Relational quantum dynamics of the black hole interior: singularity resolution and quantum bounce

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Abstract: We study the interior of the Schwarzschild black hole which is isometric to the Kantowski-Sachs cosmological model, using a fully relational and gauge-invariant quantization framework. The physical Hilbert space is constructed via refined algebraic quantization, and quantum dynamics is recovered through the Page-Wootters formalism with a covariant POVM clock built from one of the two configuration variables, whose Hamiltonian is proportional to the momentum of the said variable. Gauge-invariant relational observables for the area of 2-spheres, the Kretschmann scalar, and the expansion scalar of null geodesic are constructed via group averaging (G-twirl) and evaluated on physical states. We find that the Kretschmann and expansion scalars remain finite throughout the black hole, while the area of 2-spheres is bounded below by a minimum value proportional to the uncertainty in the system variable, which is the other configuration variable distinct from the clock variable. In particular, the expansion scalar vanishes and changes sign at the quantum bounce, establishing a black-hole-to-white-hole transition. These results hold for any general clock whose operator forms a canonical pair with the clock Hamiltonian, and require no specific quantization scheme other than the Schrodinger representation. The singularity resolution emerges directly from relationality, the Heisenberg uncertainty principle, and the structure of the physical Hilbert space.

Contents

1 Introduction

The problem of time in quantum gravity is among the most profound conceptual obstacles in theoretical physics [1–5]. In background-independent theories —where spacetime geometry is itself a dynamical variable— there is no preferred external time parameter against which evolution can be defined [6, 7]. The total Hamiltonian of the system is a sum of first-class constraints, so physical states are frozen: they are annihilated by the constraint operators and appear static with respect to any coordinate time. This is the notorious frozen formalism problem, and it renders the notion of dynamics in quantum gravity apparently meaningless.

A resolution to this problem is offered by the relational approach to quantum dynamics, in which time is not an external parameter but an internal variable: one subsystem of the universe is designated as a clock, and the evolution of the remaining degrees of freedom is described relative to it. This idea, first articulated in the context of quantum cosmology by Page and Wootters [8], has been substantially developed in recent years into a mathematically precise framework [9–11] which establishes the equivalence between the Page-Wootters conditional state approach, the Heisenberg picture of relational observables, and the Dirac quantization program. Central to this framework is the use of covariant positive operator-valued measures (POVMs) rather than self-adjoint time operators [9], which circumvents the obstruction of Pauli's theorem and allows physical clocks with semi-bounded Hamiltonians.

In parallel, the fate of classical spacetime singularities in quantum gravity remains a central open question. Classical general relativity predicts that gravitational collapse leads inevitably to singularities, where curvature invariants diverge and geodesic completeness fails —a conclusion codified in the Penrose-Hawking singularity theorems [12, 13]. Whether quantum effects resolve these singularities is a question that various approaches to quantum gravity, including loop quantum gravity, have addressed with encouraging results [14–16]. However, many existing treatments rely on specific quantization ambiguities or ad hoc modifications of the dynamics [17–20], rather than emerging from a fully relational and gauge-invariant framework.

This paper addresses both problems simultaneously. We study the interior of a Schwarzschild black hole [21–23], which is isometric to the Kantowski-Sachs cosmological model and admits a homogeneous minisuperspace description, using the relational quantization framework of [9] combined with refined algebraic quantization (RAQ) [24]. The RAQ program provides a mathematically rigorous implementation of Dirac's quantization procedure for totally constrained systems, constructing the physical Hilbert space as the space of distributional solutions to the quantum constraints via a group averaging (rigging map) procedure.

The Kantowski-Sachs Hamiltonian constraint [21], after a phase space-dependent lapse redefinition, takes a separable form $C_H = p_b + \alpha^2/p_a$, which admits a natural deparametrization: the canonical variable b serves as a relational clock with Hamiltonian $H_C = p_b$, and the areal radius variable a, which controls the area of the 2-spheres in the interior, plays the role of the system. Since the spectrum of $\hat{H}_C = \hat{p}_b$ is all of $\mathbb{R}$, the b-clock is an ideal clock in the sense of [9], admitting a covariant POVM with Dirac-orthonormal clock states and a self-adjoint time operator canonically conjugate to $\hat{H}_C$.

The central results of this paper are threefold. First, we construct the physical Hilbert space $\mathscr{H}_{\text{phys}} \cong L^2(\mathbb{R}, dp_a)$ via the RAQ rigging map [24], identify the physical states as distributions supported on the constraint surface, and show that the clock back-reaction is encoded in the uncertainty of the physical states. Second, we construct gauge-invariant relational observables for three physically motivated quantities —the area of 2-spheres, the Kretschmann curvature scalar, and the expansion scalar of null geodesic congruences— by applying the G-twirl (group averaging in the adjoint representation [25–28]) to kinematical observables conditioned on the clock reading. Third, we evaluate the expectation values of these observables on physical states, particularly, Gaussian states and demonstrate in each case that the classical singularity is resolved: the area is strictly positive and bounded below by a Planck-area quantum, and the expansion scalar remain finite throughout the spacetime, and vanishes and changes sign at the bounce. This signals a transition from a trapped non-singular black hole interior to an anti-trapped non-singular white hole geometry.

The paper is organized as follows. In Sec. 2 we review the relational evolution framework, the construction of quantum clocks from covariant POVMs, and the Page-Wootters conditional state formalism. Section 3 presents the classical Kantowski-Sachs dynamics and its Hamiltonian reduction. In Sec. 4 we construct the kinematical and physical Hilbert spaces via RAQ. In Sec. 5 we derive the gauge-invariant relational observables for the area of 2-spheres, Kretschmann scalar, and expansion scalar, where we establish the finiteness of the last two and the nonvanishing of the first one on physical states. In Sec. 6 we apply these results to explicit Gaussian states and presents numerical and analytical evidence for singularity resolution and a black-hole-to-white-hole transition. Finally, we conclude and summarize our results in Sec. 7.

2 Relational Evolution, POVMs, and Quantum Clocks

2.1 Relational evolution and physical clocks

In totally constrained systems, which includes all background-independent gravitational or gravitation plus matter systems, the total Hamiltonian of the system is just the sum of first class constraints, which in turn generate gauge transformations. This makes the entire evolution just gauge transformations. Thus, (Dirac) observables and physical states are static under this evolution with respect to any time coordinate parameter, which is the parameter of the gauge group that generates such evolution. Hence, the observed dynamics of physical systems must be recovered relationally: one subsystem must be chosen as a "clock" denoted by C to track the evolution of the remaining system denoted by S. For this method to work out, a particularly convenient realization arises when the Hamiltonian can be cast in a separable or deparametrized form $H = H_S + H_C$ (with the same separable form in the quantum theory). This separability is non-generic and typically requires a suitable choice of clock variable that leads to a deparametrized description of the system, where one of the canonical variables (the clock) vanishes from the Hamiltonian. This is achievable in homogeneous minisuperspace models. One example of such a model is the interior of a Schwarzschild black hole, which is isometric to the Kantowski-Sachs cosmological model, which is homogeneous (but not isotropic). Hence, the method just described can be applied to study the dynamics of the interior of a Schwarzschild black hole, and in particular the fate of its singularity, in the relational approach. We briefly review the classical interior dynamics in Sec. 3 and show that its Hamiltonian is indeed separable even in the absence of matter fields. In what follows we keep the discussion general, and in later chapter, we specialize it to the Schwarzschild interior.

The idea behind this relational evolution is that, in a frozen timeless universe, implied by what discussed in the previous paragraph, if the global state or density operator ρ is heavily entangled, a subsystem (the clock), described by ρ_C, can be highly correlated with another subsystem (the rest of the universe; i.e., the system), described by ρ_S. Then, based on Bayes' theorem, if an observer Alice inside the universe measures the clock and finds it reads t_1, because of entanglement, Alice can update her knowledge of the rest of the universe using quantum conditional probability yielding $\rho_S(t_1)$. Later, she looks at the clock and sees t_2, and the conditional probability yields a state $\rho_S(t_2)$ for the system. By continuously conditioning the static, global state on different readings of the internal clock, the observer pieces together a sequence of conditional states. Relational dynamics crucially relies on entanglement between clock and system; in product states, no nontrivial evolution can be reconstructed.

In classical statistics, given two random variables, S (System) and C (Clock), their joint probability distribution is $P(S, C)$. If one wants to know the state of the system given that the clock reads a specific time t, one uses the formula for conditional probability

$$P\left(S \mid C=t\right)=\frac{P(S, C=t)}{P(C=t)}. \tag{2.1}$$

Here, $P(S, C=t)$ is the joint probability of S given that C has value t, and $P(C=t)$ is the probability that the clock reads t. Quantum mechanically, the joint probability $P(S, C)$ is the global entangled density operator ρ. This contains all the correlations (entanglement) between the system and the clock. The joint probability of S and C given the clock reads t, or is in the state $|t\rangle$, is

$$P(S, C=t)=\operatorname{tr}_C\left[\left(\hat{\mathbb{I}}_S \otimes P_C(t)\right) \rho\right]. \tag{2.2}$$

The expression $\left(\hat{\mathbb{I}}_S \otimes P_C(t)\right) \rho$ is the projection of the global state ρ into the clock states with reading using t, where $P_C(t)=|t\rangle\langle t|$ is the clock projection operator, which is also called a projection-valued measure (PVM). We will see in the next section, however, that one needs to use positive operator-valued measures (POVM) instead for P_C due to mathematical reasons. The trace gives us the probability distribution solely over S given the conditioning on the clock reading $C=t$. The denominator in (2.1) quantum mechanically becomes

$$P(C=t)=\operatorname{tr}\left[\left(\hat{\mathbb{I}}_S \otimes P_C(t)\right) \rho\right], \tag{2.3}$$

which is the probability that the clock will be found in state $|t\rangle$.

2.2 POVM and quantum clocks

In order to construct the probability formula mentioned in the previous subsection, we need the clock states, and hence a clock operator $\hat{T}$. To this end, one might attempt to define a self-adjoint time operator $\hat{T}=\hat{T}^{\dagger}$ that is canonically conjugate to the clock's Hamiltonian $\hat{H}_C$, such that $\left[\hat{T}, \hat{H}_C\right]=i\hbar\hat{\mathbb{I}}_C$. However, Pauli's theorem dictates that if a self-adjoint time operator $\hat{T}$ and a Hamiltonian $\hat{H}_C$ satisfy such a global canonical commutation relation, then both operators must have an absolutely continuous spectrum equal to $\mathbb{R}$. Because any realistic physical clock must have a Hamiltonian bounded from below to ensure a stable ground state, a self-adjoint $\hat{T}$ strictly cannot be used. Hence, for a realistic physical clock where the Hamiltonian is bounded from below with a spectrum $\sigma_c=[\epsilon_{\min}, \infty)$, one must restrict the canonical commutation relation to such domain [29]. In quantum measurement, a self-adjoint operator corresponds to a PVM, which relies on a basis of mutually orthogonal states. Hence, abandoning a self-adjoint $\hat{T}$, means abandoning PVMs and their orthogonal states. One instead employs POVMs [9, 30, 31]. These are bounded positive semidefinite operators $\hat{E}$ that generalize standard quantum observables by relaxing the requirement of orthogonality, allowing for the description of unsharp or realistic measurements.

More concretely, let $\mathscr{H}_C$ be the clock Hilbert space, and let $\hat{H}_C$ be the clock Hamiltonian generating a 1-parameter unitary group of time translations, $\hat{U}_C(t) = \exp\left(-\frac{i}{\hbar} t \hat{H}_C\right)$. A quantum clock is formally defined by a POVM map $\hat{E}_T : \mathcal{B}(G) \rightarrow \mathcal{B}^+\left(\mathscr{H}_C\right)$ from the Borel subsets X of the parameter group (typically the real line, $G = \mathbb{R}$) into the space of positive semi-definite operators on $\mathscr{H}_C$. Note that this is not the clock operator. The clock operator is made of the first moment of the POVM as we will see below. To function as a valid clock, the POVM must satisfy completeness to ensure probabilities conserve to unity

$$\int_{\mathbb{R}} \hat{E}_T(dt) = \hat{\mathbb{I}}_C, \tag{2.4}$$

where dt represents the invariant Haar measure of group G. The above also implies that the time states $|t\rangle$ generating the POVM $\hat{E}_T$ span the clock Hilbert space $\mathscr{H}_C$. On the other hand, to ensure that the clock ticks uniformly with respect to the dynamics generated by $\hat{H}_C$, the POVM must satisfy the covariance condition

$$\hat{U}_C(t)\hat{E}_T(X)\hat{U}_C^{\dagger}(t) = \hat{E}_T(X+t), \quad \text{where } t + X := \{t + x \mid x \in X\}. \tag{2.5}$$

This condition guarantees that translating the clock's state forward in time is physically equivalent to shifting the measurement apparatus reading by the exact same amount. The crucial difference between POVMs and PVMs is that the former are not necessarily orthogonal. Writing

$$\hat{E}_T(dt) = \hat{e}(t)dt, \tag{2.6}$$

this means that

$$\hat{e}(t)\hat{e}\left(t'\right) \neq \delta\left(t - t'\right)\hat{e}(t). \tag{2.7}$$

As a result the time states $|t\rangle$ associated to $\hat{E}_T$ form an overcomplete resolution of identity on $\mathscr{H}_C$ and are not linearly independent, so they do not comprise a basis for $\mathscr{H}_C$. This can be seen if we write Eq. (2.6) more explicitly in terms of clock states $|t\rangle$ as

$$\hat{E}_T(dt) = \hat{e}(t)dt = \mu|t\rangle\langle t|\, dt. \tag{2.8}$$

Replacing the above in (2.7) yields

$$\hat{e}(t)\hat{e}\left(t'\right) = \mu^2|t\rangle\left\langle t \mid t'\right\rangle\left\langle t'\right| \Rightarrow \left\langle t \mid t'\right\rangle \neq \delta\left(t - t'\right). \tag{2.9}$$

In fact, this can be shown explicitly if we expand the clock states in terms of energy eigenstate $|\epsilon\rangle$. Writing $\hat{H}_C = \int_{\sigma_C} d\epsilon\, \epsilon|\epsilon\rangle\langle\epsilon|$ and by restricting the energy spectrum to $\sigma_c = [\epsilon_{\min}, \infty)$, the covariance condition (2.5) implies that the time states can be written in the form

$$|t\rangle = \int_{\epsilon_{\min}}^{\infty} d\epsilon\, e^{ig(\epsilon)} e^{-\frac{i}{\hbar}\epsilon t} |\epsilon\rangle\,, \tag{2.10}$$

where $e^{ig(\epsilon)}$ encodes the phase freedom of the clock states which parametrizes inequivalent choices of clock. Then computing $\langle t'|t\rangle$ and computing the integral by substitution $\omega = \epsilon - \epsilon_{\min}$, yields

$$\langle t' \mid t\rangle = e^{\frac{i}{\hbar}\epsilon_{\min}(t-t')} \int_0^{\infty} d\omega e^{\frac{i}{\hbar}\omega(t-t')} = e^{\frac{i}{\hbar}\epsilon_{\min}(t-t')}\hbar\left(\pi\delta\left(t-t'\right) + i\mathrm{PV}\left(\frac{1}{t-t'}\right)\right), \tag{2.11}$$

where $\mathrm{PV}\left(\frac{1}{t-t'}\right)$ denotes the Principal Value of the function $\frac{1}{t-t'}$. This explicitly demonstrates (2.9). In fact for the three cases of unbounded, semi-bounded, and bounded cases we have [9]

$$\langle t|t'\rangle \coloneqq \chi\left(t-t'\right) = \begin{cases} 2\pi\hbar\delta\left(t-t'\right), & \sigma_c = \mathbb{R} \\ e^{\frac{i}{\hbar}\epsilon_{\min}(t-t')}\hbar\left(\pi\delta\left(t-t'\right) + i\mathrm{PV}\left(\frac{1}{t-t'}\right)\right) & \sigma_c = \left(\epsilon_{\min}, +\infty\right) \\ \frac{i\hbar}{t-t'}\left(e^{\frac{i}{\hbar}\epsilon_{\min}(t-t')} - e^{\frac{i}{\hbar}\epsilon_{\max}(t-t')}\right) & \sigma_c = \left(\epsilon_{\min}, \epsilon_{\max}\right) \end{cases}. \tag{2.12}$$

In the special case of a continuous spectrum, $\sigma_c = \mathbb{R}$ above, the states form a Dirac-orthonormal family of generalized states. This would correspond to an "ideal clock" [9]. Combining this property with (2.8) reveals that the POVM elements $\hat{E}_T(dt)$ overlap, which captures the intrinsic quantum uncertainty and back-reaction of realistic clocks. In other words, because $\langle t \mid t'\rangle \neq 0$ when $t \neq t'$, the states "leak" into each other and we cannot sharply define a state perfectly localized in time if we restrict the available frequencies (energies) to only positive values. Hence, this non-orthogonality encodes an intrinsic time-energy uncertainty and limits the sharpness of temporal localization. Another crucial property of POVMs is their positivity, meaning that their eigenvalues are positive. In fact, it can be shown that the eigenvalues of discrete POVMs can be between 0 and 1, as opposed to the eigenvalues of PVMs that are strictly 0 or 1.

Given the nth moment of $\hat{E}_T(dt)$ defined as

$$\hat{T}^{(n)} = \int_G t^n \hat{E}_T(dt) = \mu \int_{\mathbb{R}} t^n\, |t\rangle\langle t| dt, \tag{2.13}$$

we can finally express the physical time (or clock) operator associated to $\hat{E}_T(dt)$ as its first moment

$$\hat{T} = \hat{T}^{(1)} = \int_{\mathbb{R}} t\hat{E}_T(dt) = \mu \int_{\mathbb{R}} t\, |t\rangle\langle t| dt. \tag{2.14}$$

The reason behind this is that the POVM $\hat{E}_T(dt)$ is an operator-valued probability measure. Hence, the integral above yields the expectation value $\left\langle \hat{T}^{(1)} \right\rangle$ which is equal to the statistical mean of the POVM measurement outcomes.

Applying the covariance condition (2.5) to the clock operator (2.14) we obtain

$$\hat{U}_C(s)\hat{T}\hat{U}_C^\dagger(s) = \hat{T} - s\hat{\mathbb{I}}_C. \tag{2.15}$$

The canonical commutation relation can be obtained by differentiating Eq. (2.15) with respect to s and evaluating at $s = 0$, using $\frac{d}{ds}\left[\hat{U}_C(s)\hat{A}\hat{U}_C^\dagger(s)\right]\Big|_{s=0} = -\frac{i}{\hbar}\left[\hat{H}_C, \hat{A}\right]$, which yields

$$\left[\hat{T}, \hat{H}_C\right] = i\hbar\mathbb{I}_C. \tag{2.16}$$

While the above operator $\hat{T}$ is symmetric, it lacks self-adjoint extensions. Consequently, the canonical commutation relation (2.16) holds only weakly on a restricted domain of states. In other words, since we must restrict the domain of $\hat{T}$ on a dense subspace of states $\mathcal{D} \subset \mathscr{H}_C$ whose wavefunctions vanish at the boundary $\epsilon_{\min}$, the above commutation relation is only valid for $\psi \in \mathcal{D}$. This resolves the tension highlighted by Pauli's theorem by abandoning the requirement of a self-adjoint time operator and instead using a covariant POVM.

For the special case of an ideal clock, the time operator is Hermitian, $\hat{T} = \hat{T}^\dagger$, and it forms a canonically conjugate pair with the clock's Hamiltonian $\hat{H}_C$. Moreover the clock states are eigenvectors of $\hat{T}$

$$\hat{T}\left|t\right\rangle = t\left|t\right\rangle, \qquad \text{for } \sigma_c = \mathbb{R}. \tag{2.17}$$

The normalization condition in (2.12) then yields

$$\mu = \frac{1}{2\pi\hbar} \tag{2.18}$$

for $\sigma_c = \mathbb{R}$.

Finally, having established the POVM for our clock, we can explicitly define the relational evolution of the system in line with what discussed in subsection 2.1. Given a global physical state $\rho = \left|\Psi\right\rangle\left\langle\Psi\right|$ with $\left|\Psi\right\rangle$ belonging to the physical Hilbert space $\mathscr{H}_S^{\text{phys}} \otimes \mathscr{H}_C^{\text{phys}}$ and satisfying $\hat{H}\left|\Psi\right\rangle = 0$, relational dynamics is obtained by conditioning this global state on the clock reading t. Using the Page-Wootters formalism [8, 9], this yields the conditional system state as

$$\rho_S(t) = \frac{\mathrm{Tr}_C\left[\left(\mathbb{I}_S \otimes \hat{E}_T(dt)\right)\rho\right]}{\mathrm{Tr}\left[\left(\mathbb{I}_S \otimes \hat{E}_T(dt)\right)\rho\right]} = \frac{\mathrm{Tr}_C\left[\left(\hat{\mathbb{I}}_S \otimes \left|t\right\rangle\!\left\langle t\right|\right)\rho\right]}{\mathrm{Tr}\left[\left(\hat{\mathbb{I}}_S \otimes \left|t\right\rangle\!\left\langle t\right|\right)\rho\right]}. \tag{2.19}$$

This defines evolution of the system relative to the clock. By enforcing the total Hamiltonian constraint $\hat{H}\left|\Psi\right\rangle = 0$, one can show that this conditional state dynamically satisfies the Schrodinger equation governed by the system Hamiltonian $\hat{H}_S$ [8, 9], thereby recovering standard unitary time evolution from a fundamentally timeless global state.

3 Black Hole Interior

As discussed before, our goal is to apply the machinery of relational evolution to the interior of the Schwarzschild black hole. Here we present the necessary ideas, and in particular show that the Hamiltonian of the model is separable and deparametrized. Hence, the relational evolution framework can be applied to this system in a rigorous way.

The general form for the interior of a static spherically symmetric black hole can be written as [21]

$$ds^2 = -g_{\lambda\lambda}(\lambda)d\lambda^2 + g_{xx}(\lambda)dx^2 + g_{\theta\theta}(\lambda)d\Omega^2 \tag{3.1}$$

where x and λ are spacelike and timelike coordinate inside the black hole, respectively. The metric in the standard Schwarzschild coordinates t, r can be derived by the transformation $t = x$ and $r = \lambda^2$. The geometry in the interior of such a black hole is dynamical and the topology of a $3D$ spatial foliation Σ_0 with fixed time-like coordinate $\lambda = \lambda_0$ is $S^2 \times \mathbb{R}$, where $\mathbb{R}$ is associated to the spacelike coordinate x and S^2 to the angular part of the metric. Hence, the interior can be pictured as a collection of cylinders having an increasing proper length and a decreasing surface area as the time-like coordinate λ runs forward [23]. Notice that such behavior holds before the singularity $\lambda = 0$ is reached [1]. We will also consider the spacelike coordinate x to have a finite range $x \in [x_{\min}, x_{\max}]$, i.e. it runs along an arbitrary finite portion of the cylinder's axis.

Plugging this metric inside the Einstein's field equation yields

$$g_{\lambda\lambda}(\lambda) = \frac{4\lambda^4}{2GM - \lambda^2}, \tag{3.2}$$

$$g_{xx}(\lambda) = \frac{2GM - \lambda^2}{\lambda^2}, \tag{3.3}$$

$$g_{\theta\theta}(\lambda) = \lambda^4, \tag{3.4}$$

where M is a constant of integration. Because of the singularity at $\lambda = 0$, λ ranges from $-\sqrt{2GM}$ to 0, which corresponds to the interior of the black hole, i.e., region II

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[1] Note that the singularity inside such a black hole is spacelike, i.e., a spatial hypersurface, not a point or local region in space.

of the Kruskal extension, but cannot be extended beyond 0. The other possible range will be $\lambda \in (0, \sqrt{2GM}]$, but because the metric components in the coordinates used here are singular at $\lambda = 0$, these two possible ranges cannot be glued together. One can now, in the same coordinate system, perform a change of configuration variables $(g_{\lambda\lambda}, g_{xx}, g_{\theta\theta}) \to (N, a, b)$ [21]

$$\begin{aligned} g_{\lambda\lambda} =& N(\lambda)^2 \frac{a(\lambda)}{b(\lambda)} \\ g_{xx} =& \frac{b(\lambda)}{a(\lambda)} \\ g_{\theta\theta} =& a(\lambda)^2 \end{aligned} \tag{3.5}$$

such that neither of the solutions of a, b, N are individually singular at $\lambda = 0$ as we will see below. As a result of the above change of variables

$$ds^2 = -N^2(\lambda)\frac{a(\lambda)}{b(\lambda)}d\lambda^2 + \frac{b(\lambda)}{a(\lambda)}dx^2 + a(\lambda)^2 d\Omega^2. \tag{3.6}$$

where as usual N is the lapse function. Inserting these variables into the first order action of general relativity yields [21]

$$S = \frac{v}{4G}\int d\lambda \left(N - \frac{\dot{a}\dot{b}}{N}\right), \tag{3.7}$$

where the dot represents derivative with respect to λ, and we have defined $v := \int_{x_{min}}^{x_{max}} dx$. The solution to the equations of motion derived from this action are $a = \lambda^2$, $b = 2GM - \lambda^2$, $N^2 = 4a$, which as claimed above are non-singular at $\lambda = 0$.

Since N is a Lagrange multiplier the only momenta of the model are $p_a = -\frac{v}{4G}\frac{\dot{b}}{N}$ and $p_b = -\frac{v}{4G}\frac{\dot{a}}{N}$. Performing a Legendre transformation leads to the Hamiltonian

$$H = -\frac{N}{\alpha}\left(\frac{v^2}{16G^2} + p_a p_b\right). \tag{3.8}$$

where we have defined $\alpha = \frac{v}{4G}$ for simplicity, with $[\alpha] = \mathrm{M} = \mathrm{L}^{-1}$ in $c = 1$ units.

Clearly, the above Hamiltonian constraint is not separable in p_a and p_b and we cannot therefore apply the methods described in Sec. 2 to find a quantum clock and define a relational evolution between a and b degrees of freedom. However, since N is just a Lagrange multiplier (gauge parameter) and its choice does not change the classical dynamics, we can redefine it as

$$\bar{N} = -\frac{N}{\alpha}p_a. \tag{3.9}$$

Classically, this redefinition is globally well-defined and non-singular because the on-shell momentum $p_a = \operatorname{sgn}(\lambda)\,\alpha$ (from the EoM) is a constant with a fixed sign along a given physical trajectory. However, $\bar{N}$ now depends on the phase-space variable p_a, and at the quantum level such dependencies might introduce operator ordering ambiguities in $\hat{H}$ and the quantum theories with N and $\bar{N}$ might not be equivalent. The above lapse rescaling leads to

$$H = \bar{N} C_H = \bar{N}\left(\frac{\alpha^2}{p_a} + p_b\right). \tag{3.10}$$

This form of the Hamiltonian constraint C_H is now clearly separable. The Hamiltonian constraint can now be written as

$$C_H = H_C\left(p_b\right) + H_S\left(p_a\right) = p_b + \frac{\alpha^2}{p_a} \tag{3.11}$$

which is in a deparametrized form. Thus the Hamiltonian

$$H_C = p_b = -H_S = H_{\text{phys}}, \tag{3.12}$$

generates evolution with respect to internal time b. Hence, one can define a quantum clock out of b using $H_C = H_b = p_b$, which results in a picture where a evolves with respect to b. We are interested in such an evolution because a is the radius of the 2-spheres in the interior of the black hole whose areas are $A = 4\pi a^2$. Since the classical singularity occurs as $A = 0$ [2], we can study the fate of singularity in this way in a relational quantum theory.

4 The Hilbert Space

In this section we apply the refined algebraic quantization (RAQ) to our system to find the physical states belonging to the physical Hilbert space $\mathscr{H}_{\text{phys}}$. RAQ is a mathematically rigorous framework designed to solve the issue of how to correctly implement Dirac's quantization program when the constraints have a continuous spectrum and 0 is in the spectrum of the operators involved, which is the case for all totally constrained systems. Since this procedure is important for the rigor of our construction, we review it in some detail in Appendix A.

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[2] The Kretschmann scalar K in the Schwarzschild metric is inversely proportional to the area, i.e. $K \propto \frac{1}{r^6} \propto \frac{1}{A^3}$.

4.1 Kinematical Hilbert Space $\mathscr{H}_{\text{kin}}$

States:

According to the RAQ steps discussed in Appendix A, the kinematical Hilbert space $\mathscr{H}_{\text{kin}} = \mathscr{H}_a \otimes \mathscr{H}_b$ of our model consists of square-integrable functions of p_a and p_b, namely

$$\mathscr{H}_{\text{kin}} = \mathscr{H}_a \otimes \mathscr{H}_b = L^2\left(\mathbb{R}^2, dp_a dp_b\right) = \left\{\psi : \mathbb{R}^2 \to \mathbb{C} \middle| \int_{\mathbb{R}^2} dp_a dp_b\, \psi^*\left(p_a, p_b\right) \psi\left(p_a, p_b\right) < \infty\right\}. \tag{4.1}$$

Operators:

We represent the classical algebra and the canonical variables in the momentum representation as

$$[\hat{a}, \hat{p}_a] = i\hbar\hat{\mathbb{I}}, \qquad \left[\hat{b}, \hat{p}_b\right] = i\hbar\hat{\mathbb{I}}, \tag{4.2}$$

and

$$\hat{a}\psi = i\hbar\frac{\partial\psi}{\partial p_a}, \qquad \hat{p}_a\psi = p_a\psi, \tag{4.3}$$

$$\hat{b}\psi = i\hbar\frac{\partial\psi}{\partial p_b}, \qquad \hat{p}_b\psi = p_b\psi. \tag{4.4}$$

Next, we need to represent the constraint (3.10). It is, however, seen that care needs to be taken in representation of the term $\frac{1}{p_a}$ in the constraint, since a naive representation of its action on kinematical wave functions with non-null support on $p_a = 0$ can be ill-defined. Moreover, the operator corresponding to the unitary representation of the gauge for the rigging map given by $\hat{U} = e^{-is\hat{C}_H}$ will contain infinite powers of the type $\left(\widehat{\frac{1}{p_a}}\right)^n$. Therefore, to be a square-integrable space, $\mathscr{H}_{\text{kin}}$ will have to contain states whose limit for $p_a \to 0$ decay to 0 faster than any power of $\frac{1}{p_a}$. The exclusion of states with null fast decay as $p_a \to 0$ has also an important physical reason: it amounts to the exclusion of frozen clocks for which $p_a = 0$. This is because, from Eq.(3.7), we have $p_a \sim \dot{b}$, i.e., p_a is classically proportional to the rate of change of the clock variable b. The $p_a = 0$ exclusion will therefore allow us to have a relational model defined with monotonic clocks. It will also exclude configurations where the relational Hamiltonian becomes singular and prevents ill-defined evolution. Later in this section, we will discuss the frozen-clock claim mathematically. Additionally, resolving this issue naturally leads to nice states for Φ as we will see below.

The treatment of inverse operators was already discussed in [32, 33]. In particular, it was proven in [32] that there exist a countable set of wave functions $\{\phi^{(n)}_{\sigma,\tilde{\sigma}}(p_a)\}_{n\in\mathbb{Z}}$

belonging to the domain of functions that decay at both $p_a = \pm\infty$ and $p_a = 0$ faster than any power of, p_a^n and $\frac{1}{p_a^n}$, respectively. Such states are given by

$$\phi^{(n)}_{\sigma,\tilde{\sigma}}(p_a) \sim p_a^n e^{-\frac{p_a^2}{2\sigma^2}} e^{-\frac{\tilde{\sigma}^2}{2p_a^2}}, \quad n \in \mathbb{Z},\ \sigma, \tilde{\sigma} \in \mathbb{R}\backslash\{0\}, \tag{4.5}$$

defined up to a normalization constant. It can be proven that the set

$$D_0\left(p_a\right) = \text{span}\left\{\phi^{(n)}_{\sigma,\tilde{\sigma}}(p_a) | n \in \mathbb{Z},\ \sigma, \tilde{\sigma} \in \mathbb{R}\backslash\{0\}\right\} \tag{4.6}$$

defined in Eq. (4.5) has the following properties [32]:

Lemma 1.

1. *D_0 is an invariant domain for any polynomial in $\hat{p}_a$, $\widehat{\frac{1}{p_a}}$, $\hat{a}$.*
2. *D_0 is dense in $L^2(\mathbb{R}, dp_a)$.*

Property 2 in Lemma 1, shows that the states (4.5) generate $L^2(\mathbb{R}, dp_a)$. More concretely, the closure of span of these wave functions is $\mathscr{H}_a$ itself,

$$\mathscr{H}_a = \overline{D_0\left(p_a\right)} = L^2(\mathbb{R}, dp_a) \tag{4.7}$$

Using these wave function $\phi^{(n)}_{\sigma,\tilde{\sigma}}$, one can represent

$$\widehat{\frac{1}{p_a}}\psi = \frac{1}{p_a}\psi, \qquad \text{for } \psi \in D_0 \tag{4.8}$$

due to their property that they fall faster than any power of, p_a^n and $\frac{1}{p_a^n}$ at $p_a = 0$.

As a consequence, the constraint operator in our case is represented as

$$\hat{C}_H = \hat{p}_b + \alpha^2 \widehat{\frac{1}{p_a}}, \tag{4.9}$$

whose kernel defines the physical states.

Inner product:

The inner product of this space is the standard one

$$\langle\psi_1|\psi_2\rangle_{\text{kin}} = \int_{\mathbb{R}^2} dp_a dp_b\, \psi_1^*\left(p_a, p_b\right) \psi_2\left(p_a, p_b\right). \tag{4.10}$$

4.2 Seed space Φ

States:

Based on the discussion in section 4.1, and since $D_0(p_a)$ has the properties mentioned in Lemma 1, the Gelfand triple in our model is

$$\Phi = D_a \otimes D_b \subset \mathscr{H}_{\text{kin}} \subset D_a^* \otimes D_b^* = \Phi^*, \tag{4.11}$$

where,

- $\Phi = D_a \otimes D_b = D_0(p_a) \otimes S_0(p_b)$, where $S_0(p_b)$ are Schwartz functions of p_b. One set of such Schwartz functions can be given by

$$S_0(p_b) \ni \chi_k^{(m)}(p_b) \sim p_b^m e^{-\frac{p_b^2}{2k^2}}, \quad m \in \mathbb{N},\ k \in \mathbb{R} \neq 0 \tag{4.12}$$

- $\Phi^* = D_a^* \otimes D_b^* = D_0^* \otimes D_b^*$. Here D_0^* is the set of all continuous linear functional on $D_a = D_0(p_a)$ which are tempered distributions acting on the $\phi_{\sigma,\tilde{\sigma}}^{(n)}(p_a)$ of Eq. (4.5). Moreover, D_b^* is the space of continuous linear functionals on $D_b = S_0(p_b)$. The space Φ^* is thus the space of all the continuous linear functionals on Φ as expected.

As a consequence of the above observations, a generic state $\phi(p_a, p_b) \in \Phi$ can be approximated arbitrarily well by finite linear combinations

$$\phi\left(p_a, p_b\right) = N_\phi \sum_{m,n} c_{nm} \phi_{\sigma,\tilde{\sigma}}^{(n)}\left(p_a\right) \chi_k^{(m)}\left(p_b\right) \tag{4.13}$$

where N_Ψ is a suitable normalization function for which

$$\int_{\mathbb{R}^2} dp_a dp_b\, \phi^*\left(p_a, p_b\right) \phi\left(p_a, p_b\right) = 1 \tag{4.14}$$

and the $c_{nm} \in \mathbb{C}$ are the coefficients of the linear combination.

Operators:

As mentioned in step 2 in Appendix A, for an operator $\hat{O}$ in $\mathscr{H}_{\text{kin}}$, the corresponding operator acting on Φ denoted by $\hat{O}'$ is defined by restricting its domain to Φ.

Inner product:

To complete our construction of the space Φ, we define its inner as

$$\langle \phi_1 | \phi_2 \rangle_\Phi := \int_{\mathbb{R}^2} dp_a dp_b\, \phi_1^*\left(p_a, p_b\right) \phi_2\left(p_a, p_b\right), \quad \forall \phi_1, \phi_2 \in \Phi. \tag{4.15}$$

4.3 Topological dual Φ^*

States:

As can be seen from step 3 of RAQ, a typical element of $\Psi_\phi \in \Phi^*$ which is a continuous linear functionals, $\Psi : \Phi \to \mathbb{C}$, on Φ, can be written as

$$\Psi_\phi(f) := \langle \phi \mid f \rangle_{\text{kin}} = \int_{\mathbb{R}^2} dp_a dp_b \, \phi^* \left(p_a, p_b\right) f \left(p_a, p_b\right). \tag{4.16}$$

Operators:

For a given operator $\hat{O}$ on $\mathscr{H}_{\text{kin}}$ or equivalently $\hat{O}'$ on Φ, the corresponding operator $\hat{O}$ on Φ^* in our case is

$$\begin{aligned}\left(\hat{O}\Psi_\phi\right)[f] :=& \Psi\left[\hat{O}'^\dagger f\right], \ \phi, \ \forall f \in \Phi, \Psi \in \Phi^* \\ =& \int_{\mathbb{R}^2} dp_a dp_b \, \phi^* \left(p_a, p_b\right) \hat{O} f \left(p_a, p_b\right),\end{aligned} \tag{4.17}$$

where we have assumed $\hat{O}'$ is self-adjoint and operationally taken as $\hat{O} = \hat{O}'$.

Inner product:

As mentioned in step 3 of RAQ, no global inner product can be defined or is needed for Φ^*.

4.4 Physical Hilbert Space

Owing to the fact that $\mathcal{V}_{\text{phys}}$ inherits the operators and inner products of Φ^*, we start this section by introducing operators, instead of states.

Operators:

The operators $\hat{O}$ on $\mathcal{V}_{\text{phys}}$, and $\forall f \in \Phi$, $\Psi_\phi^{\text{phys}} \in \mathcal{V}_{\text{phys}}$, can be written as

$$\begin{aligned}\left(\hat{O}\Psi_\phi^{\text{phys}}\right)[f] :=& \Psi_\phi^{\text{phys}}\left[\hat{O}'^\dagger f\right] \\ =& \int_{\mathbb{R}^2} dp_a dp_b \, \Psi_\phi^{\text{phys}*} \left(p_a, p_b\right) \hat{O} f \left(p_a, p_b\right)\end{aligned} \tag{4.18}$$

where as in the discussion in the Φ^* space above, we have assumed $\hat{O} = \hat{O}^\dagger = \hat{O}'^\dagger$.

States:

As discussed in step 4 Appendix A, the physical states, Ψ_ϕ^{phys}, constructed out of seeds $\phi \in \Phi$, are the ones that are in the kernel of the quantum first class constraints

$$\left(\hat{C}_I \Psi_\phi^{\text{phys}}\right)[f] = \Psi_\phi^{\text{phys}}\left[\hat{C}_I' f\right] = 0, \quad \phi, \forall f \in \Phi, \forall I. \tag{4.19}$$

This leads to the condition

$$\int_{\mathbb{R}^2} dp_a dp_b \, \Psi_\phi^{\text{phys}*}\left(p_a, p_b\right) \hat{C}_H f\left(p_a, p_b\right) = 0, \tag{4.20}$$

that should be satisfied by Ψ_ϕ^{phys}. Here again we have operationally taken $\hat{C}_H = \hat{C}_H^\dagger = \hat{C}_H'^\dagger$.

Since in our case, $\hat{U} = e^{-is\hat{C}_H}$, these physical states can be expressed as

$$\begin{aligned}
D_a^* \otimes D_b^* \ni \Psi_\phi^{\text{phys}}[f] \coloneqq \eta(\phi)[f] =& \frac{1}{2\pi}\int_G d\mu(g) \left\langle \hat{U}(g)\phi \mid f \right\rangle_{\text{kin}} \\
=& \frac{1}{2\pi}\int_G ds \left\langle \hat{U}\left(s\right)\phi \mid f \right\rangle_{\Phi} \\
=& \frac{1}{2\pi}\int_G ds \left[\int_{\mathbb{R}^2} dp_a dp_b \left(\hat{U}\left(s\right)\phi\left(p_a, p_b\right)\right)^* f\left(p_a, p_b\right)\right] \\
=& \frac{1}{2\pi}\int_{\mathbb{R}^2} dp_a dp_b \left[\left(\int_G ds \hat{U}\left(s\right)\phi\left(p_a, p_b\right)\right)^*\right] f\left(p_a, p_b\right) \\
=& \int_{\mathbb{R}^2} dp_a dp_b \, \Psi_\phi^{\text{phys}*}\left(p_a, p_b\right) f\left(p_a, p_b\right)
\end{aligned} \tag{4.21}$$

These are distributional states since $\Psi_{\text{phys}} : \Phi \to \mathbb{C}$ for all $f \in \Phi$.

Now, in case of our model, given a seed $\psi\left(p_a, p_b\right) \in \Phi$ and using Eq. (4.15), we construct a physical state in $\mathcal{V}_{\text{phys}}$ as

$$\begin{aligned}\Psi_{\psi}^{\text{phys}}[f] \coloneqq & \frac{1}{2\pi}\int_{G} d\mu(g)\left\langle \hat{U}(g)\psi \mid f\right\rangle_{\text{kin}} \\ = & \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\left\langle e^{-is\hat{C}_H}\psi \middle| f\right\rangle_{\text{kin}} \\ = & \frac{1}{2\pi}\left\langle \psi\middle|\int_{-\infty}^{\infty} d\beta\left(e^{-i\beta\hat{C}_H}\right)\middle| f\right\rangle_{\text{kin}}, \qquad \beta=-s \\ = & \left\langle \psi\middle|\delta\left(\hat{C}_H\right)\middle| f\right\rangle_{\text{kin}} \\ = & \int_{\mathbb{R}^2} dp_a dp_b\, \psi^*\left(p_a,p_b\right)\delta\left(\hat{C}_H\right) f\left(p_a,p_b\right) \\ = & \int_{\mathbb{R}^2} dp_a dp_b\, \left[\delta\left(C_H\right)\psi^*\left(p_a,p_b\right)\right] f\left(p_a,p_b\right) \\ = & \int_{\mathbb{R}} dp_a\, \left[\psi^*\left(p_a,-\frac{\alpha^2}{p_a}\right)\right] f\left(p_a,-\frac{\alpha^2}{p_a}\right),\end{aligned} \tag{4.22}$$

where we have used the fact that we are in the momentum representation so $\delta\left(\hat{C}_H\right)$ with (4.9) acts by multiplication. Comparing this with (4.21) reveals that $\Psi_{\psi}^{\text{phys}*}\left(p_a,p_b\right) = \psi^*\left(p_a,p_b\right)\delta\left(C_H\right)$ and hence we find

$$\begin{aligned}\mathcal{V}_{\text{phys}} \ni \Psi_{\psi}^{\text{phys}}\left(p_a,p_b\right) = & \psi\left(p_a,p_b\right)\delta\left(C_H\right) \\ = & \psi\left(p_a,-\frac{\alpha^2}{p_a}\right)\delta\left(p_b+\frac{\alpha^2}{p_a}\right) \\ = & \psi_{a|b}\left(p_a\right)\delta\left(p_b+\frac{\alpha^2}{p_a}\right).\end{aligned} \tag{4.23}$$

Here we have defined the physical amplitude evaluated on the constraint surface as

$$\psi_{a|b}\left(p_a\right) \coloneqq \psi\left(p_a,-\frac{\alpha^2}{p_a}\right) \in D_0\left(p_a\right) \subset L^2\left(\mathbb{R},dp_a\right), \tag{4.24}$$

which is a linear combination of the states defined in Eq. (4.5)

$$\psi_{a|b}\left(p_a\right) = \sum_n c_n \phi_{\sigma,\sigma'}^{(n)}\left(p_a\right), \tag{4.25}$$

as we will also show in more details in Eq.(4.28) below. The "$a|b$" index [9, 27] indicates that this physical states describe the evolution of a degrees of freedom (DoF) with respect to b DoFs. This form completely eliminates the p_b-dependence from the evaluated physical amplitude. This is even more clear if we write $\Psi_{\psi}^{\text{phys}}$ in (4.23) in the Dirac bracket notation (with abuse of notation) as

$$\begin{aligned}\left|\Psi_{\psi}^{\text{phys}}\right\rangle &= \int_{\mathbb{R}^2} dp_a dp_b\, \psi\left(p_a, -\frac{\alpha^2}{p_a}\right) \delta\left(p_b + \frac{\alpha^2}{p_a}\right) \left|p_a\right\rangle_a \left|p_b\right\rangle_b \\ &= \int_{\mathbb{R}} dp_a\, \psi\left(p_a, -\frac{\alpha^2}{p_a}\right) \left|p_a\right\rangle_a \left|-\frac{\alpha^2}{p_a}\right\rangle_b \\ &= \int_{\mathbb{R}} dp_a\, \psi_{a|b}\left(p_a\right) \left|p_a\right\rangle_a \left|-\frac{\alpha^2}{p_a}\right\rangle_b ,\end{aligned} \tag{4.26}$$

where we have used a notation $\left|\cdot\right\rangle_b$, etc., to keep track of which space the ket belongs to. We see that in this expression, the term containing $\left|-\frac{\alpha^2}{p_a}\right\rangle_b$ is redundant.

Moreover, we can write the state $\Psi_{\psi}^{\text{phys}}\left(p_a, p_b\right)$ in (4.23) more explicitly in terms of (4.5) and (4.12), by expressing ψ as in Eq. (4.13) (with $\phi \to \psi$)

$$\Psi_{\psi}^{\text{phys}}\left(p_a, p_b\right) = \psi\left(p_a, p_b\right) \delta\left(C_H\right) = N_{\psi} \sum_{m,n} c_{nm} \phi_{\sigma,\tilde{\sigma}}^{(n)}\left(p_a\right) \chi_k^{(m)}\left(-\frac{\alpha^2}{p_a}\right) \delta\left(p_b + \frac{\alpha^2}{p_a}\right) \tag{4.27}$$

In the same way and using Eqs. (4.13) (with $\phi \to \psi$), (4.6), and (4.12), the amplitude $\psi_{a|b}\left(p_a\right)$ in (4.24) can be written as

$$\begin{aligned}\psi_{a|b}\left(p_a\right) =& N_{\psi} \sum_{m,n} c_{nm} \phi_{\sigma,\tilde{\sigma}}^{(n)}\left(p_a\right) \chi_k^{(m)}\left(-\frac{\alpha^2}{p_a}\right) \\ \propto& \sum_{m,n} c_{nm} p_a^n e^{-\frac{p_a^2}{2\sigma^2}} e^{-\frac{\tilde{\sigma}^2}{2p_a^2}} \left(-\frac{\alpha^2}{p_a}\right)^m e^{-\frac{\alpha^4 p_a^{-2}}{2k^2}} \\ =& \sum_{m,n} c_{nm} \left(-\alpha^2\right)^m p_a^{n-m} e^{-\frac{p_a^2}{2\sigma^2}} e^{-\frac{p_a^{-2}}{2}\left(\tilde{\sigma}^2 + \frac{\alpha^4}{k^2}\right)} \\ :=& \sum_{m,n} c_{nm} \left(-1\right)^m \left(\alpha\right)^{2m} p_a^{n-m} e^{-\frac{p_a^2}{2\sigma^2}} e^{-\frac{\sigma'^2}{2p_a^2}} \\ \propto& \sum_{m,n} c'_{nm} \phi_{\sigma,\sigma'}^{(n-m)}\left(p_a\right) ,\end{aligned} \tag{4.28}$$

where we have defined

$$\sigma'^2 := \tilde{\sigma}^2 + \frac{\alpha^4}{k^2}. \tag{4.29}$$

We will comment on this important observation in Sec. 4.5.

A similar procedure applies if we decide to solve our delta $\delta\left(p_b + \frac{\alpha^2}{p_a}\right) = \frac{\alpha^2}{p_b^2} \delta\left(p_a + \frac{\alpha^2}{p_b}\right)$ for p_a instead of p_b. In this case, the new wave-function will be $\psi\left(-\frac{\alpha^2}{p_b}, p_b\right) := \psi_{b|a}\left(p_b\right)$

and the p_a dependence will be completely removed. Then the a DoFs would be the clock, with respect to which the b DoFs evolve. This leads to an important observation: the Hilbert space $\mathscr{H}_{\text{phys}}$ contains all the possible viewpoints at once. In this sense, this model is fully relational and $\mathscr{H}_{\text{phys}}$ is often denoted as a Perspective neutral space [9, 26, 27].

Inner product:

From step (4) in Appendix (A) we can see that, using Eq. (4.15), in our case the inner product on $\mathcal{V}_{\text{phys}}$ becomes

$$\begin{aligned}\left\langle \Psi_\psi^{\text{phys}} \middle| \Psi_\phi^{\text{phys}} \right\rangle_{\text{phys}} &= \left\langle \psi \middle| \delta\left(\hat{C}_H\right) \middle| \phi \right\rangle_{\text{kin}} \\ &= \int_{\mathbb{R}^2} dp_a dp_b\, \psi^*\left(p_a, p_b\right) \delta\left(\hat{C}_H\right) \phi\left(p_a, p_b\right) \\ &= \int_{\mathbb{R}^2} dp_a dp_b\, \psi^*\left(p_a, p_b\right) \delta\left(C_H\right) \phi\left(p_a, p_b\right) \\ &= \int_{\mathbb{R}} dp_a\, \psi^*_{a|b}\left(p_a\right) \phi_{a|b}\left(p_a\right)\end{aligned} \tag{4.30}$$

Finally, by eliminating p_b from the expressions and using the physical states (4.27) and their inner product (4.30), we can express the physical Hilbert space as

$$D_0\left(p_a\right) \subset \mathscr{H}_{\text{phys}} = L^2\left(\mathbb{R}, dp_a\right) \subset D_0^*\left(p_a\right). \tag{4.31}$$

4.5 Clock Hilbert space $\mathscr{H}_C$ and black hole quantum clock

We will now construct the quantum clock in our model based on the instruction in Sec. 2.2. In our system, according to (3.12), the clock Hilbert space is associated to b DoF, and hence we have

$$\mathscr{H}_C := \mathscr{H}_b = L^2\left(\mathbb{R}, dp_b\right) \tag{4.32}$$

Furthermore, due to (3.12), the clock's Hamiltonian is

$$\hat{H}_C = \hat{p}_b, \tag{4.33}$$

and its eigenstates are the momentum eigenstates [9]

$$\hat{H}_C \left|p_b\right\rangle_C = p_b \left|p_b\right\rangle_C. \tag{4.34}$$

Remember that, in the representation used so far, the momentum states $\{\left|p_a\right\rangle_a\}$ and $\{\left|p_b\right\rangle_b\}$ form a Dirac-orthonormal family of generalized eigenstates basis for $\mathscr{H}_a$ and $\mathscr{H}_b$, respectively, i.e., $\left\langle p_a' | p_a \right\rangle_a = \delta\left(p_a' - p_a\right)$ and $\left\langle p_b' | p_b \right\rangle_b = \delta\left(p_b' - p_b\right)$.

From Eq. (4.34), it is seen that the clock's spectrum is therefore given by all the possible values of p_b, i.e., $\epsilon = p_b$ and $\sigma_c = \mathbb{R}$ (see Eq. (2.12)). This means that the b DOF of the interior of a black hole serves as an ideal clock [9], as discussed in Sec. 2. Hence, according to Eq. (2.10) (but with $\epsilon_{\min} \to -\infty$) the states for this clock are given by

$$|t\rangle = \int_{-\infty}^{\infty} dp_b\, e^{ig(p_b)} e^{-\frac{i}{\hbar}p_b t} \left|p_b\right\rangle . \tag{4.35}$$

In the specific case of $g(p_b) = 0$, comparing the above with the Fourier transform $|b\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp_b\, e^{-\frac{i}{\hbar}p_b b} \left|p_b\right\rangle$ leads to the conclusion that the clock's states are directly proportional to b, namely

$$|t\rangle = \sqrt{2\pi\hbar}\, |b\rangle . \tag{4.36}$$

In this case, from Eq. (2.14) and the spectral theorem, we obtain

$$\hat{T}|_{g=0} = \frac{1}{2\pi\hbar} \int dt\, t |t\rangle\langle t| = \int db\, b |b\rangle\langle b| = \hat{b}, \tag{4.37}$$

and therefore

$$\hat{T}|_{g=0} \, |t\rangle = \hat{b} \left(\sqrt{2\pi\hbar}\, |b\rangle \right) = \sqrt{2\pi\hbar} b \, |b\rangle . \tag{4.38}$$

As mentioned before in Sec. 2.2, in general, the function $g(p_b)$ encodes the freedom in choices of inequivalent clocks or equivalently operators $\hat{T}$, which is any operator that satisfies the commutation relation (2.16) with $\hat{H}_C = \hat{p}_b$. In particular, we have

$$\begin{aligned}
\hat{T} =& \frac{1}{2\pi\hbar} \int_{\mathbb{R}} dt\, t \, |t\rangle \, \langle t| \\
=& \frac{1}{2\pi\hbar} \int dt \int dp_b \int dp_b'\, t e^{\frac{i}{\hbar}\left(p_b'-p_b\right)t} e^{ig(p_b)} \left|p_b\right\rangle \left\langle p_b'\right| e^{-ig\left(p_b'\right)} \\
=& \frac{1}{2\pi\hbar} \int dt \int dp_b \int dp_b'\, t e^{\frac{i}{\hbar}\left(p_b'-p_b\right)t} e^{ig\left(\hat{H}_C\right)} \left|p_b\right\rangle \left\langle p_b'\right| e^{-ig\left(\hat{H}_C\right)} \\
=& e^{ig\left(\hat{H}_C\right)} \left(\frac{1}{2\pi\hbar} \int dt\, t \int dp_b \int dp_b'\, e^{\frac{i}{\hbar}\left(p_b'-p_b\right)t} \left|p_b\right\rangle \left\langle p_b'\right| \right) e^{-ig\left(\hat{H}_C\right)} \\
=& e^{ig\left(\hat{H}_C\right)} \hat{T}|_{g=0} e^{-ig\left(\hat{H}_C\right)}
\end{aligned} \tag{4.39}$$

Applying the Hadamard Lemma to above yields

$$\hat{T} = \hat{T}|_{g=0} + i \left[g\left(\hat{H}_C\right), \hat{T}|_{g=0} \right] + \frac{i^2}{2!} \left[g\left(\hat{H}_C\right), \left[g\left(\hat{H}_C\right), \hat{T}|_{g=0} \right] \right] + \ldots \tag{4.40}$$

Taylor expanding $g(\hat{H}_C)$ and using (4.33) and (4.37) leads to [9]

$$\begin{aligned}\left[g\left(\hat{H}_C\right), \hat{T}|_{g=0}\right] &= \left[\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}\hat{H}_C^n, \hat{T}|_{g=0}\right] \\ &= \sum_{n=1}^{\infty}\frac{g^{(n)}(0)}{n!}(-i\hbar)n\hat{H}_C^{n-1} \\ &= -i\hbar\sum_{n=0}^{\infty}\frac{g^{(n+1)}(0)}{n!}\hat{H}_C^n \\ &= -i\hbar\sum_{n=0}^{+\infty}\frac{h^{(n)}(0)}{n!}\hat{H}_C^n \\ &= -i\hbar\, h\left(\hat{H}_C\right),\end{aligned} \tag{4.41}$$

where we have defined $h(p_b) := \frac{dg(p_b)}{dp_b}$. This means that all the higher order commutators in Eq. (4.40) are identically 0 since any two functions of the same operator commute. This allows us to rewrite Eq. (4.40) as

$$\hat{T} = \hat{T}|_{g=0} + \hbar\, h\left(\hat{H}_C\right) = \hat{b} + \hbar\, h\left(\hat{H}_C\right), \qquad h(p_b) := \frac{dg(p_b)}{dp_b}. \tag{4.42}$$

This implies that the function $g(p_b)$ inside Eq. (4.35) has the role of shifting the clock operator $\hat{b}$ by a generic function $h\left(\hat{H}_C\right)$. As a consequence, there exists an infinite number of clocks $\hat{T}$ that satisfy the commutation relation (2.16) with the clock Hamiltonian $\hat{H}_C = \hat{p}_b$, in which the "seed" clock is $\hat{b}$. These clocks correspond to different choices of time zero and different weightings of energy eigenstates in the clock state superposition. They are physically inequivalent in the sense that their POVMs assign different probabilities to the same measurement outcomes.

Having constructed the quantum clock, we can make some important observations about it from the properties of the Hilbert space. One can see that the only difference between the dense countable set of states (4.5) and the ones in (4.25) that are used as seeds to construct the physical states, is in their uncertainties, which changes from $\tilde{\sigma}$ to σ'. This can be interpreted as the clock's back-reaction, in which the clock's uncertainty (in our case Δb) influences the state of the system (in our case, any observable of (a, p_a)). This can be seen as follows: in the Gaussians of Eq. (4.12), the uncertainty in b is proportional to $\Delta b \sim \hbar/k$. Therefore, the new uncertainty σ' in p_a can be rewritten as

$$\sigma'^2 \sim \tilde{\sigma}^2 + \frac{\alpha^4}{\hbar^2}(\Delta b)^2. \tag{4.43}$$

This clearly shows how the clock's back-reaction influences our measurements on the system. In particular, $\tilde{\sigma}$ and σ' regulate the "suppression" of the Thiemann's states (4.5) around $p_a = 0$, i.e., they encode how fast the probability goes to 0 as $p_a \to 0$. The relation (4.43) clearly shows that such suppression always gets bigger as the clock's uncertainty increases. In particular, the clock's uncertainty can never reduce σ' to 0, i.e., it can never contribute to obtain states that are not rapidly decaying at $p_a = 0$ for any powers of p_a, which is clear from the positivity of the clock's uncertainty term $\frac{\alpha^4}{k^2}$ in (4.43).

It is also important to notice that because of the classical relation $p_a \sim \dot{b}$, $p_a = 0$ will correspond to a frozen clock, which is an undesired property for a relational model. In particular, Eq. (4.43) shows that, the bigger the clock's uncertainty Δb, the greater the suppression in $p_a = 0$, which suppresses contributions from configurations where the clock would be effectively frozen. Therefore, the exclusion of states that are not rapidly-enough decaying at $p_a = 0$ is justified physically and is necessary to have a well-defined relational theory that employs monotonic clocks.

5 Quantum Relational Observables

As we mentioned before, our goal in this paper is to study the fate of singularity of a Schwarzschild black hole in the relational approach. To this end, we study three relevant quantities. The first of these quantities is the area of 2-spheres inside the black hole. Classically, using the metric (3.6), this area is given by

$$A = \int d\Omega \sqrt{g_{\theta\theta} g_{\phi\phi}} = 4\pi a^2. \tag{5.1}$$

Although the nonvanishing of this area (and in fact its gauge invariant version as we will see below) may not necessarily mean that the singularity is resolved, it is a good indicator of such a phenomenon.

The second quantity which is intimately related to the singularity resolution is the invariant Kretschmann scalar

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \tag{5.2}$$

This is perhaps the most common Riemann invariant used to determine whether a spacetime is singular and its finiteness across the whole spacetime signals the lack of any singularity in that spacetime [34].

The third quantity we consider is the expansion scalar $\vartheta = \nabla_\mu k^\mu$ of null geodesics described by an affinely parametrized null vector field k^μ. This quantity describes how much the geodesics congruence focus or defocus in their motion through spacetime.

The expansion scalar is a powerful indicator in detection of singularities and is the backbone of Penrose-Hawking singularity theorems. If this quantity never diverges, particularly to $-\infty$, it is a definitive sign that the spacetime is singularity-free [35–37].

5.1 General treatment of gauge-invariant observables

In this subsection, we will summarize some of the results about gauge-invariant observable that we use in the rest of this paper. Consider a gauge-dependent operator (kinematical observable) $\hat{f}_S$ associated to the system S with a Hamiltonian operator $\hat{H}_S$, clock Hamiltonian $\hat{H}_C$ with a clock operator $\hat{T}$ and clock eigenstates $|t\rangle$ with eigenvalues t. We are considering a general clock $\hat{T} = \hat{T}|_{g=0} + \hbar\, h\left(\hat{H}_C\right)$ where $h(\epsilon) = \frac{d}{d\epsilon} g(\epsilon)$ whose states are [9]

$$|t\rangle_g = e^{ig\left(\hat{H}_C\right)}|t\rangle_{g=0} = \int_{-\infty}^{\infty} dp_b\, e^{ig(p_b)} e^{-\frac{i}{\hbar}p_b t}\, |p_b\rangle_b\,, \tag{5.3}$$

where the coefficients of the Fourier transform, or the plane waves in this case, are

$${}_b\langle p_b|t\rangle = e^{ig(p_b)} e^{-\frac{i}{\hbar}p_b t}. \tag{5.4}$$

The gauge-dependence of $\hat{f}_S$ dictates

$$\left[\hat{f}_S, \hat{C}_H\right] \neq 0. \tag{5.5}$$

To find out what the gauge-dependent system observable $\hat{f}_S$ looks like when the clock T reaches the target time τ, we must evolve the system from t to τ as

$$e^{\frac{i}{\hbar}\hat{H}_S(\tau-t)}\hat{f} e^{-\frac{i}{\hbar}\hat{H}_S(\tau-t)} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{i^n}{\hbar^n}\left(t-\tau\right)^n \left[\hat{f}_S, \hat{H}_S\right]_{(n)} \tag{5.6}$$

where

$$\left[\hat{f}_S, \hat{H}_S\right]_{(n)} = \left[\left[\hat{f}_S, \hat{H}_S\right]_{(n-1)}, \hat{H}_S\right], \left[\hat{f}_S, \hat{H}_S\right]_{(0)} := \hat{f}_S. \tag{5.7}$$

Then the gauge-invariant relational observable associated to $\hat{f}_S$ is derived by projecting the above into the clock states [9, 38–40]

$$\hat{F}_{f_S,T}\left(\tau\right) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dt|t\rangle_g\langle t|_g \sum_{n=0}^{\infty} \frac{1}{n!}\frac{i^n}{\hbar^n}\left(t-\tau\right)^n \left[\hat{f}_S, \hat{H}_S\right]_{(n)}, \tag{5.8}$$

or equivalently

$$\begin{aligned}\hat{F}_{f_S,T}(\tau) =&\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dt|t\rangle_g\langle t|_g e^{\frac{i}{\hbar}(\tau-t)\hat{H}_S}\hat{f}_S e^{-\frac{i}{\hbar}(\tau-t)\hat{H}_S}\\ =&\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dt e^{-\frac{i}{\hbar}t\left(\hat{H}_C+\hat{H}_S\right)}\left(|\tau\rangle_g\langle\tau|_g\hat{f}_S\right)e^{\frac{i}{\hbar}t\left(\hat{H}_C+\hat{H}_S\right)}\\ =&\frac{1}{2\pi}\int_R ds\,\hat{U}_{CS}(s)\left(|\tau\rangle_g\langle\tau|_g\hat{f}_S\right)\hat{U}^{\dagger}_{CS}(s)\\ =&\eta\left(|\tau\rangle_g\langle\tau|_g\hat{f}_S\right),\end{aligned} \tag{5.9}$$

where

$$\hat{U}_{CS}(s)=\exp\left[-is\hat{C}_H\right], \tag{5.10}$$

and in our case

$$\hat{C}_H=\alpha^2\widehat{\frac{1}{p_a}}+\hat{p}_b. \tag{5.11}$$

Eqs. (5.9) or (5.8) yields the gauge-invariant value associated to $\hat{f}_S$ when the clock operator $\hat{T}$ has eigenvalue τ.

The action of $\hat{F}_{f_S,T}(\tau)$ on a physical state $\left|\Psi^{\text{phys}}_{\psi}\right\rangle$ for which $\hat{U}^{\dagger}_{CS}\left|\Psi^{\text{phys}}_{\psi}\right\rangle=\left|\Psi^{\text{phys}}_{\psi}\right\rangle$ yields

$$\begin{aligned}\hat{F}_{f_S,T}(\tau)\left|\Psi^{\text{phys}}_{\psi}\right\rangle =&\frac{1}{2\pi}\int_{\mathbb{R}} ds\,\hat{U}_{CS}(s)\left\{|\tau\rangle_g\langle\tau|_g\hat{f}_S\right\}\hat{U}^{\dagger}_{CS}(s)\left|\Psi^{\text{phys}}_{\psi}\right\rangle\\ =&\hat{\Pi}_{\text{phys}}\left\{|\tau\rangle_g\langle\tau|_g\hat{f}_S\right\}\left|\Psi^{\text{phys}}_{\psi}\right\rangle,\end{aligned} \tag{5.12}$$

where the projector into the physical states $\hat{\Pi}_{\text{phys}}:\Phi\to\Phi^*$ is defined as

$$\hat{\Pi}_{\text{phys}}=\frac{1}{2\pi}\int_{\mathbb{R}} ds\,\hat{U}_{CS}(s)=\hat{\Pi}^{\dagger}_{\text{phys}}. \tag{5.13}$$

The hermiticity of $\hat{\Pi}_{\text{phys}}$ can be checked easily given the integral is over $\mathbb{R}$. Hence, Eq. (5.9) can equivalently be written as

$$\hat{F}_{f_S,T}(\tau)=\eta\left(|\tau\rangle_g\langle\tau|_g\hat{f}_S\right)\approx\hat{\Pi}_{\text{phys}}\left\{|\tau\rangle_g\langle\tau|_g\hat{f}_S\right\}, \tag{5.14}$$

where $\approx$ means that the above equality only holds for physical states. As a result the physical expectation value of $\hat{F}_{f_S,T}(\tau)$ can be written as

$$\begin{aligned}
\left\langle \Psi_\psi^{\text{phys}} \middle| \hat{F}_{f_S,T}(\tau) \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{phys}} &= \left\langle \psi \middle| \hat{\Pi}_{\text{phys}} \left\{ |\tau\rangle_g \langle\tau|_g \hat{f}_S \right\} \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{kin}} \\
&= \left\langle \psi \middle| \hat{\Pi}_{\text{phys}} \middle| \tau_g \right\rangle_{\text{kin}} \left\langle \tau_g \middle| \hat{f}_S \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{kin}} \\
&= \left\langle \psi \middle| \hat{\Pi}^\dagger_{\text{phys}} \middle| \tau_g \right\rangle_{\text{kin}} \left\langle \tau_g \middle| \hat{f}_S \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{kin}} \\
&= \left\langle \Psi_\psi^{\text{phys}} \middle| \tau_g \right\rangle \left\langle \tau_g \middle| \hat{f}_S \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{kin}} \\
&= \langle \Psi_{\text{PW}}(\tau_g) | \hat{f}_S | \Psi_{\text{PW}}(\tau_g) \rangle
\end{aligned} \tag{5.15}$$

where the last line is derived assuming $\hat{f}_S$ only includes system operators, and we have identified the Page-Wootters (PW) states as

$$|\Psi_{\text{PW}}(\tau_g)\rangle = \left\langle \tau_g \middle| \Psi_\psi^{\text{phys}} \right\rangle. \tag{5.16}$$

This shows that $\mathscr{H}_{\text{phys}}$ and the PW reduced Hilbert space $\mathscr{H}_{\text{PW}}$ are isometric as was pointed out in [9]. Consequently, the PW scalar product and matrix elements are gauge-invariant too. In our system, we can compute the terms in (5.15) more explicitly, using Eqs. (4.26) and (5.4) as

$$\begin{aligned}
\left\langle \tau_g \middle| \Psi_\psi^{\text{phys}} \right\rangle &= \langle\tau|_g \int_{\mathbb{R}} dp_b \int_{\mathbb{R}} dp_a \, \psi_{a|b}(p_a) \, \delta\left(p_b + \frac{\alpha^2}{p_a} \right) |p_a\rangle_a |p_b\rangle_b \\
&= \int_{\mathbb{R}} dp_b \int_{\mathbb{R}} dp_a \, \psi_{a|b}(p_a) \, \delta\left(p_b + \frac{\alpha^2}{p_a} \right) e^{-ig(p_b)} e^{\frac{i}{\hbar} p_b \tau} |p_a\rangle_a ,
\end{aligned} \tag{5.17}$$

For the matrix element $\left\langle \tau_g \middle| \hat{f}_S \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{kin}}$, if in general $\hat{f}_S\left(\hat{a}, \hat{p}_a, \hat{b}, \hat{p}_b\right)$, then we obtain

$$\begin{aligned}
\left\langle \tau_g \middle| \hat{f}_S \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{kin}} &= \langle\tau|_g \hat{f}_S \int_{\mathbb{R}} dp_b \int_{\mathbb{R}} dp_a \, \psi_{a|b}(p_a) \, \delta\left(p_b + \frac{\alpha^2}{p_a} \right) |p_a\rangle_a |p_b\rangle_b \\
&= \int_{\mathbb{R}} dp'_b \int_{\mathbb{R}} dp_b \int_{\mathbb{R}} dp_a \, \psi_{a|b}(p_a) \, \delta\left(p_b + \frac{\alpha^2}{p_a} \right) e^{-ig\left(p'_b\right)} e^{\frac{i}{\hbar} p'_b \tau} \\
&\quad \times {}_b\langle p'_b | \hat{f}_S | p_b \rangle_b |p_a\rangle_a
\end{aligned} \tag{5.18}$$

Notice that ${}_b\langle p'_b | \hat{f}_S | p_b \rangle_b$ would still be an operator if f depends on a, p_a. In the simplest case that $\hat{f}_S = \hat{f}_S(\hat{a}, \hat{p}_a)$ this reduces to

$$\left\langle \tau_g \middle| \hat{f}_S \middle| \Psi_\psi^{\text{phys}} \right\rangle_{\text{kin}} = \int_{\mathbb{R}} dp_b \int_{\mathbb{R}} dp_a \, \psi_{a|b}(p_a) \, \delta\left(p_b + \frac{\alpha^2}{p_a} \right) e^{-ig(p_b)} e^{\frac{i}{\hbar} p_b \tau} \hat{f}_S |p_a\rangle_a \tag{5.19}$$

Finally replacing (5.17) and (5.18) in (5.15) we obtain

$$\begin{aligned}\left\langle \Psi_{\psi}^{\text{phys}} \middle| \hat{F}_{f_S,T}(\tau) \middle| \Psi_{\psi}^{\text{phys}} \right\rangle_{\text{phys}} = & \int_{\mathbb{R}} dp''_b \int_{\mathbb{R}} dp''_a \int_{\mathbb{R}} dp'_b \int_{\mathbb{R}} dp_b \int_{\mathbb{R}} dp_a \\ & \times \psi^*_{a|b}(p''_a)\, \psi_{a|b}(p_a)\, \delta\left(p''_b + \frac{\alpha^2}{p''_a}\right) \delta\left(p_b + \frac{\alpha^2}{p_a}\right) \\ & \times e^{-i\left(g(p'_b)-g(p''_b)\right)} e^{\frac{i}{\hbar}(p'_b - p''_b)\tau} \\ & \times {}_a\langle p''_a| \left({}_b\langle p'_b| \hat{f}_S |p_b\rangle_b\right) |p_a\rangle_a \end{aligned} \tag{5.20}$$

Again if $\hat{f}_S$ depends only on a, p_a, then the above reduces to

$$\begin{aligned}\left\langle \Psi_{\psi}^{\text{phys}} \middle| \hat{F}_{f_S,T}(\tau) \middle| \Psi_{\psi}^{\text{phys}} \right\rangle_{\text{phys}} = & \int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a\, \psi^*_{a|b}(p'_a)\, \psi_{a|b}(p_a) \\ & \times e^{\frac{i}{\hbar}(\tilde{g}(p_a)-\tilde{g}(p'_a))} e^{-\frac{i\alpha^2}{\hbar}\left(\frac{1}{p_a}-\frac{1}{p'_a}\right)\tau}\, {}_a\langle p'_a| \hat{f}_S |p_a\rangle_a\,, \end{aligned} \tag{5.21}$$

where we have defined

$$\tilde{g}(p_a) := -\hbar g\left(-\frac{\alpha^2}{p_a}\right).$$

Note that if $\hat{f}_S$ depends on clock DoF $(b,\, p_b)$ then from (5.9) we get

$$\hat{F}^{\dagger}_{f_S,T}(\tau) = \frac{1}{2\pi} \int_R ds\, \hat{U}_{CS}(s) \left(\hat{f}^{\dagger}_S |\tau\rangle_g \langle\tau|_g\right) \hat{U}^{\dagger}_{CS}(s) \tag{5.22}$$

Hence, assuming a symmetric operator such that $\hat{f}_S = \hat{f}^{\dagger}_S$, then

$$\hat{F}_{f_S,T}(\tau) = \hat{F}^{\dagger}_{f_S,T}(\tau) \Leftrightarrow \left[\hat{f}_S, |\tau\rangle_g \langle\tau|_g\right] = 0. \tag{5.23}$$

Since from (5.4) $|\tau\rangle_g$ can be written as

$$|\tau\rangle_g = \int_{\mathbb{R}} dp_b\, e^{ig(p_b)} e^{-\frac{i}{\hbar}p_b\tau}\, |p_b\rangle_b\,, \tag{5.24}$$

the condition (5.23) holds for $\hat{f}_S(a, p_a)$. However, an operator $\hat{f}_S$ that also depends on $\hat{b}$ or $\hat{p}_b$ or both will not fulfill the requirement (5.23), e.g.,

$$\hat{p}_b |\tau\rangle_g = \int_{\mathbb{R}} dp_b\, e^{ig(p_b)} e^{-\frac{i}{\hbar}p_b\tau} p_b\, |p_b\rangle_b \neq \tau |\tau\rangle_g \tag{5.25}$$

and one should use

$$\begin{aligned}\left\langle \Psi_{\psi}^{\text{phys}} \middle| \frac{1}{2}\left(\hat{F}_{f_S,T}(\tau) + \hat{F}^{\dagger}_{f_S,T}(\tau)\right) \middle| \Psi_{\psi}^{\text{phys}} \right\rangle_{\text{phys}} = & \frac{1}{2} \left\langle \Psi_{\psi}^{\text{phys}} \middle| \hat{F}_{f_S,T}(\tau) \middle| \Psi_{\psi}^{\text{phys}} \right\rangle_{\text{phys}} \\ & + \frac{1}{2}\left(\left\langle \Psi_{\psi}^{\text{phys}} \middle| \hat{F}_{f_S,T}(\tau) \middle| \Psi_{\psi}^{\text{phys}} \right\rangle_{\text{phys}}\right)^* \\ = & \mathfrak{Re}\left(\left\langle \Psi_{\psi}^{\text{phys}} \middle| \hat{F}_{f_S,T}(\tau) \middle| \Psi_{\psi}^{\text{phys}} \right\rangle_{\text{phys}}\right) \end{aligned} \tag{5.26}$$

to compute the expectation value of the gauge-invaraint extension of $\hat{f}_S$.

We will use Eqs. (5.20), (5.21) and (5.26) for most of the computations in the next sections to study singularity related physical quantities.

5.2 Area of 2-spheres

In a Schwarzschild black hole, the area of two sphere A is related to the existence of a physical singularity via the Kretschmann scalar $K \propto \frac{1}{r^6} \propto \frac{1}{A^3}$. A vanishing A thus means that the Kretschmann blows up [20, 41]. Therefore, we would like to study the behavior of the quantum version of this area under the evolution in the relation approach as a means of studying the singularity. Given the classical expression of A in (5.1), one might be tempted to define the area operator as

$$\hat{f}_S = \hat{A} = 4\pi \hat{a}^2. \tag{5.27}$$

However, such an operator is not gauge-invariant since

$$\left[\hat{a}^2, \hat{C}_H\right] = \left[\hat{a}^2, \widehat{\frac{1}{p_a}}\right] \neq 0 \tag{5.28}$$

and thus cannot correspond to a physical observable. Although not necessary for our computations of the expectation value of the gauge-invariant area operator, we can construct the gauge-invariant extension of $\hat{A}$ itself using (5.8) and (5.7). To find an explicit expression for the gauge-invariant area operator, we first note that

$$\left[\hat{A}, \widehat{\frac{1}{p_a}}\right]_{(0)} := \hat{A}. \tag{5.29}$$

To compute the next $n = 1$ order commutator in Eq. (5.7), and assuming that $\widehat{\frac{1}{p_a}}$ is well-defined on the Hilbert space (which it is due to our discussion of the previous section), we can find

$$0 = \left[\hat{a}, \hat{\mathbb{I}}\right] = \left[\hat{a}, \hat{p}_a \widehat{\frac{1}{p_a}}\right] = i\hbar \widehat{\frac{1}{p_a}} + \hat{p}_a \left[\hat{a}, \widehat{\frac{1}{p_a}}\right] \tag{5.30}$$

and thus

$$\left[\hat{a}, \widehat{\frac{1}{p_a}}\right] = -i\hbar \left(\widehat{\frac{1}{p_a}}\right)^2. \tag{5.31}$$

As a result one obtains

$$\begin{aligned}
\left[\hat{A}, \alpha^2 \widehat{\frac{1}{p_a}}\right]_{(1)} =& - 4\pi\alpha^2 i\hbar \left\{ \hat{a} \left(\widehat{\tfrac{1}{p_a}}\right)^2 + \left(\widehat{\tfrac{1}{p_a}}\right)^2 \hat{a} \right\}, \\
\left[\hat{A}, \alpha^2 \widehat{\frac{1}{p_a}}\right]_{(2)} =& - 8\pi\hbar^2\alpha^4 \left(\widehat{\tfrac{1}{p_a}}\right)^4, \\
\left[\hat{A}, \alpha^2 \widehat{\frac{1}{p_a}}\right]_{(k)} =& 0, \qquad \text{for } k \geq 3,
\end{aligned} \tag{5.32}$$

Plugging these expressions into Eq. (5.8) yields

$$\begin{aligned}
\hat{F}_{A,b}(\tau) =& 4\pi \left(\hat{a}^2 + \alpha^2 \left(\hat{b} - \tau\right) \left\{ \hat{a} \left(\widehat{\tfrac{1}{p_a}}\right)^2 + \left(\widehat{\tfrac{1}{p_a}}\right)^2 \hat{a} \right\} + \alpha^4 \left(\hat{b} - \tau\right)^2 \left(\widehat{\tfrac{1}{p_a}}\right)^4 \right) \\
=& 4\pi \left(\hat{a} + \alpha^2 \left(\hat{b} - \tau\right) \left(\widehat{\tfrac{1}{p_a}}\right)^2 \right)^2 \\
=& 4\pi \hat{a}^2_{\mathrm{GI}}(\tau),
\end{aligned} \tag{5.33}$$

where we have defined the gauge-invariant extension of a as

$$\hat{a}_{\mathrm{GI}}(\tau) = \hat{a} + \alpha^2 \left(\hat{b} - \tau\right) \left(\widehat{\tfrac{1}{p_a}}\right)^2. \tag{5.34}$$

Interestingly the gauge invariant version of the area operator includes two correction terms linear and quadratic in $(\hat{b} - \tau)$. As a result, such a gauge invariant area is a parabola as a function of the relational time b, which clearly will have a minimum, and allows the possibility of a bounce scenario.

Given that $\hat{a}^2$ only depends on a, the expectation value of its gauge-invariant extension can be written, using (5.21), as

$$\begin{aligned}
\left\langle \Psi^{\mathrm{phys}}_{\psi} \middle| \hat{F}_{A,T}(\tau) \middle| \Psi^{\mathrm{phys}}_{\psi} \right\rangle_{\mathrm{phys}} =& 4\pi \int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a \, \psi^*_{a|b}(p'_a)\, \psi_{a|b}(p_a) \\
& \times e^{-\frac{i\alpha^2}{\hbar}\tau\left(\frac{1}{p_a} - \frac{1}{p'_a}\right)} e^{\frac{i}{\hbar}(\tilde{g}(p_a) - \tilde{g}(p'_a))} \left\langle p'_a \middle| \hat{a}^2 \middle| p_a \right\rangle_a .
\end{aligned} \tag{5.35}$$

Using

$$\left\langle p'_a \middle| \hat{a}^2 \middle| p_a \right\rangle_a = -\hbar^2 \frac{\partial^2}{\partial p_a^2} \delta\left(p'_a - p_a\right) \tag{5.36}$$

and assuming the functions involved decay fast enough at infinities, we get

$$\begin{aligned}\left\langle \hat{F}_{A,T}\left(\tau\right)\right\rangle_{\text{phys}} =&4\pi\int_{\mathbb{R}} dp_a\,\Bigg(\left(\frac{\partial\tilde{g}\left(p_a\right)}{\partial p_a}+\frac{\alpha^2}{p_a^2}\tau\right)^2\left|\psi_{a|b}\left(p_a\right)\right|^2+\hbar^2\left|\frac{\partial\psi_{a|b}\left(p_a\right)}{\partial p_a}\right|^2\\ &+i\hbar\left(\frac{\partial\tilde{g}\left(p_a\right)}{\partial p_a}+\frac{\alpha^2}{p_a^2}\tau\right)\left[\frac{\partial\psi^*_{a|b}\left(p_a\right)}{\partial p_a}\psi_{a|b}\left(p_a\right)-\psi^*_{a|b}\left(p_a\right)\frac{\partial\psi_{a|b}\left(p_a\right)}{\partial p_a}\right]\Bigg)\\ =&4\pi\int_{\mathbb{R}} dp_a\left|\left(\frac{\partial\tilde{g}\left(p_a\right)}{\partial p_a}+\frac{\alpha^2}{p_a^2}\tau\right)\psi_{a|b}\left(p_a\right)-i\hbar\frac{\partial\psi_{a|b}\left(p_a\right)}{\partial p_a}\right|^2,\end{aligned} \tag{5.37}$$

where the function $g\left(p_b\right)$ must be smooth and its derivative must not grow too fast as $p_a\to 0$ (i.e., as $p_b\to\infty$). We see that the second line above mimics the probability current density in quantum mechanics. More importantly, the area expectation value has a parabolic dependence on τ as $\left\langle\hat{F}_{A,b}\left(\tau\right)\right\rangle_{\text{phys}}=A\tau^2+B\tau+C$. This can be interpreted as contraction of 2-spheres in the trapped black hole region, reaching to a minimum area, followed by a bounce to the anti-trapped white hole spacetime, resulting in a black hole-to-white hole transition picture.

The result (5.37) shows that the gauge-invariant area is proportional to the integral of a modulus squared function, which is nonnegative as it should be. Below we show that it is also nonzero which makes it strictly positive. Let us for a moment consider the case where $g=0$, i.e.,

$$\left\langle\Psi_{\psi}^{\text{phys}}\left|\hat{F}_{A,b}\left(\tau\right)\right|\Psi_{\psi}^{\text{phys}}\right\rangle_{\text{phys}}=4\pi\int_{\mathbb{R}}dp_a\left|\frac{\alpha^2}{p_a^2}\psi_{a|b}\left(p_a\right)\tau-i\hbar\frac{d\psi_{a|b}\left(p_a\right)}{dp_a}\right|^2 \tag{5.38}$$

This will only vanish if for some $\tau=\tau_0$, the integrand vanishes. This would correspond to the classical singularity. The vanishing of this integrand yields a differential equation

$$\frac{d\psi_{a|b}\left(p_a\right)}{dp_a}=-\frac{i}{\hbar}\frac{\alpha^2\tau}{p_a^2}\psi_{a|b}\left(p_a\right) \tag{5.39}$$

with a solution

$$\psi_{a|b}\left(p_a;\tau_0\right)=Ce^{\frac{i}{\hbar}\frac{\alpha^2}{p_a}\tau_0}, \tag{5.40}$$

with C being an integration constant. But such a $\psi_{a|b}$ is non-normalizable and hence does not belong to $\mathscr{H}_{\text{phys}}$. One might try to construct a normalizable state out of the above functions as

$$\Upsilon\left(p_a\right)=\int d\tau'\,\xi\left(\tau'\right)e^{\frac{i\alpha^2}{\hbar p_a}\tau'}. \tag{5.41}$$

Plugging this in (5.39) leads to the condition $(\tau'-\tau_0)\xi(\tau')=0$, $\forall\tau'$. The only nontrivial function satisfying this condition is $\xi(\tau')=C\delta(\tau'-\tau_0)$. Plugging this back into (5.41) again yields (5.40). We thus conclude that there is no physical state or solution for which $\left\langle \hat{F}_{A,b}\left(\tau\right)\right\rangle_{\text{phys}} = 0$. This result holds not only for the special case in which $T = b$, but it is valid for any clock (4.42): the expectation value (5.37) will vanish only for the non-normalizable states

$$\psi_{a|b}\left(p_a;\tau_0\right) = Ce^{\frac{i}{\hbar}\left(\frac{\alpha^2\tau_0}{p_a}-\tilde{g}(p_a)\right)} \tag{5.42}$$

for a generic function $\tilde{g}\left(p_a\right) := -\hbar g\left(-\frac{\alpha^2}{p_a}\right)$. This means that the vanishing of the expectation value of the area of 2-spheres in this model is valid for any choice of a quantum clock (4.42).

This is a strong hint of singularity resolution, although not a direct proof. Hence in what follows, we prove this statement by studying the gauge-invariant extensions of the Kretschmann scalar and the expansion scalar.

5.3 Expansion scalar

The expansion scalar ϑ, particularly for null geodesics, is one of the most important indicators of whether or not a spacetime is singular. It describes whether the geodesics during their motion through spacetime get focused or defocused. An infinite focusing, corresponding to $\vartheta \to -\infty$ in some region of spacetime, signals the presence of singularity in that region.

Given a congruence of null geodesics described by a null tangent vector field k^μ, the expansion scalar for such a congruence that is affinely parametrized, i.e.,

$$k^\mu\nabla_\mu k^\nu = 0, \qquad k_\mu k^\mu = 0 \tag{5.43}$$

can be computed as

$$\vartheta = \nabla_\mu k^\mu \tag{5.44}$$

where ∇_μ is the covariant derivative associated to the Levi-Civita connection of the spacetime.

For a radial null geodesic

$$k^\mu = \left(k^0\left(x,\lambda\right), k^1\left(x,\lambda\right), 0, 0\right) \tag{5.45}$$

in the spacetime described by the metric (3.6) we obtain

$$\vartheta = \frac{1}{N}\frac{\dot{A}}{A} = -\frac{2}{\alpha}\frac{p_b}{a}, \tag{5.46}$$

where $\dot{A} := \frac{d}{d\lambda}\left(4\pi a(\lambda)^2\right)$. This classical gauge-dependent expression is different from the expression of the area of 2-spheres in that it depends on clock DoF p_b. As discussed in 5.1, we should consider the real part of its expectation value.

The gauge-dependent expansion scalar operator is

$$\hat{f}_S = \hat{\vartheta} = -\frac{2}{\alpha}\hat{p}_b\widehat{\frac{1}{a}}. \tag{5.47}$$

The expectation value of the gauge-invariant extension of this operator is derived using (5.20) as

$$\begin{aligned}\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau\right)\right\rangle_{\text{phys}} =&\mathfrak{Re}\left\{-\frac{2}{\alpha}\int_{\mathbb{R}} dp'_b\int_{\mathbb{R}} dp'_a\int_{\mathbb{R}} dp_b\int_{\mathbb{R}} dp_a\right.\\ &\times p_b\psi^*_{a|b}\left(p'_a\right)\psi_{a|b}\left(p_a\right)\delta\left(p'_b+\frac{\alpha^2}{p'_a}\right)\delta\left(p_b+\frac{\alpha^2}{p_a}\right)\\ &\times e^{-i\left(g(p_b)-g\left(p'_b\right)\right)}e^{\frac{i}{\hbar}\left(p_b-p'_b\right)\tau}\\ &\left.\times_a\left\langle p'_a\right|\widehat{\frac{1}{a}}\left|p_a\right\rangle_a\right\}\end{aligned} \tag{5.48}$$

To derive a general expression for $\left\langle p'_a\left|\widehat{\frac{1}{a^n}}\right|p_a\right\rangle$ we consider

$$\hat{a}^n\widehat{\frac{1}{a^n}} = \mathbb{I}, \tag{5.49}$$

and compute its matrix element as

$$\left\langle p'_a\left|\hat{a}^n\widehat{\frac{1}{a^n}}\right|p_a\right\rangle = \delta\left(p'_a-p_a\right). \tag{5.50}$$

Using $\left\langle p'_a\right|\hat{a} = i\hbar\frac{\partial}{\partial p'_a}\left\langle p'_a\right|$ yields

$$\left(i\hbar\right)^n\frac{\partial^n}{\partial p'^n_a}\left\langle p'_a\left|\widehat{\frac{1}{a^n}}\right|p_a\right\rangle = \delta\left(p'_a-p_a\right) \tag{5.51}$$

which is an n-th order differential equation for the Green's function $G_n\left(p'_a-p_a\right)$. The solution to this equation is

$$\begin{aligned}\left\langle p'_a\left|\widehat{\frac{1}{a^n}}\right|p_a\right\rangle &= \left(\frac{1}{i\hbar}\right)^n G_n\left(p'_a-p_a\right)\\ &= \left(\frac{1}{i\hbar}\right)^n\frac{\left(p'_a-p_a\right)^{n-1}\operatorname{sgn}\left(p'_a-p_a\right)}{2\left(n-1\right)!}\\ &= \left(\frac{1}{i\hbar}\right)^n\frac{\left|p'_a-p_a\right|^{n-1}\left[\operatorname{sgn}\left(p'_a-p_a\right)\right]^n}{2\left(n-1\right)!}.\end{aligned} \tag{5.52}$$

Notice that from this computation, there is no mathematical condition that exclude states with non-null support at $a = 0$. Also, unlike the exclusion of states with $p_a = 0$, which ensures the monotonicity of the clock, the $a = 0$ exclusion is not supported by any physical reason, although it could instead force the singularity resolution by simply not considering the cases in which $a = 0$. Thus the black hole is allowed to have states with support in $a = 0$.

Plugging in the (5.52) into yields

$$\begin{aligned}\mathfrak{Re}\left\langle \Psi_\psi^{\mathrm{phys}} \middle| \hat{F}_{\vartheta,T}(\tau) \middle| \Psi_\psi^{\mathrm{phys}} \right\rangle_{\mathrm{phys}} =& \mathfrak{Re}\left\{ \frac{i\alpha}{\hbar} \int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a\, \psi^*_{a|b}(p'_a) \frac{1}{p_a} \psi_{a|b}(p_a) \right. \\ & \left. \times e^{\frac{i}{\hbar}(\tilde{g}(p_a) - \tilde{g}(p'_a))} e^{\frac{i}{\hbar}\left(-\frac{\alpha^2}{p_a} + \frac{\alpha^2}{p'_a}\right)\tau} \operatorname{sgn}(p_a - p'_a) \right\}\end{aligned} \tag{5.53}$$

In the classical regime, a singularity is present whenever $\vartheta \to -\infty$. As a result we want to check whether in the quantum relational approach, we still have such behavior, i.e., whether $\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}(\tau) \right\rangle_{\mathrm{phys}} \to -\infty$ for some τ. To this end we note that

$$\left| \mathfrak{Re}\left\langle \hat{F}_{f_S,T}(\tau) \right\rangle_{\mathrm{phys}} \right| \leq \left| \left\langle \hat{F}_{f_S,T}(\tau) \right\rangle_{\mathrm{phys}} \right| = \left| \int dp'_a dp_a \dots \right| \leq \int dp'_a dp_a \left| \dots \right| \tag{5.54}$$

Applying this to (5.53) yields

$$\left| \mathfrak{Re}\left\langle \Psi_\psi^{\mathrm{phys}} \middle| \hat{F}_{\vartheta,T}(\tau) \middle| \Psi_\psi^{\mathrm{phys}} \right\rangle_{\mathrm{phys}} \right| \leq \frac{\alpha}{\hbar} \int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a \left| \psi^*_{a|b}(p'_a) \right| \frac{1}{|p_a|} \left| \psi_{a|b}(p_a) \right| < \infty. \tag{5.55}$$

In obtaining this result, i.e., non-divergence of the integral, we have used the properties of the Thiemann states (4.25) and the physical Hilbert space $\mathscr{H}_{\mathrm{phys}}$ as discussed before. In other words, since $\psi_{a|b} \in D_0(p_a)$ decays faster than any power of p_a as $p_a \to 0$, the integral $\int dp_a \frac{|\psi_{a|b}(p_a)|}{|p_a|}$ is finite. More concretely, $\left|\psi_{a|b}(p_a)\right| \sim |p_a|^n$ for all n as $p_a \to 0$. The finiteness of the gauge-invariant expansion parameter is therefore already ensured by the request of having a monotonic clock. As a result, the expansion scalar remains finite for all τ, which is a strong indicator of singularity resolution and avoidance of the conditions of the Penrose-Hawking theorems.

5.4 Kretschmann scalar

The Kretschmann scalar $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is one of the Riemann invariant whose divergence signals the presence of similarities. It classical expression for our metric (3.6) becomes

$$
\begin{aligned}
K =& \frac{4}{a^6}\left(a+b\frac{p_b^2}{\alpha^2}\right)^2+\frac{2p_b^2}{a^6\alpha^4}\left(ap_a-bp_b\right)^2\\
&+\frac{2}{N^4a^6}\left[2ab\left(\ddot{a}+\frac{\dot{N}}{\alpha}p_b\right)+\frac{N^2}{\alpha^2}p_b\left(ap_a-bp_b\right)\right]^2\\
&+\frac{1}{N^4a^6}\left[a\left(a\ddot{b}-b\ddot{a}\right)+\frac{1}{\alpha}\left(ap_a-bp_b\right)\left(a\dot{N}-2\frac{N^2}{\alpha}p_b\right)\right]^2
\end{aligned}
\tag{5.56}
$$

Using the definition of momenta $p_a=-\alpha\frac{\dot{b}}{N}$ and $p_b=-\alpha\frac{\dot{a}}{N}$ and the two EOM $\frac{d}{d\tau}\frac{\dot{a}}{N}=0=\frac{d}{d\tau}\frac{\dot{b}}{N}$ (see [21]), the expression simplifies

$$
K=\frac{4}{a^6}\left(a+b\frac{p_b^2}{\alpha^2}\right)^2+\frac{8p_b^2}{a^6\alpha^4}\left(ap_a-bp_b\right)^2. \tag{5.57}
$$

The operator expression for the above

$$
\hat{K}=4\widehat{\frac{1}{a^4}}+\frac{12}{\alpha^4}\hat{b}^2\hat{p}_b^4\widehat{\frac{1}{a^6}}+\frac{8}{\alpha^2}\hat{b}\hat{p}_b^2\widehat{\frac{1}{a^5}}-\frac{16}{\alpha^4}\hat{b}\hat{p}_b^3\hat{p}_a\widehat{\frac{1}{a^5}}+\frac{8}{\alpha^4}\hat{p}_b^2\hat{p}_a^2\widehat{\frac{1}{a^4}} \tag{5.58}
$$

is not symmetric and more importantly depends on both $\hat{b}$ and $\hat{p}_b$ in addition to its dependence of the system variables $\hat{a}$, $\hat{p}_a$. Hence, once again we need to use Eqs. (5.26) and (5.20) to derive the expectation value of its gauge-invariant extension. Using these equations, and considering $\hat{b}\left|p_b\right\rangle=i\hbar\frac{\partial}{\partial p_b}\left|p_b\right\rangle$, and Eqs. (5.52) and (5.54), we obtain

$$
\begin{aligned}
\left|\left\langle\hat{F}_{K,T}\left(\tau\right)\right\rangle_{\text{phys}}\right| \leq& \frac{1}{\hbar^4}\int_{\mathbb{R}}dp_a'\int_{\mathbb{R}}dp_a\,\left|\psi_{a|b}^*\left(p_a'\right)\right|\left|\psi_{a|b}\left(p_a\right)\right|\left|\left(p_a'-p_a\right)^3\right|\\
&\times\left|\frac{1}{3}+\frac{2}{3}\frac{p_a'^2}{p_a^2}\right.\\
&-\frac{\alpha^4}{20}\frac{\left(p_a'-p_a\right)^2}{p_a^4}\left(\left(\frac{\partial g\left(p_b\right)}{\partial p_b}-\frac{\tau}{\hbar}\right)^2+i\frac{\partial^2g\left(p_b\right)}{\partial p_b^2}\right)_{p_b=-\frac{\alpha^2}{p_a}}\\
&\left.+i\frac{\alpha^2}{6}\frac{\left(p_a'-p_a\right)}{p_a^2}\left(1+2\frac{p_a'}{p_a}\right)\left(\frac{\partial g\left(p_b\right)}{\partial p_b}-\frac{\tau}{\hbar}\right)_{p_b=-\frac{\alpha^2}{p_a}}\right|
\end{aligned}
\tag{5.59}
$$

and using the triangle inequality we get

$$\begin{aligned}\left|\left\langle \hat{F}_{K,T}\left(\tau\right)\right\rangle_{\text{phys}}\right| \leq & \frac{1}{\hbar^4}\int_{\mathbb{R}} dp'_a \int_{\mathbb{R}} dp_a \, \left|\psi^*_{a|b}\left(p'_a\right)\right| \left|\psi_{a|b}\left(p_a\right)\right| \left|\left(p'_a - p_a\right)^3\right| \\ & \times \left\{\frac{1}{3} + \frac{2}{3}\frac{p'^2_a}{p^2_a}\right. \\ & + \frac{\alpha^4}{20}\frac{\left(p'_a - p_a\right)^2}{p^4_a}\left(\left(\frac{\partial g\left(p_b\right)}{\partial p_b} - \frac{\tau}{\hbar}\right)^2 + \left|\frac{\partial^2 g\left(p_b\right)}{\partial p^2_b}\right|\right)_{p_b = -\frac{\alpha^2}{p_a}} \\ & \left. + \frac{\alpha^2}{6}\frac{\left|p'_a - p_a\right|}{p^2_a}\left|\left(1 + 2\frac{p'_a}{p_a}\right)\right|\left|\left(\frac{\partial g\left(p_b\right)}{\partial p_b} - \frac{\tau}{\hbar}\right)_{p_b = -\frac{\alpha^2}{p_a}}\right|\right\}\end{aligned} \tag{5.60}$$

Again we note that, since $\psi_{a|b} \in D_0\left(p_a\right)$ decays faster than any power of p_a as $p_a \to 0$ and $p_a \to \pm\infty$, the integral above remains finite for all finite values of τ. Thus this equation establishes a strict upper bound on the physical expectation value of the Kretschmann scalar. Consequently, for any finite relational time τ, the expectation value $\left|\left\langle \hat{F}_{K,T}(\tau)\right\rangle_{\text{phys}}\right|$ is bounded by a finite, purely quadratic polynomial in τ. This demonstrates that the Kretschmann scalar never diverges to infinity, confirming that the classical curvature singularity is successfully resolved in the quantum relational framework.

6 Example: Gaussian states

In this section we present some concrete results related to the general framework we considered until now. To this end and to be able to perform the computations analytically, we consider physical Gaussian states in the following form

$$\psi^{(n)}_{a|b}\left(p_a\right) = N_G \Theta\left(-p_a\right) p^n_a e^{-\frac{p^2_a}{2\sigma^2}} \tag{6.1}$$

where N_G is a normalization constant, $n \in \mathbb{N}\backslash\{0\}$, and σ is the width of the Gaussian. This quantum state is real, $\psi^{(n)}_{a|b} = \psi^{(n)*}_{a|b}$ and represents the initial condition of the black hole interior, in which the momentum p_a is sharply distributed around the only peak of the Gaussian at $\bar{p}_a = -\sigma\sqrt{n} < 0$. The fact that such states are centered around a negative value for p_a is consistent with the classical solutions. Recall that the EOM resulting from Eq. (3.7) yield $p_a = \text{sgn}(\lambda)\alpha$, and because the black hole solution corresponds to the case in which $\lambda \in [-\sqrt{2GM}, 0)$, we conclude $p_a = -\alpha < 0$ in the black hole region. Moreover, such states have null support in $p_a = 0$, which was one of the fundamental requirements for the construction of the physical Hilbert space, which also assures the monotonicity of the quantum clock.

The states (6.1) are not the exact wave functions of (4.5) belonging to $D_0\left(p_a\right)$ in (4.6). However, they belong to the closure of their span, i.e, the same physical Hilbert space $\mathscr{H}_{\text{phys}}=\overline{D_0\left(p_a\right)}=L^2\left(\mathbb{R},dp_a\right)$. Moreover, they share the relevant features such as rapid decay for $p_a\to 0$ and $p_a\to\pm\infty$, ensuring the convergence of the integrals involved with a suitable choice of $n\in\mathbb{N}\backslash\{0\}$. Also, notice that the states (6.1) allow the case for which $a=0$. This is seen by writing the Fourier transform of $\psi^{(n)}_{a|b}\left(p_a\right)$ as

$$\tilde{\psi}^{(n)}_{a|b}\left(a\right)=\int_{\mathbb{R}}dp_aN_G\Theta\left(-p_a\right)p_a^ne^{-\frac{p_a^2}{2\sigma^2}}e^{\frac{i}{\hbar}ap_a} \tag{6.2}$$

which implies

$$\begin{aligned}\tilde{\psi}^{(n)}_{a|b}\left(0\right)=&\int_{\mathbb{R}}dp_a\,N_G\Theta\left(-p_a\right)p_a^ne^{-\frac{p_a^2}{2\sigma^2}}\\=&N_G(-1)^n2^{\frac{n-1}{2}}\sigma^{n+1}\Gamma\left(\frac{n+1}{2}\right)\\\neq&0\end{aligned} \tag{6.3}$$

Thus the results that we will obtain in this section hold even if the black hole is in a quantum state with $a=0$, which classically corresponds to a singularity.

Since

$$\psi_{\text{kin}}\left(p_a,p_b\right)\Big|_{C_H=0}=\psi_{\text{kin}}\left(p_a,-\frac{\alpha^2}{p_a}\right)=\psi_{a|b}\left(p_a\right) \tag{6.4}$$

we can express such a kinematical state as

$$\psi^{(n)}_{\text{kin}}\left(p_a,p_b\right)=N_G\Theta\left(-p_a\right)p_a^ne^{-\frac{p_a^2}{2\sigma^2}}e^{-\frac{\left(p_b+\frac{\alpha^2}{p_a}\right)^2}{2\delta^2}}\in L^2(\mathbb{R}^2,dp_adp_b)=\mathscr{H}_{\text{kin}}, \tag{6.5}$$

for some parameter $\delta\in\mathbb{R}$. We can also see that

$$\psi^{(n)}_{\text{kin}}\left(p_a,p_b\right)\Big|_{C_H=0}=\psi^{(n)}_{\text{kin}}\left(p_a,-\frac{\alpha^2}{p_a}\right)=N_G\Theta\left(-p_a\right)p_a^ne^{-\frac{p_a^2}{2\sigma^2}}=\psi^{(n)}_{a|b}\left(p_a\right) \tag{6.6}$$

as desired.

6.1 Area of 2-spheres

To study the area of 2-spheres using the states (6.1), we choose the quantum relational clock operator to be $\hat{T}=\hat{b}$, or equivalently, $g(p_b)=0$. Using this condition, and looking at the first two lines of (5.37) and recalling that the states are real, $\psi^{(n)*}_{a|b}=\psi^{(n)}_{a|b}$, we obtain

$$\left\langle\hat{F}_{A,T}\left(\tau\right)\right\rangle_{\text{phys}}=4\pi\int_{\mathbb{R}}dp_a\frac{\alpha^4}{p_a^4}\tau^2\left|\psi^{(n)}_{a|b}\left(p_a\right)\right|^2+4\pi\hbar^2\int_{\mathbb{R}}dp_a\left|\frac{\partial\psi^{(n)}_{a|b}\left(p_a\right)}{\partial p_a}\right|^2\quad. \tag{6.7}$$

Given the form of the first term in the above, the states (6.1) that will yield well-defined results are the ones with $n \geq 2$, so we will restrict our attention to this case throughout this section.

It is evident from Eq. (6.7) that at $\tau = 0$ one obtains the minimum area of 2-spheres

$$A_{\text{min}} = 4\pi\hbar^2 \int_{\mathbb{R}} dp_a \left| \frac{\partial \psi^{(n)}_{a|b}(p_a)}{\partial p_a} \right|^2 \tag{6.8}$$

which is strictly positive, even if the state corresponds to $a = 0$, as discussed in Eq. (6.3). As we will see below, the nonvanishing of this minimum area is a direct consequence of the quantum fluctuations resulted from the uncertainty in $\hat{a}$. This can be seen more clearly by writing the minimum area (6.8) in terms of the uncertainty in $\hat{a}$, namely, $\Delta a = \sqrt{\langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2}$. In the momentum representation $\hat{a} = i\hbar \frac{\partial}{\partial p_a}$, we ca wite

$$\left\langle \hat{a}^2 \right\rangle = \int_{\mathbb{R}} dp_a\, \psi^{(n)*}_{a|b}(p_a) \left(-\hbar^2 \frac{\partial^2}{\partial p_a^2} \right) \psi^{(n)}_{a|b}(p_a) = \hbar^2 \int_{\mathbb{R}} dp_a \left| \frac{\partial \psi^{(n)}_{a|b}(p_a)}{\partial p_a} \right|^2, \tag{6.9}$$

where boundary terms have been assumed to vanish due to decay of $\psi^{(n)}_{a|b}(p_a)$ at infinities. On other hand, since the states (6.1) are real, we also have the well-known result

$$\langle \hat{a} \rangle = \int_{\mathbb{R}} dp_a\, \psi^{(n)*}_{a|b}(p_a)\, i\hbar \frac{\partial \psi^{(n)}_{a|b}(p_a)}{\partial p_a} = \frac{i\hbar}{2} \int_{\mathbb{R}} dp_a \frac{\partial}{\partial p_a} \left(\psi^{(n)}_{a|b}(p_a) \right)^2 = 0 \tag{6.10}$$

where we have again assumed that the states vanish at the boundaries. From these it is evident that we can write

$$A_{\text{min}} = 4\pi \left(\Delta a \right)^2. \tag{6.11}$$

This very illuminating result shows that the minimum non-zero area is rooted in quantum effects as expected. This is also consistent with what we showed in Eq. (5.40), that it is not possible to have $\Delta a = 0$, as it will result in non-normalizable states, i.e., states that do not belong to the physical Hilbert space $\mathscr{H}_{\text{phys}}$.

To find a more explicit expression for (6.8), we first find the normalization constant N_G using the normalization condition $\int_{\mathbb{R}} dp_a |\psi^{(n)}_{a|b}(p_a)|^2 = 1$, as

$$N_G = \sqrt{\frac{2}{\sigma^{2n+1}\Gamma\left(n + \frac{1}{2}\right)}} \tag{6.12}$$

where we have used

$$\int_0^\infty dp_a\, p_a^{2k} e^{-\frac{p_a^2}{\sigma^2}} = \frac{\sigma^{2k+1}}{2} \Gamma\left(k + \frac{1}{2}\right). \tag{6.13}$$

Replacing this back into (6.1) and plugging it in (6.8) yields

$$A_{\min} = 4\pi\hbar^2 \frac{1}{\sigma^2}\left(\frac{4n-1}{4n-2}\right). \tag{6.14}$$

Once again we can express the above result in terms of the uncertainty in configuration variable a. Using $\hat{a} = i\hbar\frac{\partial}{\partial p_a}$ and computing the corresponding uncertainty over the states (6.1) we obtain

$$\Delta a = \sqrt{\langle \hat{a}^2\rangle - \langle \hat{a}\rangle^2} \quad = \frac{\hbar}{\sigma}\sqrt{\frac{4n-1}{4n-2}}. \tag{6.15}$$

which again leads to (6.11).

In Appendix B, we will show that the result $A_{\min} > 0$ for Gaussian states holds not only for the specific clock choice $\hat{T} = \hat{b}$, but for all $\hat{T}$ that satisfy $[\hat{T}, \hat{H}_C] = i\hbar$ which are defined up to $\hat{T} \to \hat{T} + h(\hat{H}_C)$. We will also extend such result to any real state $\psi_{a|b}(p_a)$, without any particular restriction to Gaussian functions.

Another observation about $A_{\min}$ is the following. We stated before that the peak of the Gaussian (6.1) is at $\bar{p}_a \propto \sigma$ and thus σ has dimensions of momentum. Using

$$l_P = \sqrt{\hbar G}, \quad p_P = \frac{\hbar}{l_P} = \sqrt{\frac{\hbar}{G}} \tag{6.16}$$

in units where speed of light $c = 1$ and replacing one $\hbar$ from ℓ_p and the other $\hbar$ from p_p in (6.14) yields

$$A_{\min} = 4\pi\ell_p^2\left(\frac{p_p}{\sigma}\right)^2\left(\frac{4n-1}{4n-2}\right), \tag{6.17}$$

where $4\pi\ell_p^2 = A_p$ is the Planck area. From this we see that, for a spread $\sigma \gg p_p$, the minimum area goes to zero. In the same way, for large n and $\sigma \sim p_p$, we get $A_{\min} = A_p$.

If one wants to set the area (6.14) equal to the loop quantum gravity area gap, $A_{\text{LQG}} = 8\pi\gamma l_P^2$, one obtains

$$\sigma = \frac{\hbar}{l_P}\sqrt{\frac{1}{2\gamma}\left(\frac{n-\frac{1}{4}}{n-\frac{1}{2}}\right)}, \tag{6.18}$$

which for large n reduces to

$$\sigma = \frac{\hbar}{\sqrt{2\gamma}l_P} = \frac{p_P}{\sqrt{2\gamma}}. \tag{6.19}$$

6.2 Expansion scalar

Let us turn our attention to the expansion scalar (5.53). Choosing $g = 0$ or equivalently $\hat{T} = \hat{b}$, we obtain

$$\begin{aligned}\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau\right)\right\rangle_{\text{phys}} =&\mathfrak{Re}\left\{\frac{i\alpha}{\hbar}N_G^2\int_0^\infty dp'_a\int_0^\infty dp_a\ \left(-1\right)^{2n-1}p'^n_a p^{n-1}_a\right.\\ &\left.\times e^{-\frac{1}{2\sigma^2}\left(p_a^2+p'^2_a\right)}e^{-\frac{i}{\hbar}\left(-\frac{\alpha^2}{p_a}+\frac{\alpha^2}{p'_a}\right)\tau}\operatorname{sgn}\left(p'_a-p_a\right)\right\}\end{aligned} \tag{6.20}$$

Using Euler's formula $e^{i\phi} = \cos\left(\phi\right) + i\sin\left(\phi\right)$ in the above yields

$$\begin{aligned}\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau\right)\right\rangle_{\text{phys}} =&\frac{\alpha}{\hbar}N_G^2\int_0^\infty dp'_a\int_0^\infty dp_a\ \left(-1\right)^{2n-1}p'^n_a p^{n-1}_a\\ &\times e^{-\frac{1}{2\sigma^2}\left(p_a^2+p'^2_a\right)}\left[\sin\left(\frac{\alpha^2\tau}{\hbar}\left(\frac{1}{p'_a}-\frac{1}{p_a}\right)\right)\right]\\ &\times\operatorname{sgn}\left(p'_a-p_a\right)\end{aligned} \tag{6.21}$$

It is immediately seen from the above that

$$\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau=0\right)\right\rangle_{\text{phys}} = 0. \tag{6.22}$$

Moreover, the derivative of the expansion scalar is

$$\begin{aligned}\frac{d}{d\tau}\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau\right)\right\rangle_{\text{phys}} =&\frac{\alpha}{\hbar}N_G^2\int_0^\infty dp'_a\int_0^\infty dp_a\ \left(-1\right)^{2n-1}p'^n_a p^{n-1}_a\\ &\times e^{-\frac{1}{2\sigma^2}\left(p_a^2+p'^2_a\right)}\left[\cos\left(\frac{\alpha^2\tau}{\hbar}\left(\frac{1}{p'_a}-\frac{1}{p_a}\right)\right)\right]\\ &\times\frac{\alpha^2}{\hbar}\left(\frac{1}{p'_a}-\frac{1}{p_a}\right)\operatorname{sgn}\left(p'_a-p_a\right).\end{aligned} \tag{6.23}$$

Given the limits of the integral, the term $p'^n_a p^{n-1}_a$ in the integrand above is strictly positive, while the term $\left(-1\right)^{2n-1}$ is strictly negative. The last line can be written as $-\frac{|p'_a-p_a|}{p_a p'_a}$ which again given the integral limits is strictly negative. Hence, at $\tau = 0$ we get

$$\frac{d}{d\tau}\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau\right)\right\rangle_{\text{phys}} > 0. \tag{6.24}$$

The two results (6.22) and (6.24) show that at $\tau = 0$, which corresponds to the classical singularity, the expectation value of the physical expansion scalar vanishes and thus the singularity does not exist anymore. Moreover, these two equations show that a bounce from the black hole spacetime to the white hole spacetime happens at $\tau = 0$. Furthermore, we see from (6.21) that since sin is an odd function of τ we can conclude

$$\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau\right)\right\rangle_{\text{phys}} = -\mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(-\tau\right)\right\rangle_{\text{phys}} . \tag{6.25}$$

This result further affirms that $\tau = 0$ corresponds to the bounce from the a trapped to an antitrapped region.

It is also illuminating to study the asymptotic behavior of the gauge-invariant expansion scalar for $\tau \to \pm\infty$. For this, we will employ the Riemann-Lebesgue lemma which states that, given an integrable function $f \in L^1[0,\infty)$ such that $\int_{\mathbb{R}^n} |f(x)|dx < \infty$, its Fourier transform vanishes at infinity, i.e., $\left|\int_{\mathbb{R}^n} f(x)\mathrm{e}^{-\mathrm{i}x\xi}dx\right| \to 0$ as $|\xi| \to \infty$. Let us look at the p_a integral in (6.20)

$$I_{p_a} = \int_0^\infty dp_a\, p_a^{n-1} e^{-\frac{p_a^2}{2\sigma^2}} e^{i\frac{\alpha^2}{\hbar p_a}\tau} \mathrm{sgn}\left(p'_a - p_a\right) \tag{6.26}$$

Making a substitution $x = \frac{\alpha^2}{\hbar p_a} \Rightarrow dx = -\frac{\alpha^2}{\hbar p_a^2} dp_a$ yields

$$I_{p_a} = \int_0^\infty dx \left(\frac{\alpha^2}{\hbar}\right)^n \frac{1}{x^{n+1}} e^{-\frac{\alpha^4}{2\hbar^2 x^2\sigma^2}} \mathrm{sgn}\left(p'_a - \frac{\alpha^2}{\hbar x}\right) e^{ix\tau} \quad . \tag{6.27}$$

The function $\left(\frac{\alpha^2}{\hbar}\right)^n \frac{1}{x^{n+1}} e^{-\frac{\alpha^4}{2\hbar^2 x^2\sigma^2}} \mathrm{sgn}\left(p'_a - \frac{\alpha^2}{\hbar x}\right)$ belongs to $L^1[0,\infty)$ for each of its branches corresponding to $\mathrm{sgn}\left(p'_a - \frac{\alpha^2}{\hbar x}\right)$ being ± 1, i.e.,

$$\left|\int_0^\infty dx \left(\frac{\alpha^2}{\hbar}\right)^n \frac{1}{x^{n+1}} e^{-\frac{\alpha^4}{2\hbar^2 x^2\sigma^2}}\right| < \infty. \tag{6.28}$$

This is because at $x \to 0^+$ the Gaussian $e^{-\frac{\alpha^4}{2\hbar^2 x^2\sigma^2}} \to 0$ faster than any power of x so the integral vanishes, and at $x \to \infty$ the term $x^{-(n+1)} \to 0$ for $n \geq 0$. Hence, by Riemann-Lebesgue lemma we conclude that its Fourier transform (6.26) decays to zero as $\tau \to \pm\infty$, which consequently means that

$$\lim_{\tau \to \pm\infty} \mathfrak{Re}\left\langle \hat{F}_{\vartheta,T}\left(\tau\right)\right\rangle_{\text{phys}} = 0. \tag{6.29}$$

So far we have obtained the behavior of the expansion scalar at $\tau \to 0$ and $\tau \to \pm\infty$, and have noticed that it is an odd function in τ. Computing an explicit expression for the expansion parameter for a general τ, however, is not possible given the complicated form of the integral (6.20). One can, however, compute this integral numerically. The plot of the numerical computation of the expansion scalar is presented in Fig. 1. It

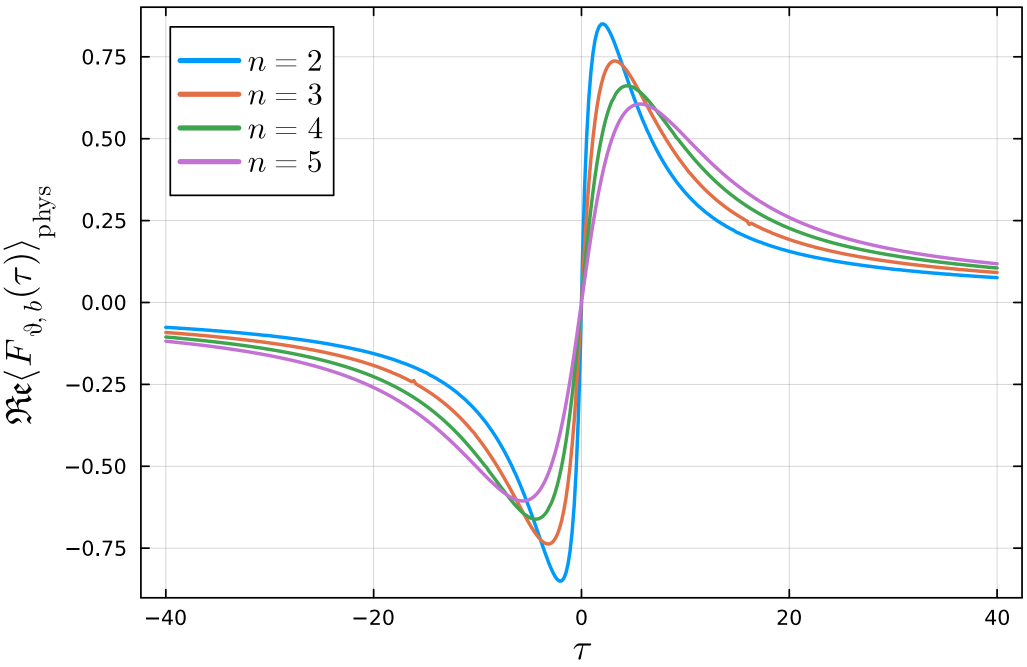

Figure 1. Expectation value of the (gauge-invariant) expansion parameter evolution, for the quantum clock $\hat{T} = \hat{b}$ and the state $\psi^{(n)}_{a|b}(p_a) = N_G \Theta(-p_a)\, p_a^n e^{-\frac{p_a^2}{2\sigma^2}}$. In the specific case presented here, for the plot we have used $\hbar = \sigma = \alpha = 1$.

is clearly seen from this plot that the expansion parameter inside the black hole for finite negative values of τ is negative, but near the deep quantum regime, i.e., $\tau \to 0$, it reverses behavior and vanishes at $\tau = 0$. This is distinctly different from the classical behavior where the expansion scalar goes to $-\infty$ as $\tau \to 0$, where the singularity resides. Moreover, in the quantum theory, the expansion scalar bounces to positive values and remains positive and finite for finite positive values of τ, which is the characteristic behavior of an anti-trapped region or a white hole.

It is worth mentioning that the same behavior as the one in Fig. 1 can be reproduced as $\vartheta(\tau)$ by considering an area function $A = A_{\text{min}} + \beta\tau^2$ whose expansion becomes

$$\vartheta(\tau) = \frac{1}{A(\tau)} \frac{dA(\tau)}{d\tau} = \frac{2\beta\tau}{A_{\text{min}} + \beta\tau^2}, \tag{6.30}$$

where $A_{\text{min}} > 0$ and $\beta \in \mathbb{R}$ is some acceleration parameter. This also shows that the classical behavior (when $A_{\text{min}} = 0$ for $\hbar \to 0$; see Eq. (6.14)) yields $\vartheta(\tau) \propto \frac{1}{\tau}$ which is the qualitative behavior one expects from a classical expansion scalar in the interior of the Schwarzschild black hole [42, 43].

These results explicitly show that the singularity is resolved in this framework as a direct consequence of quantum effects, including the uncertainty in metric components, i.e., in a, and associated quantum fluctuations.

7 Conclusions

In this work, we have studied the dynamics of the interior of a Schwarzschild black hole, which is isometric to the Kantowski-Sachs metric, in the quantum relational formalism. The model has a four dimensional phase space with configuration variables a, b and conjugate momenta p_a, p_b. Since the Hamiltonian of the model can be written in a separable way as the Hamiltonian of the clock plus the Hamiltonian of the system, i.e., $C_H = p_b + \frac{\alpha^2}{p_c}$, we were able to implement the Page-Wootters formalism and construct quantum clocks corresponding to b. The physical Hilbert space of the model $\mathscr{H}_{\text{phys}} \cong L^2(\mathbb{R}, dp_a)$ was then constructed using the refined algebraic quantization. The physical states were written down using Neuser-Thiemann's states [32, 33] which decay fast enough at $p_a \to 0$ and $p_a \to \pm\infty$. This ensured the monotonicity of the quantum clock and allowed the Hamiltonian constraint to have a well-defined action in the standard Schrodinger representation.

After setting this stage, we studied the general form and properties of the expectation values of three gauge-invariant quantum observables, pertinent to the singularity resolution, namely the area of 2-speheres, the expansion scalar, and the Kretschmann scalar. We computed these expectation values as a function of the relational time τ.

For the expectation area of 2-spheres inside the black hole, we showed that i) it follows a quadratic evolution equation in τ which points to a black-to-white hole bounce, ii) it has a nonzero minimum A_{min}.

For the expansion scalar of radial null geodesics, we found results consistent with the above. In particular, we found that it is finite everywhere in the interior.

Similarly, we found that the bound on the expectation value of the Kretschmann scalar is also quadratic in τ, and thus the Kretschmann is finite everywhere inside the black hole.

These general results imply that the singularity of the black hole is resolved in this approach. To show these results in a more concrete way, we chose specific physical states which were real and Gaussian, and computed the above expectation values specific for such states.

Our analysis of the area of 2-spheres for these states revealed that the minimum area is actually proportional to the uncertainty in a, namely the quantum fluctuations, and the associated uncertainty in a prohibits the minimum area to vanish.

We also studied the expansion scalar for these states and analytically showed that it vanishes at $\tau \to \pm\infty$ and at $\tau = 0$, where its derivative with respect to τ is positive. Moreover, it is an odd function of τ around $\tau = 0$ and never diverges inside the black hole. These results were also confirmed by directly computing the expansion scalar numerically. Thus, once again we confirmed that the singularity of the black hole is resolved, and there is a bounce from the black hole to a white hole at $\tau = 0$, which used to be a classical singularity.

As a result of all of the above observations, we conclude that applying the relational framework and the usual Schrodinger representation to the interior of the Schwarzschild black hole resolves its singularity.

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A Review of RAQ

In constrained systems with continuous spectrum, a set of first class constraints $\hat{C}_I$ and a kinematical unconstrained square integrable Hilbert space $\mathscr{H}_{\text{kin}}$, the physical states would be the ones that satisfy $\hat{C}_I \left|\psi\right\rangle = 0$. As mentioned above, for such systems with 0 eigenvalue, the physical states lie outside of $\mathscr{H}_{\text{kin}}$. An example is a system whose momentum p is constrained to vanish. Therefore, physical states satisfy $\hat{p}\Psi_{\text{phys}}(p) = p\Psi_{\text{phys}}(p) = 0$ and thus we get $\Psi_{\text{phys}}(p) = \delta(p)$. This is a distribution and is not normalizable, so it lies outside of $\mathscr{H}_{\text{kin}}$. To find the physical states which lie in a larger space, one applies RAQ.

The arena of RAQ includes three spaces, the so called, Gelfand triple

$$\Phi \subset \mathscr{H}_{\text{kin}} \subset \Phi^*, \tag{A.1}$$

where

- $\mathscr{H}_{\text{kin}}$ is the kinematical Hilbert space of square integrable functions on which the classical unconstrained system is represented,

- $\Phi \subset \mathscr{H}_{\text{kin}}$ is a dense subspace of test states whose states are infinitely differentiable and all derivatives rapidly decay (nuclear Fréchet space), which is a "seed" space for physical states,
- Φ^* is the topological dual of Φ, consisting of all continuous linear functionals on Φ. Physical states live in Φ^*.

RAQ has the following steps:

1. Kinematical Hilbert space:
    (a) **States:** This is a square integrable space

    $$L^2(X, d\mu) = \left\{ \psi : X \to \mathbb{C} | \int_X \psi^*(x) \psi(x) d\mu(x) < \infty \right\}. \tag{A.2}$$

    (b) **Operators:** The classical algebra of the unconstrained system is represented on the kinematical Hilbert space $\mathscr{H}_{\text{kin}}$. Classical operators and constraints C_I are represented as self-adjoint operators $\hat{C}_I$ on $\mathscr{H}_{\text{kin}}$.

    (c) **Inner product:** The kinematical inner product is defined as

    $$\langle \psi_1 | \psi_2 \rangle_{\text{kin}} = \int_X \psi_1^*(x) \psi_2(x) d\mu(x). \tag{A.3}$$

2. Space of test functions Φ:
    (a) **States:** elements (or test states) of a dense subspace, $\Phi \subset \mathscr{H}_{\text{kin}}$, which is a nuclear Fréchet space. The functions are typically infinitely differentiable and fall off quickly at infinity and possibly other points, e.g., at zero.

    (b) **Operators:** If $\hat{O}$ is an operator densely defined on $\mathscr{H}_{\text{kin}}$, the corresponding operator acting on Φ is then denoted by $\hat{O}'$ and its action on Φ is defined by restricting its domain to Φ. We also require Φ to have three properties: be invariant under the action of the quantum constraint algebra, i.e., $\hat{C}_I \phi \in \Phi$ for all $\phi \in \Phi$, be stable under the observable algebra $\hat{O}\phi \in \Phi$, $\forall \phi \in \Phi$ for $\left[\hat{O}, \hat{C}_I\right] = 0$ on Φ, and be equipped with a finer topology than $\mathscr{H}_{\text{kin}}$. The action of any operator $\hat{O}$ should not take us out of Φ, introduce singularities, or ruin the decay properties of the state.

    (c) **Inner product:** operationally we use the same inner product on Φ, as the one on $\mathscr{H}_{\text{kin}}$.

3. Space of linear functionals Φ^*:

(a) **States:** The space Φ^* is the space of all continuous linear functionals, $\Psi : \Phi \to \mathbb{C}$, on Φ. For a state $\phi \in \Phi$, a corresponding $\Psi_\phi \in \Phi^*$ is a linear functional

$$\begin{aligned} \Psi_\phi : & \Phi \to \mathbb{C} \\ & f \mapsto \langle \phi \mid f \rangle_{\text{kin}} \end{aligned} \tag{A.4}$$

in Φ^* where

$$\Psi_\phi(f) := \langle \phi \mid f \rangle_{\text{kin}} = \int_X \phi^* (x) f(x)\, d\mu(x)\,. \tag{A.5}$$

Note that not all elements in Φ^* look like Ψ_ϕ since Φ^* is strictly larger than Φ (and $\mathscr{H}_{\text{kin}}$) and contains distributions, such as the Dirac delta, that cannot be written as $\langle \phi \mid \cdot \rangle_{\text{kin}}$ for any square-integrable ϕ.

(b) **Operators:** The action of $\hat{O}$ or equivalently $\hat{O}'$ on Φ^* is defined, under certain conditions[3], as

$$\left(\hat{O}\Psi_\phi\right)[f] := \Psi\left[\hat{O}'^\dagger f\right],\ \forall \phi, f \in \Phi, \Psi \in \Phi^*. \tag{A.6}$$

(c) **Inner product:** There is no global inner product on the entirety of Φ^* and in fact we do not need one (see next item).

4. Physical Hilbert space $\mathscr{H}_{\text{phys}}$:

(a) **States:** The physical states are defined as distributions $\Psi_{\text{phys}} \in \Phi^*$, such that assuming self-adjointness, $\hat{C}_I'^\dagger = \hat{C}_I'$,

$$\left(\hat{C}_I \Psi_\phi^{\text{phys}}\right)[f] = \Psi_\phi^{\text{phys}}\left[\hat{C}_I'^\dagger f\right] = \Psi_\phi^{\text{phys}}\left[\hat{C}_I' f\right] = 0, \quad \forall \phi, f \in \Phi. \tag{A.7}$$

Note that for non-self-adjoint operators, for this to be mathematically well-defined, the kinematic adjoint $\hat{O}'^\dagger$ must leave the test space invariant, $\hat{O}^\dagger \Phi \subset \Phi$. In any case, we denote the space of all physical solutions as

$$\mathcal{V}_{\text{phys}} := \left\{\Psi_\phi^{\text{phys}} \in \Phi^* \mid \left(\hat{C}_I \Psi_{\text{phys}}\right)[f] = 0,\ \forall \phi, f \in \Phi, \forall I\right\}. \tag{A.8}$$

These physical states are defined as

$$\begin{aligned} \Psi_\phi^{\text{phys}} : & \Phi \to \mathbb{C} \\ & f \mapsto \eta(\phi)[f] \end{aligned} \tag{A.9}$$

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[3] One defines a $\star$ that mimics the Hermitian conjugate operation for observables, and then $\left(\hat{A}'\Psi\right)[\phi] = \Psi\left[\hat{A}^\star \phi\right]$. But if the representation π of operators on Φ is faithful, i.e., $\pi(A^\star) = \pi(A)^\dagger$, then $\left(\hat{A}'\Psi\right)[\phi] = \Psi\left[\hat{A}^\dagger \phi\right]$

where, given $\hat{U}(g) = e^{i\lambda^I \hat{C}_I}$, the unitary representation of the group G of the constraint Lie algebra with λ^I, $I = 1, \ldots, N$ being real-valued group coordinates/parameters, we have

$$\Psi_\phi^{\text{phys}}[f] \coloneqq \eta(\phi)[f] = \frac{1}{(2\pi)^N} \int_G d\mu(g) \langle \hat{U}(g)\phi \mid f \rangle_{\text{kin}}. \tag{A.10}$$

The above map

$$\begin{aligned} \eta : &\Phi \to \mathcal{V}_{\text{phys}} \\ &\phi \mapsto \Psi_\phi^{\text{phys}}[\cdot] \end{aligned} \tag{A.11}$$

is called a rigging map. It takes a kinematic test state and projects it onto a physical solution. One can show that unimodularity of G implies that η is compatible with the adjointness relations of the observable algebra, and $\left\langle \Psi_{\phi_1}^{\text{phys}} \mid \Psi_{\phi_2}^{\text{phys}} \right\rangle_{\text{phys}} = \left\langle \Psi_{\phi_2}^{\text{phys}} \mid \Psi_{\phi_1}^{\text{phys}} \right\rangle_{\text{phys}}^*$ or equivalently $\eta(\phi_1)[\phi_2] = \eta(\phi_2)[\phi_1]^*$. Note that $\eta(\phi)[\cdot]$ defined as

$$\begin{aligned} \eta(\phi) : &\Phi \to \mathbb{C} \\ &f \mapsto \Psi_\phi^{\text{phys}}[f] \end{aligned} \tag{A.12}$$

is an element of $\mathcal{V}_{\text{phys}}$, while $\eta(\cdot)$ itself is the above rigging map.

(b) **Operators:** The operators act similarly to their action on the generic Φ^* space; we have

$$\left(\hat{O}\Psi_\phi^{\text{phys}}\right)[f] \coloneqq \Psi_\phi^{\text{phys}}\left[\hat{O}'^\dagger f\right], \quad \forall f \in \Phi,\ \Psi_\phi^{\text{phys}} \in \mathcal{V}_{\text{phys}}. \tag{A.13}$$

(c) **Inner product:** The physical inner product is now defined as

$$\begin{aligned} \left\langle \Psi_{\phi_1}^{\text{phys}} | \Psi_{\phi_2}^{\text{phys}} \right\rangle_{\text{phys}} &= \langle \eta(\phi_1) \mid \eta(\phi_2) \rangle_{\text{phys}} \\ &\coloneqq \eta(\phi_1)[\phi_2] \\ &= \frac{1}{(2\pi)^N} \int_G d\mu(g) \left\langle \hat{U}(g)\phi_1 \mid \phi_2 \right\rangle_{\text{kin}}, \end{aligned} \tag{A.14}$$

where we demand $\eta(\phi)[\phi] \geq 0$. Notice that, given that we are working in a unitary representation of G where $\hat{U}^\dagger(g) = \hat{U}(g^{-1})$ and if G is a unimodular group, one can also write the above as

$$\begin{aligned} \left\langle \Psi_{\phi_1}^{\text{phys}} | \Psi_{\phi_2}^{\text{phys}} \right\rangle_{\text{phys}} &= \frac{1}{(2\pi)^N} \int_G d\mu(g) \left\langle \hat{U}(g)\phi_1 \mid \phi_2 \right\rangle_{\text{kin}} \\ &= \frac{1}{(2\pi)^N} \int_G d\mu(g) \left\langle \phi_1 \mid \hat{U}^\dagger(g)\phi_2 \right\rangle_{\text{kin}} \\ &= \frac{1}{(2\pi)^N} \int_G d\mu(g) \left\langle \phi_1 \mid \hat{U}\left(g^{-1}\right)\phi_2 \right\rangle_{\text{kin}} \end{aligned} \tag{A.15}$$

Now if we change the variables as $h = g^{-1}$, and since for unimodular groups, the Haar measure is invariant under inversion, i.e., $d\mu(g) = d\mu\left(g^{-1}\right) = d\mu(h)$, we can write this as (since ϕ_1 does not depend on h or equivalently the group parameters λ^I)

$$\begin{aligned}\left\langle \Psi^{\text{phys}}_{\phi_1} | \Psi^{\text{phys}}_{\phi_2}\right\rangle_{\text{phys}} &= \frac{1}{(2\pi)^N}\int_G d\mu(h)\left\langle \phi_1 \mid \hat{U}\left(h\right)\phi_2\right\rangle_{\text{kin}} \\ &= \frac{1}{(2\pi)^N}\left\langle \phi_1 \middle| \left(\int_G d\mu(h)\, \hat{U}\left(h\right)\phi_2\right)\right\rangle_{\text{kin}} \\ &= \left\langle \phi_1 \middle| \Psi^{\text{phys}}_{\phi_2}\right\rangle_{\text{kin}}.\end{aligned}\tag{A.16}$$

Furthermore, given that $\hat{U} = e^{-i\lambda^I \hat{C}_I}$, $I = 1, \ldots, N$, we can write the second line above as

$$\begin{aligned}\left\langle \phi_1 \middle| \left(\frac{1}{(2\pi)^N}\int_G d\mu(h)\, \hat{U}\left(h\right)\phi_2\right)\right\rangle_{\text{kin}} &= \left\langle \phi_1 \middle| \left(\frac{1}{(2\pi)^N}\int_G d^N\lambda\, \hat{U}\left(C_I\right)\phi_2\right)\right\rangle_{\text{kin}} \\ &= \left\langle \phi_1 \middle| \left(\frac{1}{(2\pi)^N}\int_G d^N\lambda\, e^{-i\lambda^I \hat{C}_I}\phi_2\right)\right\rangle_{\text{kin}} \\ &= \left\langle \phi_1 \middle| \left(\frac{1}{(2\pi)^N}\int_G d^N\lambda\, e^{-i\lambda^I \hat{C}_I}\right)\phi_2\right\rangle_{\text{kin}} \\ &= \left\langle \phi_1 \middle| \delta^N\left(\hat{C}_I\right)\middle| \phi_2\right\rangle_{\text{kin}}\end{aligned}\tag{A.17}$$

(d) There may be states ϕ_{null} that yield zero norm: $\|\eta\left(\phi_{\text{null}}\right)\|_{\text{phys}} = \eta\left(\phi_{\text{null}}\right)\left[\phi_{\text{null}}\right] = 0$. We define the null space $\mathcal{N} \subset \mathcal{V}_{\text{phys}}$ as the set of all such zero-norm states. To satisfy the axioms of a true inner product, we quotient out this null space and form the quotient inner product space $\mathcal{V}_{\text{phys}}/\mathcal{N}$. The final physical Hilbert space $\mathscr{H}_{\text{phys}}$ is defined as the Cauchy completion of this quotient space with respect to the physical norm $\|\eta(\phi)\|_{\text{phys}} = \sqrt{\eta(\phi)[\phi]}$,

$$\mathscr{H}_{\text{phys}} := \overline{(\mathcal{V}_{\text{phys}}/\mathcal{N})}.$$

B Minimum area for real states

In this subsection, we will show that the result found in Sec. 6.1 $A_{\min} > 0$ holds not only for the specific case when $\hat{T} = \hat{b}$, but i) for all the clocks that satisfy $[\hat{T}, \hat{H}_C] = i\hbar$, i.e., for a generic function $g(p_b)$, and ii) any real state $\psi_{a|b}(p_a)$ [4] and not just the Gaussian ones.

To this end we consider the expression for $\left\langle \hat{F}_{A,T}\left(\tau\right)\right\rangle_{\text{phys}}$ from Eq.(5.37) with $g \neq 0$

$$\begin{aligned}\left\langle \hat{F}_{A,T}\left(\tau\right)\right\rangle_{\text{phys}} =& 4\pi\alpha^4\tau^2 \int_{\mathbb{R}} dp_a \, \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^4} + 8\pi\alpha^2\tau \int_{\mathbb{R}} dp_a \, \frac{\partial \tilde{g}\left(p_a\right)}{\partial p_a} \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^2} \\ &+ 4\pi \int_{\mathbb{R}} dp_a \left\{ \hbar^2 \left[\frac{\partial \psi_{a|b}\left(p_a\right)}{\partial p_a}\right]^2 + \left[\frac{\partial \tilde{g}\left(p_a\right)}{\partial p_a}\right]^2 \psi_{a|b}\left(p_a\right)^2 \right\}\end{aligned} \tag{B.1}$$

This is in the parabolic form in τ,

$$\left\langle \hat{F}_{A,T}\left(\tau\right)\right\rangle_{\text{phys}} = \mathcal{A}\left(\alpha^2\tau\right)^2 + \mathcal{B}\left(\alpha^2\tau\right) + \mathcal{C} \tag{B.2}$$

where

$$\begin{aligned}\mathcal{A} =& 4\pi \int_{\mathbb{R}} dp_a \, \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^4}, \\ \mathcal{B} =& 8\pi \int_{\mathbb{R}} dp_a \, \frac{\partial \tilde{g}\left(p_a\right)}{\partial p_a} \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^2}, \\ \mathcal{C} =& 4\pi \int_{\mathbb{R}} dp_a \left\{ \hbar^2 \left[\frac{\partial \psi_{a|b}\left(p_a\right)}{\partial p_a}\right]^2 + \left[\frac{\partial \tilde{g}\left(p_a\right)}{\partial p_a}\right]^2 \psi_{a|b}\left(p_a\right)^2 \right\}.\end{aligned} \tag{B.3}$$

Since $\mathcal{A} > 0$, to show that the minimum area is strictly positive for all $g(p_a) \neq 0$, a sufficient condition is to show that the discriminant $\Delta = \mathcal{B}^2 - 4\mathcal{A}\mathcal{C} < 0$, or $\mathcal{B}^2 < 4\mathcal{A}\mathcal{C}$. However,

$$\mathcal{B}^2 = 64\pi^2 \left(\int_{\mathbb{R}} dp_a \, \frac{\partial \tilde{g}\left(p_a\right)}{\partial p_a} \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^2} \right)^2 \tag{B.4}$$

and using the Cauchy-Schwarz inequality $\left(\int fg\right)^2 \leq \left(\int f^2\right)\left(\int g^2\right)$ for $f = \sqrt{8\pi}\frac{\psi_{a|b}(p_a)}{p_a^2}$ and $g = \sqrt{8\pi}\psi_{a|b}\left(p_a\right)\frac{\partial \tilde{g}(p_a)}{\partial p_a}$, we obtain

$$\begin{aligned}\mathcal{B}^2 \leq& 64\pi^2 \int_{\mathbb{R}} dp_a \, \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^4} \int_{\mathbb{R}} dp'_a \left[\frac{\partial \tilde{g}\left(p'_a\right)}{\partial p'_a}\right]^2 \psi_{a|b}\left(p'_a\right)^2 \\ =& 64\pi^2 \int_{\mathbb{R}} dp_a \int_{\mathbb{R}} dp'_a \, \frac{\psi_{a|b}\left(p_a\right)^2 \psi_{a|b}\left(p'_a\right)^2}{p_a^4} \left[\frac{\partial \tilde{g}\left(p'_a\right)}{\partial p'_a}\right]^2\end{aligned} \tag{B.5}$$

This means that we can also write

$$\mathcal{B}^2 < 64\pi^2 \int_{\mathbb{R}} dp_a \int_{\mathbb{R}} dp'_a \, \frac{\psi_{a|b}\left(p_a\right)^2 \psi_{a|b}\left(p'_a\right)^2}{p_a^4} \left[\frac{\partial \tilde{g}\left(p'_a\right)}{\partial p'_a}\right]^2 + J \tag{B.6}$$

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[4] Equivalently, a state with a global and constant complex phase, i.e. $\psi_{a|b}(p_a) = |\psi_{a|b}(p_a)|e^{i\alpha}$ with $\alpha =$const.

where $J > 0$ is any strictly positive expression. On the other hand we have

$$\begin{aligned}4\mathcal{A}\mathcal{C} =& 64\pi^2 \int_{\mathbb{R}} dp_a \int_{\mathbb{R}} dp_a' \, \hbar^2 \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^4} \left[\frac{\partial \psi_{a|b}\left(p_a'\right)}{\partial p_a'}\right]^2 \\ &+ 64\pi^2 \int_{\mathbb{R}} dp_a \int_{\mathbb{R}} dp_a' \left[\frac{\partial \tilde{g}\left(p_a'\right)}{\partial p_a'}\right]^2 \frac{\psi_{a|b}\left(p_a\right)^2}{p_a^4} \psi_{a|b}\left(p_a'\right)^2\end{aligned} \tag{B.7}$$

Since the last line in the above is strictly positive, and by comparing (B.6) and (B.7), we conclude that indeed $\mathcal{B}^2 < 4\mathcal{A}\mathcal{C}$ and thus $\Delta < 0$. As a result, we have proven that $A_{\min} > 0$ is valid for any generic function $g(p_b)$, and any real state $\psi_{a|b}(p_a)$.

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