic Field Linearisation Change of bases WKB Approximation Magnetic Background Plane Wave

"Schrödinger-like equation"

$$i\frac{d}{\underset{t}{dz}}\overset{\Psi}{\begin{pmatrix} A_\perp \\ h_+ \\ A_\parallel \\ h_\times \end{pmatrix}} = \underbrace{\left[\omega\mathbb{I} + \begin{pmatrix} \Delta_\perp & \Delta_M & \Delta_\Phi & 0 \\ \Delta_M & 0 & 0 & 2\Delta_\Phi \\ \Delta_\Phi & 0 & \Delta_\parallel & \Delta_M \\ 0 & 2\Delta_\Phi & \Delta_M & 0 \end{pmatrix}\right]}_{\mathcal{H}}\overset{\Psi}{\begin{pmatrix} A_\perp \\ h_+ \\ A_\parallel \\ h_\times \end{pmatrix}}$$

with

$$\Delta_\perp = \frac{2\alpha_{EM}\omega}{45\pi}\left(\frac{B_t}{B_{\text{crit}}}\right)^2 \qquad \Delta_\parallel = \frac{7\alpha_{EM}\omega}{90\pi}\left(\frac{B_t}{B_{\text{crit}}}\right)^2 \qquad \Delta_{\text{M}} = \frac{B_t}{\sqrt{2}m_P} \qquad \Delta_\Phi = \frac{d\Phi}{dz}$$

with $B_t = B\cos\Theta$ and $B_{\text{crit}} = m_e^2/e$

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Equations of Motion

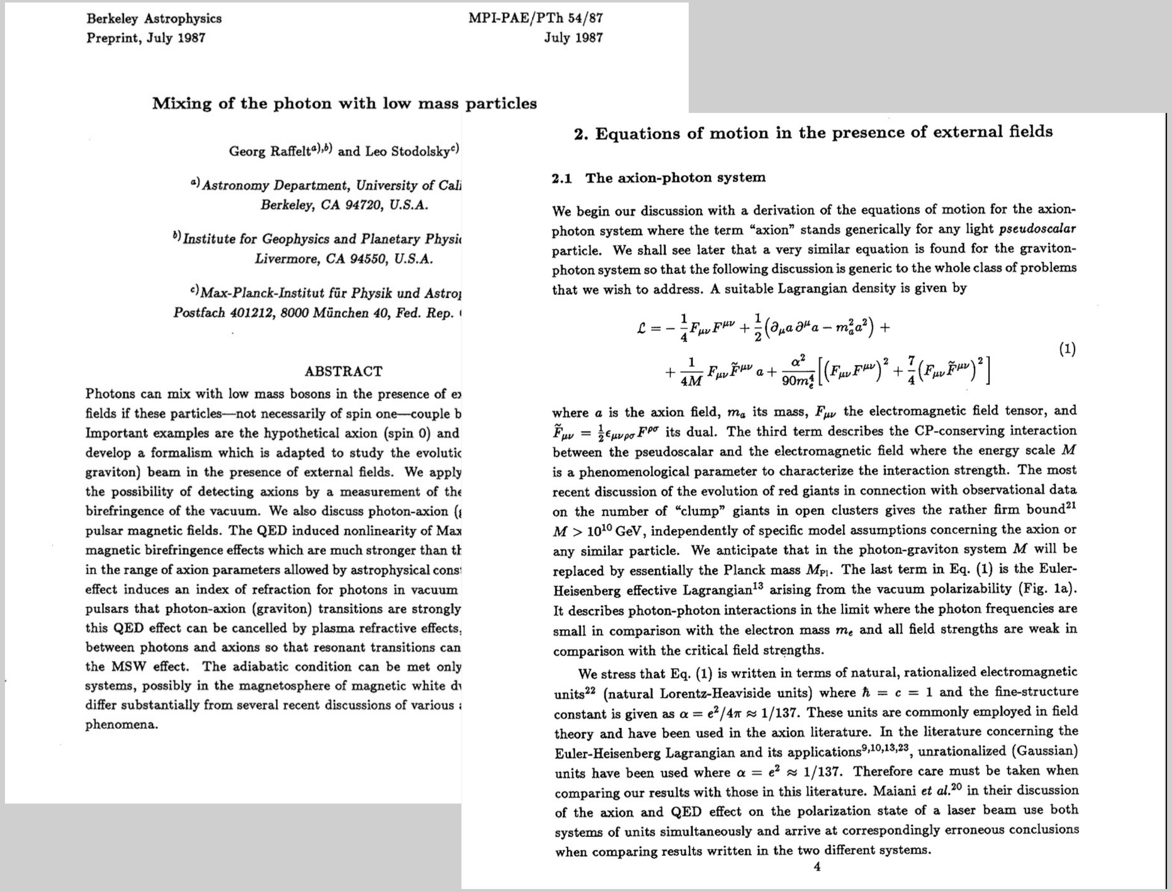
$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^\mu h^{\rho\sigma})(\partial_\mu h_{\rho\sigma}) + \frac{1}{m_P}h_{\mu\nu}F^{\mu\rho}\,F^{\nu}{}_{\rho} + \frac{\alpha_{EM}^2}{90m_e^4}\left[\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + \frac{7}{4}\left(F^{\mu\nu}\tilde{F}_{\mu\nu}\right)^2\right]$$

Gauge fixing Euler-Lagrange Equations Electromagnetic Field Linearisation Change of bases WKB Approximation

Magnetic Background Plane Wave

"*Mixing of the photon with low-mass particles*"

- Published in 1987 by G. Raffelt & L. Stodolsky
- Focused on axion-photon mixing
- Formalism for propagating wave packets in a magnetic background
- Similarities between axion-photon and GW-photon systems



Berkeley Astrophysics
Preprint, July 1987

MPI-PAE/PTh 54/87
July 1987

Mixing of the photon with low mass particles

Georg Raffelt[a),b)] and Leo Stodolsky[c)]

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[b)] *Institute for Geophysics and Planetary Physi*
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ABSTRACT

Photons can mix with low mass bosons in the presence of e
fields if these particles—not necessarily of spin one—couple b
Important examples are the hypothetical axion (spin 0) and
develop a formalism which is adapted to study the evoluti
graviton) beam in the presence of external fields. We apply
the possibility of detecting axions by a measurement of th
birefringence of the vacuum. We also discuss photon-axion (
pulsar magnetic fields. The QED induced nonlinearity of Ma
magnetic birefringence effects which are much stronger than t
in the range of axion parameters allowed by astrophysical cons
effect induces an index of refraction for photons in vacuum
pulsars that photon-axion (graviton) transitions are strongly
this QED effect can be cancelled by plasma refractive effects,
between photons and axions so that resonant transitions can
the MSW effect. The adiabatic condition can be met only
systems, possibly in the magnetosphere of magnetic white d
differ substantially from several recent discussions of various
phenomena.

2. Equations of motion in the presence of external fields

2.1 The axion-photon system

We begin our discussion with a derivation of the equations of motion for the axion-photon system where the term "axion" stands generically for any light *pseudoscalar* particle. We shall see later that a very similar equation is found for the graviton-photon system so that the following discussion is generic to the whole class of problems that we wish to address. A suitable Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu a\,\partial^\mu a - m_a^2 a^2\right) + \\ + \frac{1}{4M}F_{\mu\nu}\tilde{F}^{\mu\nu}a + \frac{\alpha^2}{90m_e^4}\left[\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{7}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right] \tag{1}$$

where a is the axion field, m_a its mass, $F_{\mu\nu}$ the electromagnetic field tensor, and $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ its dual. The third term describes the CP-conserving interaction between the pseudoscalar and the electromagnetic field where the energy scale M is a phenomenological parameter to characterize the interaction strength. The most recent discussion of the evolution of red giants in connection with observational data on the number of "clump" giants in open clusters gives the rather firm bound[21] $M > 10^{10}$ GeV, independently of specific model assumptions concerning the axion or any similar particle. We anticipate that in the photon-graviton system M will be replaced by essentially the Planck mass $M_{\rm Pl}$. The last term in Eq. (1) is the Euler-Heisenberg effective Lagrangian[13] arising from the vacuum polarizability (Fig. 1a). It describes photon-photon interactions in the limit where the photon frequencies are small in comparison with the electron mass m_e and all field strengths are weak in comparison with the critical field strengths.

We stress that Eq. (1) is written in terms of natural, rationalized electromagnetic units[22] (natural Lorentz-Heaviside units) where $\hbar = c = 1$ and the fine-structure constant is given as $\alpha = e^2/4\pi \approx 1/137$. These units are commonly employed in field theory and have been used in the axion literature. In the literature concerning the Euler-Heisenberg Lagrangian and its applications[9,10,13,23], unrationalized (Gaussian) units have been used where $\alpha = e^2 \approx 1/137$. Therefore care must be taken when comparing our results with those in this literature. Maiani *et al.*[20] in their discussion of the axion and QED effect on the polarization state of a laser beam use both systems of units simultaneously and arrive at correspondingly erroneous conclusions when comparing results written in the two different systems.

4

“Schrödinger-like equation”

$$i\frac{d}{dz}\begin{pmatrix} A_\perp \\ h_+ \\ A_\parallel \\ h_\times \end{pmatrix} = \left[\omega\mathbb{I} + \begin{pmatrix} \Delta_\perp & \Delta_M & \Delta_\Phi & 0 \\ \Delta_M & 0 & 0 & 2\Delta_\Phi \\ \Delta_\Phi & 0 & \Delta_\parallel & \Delta_M \\ 0 & 2\Delta_\Phi & \Delta_M & 0 \end{pmatrix}\right]\begin{pmatrix} A_\perp \\ h_+ \\ A_\parallel \\ h_\times \end{pmatrix}$$

- Δ_Φ was neglected by G. Raffelt & L. Stodolsky
- Justified for axion-photon mixing since $\Delta_\Phi \ll \Delta_M$

$$\Delta_\Phi = 0$$

$$i\frac{d}{dz}\begin{pmatrix} A_j \\ h_j \end{pmatrix} = \left[\omega\mathbb{I} + \begin{pmatrix} \Delta_\mathrm{j} & \Delta_M \\ \Delta_M & 0 \end{pmatrix}\right]\begin{pmatrix} A_j \\ h_j \end{pmatrix}$$

- Two uncoupled 2×2 systems: $\{A_\perp, h_+\}$ and $\{A_\parallel, h_\times\}$
- Not justified for GW-photon mixing since $\Delta_\Phi \gg \Delta_M$

For some geometries
$\Delta_\Phi = 0$ exactly

- Analysis restricted to two specific geometries
- Approximation is not valid outside these configurations

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arXiv : 2606.12546

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$$i\frac{d}{dz}\begin{pmatrix}A_j\\ h_j\end{pmatrix} = \left[\omega\mathbb{I} + \begin{pmatrix}\Delta_\mathrm{j} & \Delta_M\\ \Delta_M & 0\end{pmatrix}\right]\begin{pmatrix}A_j\\ h_j\end{pmatrix}$$

Solutions: Free + Interacting part

$$\mathcal{H}_0 = \omega\mathbb{I} + \begin{pmatrix}\Delta_\mathrm{j} & 0\\ 0 & 0\end{pmatrix} \qquad \delta\mathcal{H} = \begin{pmatrix}0 & \Delta_M\\ \Delta_M & 0\end{pmatrix} \qquad \mathcal{U}(z, z_0) = \mathcal{U}_0(z, z_0)\mathcal{U}_\mathrm{int}(z, z_0)$$

Evolution operators

$$\mathcal{U}_0(z, z_0) \ = \exp\left[-i\int_{z_0}^{z} dz'\,\mathcal{H}_0(z')\right] \qquad \mathcal{U}_\mathrm{int}(z, z_0) \ = \mathcal{P}\left\{\exp\left[-i\int_{z_0}^{z} dz'\,\mathcal{H}_\mathrm{int}(z')\right]\right\}$$

Measurable quantities, e.g.,

$$P_{h_j\to\gamma_j} = \left|\int_{z_0}^{z}\mathrm{dz'}\ \Delta_M(z')\exp\left[-i\int_{z_0}^{z'} dz''\ \Delta_\mathrm{j}(z'')\right]\right|^2$$

$$\delta\phi_{h_j} = \mathrm{Im}\left\{\int_{z_0}^{z}\mathrm{dz'}\int_{z_0}^{z'}\mathrm{dz''}\ \Delta_M(z')\Delta_M(z'')\exp\left[-i\int_{z''}^{z'} dz'''\ \Delta_\mathrm{j}(z''')\right]\right\}$$

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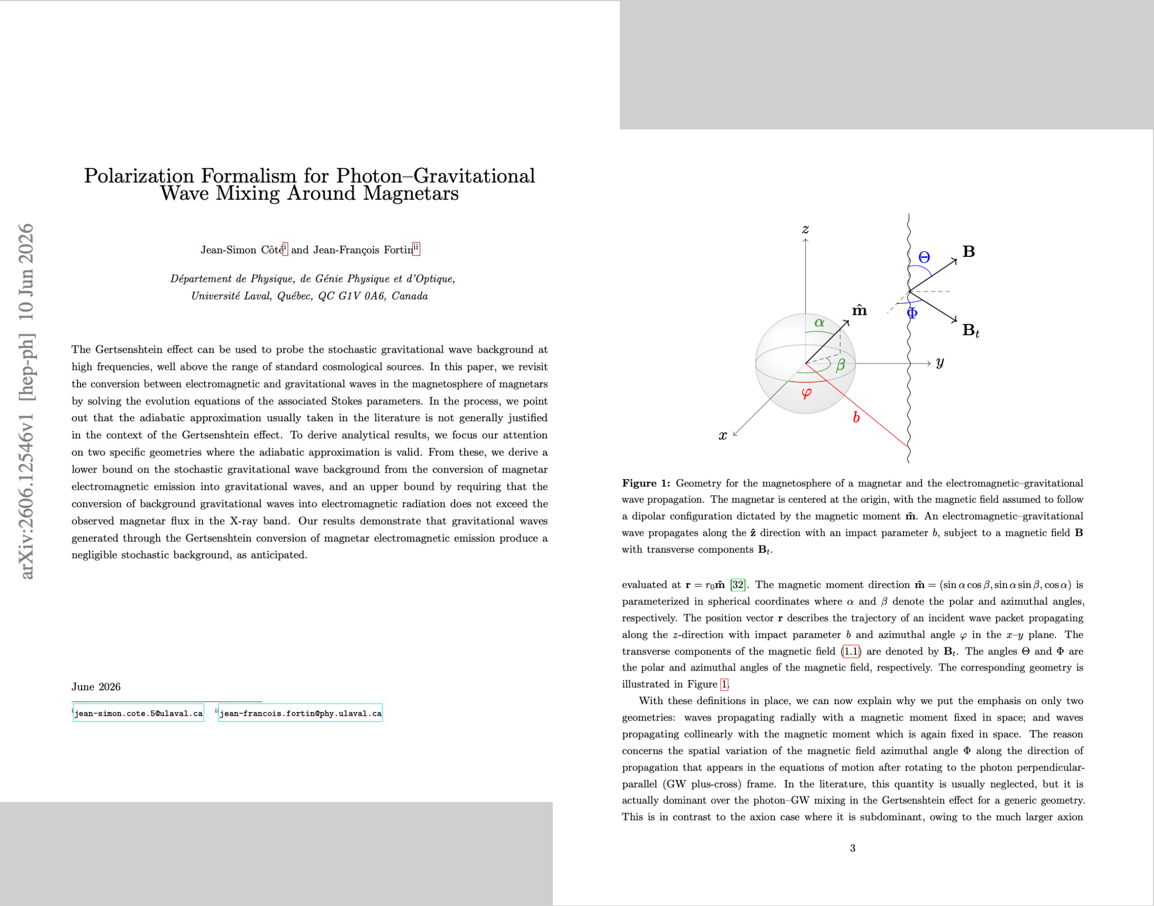
Polarization Formalism for Photon–Gravitational Wave Mixing Around Magnetars

- Stokes parameters evolution expressed uniquely in terms of measurable quantities (e.g., $P_{h_j \to \gamma_j}$, $\delta\phi_{h_j}$, etc.)
- Conversion probabilities and phase shifts computed in two geometries
- Upper and lower bounds on the SGWB at X-ray frequencies
- arXiv: 2606.12546



arXiv:2606.12546v1 [hep-ph] 10 Jun 2026

Polarization Formalism for Photon–Gravitational Wave Mixing Around Magnetars

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The Gertsenshtein effect can be used to probe the stochastic gravitational wave background at high frequencies, well above the range of standard cosmological sources. In this paper, we revisit the conversion between electromagnetic and gravitational waves in the magnetosphere of magnetars by solving the evolution equations of the associated Stokes parameters. In the process, we point out that the adiabatic approximation usually taken in the literature is not generally justified in the context of the Gertsenshtein effect. To derive analytical results, we focus our attention on two specific geometries where the adiabatic approximation is valid. From these, we derive a lower bound on the stochastic gravitational wave background from the conversion of magnetar electromagnetic emission into gravitational waves, and an upper bound by requiring that the conversion of background gravitational waves into electromagnetic radiation does not exceed the observed magnetar flux in the X-ray band. Our results demonstrate that gravitational waves generated through the Gertsenshtein conversion of magnetar electromagnetic emission produce a negligible stochastic background, as anticipated.

June 2026

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Figure 1: Geometry for the magnetosphere of a magnetar and the electromagnetic–gravitational wave propagation. The magnetar is centered at the origin, with the magnetic field assumed to follow a dipolar configuration dictated by the magnetic moment $\hat{\mathbf{m}}$. An electromagnetic–gravitational wave propagates along the $\hat{\mathbf{z}}$ direction with an impact parameter b, subject to a magnetic field $\mathbf{B}$ with transverse components $\mathbf{B}_t$.

evaluated at $\mathbf{r} = r_0\hat{\mathbf{m}}$ [32]. The magnetic moment direction $\hat{\mathbf{m}} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$ is parameterized in spherical coordinates where α and β denote the polar and azimuthal angles, respectively. The position vector $\mathbf{r}$ describes the trajectory of an incident wave packet propagating along the z-direction with impact parameter b and azimuthal angle φ in the x–y plane. The transverse components of the magnetic field (1.1) are denoted by $\mathbf{B}_t$. The angles Θ and Φ are the polar and azimuthal angles of the magnetic field, respectively. The corresponding geometry is illustrated in Figure 1.

With these definitions in place, we can now explain why we put the emphasis on only two geometries: waves propagating radially with a magnetic moment fixed in space; and waves propagating collinearly with the magnetic moment which is again fixed in space. The reason concerns the spatial variation of the magnetic field azimuthal angle Φ along the direction of propagation that appears in the equations of motion after rotating to the photon perpendicular-parallel (GW plus-cross) frame. In the literature, this quantity is usually neglected, but it is actually dominant over the photon–GW mixing in the Gertsenshtein effect for a generic geometry. This is in contrast to the axion case where it is subdominant, owing to the much larger axion

Stokes Parameters

$$I_\gamma(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \left[|A_\perp(z)|^2 + \left|A_\parallel(z)\right|^2 \right] \qquad U_\gamma(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \left[A_\perp(z) A_\parallel^*(z) + A_\perp^*(z) A_\parallel(z) \right]$$

$$Q_\gamma(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \left[|A_\perp(z)|^2 - \left|A_\parallel(z)\right|^2 \right] \qquad V_\gamma(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} i\left[A_\perp(z) A_\parallel^*(z) - A_\perp^*(z) A_\parallel(z) \right]$$

$$I_h(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \left[|h_+(z)|^2 + |h_\times(z)|^2 \right] \qquad U_h(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \left[h_+(z) h_\times^*(z) + h_+^*(z) h_\times(z) \right]$$

$$Q_h(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \left[|h_+(z)|^2 - |h_\times(z)|^2 \right] \qquad V_h(z) = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} i\left[h_+(z) h_\times^*(z) - h_+^*(z) h_\times(z) \right]$$

- Convenient description of polarization (I: intensity, $Q + U$: linear polarization, V: circular polarization)
- Any measurement involves averaging
- Integral over the arbitrary parameter $\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}$

arXiv : 2606.12546

Stokes Parameters

Density matrix

$$\rho = \begin{pmatrix} |A_\perp|^2 & A_\perp h_+^* & A_\perp A_\parallel^* & A_\perp h_\times^* \\ h_+ A_\perp^* & |h_+|^2 & h_+ A_\parallel^* & h_+ h_\times^* \\ A_\parallel A_\perp^* & A_\parallel h_+^* & |A_\parallel|^2 & A_\parallel h_\times^* \\ h_\times A_\perp^* & h_\times h_+^* & h_\times A_\parallel^* & |h_\times|^2 \end{pmatrix}$$

$$\int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \left[\rho = \mathcal{U}(z, z_0)\rho_0 \mathcal{U}^\dagger(z, z_0) \right]$$

Initial averaged density matrix

$$\bar{\rho}_0 = \int_0^{2\pi} \frac{\mathrm{d}\Delta\phi_{\gamma_{\perp 0} h_{\times 0}}}{2\pi} \begin{pmatrix} |A_\perp|^2 & A_\perp h_+^* & A_\perp A_\parallel^* & A_\perp h_\times^* \\ h_+ A_\perp^* & |h_+|^2 & h_+ A_\parallel^* & h_+ h_\times^* \\ A_\parallel A_\perp^* & A_\parallel h_+^* & |A_\parallel|^2 & A_\parallel h_\times^* \\ h_\times A_\perp^* & h_\times h_+^* & h_\times A_\parallel^* & |h_\times|^2 \end{pmatrix} \Bigg|_{z=z_0} = \frac{1}{2} \begin{pmatrix} I_{\gamma 0} + Q_{\gamma 0} & 0 & U_{\gamma 0} - iV_{\gamma 0} & 0 \\ 0 & I_{h0} + Q_{h0} & 0 & U_{h0} - iV_{h0} \\ U_{\gamma 0} + iV_{\gamma 0} & 0 & I_{\gamma 0} - Q_{\gamma 0} & 0 \\ 0 & U_{h0} + iV_{h0} & 0 & I_{h0} - Q_{h0} \end{pmatrix}$$

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Stokes Parameters

$$\overline{\rho} = \mathcal{U}(z,z_0)\overline{\rho}_0\mathcal{U}^\dagger(z,z_0)$$

$$\overline{\rho}_0 = \frac{1}{2}\begin{pmatrix} I_{\gamma 0}+Q_{\gamma 0} & 0 & U_{\gamma 0}-iV_{\gamma 0} & 0 \\ 0 & I_{h0}+Q_{h0} & 0 & U_{h0}-iV_{h0} \\ U_{\gamma 0}+iV_{\gamma 0} & 0 & I_{\gamma 0}-Q_{\gamma 0} & 0 \\ 0 & U_{h0}+iV_{h0} & 0 & I_{h0}-Q_{h0} \end{pmatrix}$$

- Evolution operator $\mathcal{U} = \mathcal{U}_0\mathcal{U}_{\text{int}}$

Amplitude conservation
+
$\text{tr}[\mathcal{H}_{\text{int}}] = 0$

$\mathcal{U}_{\text{int}}(z,z_0) \in SU(2)$

$$\mathcal{U}_{\text{int}}(z,z_0) = \begin{pmatrix} \sqrt{1-P_{h_j\to\gamma_j}}e^{i\delta\phi_{h_j}} & \sqrt{P_{h_j\to\gamma_j}}e^{-i\delta\phi_{h_j\to\gamma_j}} \\ -\sqrt{P_{h_j\to\gamma_j}}e^{i\delta\phi_{h_j\to\gamma_j}} & \sqrt{1-P_{h_j\to\gamma_j}}e^{-i\delta\phi_{h_j}} \end{pmatrix}$$

$$\delta\phi_{\text{QED}} = \int_{z_0}^{z}\text{dz'}[\Delta_\perp(z') - \Delta_\parallel(z')] \qquad \Delta\phi_h = \delta\phi_{h_+} - \delta\phi_{h_\times} \qquad \Delta\phi_{h\to\gamma} = \delta\phi_{h_+\to\gamma_\perp} - \delta\phi_{h_\times\to\gamma_\parallel}$$

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Stokes Parameters

Photon

$$I_\gamma = I_{\gamma 0} + \left(\frac{I_{h0} + Q_{h0}}{2} - \frac{I_{\gamma 0} + Q_{\gamma 0}}{2}\right) P_{h_+ \to \gamma_\perp} + \left(\frac{I_{h0} - Q_{h0}}{2} - \frac{I_{\gamma 0} - Q_{\gamma 0}}{2}\right) P_{h_\times \to \gamma_\parallel}$$

$$Q_\gamma = Q_{\gamma 0} + \left(\frac{I_{h0} + Q_{h0}}{2} - \frac{I_{\gamma 0} + Q_{\gamma 0}}{2}\right) P_{h_+ \to \gamma_\perp} - \left(\frac{I_{h0} - Q_{h0}}{2} - \frac{I_{\gamma 0} - Q_{\gamma 0}}{2}\right) P_{h_\times \to \gamma_\parallel}$$

$$\begin{aligned} U_\gamma = \ & \left[U_{\gamma 0} \cos\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_h\right) - V_{\gamma 0} \sin\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_h\right)\right] \sqrt{\left(1 - P_{h_+ \to \gamma_\perp}\right)\left(1 - P_{h_\times \to \gamma_\parallel}\right)} \\ & + \left[U_{h0} \cos\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_{h \to \gamma}\right) - V_{h0} \sin\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_{h \to \gamma}\right)\right] \sqrt{P_{h_+ \to \gamma_\perp} P_{h_\times \to \gamma_\parallel}} \end{aligned}$$

$$\begin{aligned} V_\gamma = \ & \left[V_{\gamma 0} \cos\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_h\right) + U_{\gamma 0} \sin\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_h\right)\right] \sqrt{\left(1 - P_{h_+ \to \gamma_\perp}\right)\left(1 - P_{h_\times \to \gamma_\parallel}\right)} \\ & + \left[V_{h0} \cos\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_{h \to \gamma}\right) + U_{h0} \sin\left(\delta\phi_{\mathrm{QED}} - \Delta\phi_{h \to \gamma}\right)\right] \sqrt{P_{h_+ \to \gamma_\perp} P_{h_\times \to \gamma_\parallel}} \end{aligned}$$

Stokes Parameters

Gravitational wave

$$I_h = I_{h0} - \left(\frac{I_{h0} + Q_{h0}}{2} - \frac{I_{\gamma 0} + Q_{\gamma 0}}{2}\right) P_{h_+ \to \gamma_\perp} - \left(\frac{I_{h0} - Q_{h0}}{2} - \frac{I_{\gamma 0} - Q_{\gamma 0}}{2}\right) P_{h_\times \to \gamma_\parallel}$$

$$Q_h = Q_{h0} - \left(\frac{I_{h0} + Q_{h0}}{2} - \frac{I_{\gamma 0} + Q_{\gamma 0}}{2}\right) P_{h_+ \to \gamma_\perp} + \left(\frac{I_{h0} - Q_{h0}}{2} - \frac{I_{\gamma 0} - Q_{\gamma 0}}{2}\right) P_{h_\times \to \gamma_\parallel}$$

$$\begin{aligned} U_h &= [U_{h0} \cos \Delta \phi_h - V_{h0} \sin \Delta \phi_h] \sqrt{(1 - P_{h_+ \to \gamma_\perp})(1 - P_{h_\times \to \gamma_\parallel})} \\ &\quad + \left[U_{\gamma 0} \cos \Delta \phi_{h \to \gamma} - V_{\gamma 0} \sin \Delta \phi_{h \to \gamma}\right] \sqrt{P_{h_+ \to \gamma_\perp} P_{h_\times \to \gamma_\parallel}} \end{aligned}$$

$$\begin{aligned} V_h &= [V_{h0} \cos \Delta \phi_h + U_{h0} \sin \Delta \phi_h] \sqrt{(1 - P_{h_+ \to \gamma_\perp})(1 - P_{h_\times \to \gamma_\parallel})} \\ &\quad + \left[V_{\gamma 0} \cos \Delta \phi_{h \to \gamma} + U_{\gamma 0} \sin \Delta \phi_{h \to \gamma}\right] \sqrt{P_{h_+ \to \gamma_\perp} P_{h_\times \to \gamma_\parallel}} \end{aligned}$$

Probabilities and Phase Shifts

- Analytical computation of $P_{h_j \to \gamma_j}, \delta\phi_{h_j \to \gamma_j}, \delta\phi_{h_j}$ and $\delta\phi_{\text{QED}}$
- Two geometries of interest ($\Delta_\Phi = 0$):
  - radial case ($b = 0$)
  - parallel case ($\alpha = 0$)

Magnetic dipole

$$\mathbf{B} = \frac{B_0}{2}\left(\frac{r_0}{r}\right)^3 [3\hat{\mathbf{r}}(\hat{\mathbf{m}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{m}}]$$

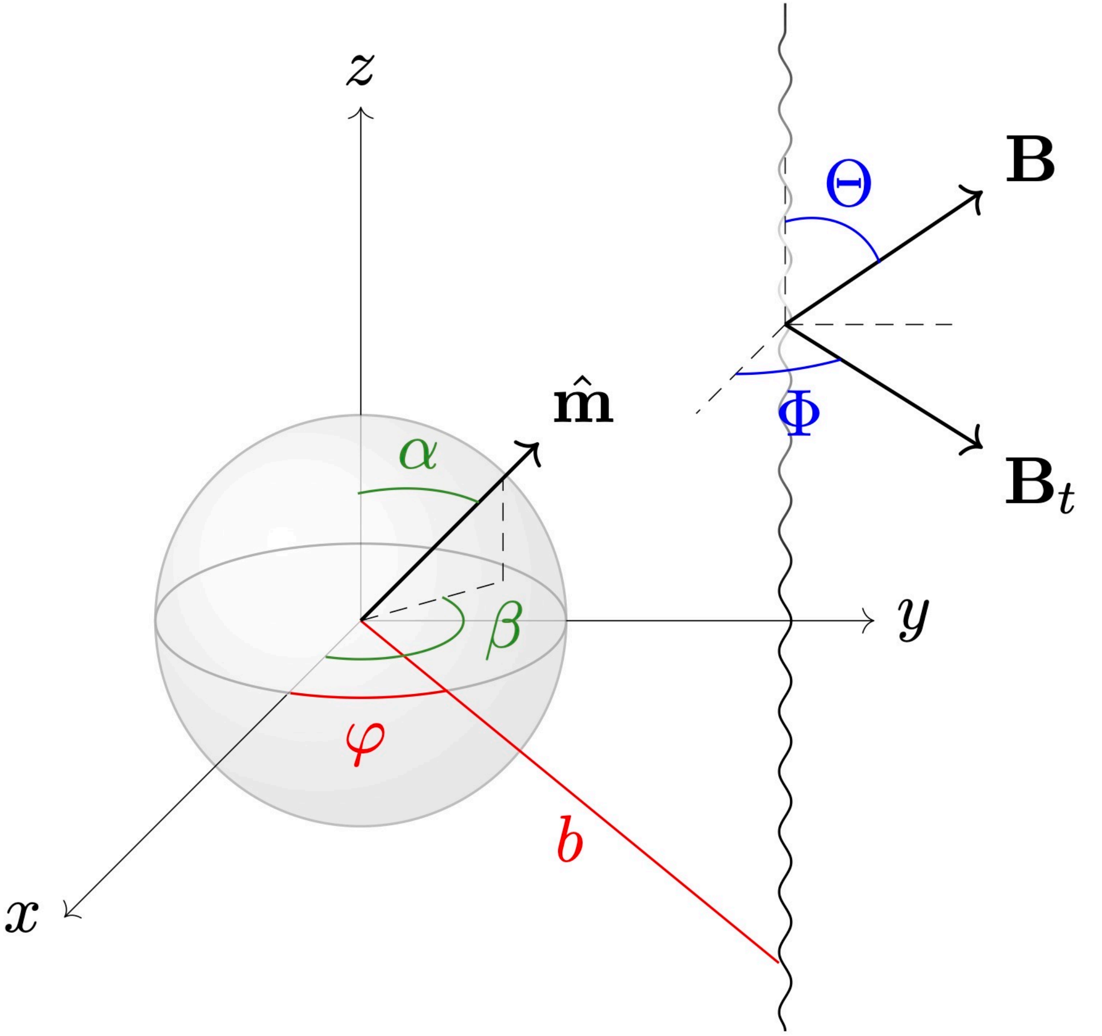
r_0 : Radius of magnetar

$\mathbf{r}$: Wave-packet position

B_0 : Surface magnetic field

$\mathbf{m}$: Magnetic moment

⇒ Computation of the EOM quantities ($\Delta_\parallel, \Delta_\perp, \Delta_M$)



arXiv : 2606.12546

Probabilities and Phase Shifts

Large parameter : $\Delta_{j0}r_0 \sim 10^{12}$

Radial case

$$P_{h_j\to\gamma_j}\Big|_{b=0} = \frac{(\Delta_{M0}r_0)^2(\sin\alpha)^{\frac{2}{5}}}{25\left(\Delta_{j0}r_0/5\right)^{\frac{4}{5}}}\left|\left|\left[\int_0^\infty - \int_{-i\Delta_{j0}r_0\sin^2\alpha/5}^{\infty}\right]\mathrm{d}x'x'^{\frac{2}{5}-1}e^{-x'}\right|^2$$

$$\delta\phi_{h_j}\Big|_{b=0} = (\Delta_{M0}r_0\sin\alpha)^2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}d\vartheta'\int_{\frac{\pi}{2}}^{\vartheta'}d\vartheta''\cot\vartheta'\csc^2\vartheta'\cot\vartheta''\csc^2\vartheta''\sin\left(\Delta_{j0}r_0\sin^2\alpha\int_{\vartheta''}^{\vartheta'}d\vartheta'''\cot^4\vartheta'''\csc^2\vartheta'''\right)$$

Parallel case

$$P_{h_j\to\gamma_j}\Big|_{\alpha=0} = \frac{(6\Delta_{M0}r_0)^2}{\overline{b}^{\,4}}\left|\int_0^{\frac{\pi}{2}}d\vartheta'\sin\vartheta'\cos^2\vartheta'\sin\left(\frac{9\Delta_{j0}r_0}{\overline{b}^{\,5}}\int_0^{\vartheta'}d\vartheta''\sin^2\vartheta''\cos^6\vartheta''\right)\right|^2$$

$$\delta\phi_{h_j}\Big|_{\alpha=0} = \frac{(6\Delta_{M0}r_0)^2}{\overline{b}^{\,4}}\left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\vartheta'\int_{-\frac{\pi}{2}}^{\vartheta'}d\vartheta''\sin\vartheta''\cos^2\vartheta''\sin\vartheta''\cos^2\vartheta''\sin\left(\frac{9\Delta_{j0}r_0}{\overline{b}^{\,5}}\int_{\vartheta''}^{\vartheta'}d\vartheta'''\sin^2\vartheta'''\cos^6\vartheta'''\right)\right|^2$$

change of variables, initial/final conditions and $\overline{b} = b/r_0$

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Probabilities and Phase Shifts

Large parameter : $\Delta_{j0} r_0 \sim 10^{12}$

$$\lambda = \frac{\Delta_{j0} r_0 \sin^2 \alpha}{5}$$

Radial case

$$P_{h_j \to \gamma_j}\Big|_{b=0} \approx \frac{\Gamma\left(\frac{2}{5}\right)^2 (\Delta_{M0} r_0)^2 \sin^2 \alpha}{25 \lambda^{\frac{4}{5}}} \left\{1 + \mathcal{O}\left[\lambda^{-\frac{3}{5}}\right]\right\}$$

$$\delta\phi_{h_j}\Big|_{b=0} \approx (\Delta_{M0} r_0 \sin \alpha)^2 \left\{-\frac{\sqrt{5+2\sqrt{5}}\,\Gamma\left(\frac{2}{5}\right)^2}{50 \lambda^{\frac{4}{5}}} + \frac{1}{5\lambda} + \frac{\Gamma\left(\frac{2}{5}\right) \cos\left(\lambda - \frac{\pi}{5}\right)}{25 \lambda^{\frac{7}{5}}} + \mathcal{O}\left[\lambda^{-\frac{8}{5}}\right]\right\}$$

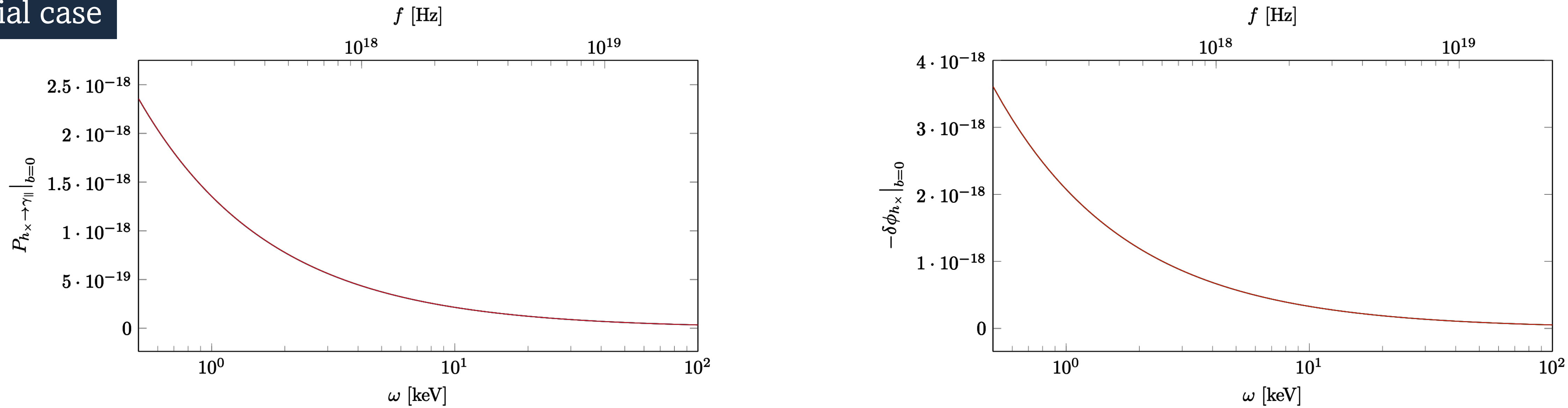
$$\lambda = \frac{45 \pi \Delta_{j0} r_0}{256\, \bar{b}^{5}}$$

Parallel case

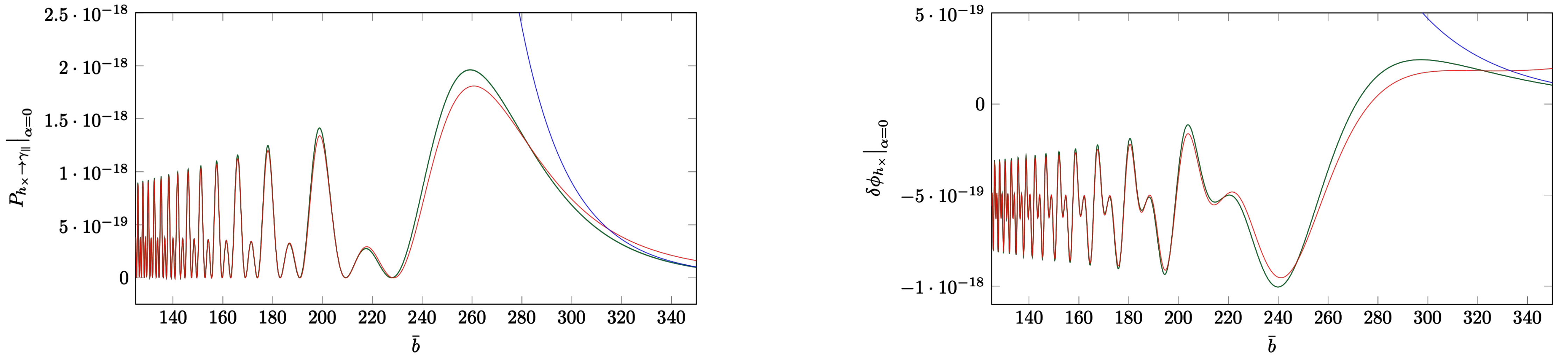
$$P_{h_j \to \gamma_j}\Big|_{\alpha=0} \approx \frac{(\Delta_{M0} r_0)^2}{\bar{b}^{4}} \left| \frac{3(35\pi/2)^{\frac{3}{7}} \Gamma\left(\frac{3}{7}\right) \cos\left(\lambda - \frac{5\pi}{7}\right)}{28 \lambda^{\frac{3}{7}}} + \frac{3^{\frac{1}{2}} (30\pi)^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)}{64 \lambda^{\frac{2}{3}}} - \frac{5(35\pi/2)^{\frac{5}{7}} \Gamma\left(\frac{5}{7}\right) \cos\left(\lambda + \frac{\pi}{7}\right)}{672 \lambda^{\frac{5}{7}}} + \mathcal{O}[\lambda^{-1}] \right|^2$$

$$\delta\phi_{h_j}\Big|_{\alpha=0} \approx -\frac{(3\Delta_{M0} r_0)^2}{\bar{b}^{4}} \left\{ \frac{(35\pi/2)^{\frac{6}{7}} \Gamma\left(\frac{3}{7}\right)^2 \left[\tan\left(\frac{3\pi}{7}\right) - \sin\left(2\lambda - \frac{3\pi}{7}\right)\right]}{3136 \lambda^{\frac{6}{7}}} - \frac{(35\pi/2)^{\frac{3}{7}} (30\pi)^{\frac{2}{3}} \Gamma\left(\frac{3}{7}\right) \Gamma\left(\frac{2}{3}\right) \sin\left(\lambda + \frac{2\pi}{7}\right)}{1792 \cdot 3^{\frac{1}{2}} \lambda^{\frac{23}{21}}} \right.$$
$$\left. - \frac{5(35\pi/2)^{\frac{8}{7}} \Gamma\left(\frac{3}{7}\right) \Gamma\left(\frac{5}{7}\right) \left[\cos\left(\frac{\pi}{7}\right) \tan\left(\frac{3\pi}{7}\right) + \sin\left(2\lambda - \frac{4\pi}{7}\right)\right]}{112\,896 \lambda^{\frac{8}{7}}} - \frac{(30\pi)^{\frac{4}{3}} \Gamma\left(\frac{2}{3}\right)^2}{8192 \cdot 3^{\frac{1}{2}} \lambda^{\frac{4}{3}}} + \mathcal{O}\left[\lambda^{-\frac{29}{21}}\right] \right\}$$

Radial case



Parallel case



*For the benchmark magnetar 1E 1547.0-5408 : $B_0 = 3.2 \times 10^{14}$ G, $r_0 = 10$ km and $\omega = 10$ keV

Lower bounds

General idea

Magnetars emit X-rays from their surfaces, which are converted into high-frequency gravitational waves via the Gertsenshtein effect in their magnetospheres.

$$I_h^{(\mathrm{x-y})} = \frac{1}{4\pi}\int d\Omega_{\hat{\mathbf{m}}} \int_0^{2\pi} \frac{\Delta\phi_{\gamma_{\perp 0}}}{2\pi}\left[|h_{xx}|^2 + \left|h_{xy}\right|^2\right] \qquad \Rightarrow \qquad \frac{m_P^2}{2}\omega h_c^2 = \overline{P}_{h\to\gamma}^{(\mathrm{rad}}\ \frac{\nu F_\nu}{\omega}$$

Lower bounds

$$h_c > 3.8\times 10^{-53}\cdot\left(\frac{B_0}{B_{\mathrm{crit}}}\right)^{\frac{1}{5}}\left(\frac{\omega}{10\ \mathrm{keV}}\right)^{-\frac{7}{5}}\left(\frac{r_0}{10\ \mathrm{km}}\right)^{\frac{3}{5}}\left(\frac{\nu F_\nu}{10^{-2}\,\frac{\mathrm{keV}^2\cdot\mathrm{Photons}}{\mathrm{cm}^2\cdot\mathrm{s}\cdot\mathrm{keV}}}\right)^{\frac{1}{2}}$$

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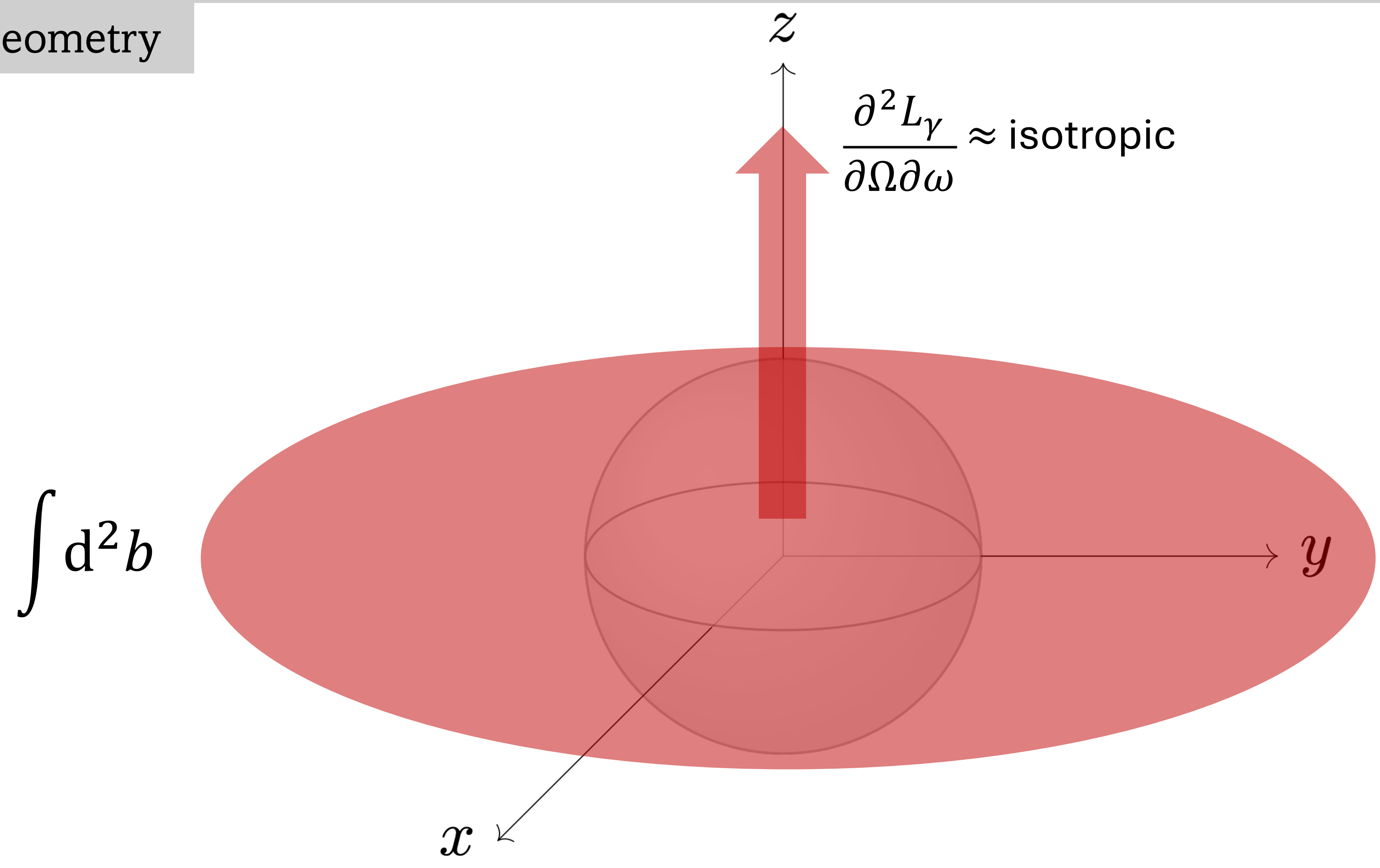
Upper bounds

General idea An existing SGWB is partially converted into X-rays in the vicinity of a magnetar. The converted light then propagates and can be detected on Earth.

$$I_\gamma^{(\mathrm{x-y})} = \int_0^{2\pi} \frac{\Delta\phi_{\gamma_{\perp 0}}}{2\pi} \left[|A_x|^2 + \left|A_y\right|^2 \right]$$

Geometry



arXiv : 2606.12546

Upper bounds

General idea An existing SGWB is partially converted into X-rays in the vicinity of a magnetar. The converted light then propagates and can be detected on Earth.

$$I_\gamma^{(\mathrm{x-y})} = \int_0^{2\pi} \frac{\Delta\phi_{\gamma_{\perp 0}}}{2\pi} \left[|A_x|^2 + \left|A_y\right|^2 \right] \qquad \Rightarrow$$

$$\frac{\partial F_\gamma}{\partial\omega} = \frac{\sigma_{\mathrm{eff}}}{4\pi d^2} \frac{\partial F_{h0}}{\partial\omega}$$

$$\sigma_{\mathrm{eff}} = \frac{1}{2} \sum_j \int \mathrm{d}^2 b \left[P_{h_j \to \gamma_j} \right|_{\alpha=0}$$

Upper bounds

$$h_c < 3.5 \times 10^{-20} \cdot \left(\frac{B_0}{B_{\mathrm{crit}}} \right)^{-\frac{3}{5}} \left(\frac{\omega}{10\ \mathrm{keV}} \right)^{-\frac{4}{5}} \left(\frac{r_0}{10\ \mathrm{km}} \right)^{-\frac{9}{5}} \left(\frac{d}{5\ \mathrm{kpc}} \right) \left(\frac{\nu F_\nu}{10^{-2} \frac{\mathrm{keV}^2 \cdot \mathrm{Photons}}{\mathrm{cm}^2 \cdot \mathrm{s} \cdot \mathrm{keV}}} \right)^{\frac{1}{2}}$$

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Lower bounds

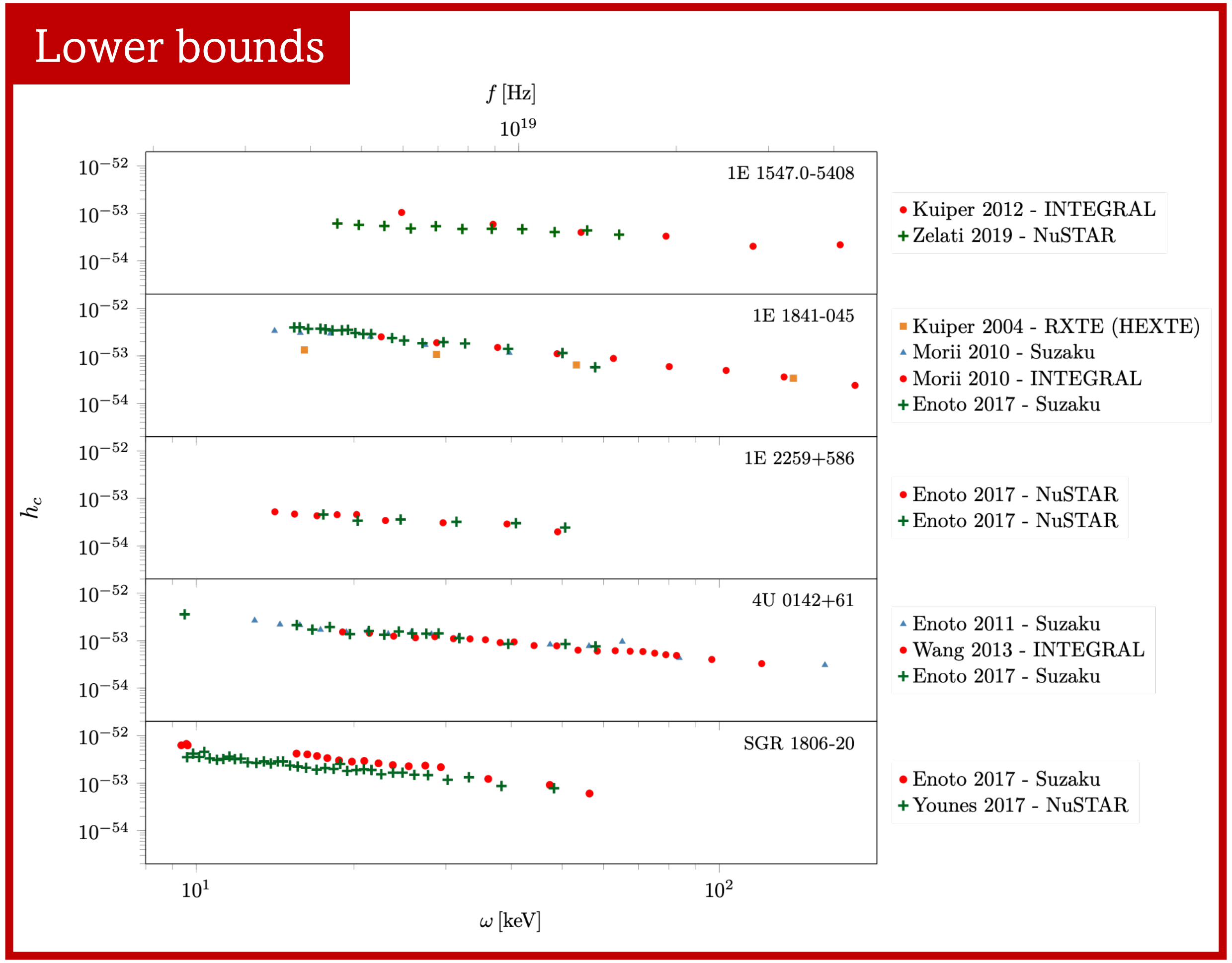

Figure 1 - Characteristic strain measured at Earth from gravitational waves produced through Gertsenshtein conversion of radially propagating X-rays in magnetars' magnetospheres.

Upper bounds

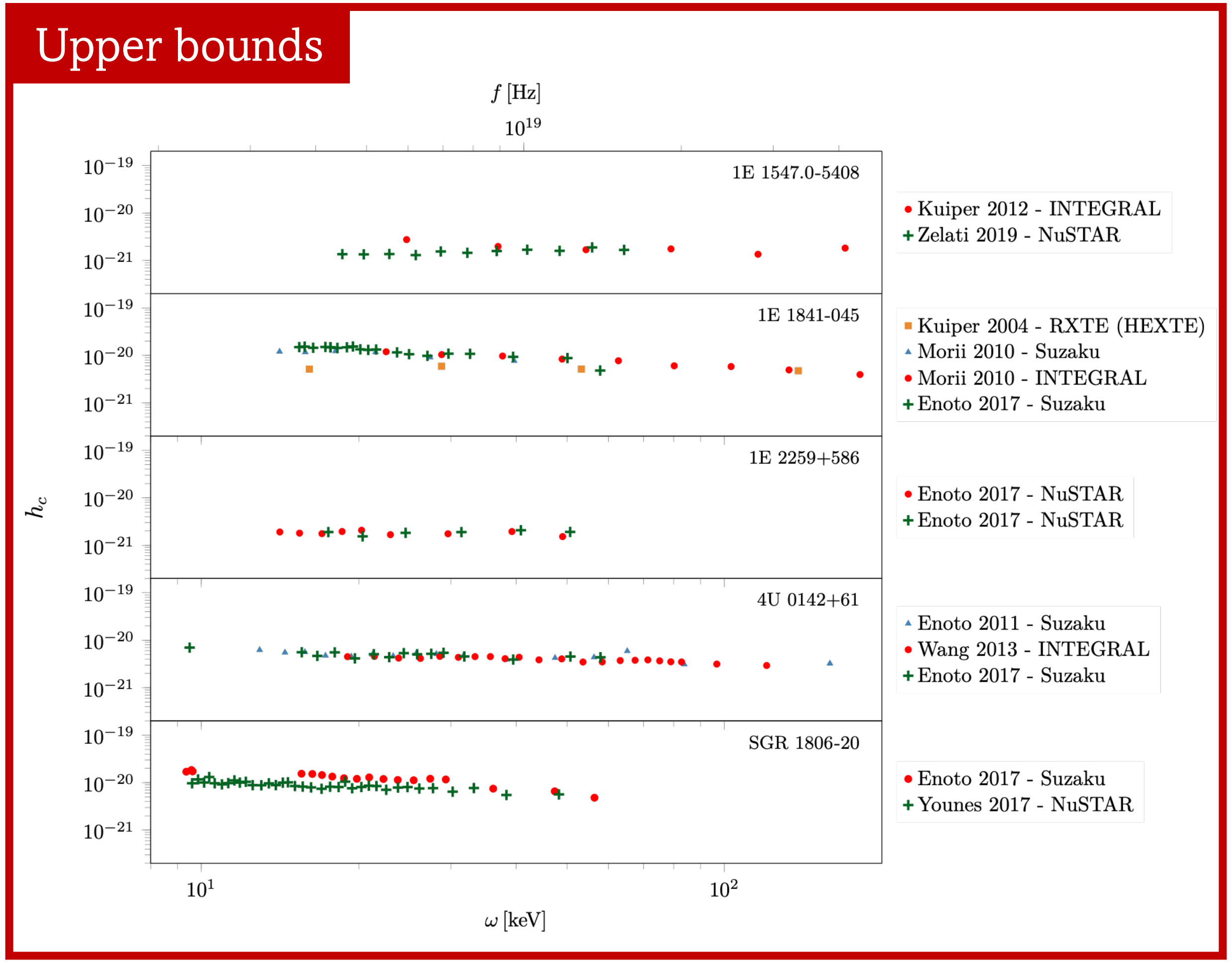

Figure 2 - Upper bounds on the characteristic strain of the gravitational-wave background derived from Gertsenshtein conversion in magnetars' magnetospheres.

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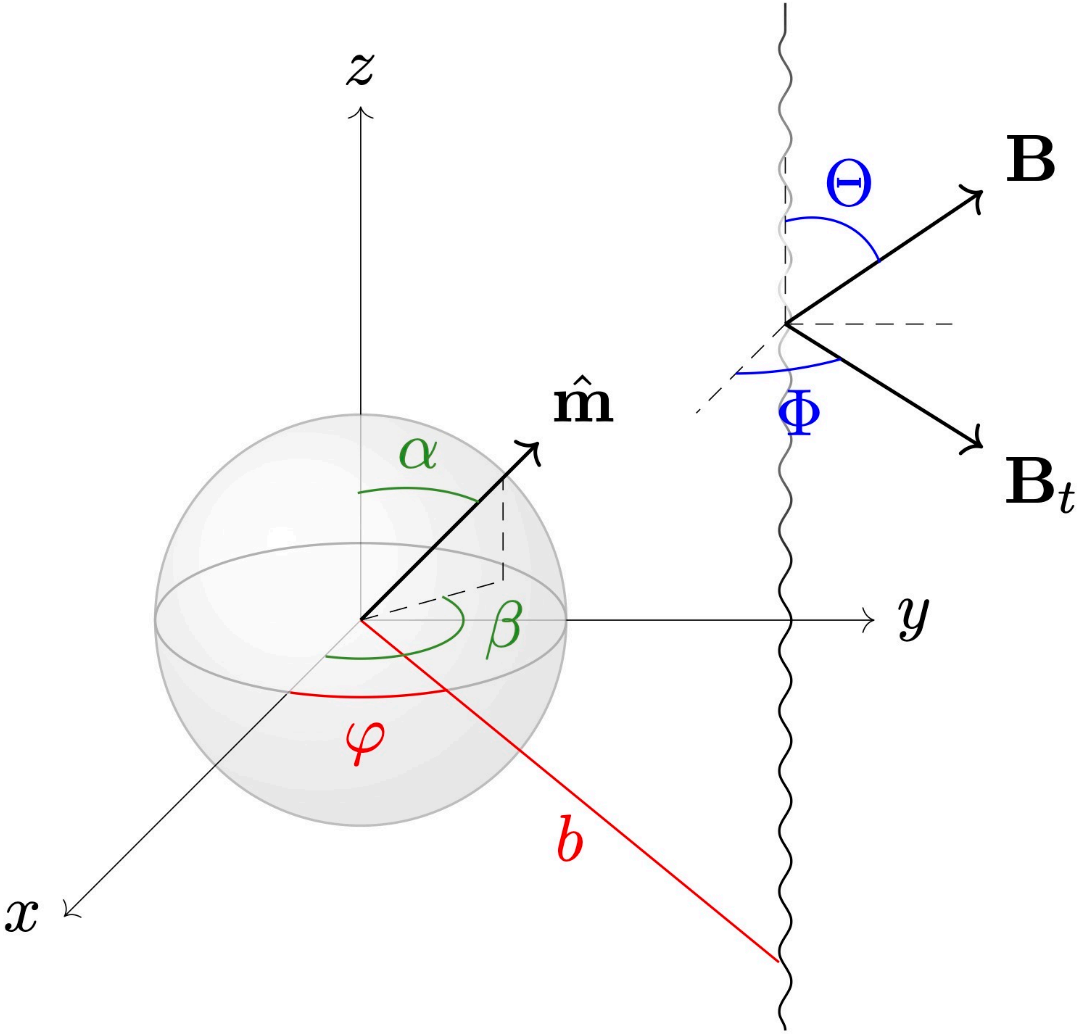
Conclusion

- Formalism developped by Raffelt & Stodolsky (1987)
- EOM of the GW−photon system
- Evolution of Stokes parameters
- Conversion probabilities and phase shifts computed and approximated in two geometries
- Upper and lower bounds on the SGWB in X-ray frequency range



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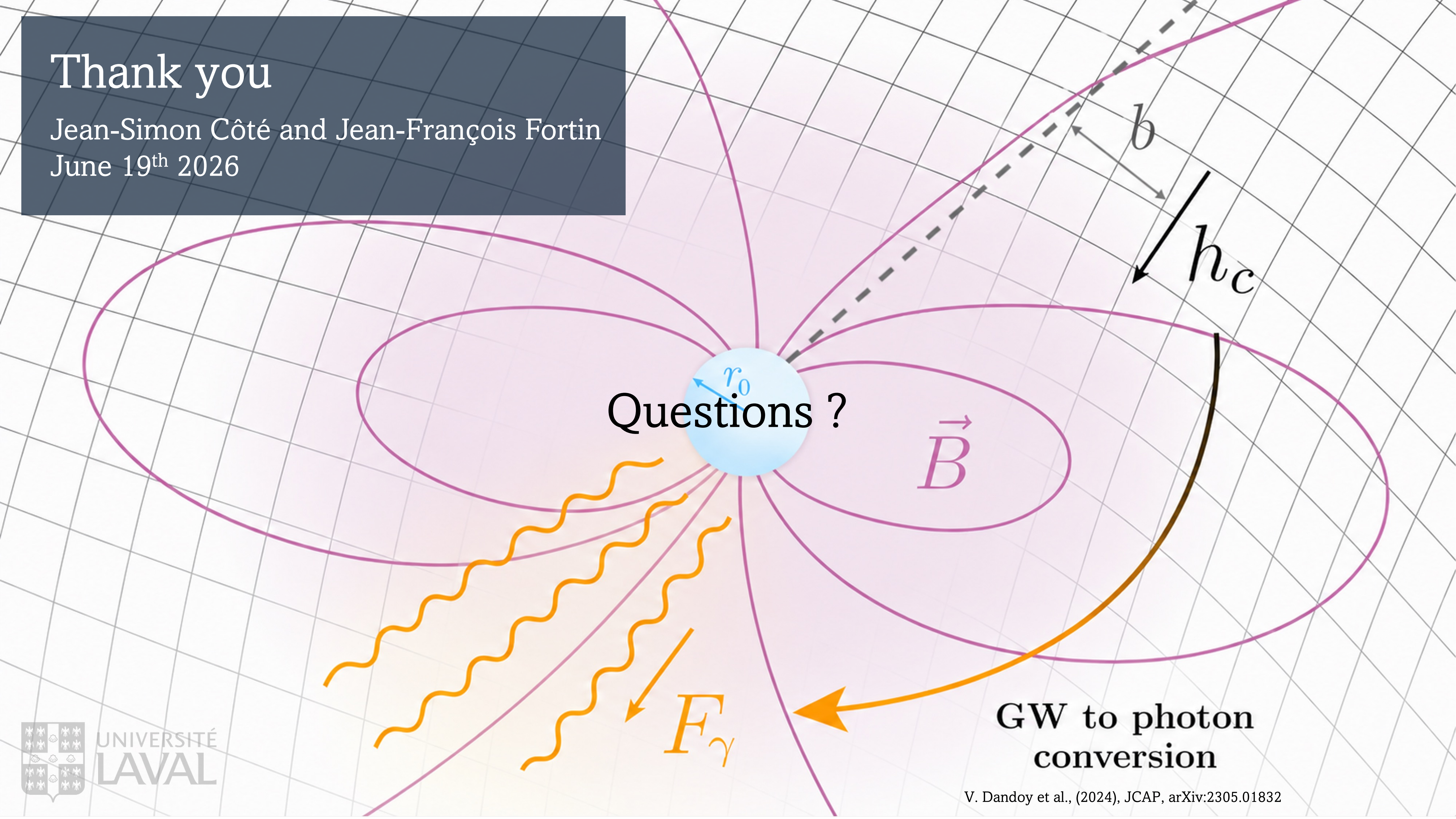
Thank you

Jean-Simon Côté and Jean-François Fortin
June 19th 2026



Questions ?

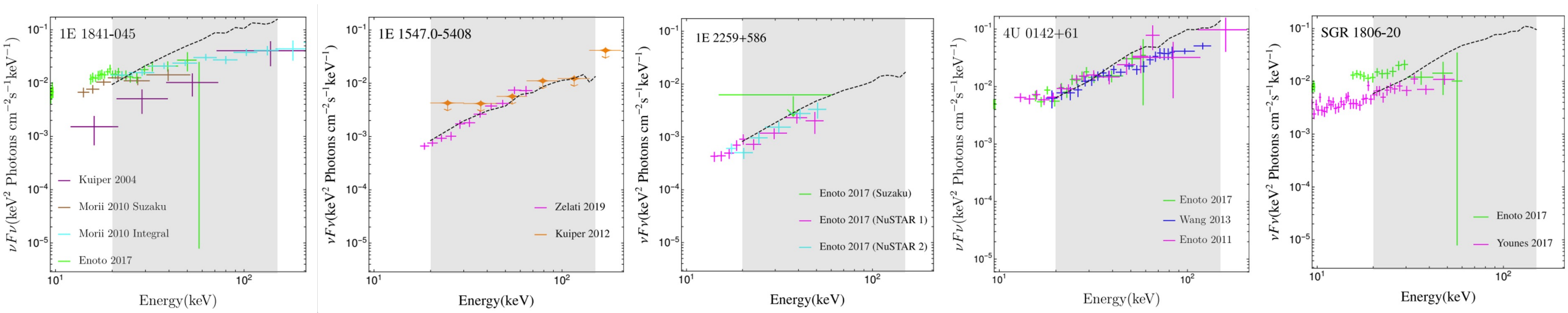
V. Dandoy et al., (2024), JCAP, arXiv:2305.01832

Appendix - Magnetars Parameters

Magnétars	B_0 $[10^{14}$ G$]$	d [kpc]
1E 1547.0-5408	3,2	4,5
1E 1841-045	7,0	8,5
1E 2259+586	0,59	3,2
4U 0142+61	1,3	3,6
SGR 1806-20	7,7	8,7

Table 1 — Magnetar parameters used in Figures 1 and 2. In all cases, the neutron-star radius is taken to be $r_0 = 10$ km. All parameters used are from the *McGill Online Magnetar Catalog* except for SGR 1806-20.

S. A. Olausen & V. M. Kaspi, (2014), ApJS, arXiv:1309.4167
G. Younes et al., (2017), ApJ, arXiv :1711.00034
J. L. Bibby et al. (2008), MNRAS, arXiv:0802.0815



Appendix - Stationary Phase Method for Degenerate Critical Points

$$\mathcal{J}_1(\vartheta_*) \ = \int_{\vartheta_*}^{\delta} d\vartheta' \, g(\vartheta') e^{i\lambda f(\vartheta')},$$

$$\mathcal{J}_2(\vartheta_*) \ = \int_{\vartheta_*}^{\delta} d\vartheta' \int_{\vartheta_*}^{\vartheta'} d\vartheta'' \, g(\vartheta') g(\vartheta'') e^{i\lambda[f(\vartheta')-f(\vartheta'')]}$$

Oscillation frequency scales as $\lambda \propto \Delta_{j0} r_0$

$\Delta_{j0} r_0 \sim 10^{12}$ for an X-ray close to a magnetar

Rapid oscillations ⇒ destructive interference

$$f(\vartheta') \simeq f_0 + f_1 |\vartheta' - \vartheta_*|^{\ell} + f_2 |\vartheta' - \vartheta_*|^{\ell+k} + \mathcal{O}\left[|\vartheta' - \vartheta_*|^{\ell+2k}\right]$$

$$g(\vartheta') \simeq g_1 |\vartheta' - \vartheta_*|^{n-1} + g_2 |\vartheta' - \vartheta_*|^{n+m-1} + \mathcal{O}[|\vartheta' - \vartheta_*|^{n+2m-1}]$$

$$\mathcal{J}_1(\vartheta_*) \underset{\lambda \to \infty}{\sim} \frac{g_1 e^{i f_0 \lambda + \frac{i\pi n}{2\ell}}}{\ell (f_1 \lambda)^{\frac{n}{\ell}}} \left[\Gamma\left(\frac{n}{\ell}\right) + \frac{g_2}{g_1} \frac{\Gamma\left(\frac{n+m}{\ell}\right) e^{\frac{i\pi m}{2\ell}}}{(f_1\lambda)^{\frac{m}{\ell}}} - \frac{f_2}{f_1} \frac{\Gamma\left(\frac{n+k}{\ell}+1\right) e^{\frac{i\pi k}{2\ell}}}{(f_1\lambda)^{\frac{k}{\ell}}} \right] + \mathcal{O}\left[\lambda^{-\frac{n+2\ \min\{\mathrm{k,m}\}}{\ell}}\right]$$

$$\mathcal{J}_2(\vartheta_*) \underset{\lambda \to \infty}{\sim} \frac{g_1^2 e^{\frac{in\pi}{\ell}}}{\ell^2 (f_1\lambda)^{\frac{2n}{\ell}}} \left[\frac{\Gamma\left(\frac{n}{\ell}\right)^2}{2\cos\left(\frac{n\pi}{\ell}\right)} + \frac{g_2}{g_1} \frac{\Gamma\left(\frac{n}{\ell}\right)\Gamma\left(\frac{n+m}{\ell}\right)\cos\left(\frac{m\pi}{2\ell}\right) e^{\frac{im\pi}{2\ell}}}{(f_1\lambda)^{\frac{m}{\ell}} \cos\left(\frac{n\pi}{\ell}+\frac{m\pi}{2\ell}\right)} - \frac{f_2}{f_1} \frac{\Gamma\left(\frac{n}{\ell}\right)\Gamma\left(\frac{n+k}{\ell}+1\right)\cos\left(\frac{k\pi}{2\ell}\right) e^{\frac{ik\pi}{2\ell}}}{(f_1\lambda)^{\frac{k}{\ell}} \cos\left(\frac{n\pi}{\ell}+\frac{k\pi}{2\ell}\right)} \right] + \mathcal{O}\left[\lambda^{-\frac{2(n+\min\{\mathrm{k,m}\})}{\ell}}\right]$$

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