



Western

Energy-Momentum Tensor Uniqueness for Scalar Fields

Sean Snider

Supervisor: Mark Baker

Project Objective(s)/Overview

- Inconsistencies with Energy-Momentum tensors as defined via Noether's Theorem^[1]
- Wish to resolve these, connect ideas to other definitions
- Obtain general formulas for currents that work without improvements

- This presentation is on work done for scalar field theories
- Derivations are as abstract as possible to avoid “ad-hoc” tricks for specific field theories
- Abstraction makes the math harder, but results are more robust

Noether's First Theorem

- Smooth symmetries of the Lagrangian imply conserved currents^[2]

$$0 = E(L)q + \nabla_{\mu} J^{\mu}$$

- Translation symmetry associated to a collection of such currents, stacked into the (canonical) Energy-Momentum Tensor (EMT)

$$T_C^{\mu\nu} = \frac{\partial L}{\partial(\nabla_{\mu}\phi)} \nabla^{\nu}\phi + \frac{\partial L}{\partial(\nabla_{\mu}\nabla_{\rho}\phi)} \nabla^{\nu}\nabla_{\rho}\phi - \nabla_{\rho} \left(\frac{\partial L}{\partial(\nabla_{\mu}\nabla_{\rho}\phi)} \right) \nabla^{\nu}\phi - g^{\mu\nu} L$$

- Two Lagrangians which give the same equations of motion can give different EMTs, and in more complicated cases results can fail to be gauge invariant^[1]

Klein-Gordon and Energy-Momentum

- Consider the following two Lagrangians which give the same equations of motion

$$L_1 = \nabla_\mu \phi \nabla^\mu \phi \qquad L_2 = -\phi \nabla_\mu \nabla^\mu \phi$$

- Canonical tensors are fundamentally different, even on-shell

$$T_C^{\mu\nu} = \frac{\partial L}{\partial(\nabla_\mu \phi)} \nabla^\nu \phi + \frac{\partial L}{\partial(\nabla_\mu \nabla_\rho \phi)} \nabla^\nu \nabla_\rho \phi - \nabla_\rho \left(\frac{\partial L}{\partial(\nabla_\mu \nabla_\rho \phi)} \right) \nabla^\nu \phi - g^{\mu\nu} L$$

$$T_1^{\mu\nu} = 2\nabla^\mu \phi \nabla^\nu \phi - g^{\mu\nu} \nabla_\rho \phi \nabla^\rho \phi$$

$$T_2^{\mu\nu} = \nabla^\mu \phi \nabla^\nu \phi - \phi \nabla^\nu \nabla^\mu \phi + g^{\mu\nu} \phi \nabla_\rho \nabla^\rho \phi$$

- Along with the “canonical” there are other definitions involving variation of the metric tensor or contributions from spin angular momentum of the fields

Alternate Formulae

- Metric variation gives the “Hilbert Tensor”. Gives the right hand side of the Einstein field equations, and produces the same tensors for L1 and L2

$$T_1^{\mu\nu} = 2\nabla^\mu\phi\nabla^\nu\phi - g^{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi$$

- Angular momentum / symmetrization of T gives the “Belinfante-Rosenfeld” tensor. Adding a “superpotential” that doesn't affect conservation but does shift values

$$T_B^{\mu\nu} = T_C^{\mu\nu} + \nabla_\rho b^{\mu\rho\nu}$$

- For scalars however, no intrinsic spin so is b zero? Superpotential formula^[3] gives nonzero b, and once added the result agrees with Hilbert
- Ideally there should be a way to obtain an expression directly from Noether’s first theorem that needs no extra arguments or improvements to obtain, and for that an equation to directly solve for symmetries was derived

Symmetry Equation

- There is a way to directly solve for symmetries^[4] (multiple equivalent ways to derive, variational or through jet bundle)

$$\mathbf{Symm}(L) = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\nabla_{\mu} \phi)} \nabla_{\mu} \delta \phi + \frac{\partial L}{\partial (\nabla_{\mu} \nabla_{\nu} \phi)} \nabla_{\mu} \nabla_{\nu} \delta \phi - \nabla_{\mu} (\delta x^{\mu} L)$$

- First order Klein-Gordon Lagrangian L1 obtains 2D conformal symmetry and shifting the fields by a constant as simple symmetries

$$0 = 2 \nabla^{\mu} \phi \nabla_{\mu} \delta \phi + \nabla^{\mu} \phi \nabla^{\nu} \phi (\nabla_{\mu} \delta x_{\nu} + \nabla_{\nu} \delta x_{\mu} - (\nabla_{\rho} \delta x^{\rho}) g_{\mu\nu})$$

- Transformations that leave a total derivative also constitute symmetries^[4] (often called divergence, inexact, or quasi symmetries)

$$\mathbf{Symm}(L) = \nabla_{\mu} f^{\mu} \quad \rightarrow \quad 0 = E(L)q + \nabla_{\mu} (J^{\mu} - f^{\mu})$$

Conformal Symmetry and Tracelessness

- The symmetry equation can be written as a contraction

$$\text{Symm}(L) = T^{\mu\nu} \nabla_{\mu} \delta x_{\nu}$$

- Using the conformal Killing equation, this implies that the trace of the EMT is zero.

$$0 = T^{\mu\nu} \nabla_{\mu} \delta x_{\nu} \quad \rightarrow \quad 0 \propto (\nabla_{\mu} \delta x^{\mu}) g_{\mu\nu} T^{\mu\nu}$$

- Another proposed EMT for the Klein-Gordon field, proposed by Callan Coleman and Jackiw, was constructed to obtain on-shell tracelessness in 4D space^[5]

$$T_{CCJ}^{\mu\nu} = \nabla^{\mu} \phi \nabla^{\nu} \phi - \frac{1}{2} g^{\mu\nu} \nabla_{\rho} \phi \nabla^{\rho} \phi - \frac{1}{6} (\nabla^{\mu} \nabla^{\nu} - g^{\mu\nu} \nabla_{\rho} \nabla^{\rho}) \phi^2$$

- However, since symmetry conditions are off-shell, conformal symmetry is only connected with off-shell tracelessness

Modified Klein-Gordon

- Example model uses a linear combination of L1,L2

$$L = A\nabla_{\mu}\phi\nabla^{\mu}\phi + B\phi\nabla_{\mu}\nabla^{\mu}\phi$$

- Symmetry equation produces a linear multiple of the original equation plus a large total derivative (same symmetries up to div)
- Using this total derivative to modify the Noether current, the result is

$$2(A - B)\nabla^{\mu}\phi\delta x^{\rho}\nabla_{\rho}\phi - (A - B)\delta x^{\mu}\nabla_{\sigma}\phi\nabla^{\sigma}\phi$$

- From which the EMT can be extracted as a contraction with δx

$$T^{\mu\nu} = 2(A - B)\nabla^{\mu}\phi\nabla^{\nu}\phi - (A - B)g^{\mu\nu}\nabla_{\rho}\phi\nabla^{\rho}\phi$$

- This agrees with Hilbert and Belinfante-Rosenfeld, but was derived *directly* from Noether's theorem

Same Dynamics, Same Symmetries

- Symmetry equation is linear in L , so it suffices to look at Lagrangians with Euler equation zero (null Lagrangians)

$$\text{Symm}(L) = E(L)q + \nabla_{\mu}J^{\mu}$$

- $E(L) = 0$, so the equation can be rewritten

$$\text{Symm}(L) = \nabla_{\mu}J^{\mu}$$

- All transformations of coordinates and/or fields are symmetries for null Lagrangians
- If a transformation is a symmetry of two Lagrangians, it is a symmetry of their sum
- Thus all Lagrangians with the same equations of motion have the same set of divergence symmetries

Arbitrary Scalar Field

- The modified Klein-Gordon calculation is cute, but does this work in general?
- Should find result for an arbitrary scalar field, get formula for EMT

$$J^\mu = T^{\mu\nu} \delta x_\nu$$

- The starting point allowed for certain terms to be rewritten in terms of Lie derivatives of other variables, and produces an expression equivalent to the Hilbert and Belinfante-Rosenfeld formulae

$$T^{\mu\nu} = \frac{\partial L}{\partial \nabla_\mu \phi} \nabla^\nu \phi + \frac{\partial L}{\partial (\nabla_\rho \nabla_\mu \phi)} \nabla_\rho \nabla^\nu \phi - \nabla_\rho \left(\frac{\partial L}{\partial (\nabla_\mu \nabla_\rho \phi)} \right) \nabla^\nu \phi - g^{\mu\nu} L \\ + \nabla_\sigma \left(\frac{\partial L}{\partial (\nabla_\mu \nabla_\nu \phi)} \nabla^\sigma \phi - \frac{\partial L}{\partial (\nabla_\sigma \nabla_\nu \phi)} \nabla^\mu \phi \right)$$

- Produces the same result as the previous calculation for the modified Klein-Gordon Lagrangian!

$$T^{\mu\nu} = 2(A - B) \nabla^\mu \phi \nabla^\nu \phi - (A - B) g^{\mu\nu} \nabla_\rho \phi \nabla^\rho \phi$$

Future Work

- The same procedure used to obtain an EMT formula for scalars has also been applied to vectors with great success, and work continues on generalizing the procedure
- So far, the spacetime has been a “static background”. Soon, the metric will be varied in order to apply these methods to linearized and Einstein-Hilbert gravity
- Further mathematical rigour is desired, which will be achieved from studying the Variational Bicomplex. This mathematical structure uses powerful analogies with differential forms to do calculus of variations^[6]

Results Summary and Conclusion

- Same equations of motion, implies same symmetries (up to divergence terms)
- These divergence terms exactly cancel out the changes other Lagrangians induce in the EMT for the Klein-Gordon field, producing a unique result
- Starting from the symmetry equation, an expression can be derived that automatically accounts for this
- Conformal symmetry is only connected to off-shell tracelessness of the EMT

Thanks for listening!