Eigenstate thermalization hypothesis as a quantum resource for linear algebra problems

Thomas E. Baker
Department of Physics & Astronomy
Department of Chemistry
University of Victoria, Canada

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Linear algebra (Classical)

- Matrix **A**
  - Provided **b**
  - Find **x**
- Classical methods:
  - Invert **A** and apply (or LU with back-substitution)
  - $O(N^3)$ by LU decomposition

$$\mathbf{A}x = b$$

A. Dayton, et. al., arxiv: 2604.25625

Quantum methods

$$\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho \mathcal{O}) = \langle \Psi | \mathcal{O} | \Psi \rangle$$

HHL (2009)

with $\rho = |\Psi\rangle\langle\Psi|$ and $\hat{\mathcal{O}} = f(\hat{A})$

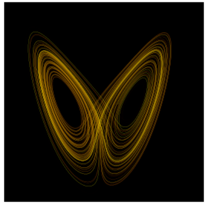
$$\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho \mathcal{O}) = \frac{1}{N} \sum_k f(E_k) \langle k | \rho | k \rangle$$

Dual algorithm

T.E. Baker, arxiv: 2504.19185

How to generate an equal superposition of eigenstates?

Chaos



Classical

- Non-linear equations
  - Small deviations in initial conditions
  - Big changes in outcomes
- Liouville's theorem

$$\langle O \rangle_t = \frac{1}{T}\int_0^T O(t,x)dt = \frac{1}{N}\sum_{i=1}^{N} O(x,t_i)$$

Quantum

- Not really well defined!
- Unitarity of time-dynamics

$$\mathcal{U}(t)\,|\,\psi(0)\rangle = |\,\psi(t)\rangle$$

$$\mathcal{U}^\dagger \mathcal{U} = \mathbb{I}$$

  - Reversible

Is there no chaos in quantum physics?

Random matrix theory

- Think of energy levels as random
  - Energy-time uncertainty

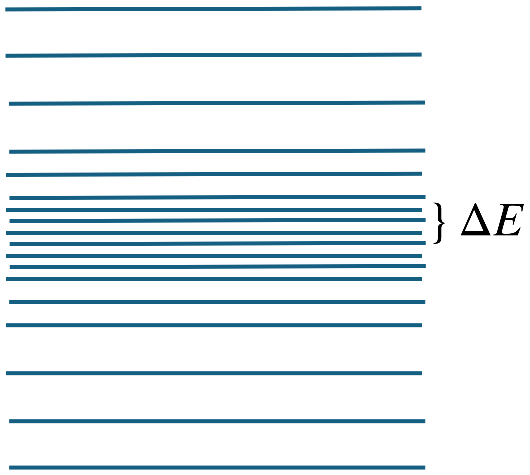
$$\Delta E \Delta t \geq \hbar/2$$

- Prediction:

$$\mathcal{O}_{mn} = \mathcal{O}(\bar{E})\delta_{mn} + e^{-S(\bar{E})} g(\bar{E},\omega) R_{mn}$$

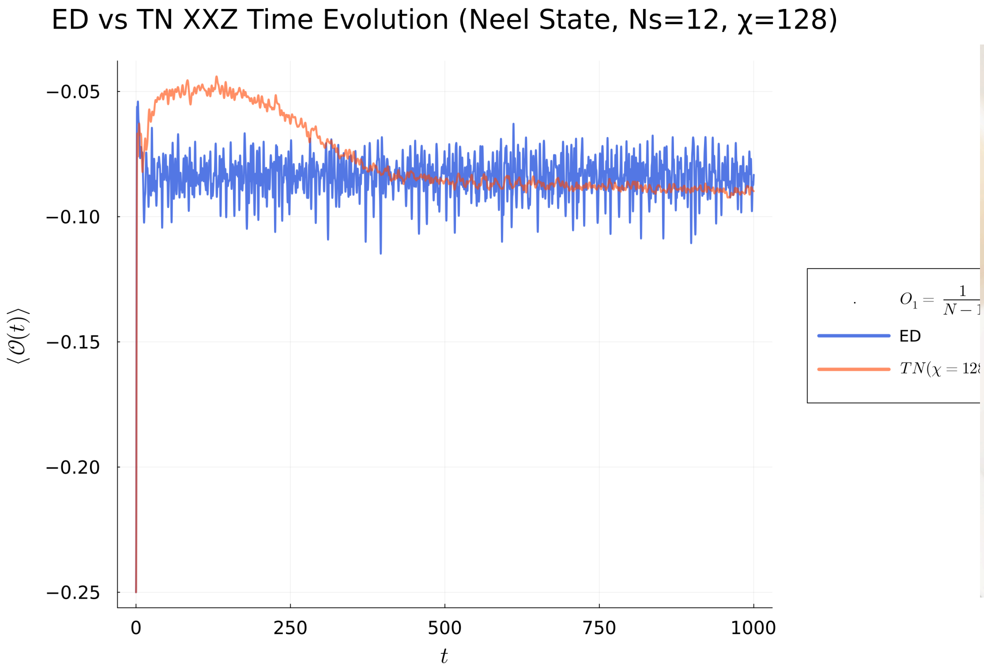
- Eigenstate thermalization hypothesis

$$\overline{\langle \Psi(t) | \hat{\mathcal{O}} | \Psi(t) \rangle} = \mathrm{Tr}(\Gamma \hat{\mathcal{O}})$$



T.E. Baker, arxiv: 2504.19185

Thermalization

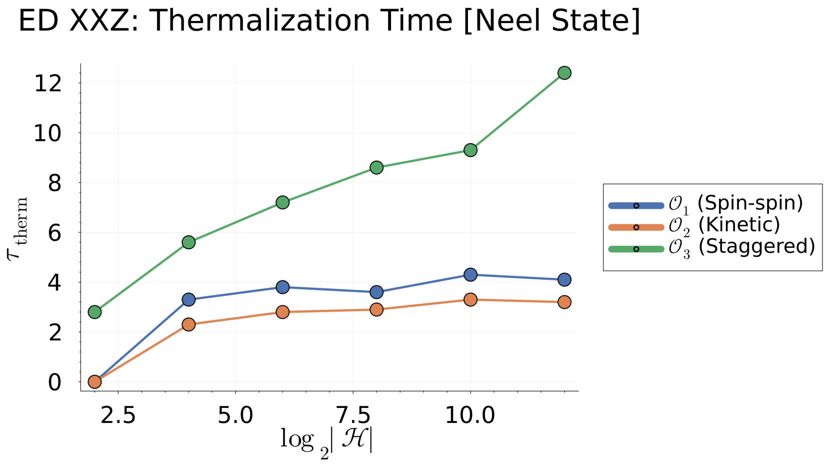


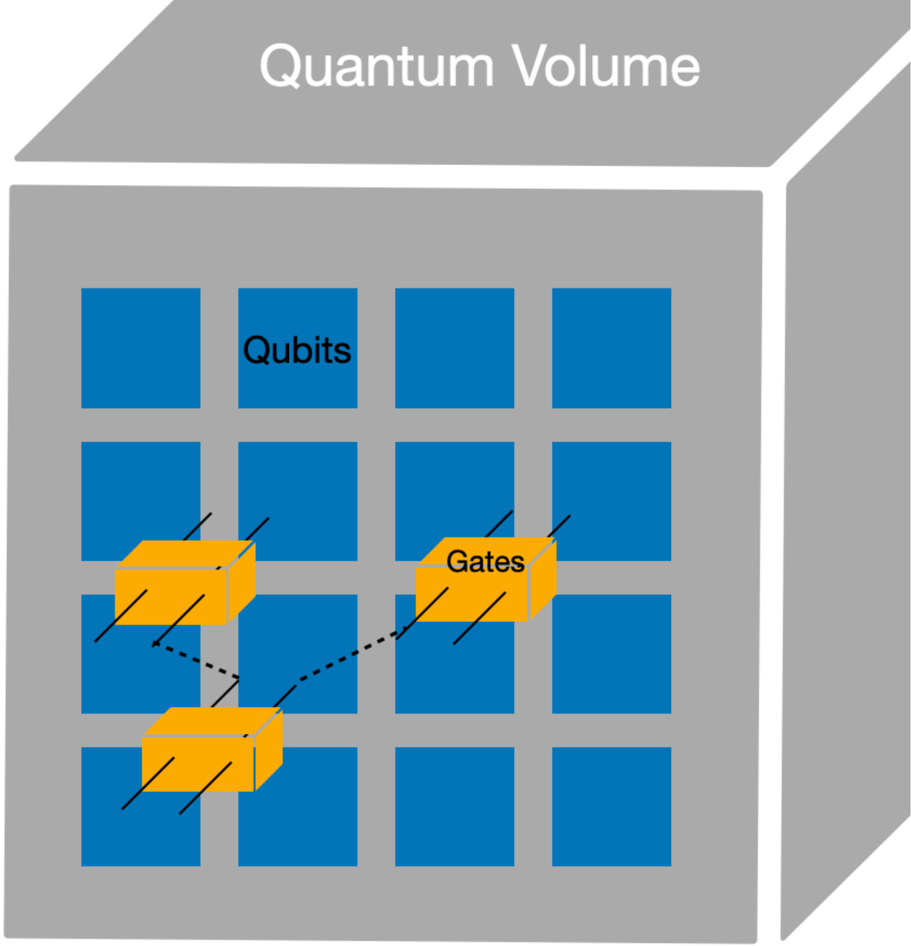
J. Morais, *et. al. (coming soon)*

Quantum computers

- Thermalization times
  - Increases like qubits
  - Exception: many-body localization

$$O\left(\frac{\tau}{\varepsilon}\log_2 N\right)$$





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ETH-summation

- Operator

$$\check{\mathcal{O}} = \hat{\Delta} \otimes \hat{\Upsilon} \qquad \left(e.g., \hat{\Delta} = \hat{\rho} \quad \text{and} \quad \hat{\Upsilon} = \text{diag}\left(E_1^{-1}, E_2^{-1}, \ldots E_M^{-1}\right)\right)$$

- Quantum phase estimation

$$\text{QPE}\,|\,k\rangle\,|\,0\rangle^{\otimes m} = |\,k\rangle\,|\,E_k\rangle$$

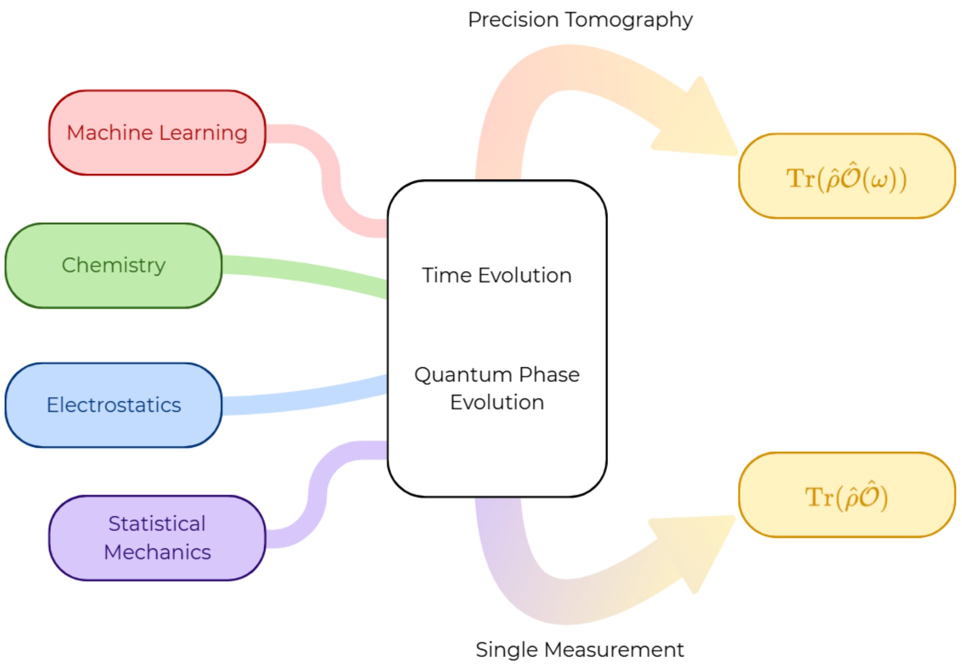
- Circuit

$$\langle\mathcal{O}\rangle = \overline{\langle\Psi(t)\,|\,\langle 0\,|^{\otimes m}(\text{QPE}^\dagger)\check{\mathcal{O}}(\text{QPE})\,|\,\Psi(t)\rangle\,|\,0\rangle^{\otimes m}} = \text{Tr}\left(\hat{\Gamma}\check{\mathcal{O}}\right)$$

*recovery of the microcanonical ensemble

T.E. Baker, arxiv: 2504.19185

Preview: thermalization



- Flexible
  - Extends to integrable systems
    - Generalized Gibbs ensemble
  - Solves all wavefunctions in one set of measurements
  - Generalisable

L.D. Adams, *et. al.*, *submitted*

Logarithm-determinants

Quantum field theory

$$Z = \int \mathcal{D}\varphi \; e^{i\mathcal{S}[\varphi]}$$

Statistical thermodynamics

$$E = -\frac{\partial}{\partial \beta} \ln Z$$

Correlation functions

$$\frac{\partial}{\partial J(x)} \left(\frac{\partial}{\partial J(y)} \ln Z \right) \bigg|_{J=0} = \langle \hat{c}^\dagger_{i\sigma}(x) \hat{c}_{j\sigma}(y) \rangle$$

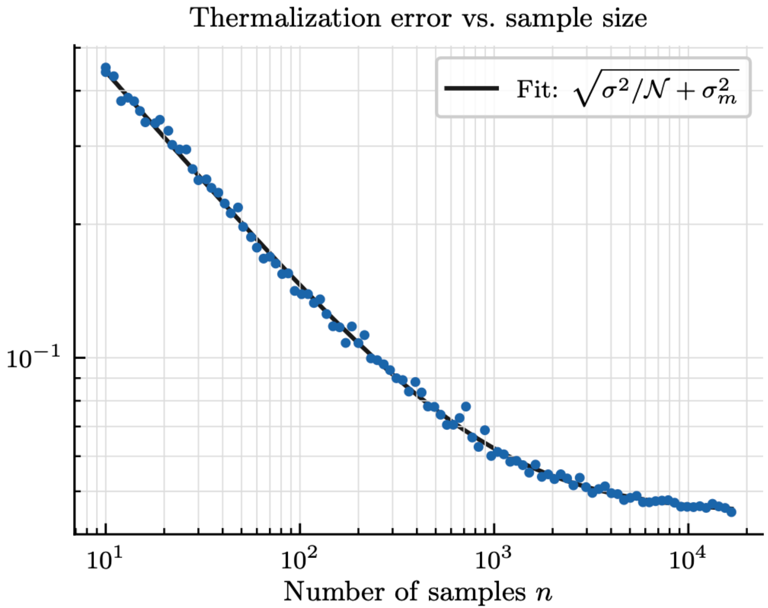
Matrix inverses

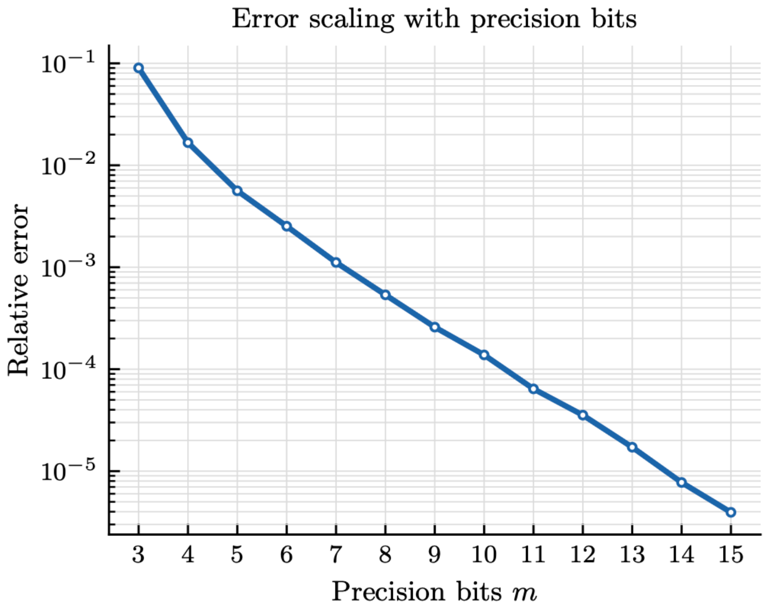
$$y_{ij} = \frac{\partial}{\partial x_{ij}} \ln \det(X)$$

T.E. Baker, J. Greasley, arxiv: 2501.09413

Gradients of logarithm-determinant

$$\sum_k \langle k \,|\, \hat{\Delta} \,|\, k \rangle = \nabla \ln \det(\hat{A})$$





L.D. Adams, *et. al.*, *submitted*

Conclusion

- ETH-summation
  - Thermalization
  - Scaling like BQP
  - Applies to a wide variety of problems
- Runs on real hardware
  - PINQ2
  - NVIDIA
- BC's quantum algorithm hub
  - 2026 Quantum Days

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