Relational Quantum Black Holes: Bounce and Singularity Resolution

Paolo Fragolino

University of Alberta
Supervisor: Prof. Saeed Rastgoo, Don Page
Theory Canada 18
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Research interests and key concepts

- i)**Problem of Time** in QG, ii) **Quantum Reference Frames** (QRF), iii) **Black Holes**.
- There is a connection between QRF and the Problem of Time! (see arXiv:**1809.00556v4** and arXiv:**1912.00033v3**).
- In general, **gauge** symmetries in quantum gravity *call* for a **relational** description of reality. $\hat{H}\,|\Psi\rangle_{\text{phys}} = 0$ expresses independence on redundant gauge DOFs (e.g. coordinates) and contains all the possible points of view at once $\rightarrow \mathcal{H}_{\text{phys}}$ is **Perspective-Neutral**.
- **Black Holes**: The interior ($r_s < 2m$) is *dynamical* with topology $\mathbb{R} \times S^2$. **Mini-superspace models** (Kantowski-Sachs metric): *fixed* $3D$ space coordinates, leaves freedom to change time coordinate. Lapse function N as lagrangian multiplier $\rightarrow$ gives rise to a constraint.

Quantum Clocks and Page-Wootters for Black Holes

- **Page-Wootters**: Total Hamiltonian constraint H is **separable** (System + Clock), i.e. $\hat{H} = \hat{H}_S + \hat{H}_C$.
- **Clock states**: From $\hat{H}_C |\epsilon\rangle = \epsilon |\epsilon\rangle$ we have
$$\boxed{|\tau\rangle = \int_{\sigma_C} d\epsilon\; e^{ig(\epsilon)} e^{-\frac{i}{\hbar}\epsilon\tau} |\epsilon\rangle}.$$
- **Quantum clock operator**: $\hat{T} = \mu \int_G dt\; t\, |t\rangle \langle t|$, such that $[\hat{T}, \hat{H}_C] = i\hbar$.
- **Black Holes**: $\boxed{ds^2 = N(\lambda)^2 \frac{a(\lambda)}{b(\lambda)} d\lambda^2 - \frac{b(\lambda)}{a(\lambda)} dx^2 - a(\lambda)^2 d\Omega^2}$.
- $L = N - \frac{\dot{a}\dot{b}}{N} \rightarrow \boxed{H = N' \left(\frac{\alpha^2}{p_a} + p_b \right)} \rightarrow$ **Separable!**
- In this way, we can define the b's DOF as our Clock, w.r.t. which the a's DOF evolves. **Singularity** is in $a = 0$.

Quantum Clocks and physical states

- Standard Schrodinger representation in $\mathscr{H}_{\text{kin}} = \mathscr{H}_a \otimes \mathscr{H}_b$:
  $\hat{a} = i\hbar\frac{\partial}{\partial p_a},\ \hat{b} = i\hbar\frac{\partial}{\partial p_b},\ \hat{p}_a = p_a\hat{\mathbb{I}},\ \hat{p}_b = p_b\hat{\mathbb{I}},\ [\hat{a},\hat{p}_a] = i\hbar\hat{\mathbb{I}} = \left[\hat{b},\hat{p}_b\right]$.
- **Physical states**: $\hat{C}_H\left|\Psi\right\rangle_{phys} = \left(\alpha^2\widehat{\frac{1}{p_a}} + \hat{p}_b\right)\left|\Psi\right\rangle_{phys} = 0.$
- $\widehat{\frac{1}{p_a}}$ is not defined in $p_a = 0$. From $\boxed{p_a \sim \dot{b}}$, $p_a = 0$ corresponds to a **frozen clock**, it is natural to exclude it.
- We use WFs $\phi^{(n)}_{\sigma,\tilde{\sigma}}(p_a = 0) = 0$ *faster* than any p_a^n:

$$\boxed{\phi^{(n)}_{\sigma,\tilde{\sigma}}(p_a) \sim p_a^n e^{-\frac{p_a^2}{2\sigma^2}} e^{-\frac{\tilde{\sigma}^2}{2p_a^2}}},\quad n \in \mathbb{Z},\ \sigma,\tilde{\sigma} \in \mathbb{R}\backslash\{0\}, \tag{1}$$

  They are **dense** in $L^2(\mathbb{R}, dp_a)$.
- After constraint: $\{\phi^{(n)}_{\sigma,\tilde{\sigma}}(p_a)\} \to \{\phi^{(n)}_{\sigma,\sigma'}(p_a)\}$, with $\sigma'^2 = \tilde{\sigma}^2 + \frac{\alpha^4}{\hbar^2}(\Delta b)^2$
- **BH clocks**: $\hat{H}_C = \hat{p}_b \to$ eigenstates: $\hat{H}_C\left|p_b\right\rangle_C = p_b\left|p_b\right\rangle_C$.

$$\boxed{\hat{T} = \hat{b} + \hbar\, h\left(\hat{H}_C\right)},\ h(p_b) := \frac{dg(p_b)}{dp_b}$$

Gauge-Invariant Observables

- **Relational Observables**: "value of O when $T=\tau$"

$$\boxed{\hat{F}_{O,T}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; e^{-it\hat{C}_H}\Big(\left|\tau\right\rangle\left\langle\tau\right| \otimes \hat{O}\Big)e^{it\hat{C}_H}}.$$

- It can be proven that $[\hat{F}_{O,T}(\tau), \hat{C}_H] = 0$. **Gauge-invariant extension** of gauge-dependent observables.
- The **matrix elements** are given by: $\langle \hat{F}_{A,T}(\tau)\rangle_{ph} \equiv \langle \Psi_{ph}|\, \hat{F}_{A,T}(\tau)\, |\Psi_{ph}\rangle_{ph} = \langle \Psi_{kin}|\, \hat{F}_{A,T}(\tau)\, |\Psi_{ph}\rangle_{kin}$.
- **Area**: $\hat{O} = \hat{A} = 4\pi\hat{a}^2$. Choosing $\hat{T} = \hat{b}$ we obtain:

$$\hat{F}_{A,b}(\tau) = 4\pi\left(\hat{a} + \alpha^2\left(\hat{b}-\tau\right)\left(\widehat{\frac{1}{p_a}}\right)^2\right)^2 \tag{2}$$

Bounce structure *already* encoded in the relational (gauge-invariant) evolution.

Area's matrix elements

- Matrix elements are given by:

$$\left\langle \hat{F}_{A,T}(\tau) \right\rangle_{\text{phys}} = \langle \Psi_a(\tau)|\hat{A}|\Psi_a(\tau)\rangle_{\text{kin}} = \\ = 4\pi \int_{\mathbb{R}} dp_a \left| \left(\frac{\partial \tilde{g}(p_a)}{\partial p_a} + \frac{\alpha^2}{p_a^2}\tau \right) \psi_{a|b}(p_a) - i\hbar \frac{\partial \psi_{a|b}(p_a)}{\partial p_a} \right|^2 \tag{3}$$

with $\tilde{g}(p_a) := -\hbar g\left(-\frac{\alpha^2}{p_a}\right)$.

- $\left\langle \hat{F}_{A,T}(\tau_0) \right\rangle_{\text{phys}} = 0$ for some τ_0 only if:

$$\psi_{a|b}(p_a;\tau_0) = Ce^{\frac{i}{\hbar}\left(\frac{\alpha^2\tau_0}{p_a} - \tilde{g}(p_a)\right)} \tag{4}$$

which is **non-normalizable** and therefore $\psi_{a|b}(p_a;\tau_0) \notin \mathscr{H}_{\text{phys}}$.

- $A_{min} \neq 0$ good **hint** for **singularity resolution**.

Expansion parameter

- Singularity $\leftrightarrow$ **Geodesic incompleteness** $\leftrightarrow \theta \to \infty$.
- Matrix elements for **radial null geodesics** $\theta = -\frac{2}{\alpha}\frac{p_b}{a}$:

$$\left\langle \hat{F}_{\theta,T}(\tau) \right\rangle_{\text{phys}} = \text{Re}\left\{ \frac{i\alpha}{\hbar} \int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a \, \psi^*_{a|b}(p'_a) \frac{1}{p_a} \psi_{a|b}(p_a) \times e^{\frac{i}{\hbar}(\tilde{g}(p_a)-\tilde{g}(p'_a))} e^{\frac{i}{\hbar}\left(-\frac{\alpha^2}{p_a}+\frac{\alpha^2}{p'_a}\right)\tau} \text{sgn}(p_a - p'_a) \right\} \quad (5)$$

- Using $|\text{Re} \int f| \leq |\int f| \leq \int |f|$:

$$\left| \left\langle \hat{F}_{\theta,T}(\tau) \right\rangle_{\text{phys}} \right| \leq \frac{\alpha}{\hbar} \int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a \left| \psi^*_{a|b}(p'_a) \right| \frac{1}{|p_a|} \left| \psi_{a|b}(p_a) \right| < \infty. \quad (6)$$

- **Finiteness** *already* ensured by **monotonic clock**.
- $\langle \hat{\theta}_{GI}(\tau) \rangle < \infty$, $\forall \tau \to$ (relational) geodesic completeness $\to$ **Singularity resolution**! For any clock (i.e. any $\tilde{g}$)!

Kretschmann scalar

- $K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ becomes:

$$K = \frac{4}{a^6}\left(a + b\frac{p_b^2}{\alpha^2}\right)^2 + \frac{8p_b^2}{a^6\alpha^4}\left(ap_a - bp_b\right)^2 . \tag{7}$$

- Matrix elements:

$$\begin{aligned}\left|\left\langle \hat{F}_{K,T}\left(\tau\right)\right\rangle_{\text{phys}}\right| \leq \frac{1}{\hbar^4}\int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a \left|\psi^*_{a|b}\left(p'_a\right)\right| \left|\psi_{a|b}\left(p_a\right)\right| \left|\left(p'_a - p_a\right)^3\right| \\ \times\left\{\frac{1}{3} + \frac{2}{3}\frac{p'^2_a}{p_a^2} + \frac{\alpha^4}{20}\frac{\left(p'_a - p_a\right)^2}{p_a^4}\left(\left(\frac{\partial g\left(p_b\right)}{\partial p_b} - \frac{\tau}{\hbar}\right)^2 + \left|\frac{\partial^2 g\left(p_b\right)}{\partial p_b^2}\right|\right)_{p_b = -\frac{\alpha^2}{p_a}}\right. \\ \left. + \frac{\alpha^2}{6}\frac{\left|p'_a - p_a\right|}{p_a^2}\left|\left(1 + 2\frac{p'_a}{p_a}\right)\right|\left|\left(\frac{\partial g\left(p_b\right)}{\partial p_b} - \frac{\tau}{\hbar}\right)_{p_b = -\frac{\alpha^2}{p_a}}\right|\right\}\end{aligned} \tag{8}$$

- Finite $\forall\tau \rightarrow$ no singularity!

Gaussian States

- Consider $\boxed{\psi^{(n)}_{a|b}(p_a) = N_G \Theta(-p_a)\, p_a^n e^{-\frac{p_a^2}{2\sigma^2}}}$ and $\hat{T} = \hat{b}$ as a clock.
- Peak at $\bar{p}_a = -\sigma\sqrt{n} < 0 \to$ **initial Black Hole** (from classical EOM).
- Not exactly Thiemanns WFs, but mimics behavior in $p_a = 0$, with suitable choice of $n > 0$.
- Allows state in $a = 0$!

$$\begin{aligned} \tilde{\psi}^{(n)}_{a|b}(a=0) &= \int_{\mathbb{R}} dp_a\, N_G \Theta(-p_a)\, p_a^n e^{-\frac{p_a^2}{2\sigma^2}} \\ &= N_G(-1)^n 2^{\frac{n-1}{2}} \sigma^{n+1} \Gamma\left(\frac{n+1}{2}\right) \neq 0 \end{aligned} \tag{9}$$

Area

- From Eq.(3), we find a **Minimum Area** in $\tau = 0$ (**bounce**!):

$$A_{\min} = 4\pi\hbar^2 \int_{\mathbb{R}} dp_a \left| \frac{\partial\psi^{(n)}_{a|b}(p_a)}{\partial p_a} \right|^2 = \boxed{4\pi\left(\Delta a\right)^2} \tag{10}$$

Quantum fluctuations in a **avoid the collapse**!

- Using $l_P = \sqrt{\hbar G},\ p_P = \frac{\hbar}{l_P} = \sqrt{\frac{\hbar}{G}}$ we have:

$$A_{\min} = 4\pi\hbar^2 \frac{1}{\sigma^2}\left(\frac{4n-1}{4n-2}\right) = \boxed{4\pi\ell_p^2\left(\frac{p_p}{\sigma}\right)^2\left(\frac{4n-1}{4n-2}\right)}, \tag{11}$$

where $A_P := 4\pi\ell_p^2$ **Planck's Area**!

Expansion Parameter

- Matrix elements:

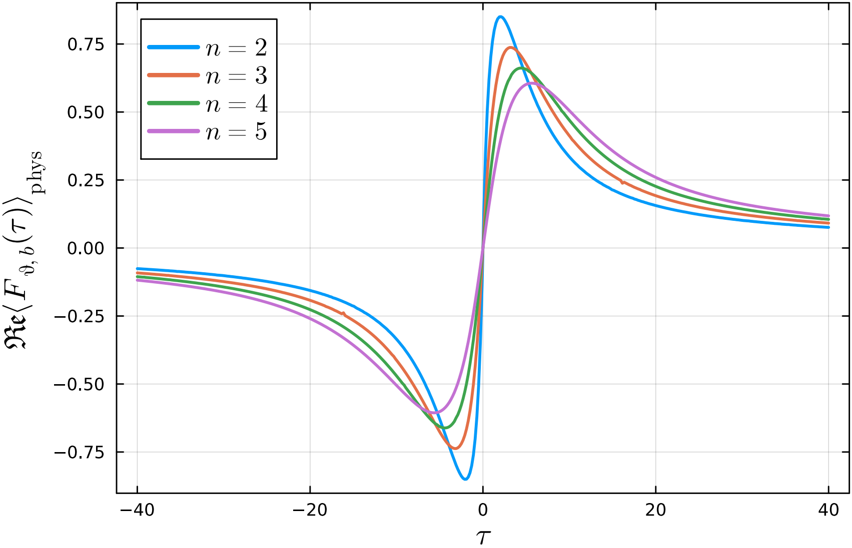
$$\langle \hat{F}_{\theta,b}(\tau)\rangle_{\text{phys}} = \frac{\alpha}{\hbar} N_G^2 \int_0^\infty dp'_a \int_0^\infty dp_a \, (-1)^{2n-1} \, p_a'^n p_a^{n-1} \\ \times e^{-\frac{1}{2\sigma^2}\left(p_a^2 + p_a'^2\right)} \left[\sin\left(\frac{\alpha^2\tau}{\hbar}\left(\frac{1}{p'_a} - \frac{1}{p_a}\right)\right)\right] \text{sgn}\left(p'_a - p_a\right) \tag{12}$$

- Properties:

$$\begin{cases} 1)\, \langle \hat{F}_{\theta,b}(\tau=0)\rangle_{\text{phys}} = 0 \\ 2)\, \frac{d}{d\tau} \langle \hat{F}_{\theta,b}(\tau)\rangle_{\text{phys}} |_{\tau=0} > 0 \\ 3)\, \lim_{\tau\to\pm\infty} \langle \hat{F}_{\theta,b}(\tau)\rangle_{\text{phys}} = 0 \end{cases} \tag{13}$$

1) gives **singularity resolution**, and 2) gives a **bounce**!

Expansion Parameter



Conclusions

- Fully Quantum and Relational description of a Black Hole **solves the singularity**.
- No loops, no polymer quantization, no "ad-hoc" modifications of the dynamics. Everything in standard continuous Schrodinger representation.
- Might suggests that fundamental mechanism for singularity resolution in QG could be **relationality**, *not* necessarily spacetime **discreteness**.
- Explicit **bridge** between the **QRF** and **LQG**/LQC communities.
- **Future applications**: FRLW **cosmology** (+ massless KG field ϕ); **BH information paradox** and Page curve.