General Set of Noetherian Energy-Momentum Tensors in Linearized Gravity

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Linearized Gravity:

This presentation is centered around work done in [1].

Noether's theorem utilizes symmetries of the coordinates and fields:

- GR has no well-defined group of finite global coordinate symmetries
- Minkowski spacetime has the 10-parameter Poincaré group

$$\delta x_\lambda = a_\lambda$$

The linearization of the vacuum Einstein-Hilbert action around a flat Minkowski background. This linearization is:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \epsilon h_{\mu\nu}$$

[1] Taylor, L. B., & Baker, M. R. (2024). The general set of Noetherian energy-momentum tensors in linearized gravity: mathematical framework. *Physica Scripta*, *99*(3), 035258.

Non-Uniqueness Problem in Linearized Gravity:

Energy-momentum tensors are foundational objects:

- Uniquely defined in standard theories
- Not uniquely defined for linearized gravity
- Uniqueness related to the gauge invariance of the action

Numerous distinct energy-momentum expressions [2]:

- Infinitely many with super-potential improvement method

Padmanabhan explicitly discusses this non-uniqueness [3]:

- Is it possible to recover Einstein-Hilbert action from a linear theory

[2] Baker, M. R. (2021). Canonical Noether and the energy–momentum non-uniqueness problem in linearized gravity. *Classical and Quantum Gravity*, *38*(9), 095007.
[3] Padmanabhan, T. (2008). From gravitons to gravity: Myths and reality. *International Journal of Modern Physics D*, *17*(03n04), 367-398.

Noether's Theorem:

Symmetries

$$\delta x_\nu, \delta h_{\mu\nu}$$

⟷ Connected ⟷

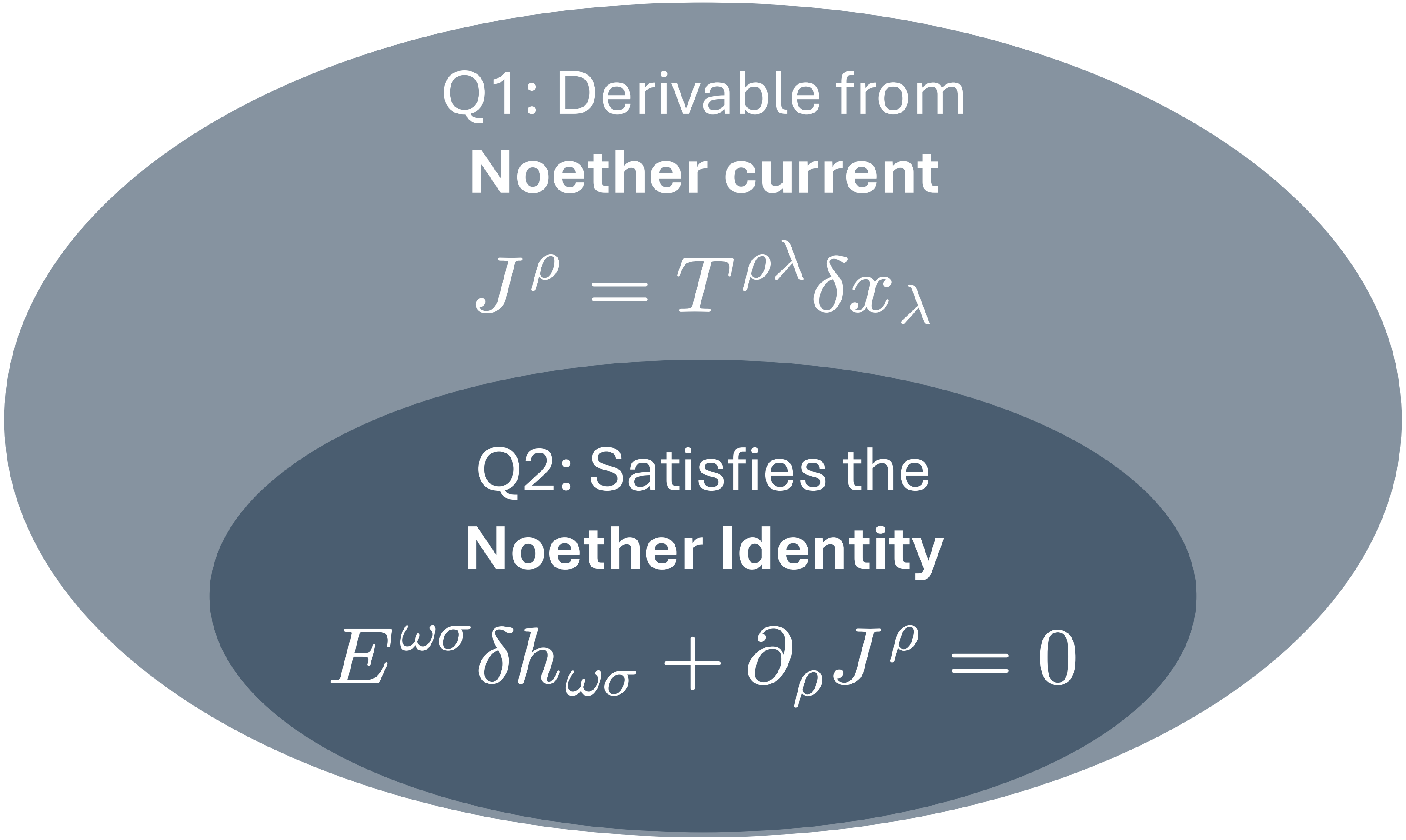
Conservation Laws

$$\partial_\rho T^{\rho\lambda} = 0$$

The **Noether identity** is derived from asserting invariance of the action under infinitesimal changes in the symmetries.

$$E^{\omega\sigma}\delta h_{\omega\sigma} + \partial_\rho J^\rho = 0$$

Noetherian Energy-Momentum Tensors:



Noether Current Internal Freedoms:

1) Generalized Lagrangian density,

$$\mathcal{L} = C_1\partial_\beta h_{\mu\nu}\partial^\beta h^{\mu\nu} + C_2\partial_\beta h^\mu_\mu\partial^\beta h^\nu_\nu + C_3\partial_\beta h^{\beta\mu}\partial_\mu h^\nu_\nu + C_4\partial_\beta h^{\beta\mu}\partial^\nu h_{\mu\nu} + C_5\partial_\mu h_{\nu\beta}\partial^\nu h^{\mu\beta}$$
$$+ D_1 h_{\mu\nu}\partial^\mu\partial^\nu h^\beta_\beta + D_2 h_{\mu\nu}\partial^\mu\partial_\beta h^{\nu\beta} + D_3 h_{\mu\nu}\partial^\beta\partial_\beta h^{\mu\nu} + D_4 h^\mu_\mu\partial^\beta\partial_\beta h^\nu_\nu + D_5 h^\beta_\beta\partial^\mu\partial^\nu h_{\mu\nu}$$

2) Generalized field transformations,

$$\delta h_{\omega\sigma} = B_1\partial^\alpha h_{\omega\sigma}a_\alpha + B_2\partial^\alpha h_{\alpha\sigma}a_\omega + B_3\partial^\alpha h_{\omega\alpha}a_\sigma + B_4\partial_\omega h_{\alpha\sigma}a^\alpha$$
$$+ B_5\partial_\sigma h_{\omega\alpha}a^\alpha + B_6\partial_\omega h^\alpha_\alpha a_\sigma + B_7\partial_\sigma h^\alpha_\alpha a_\omega + B_8\eta_{\omega\sigma}\partial^\alpha h_{\alpha\gamma}a^\gamma + B_9\eta_{\omega\sigma}\partial^\alpha h^\gamma_\gamma a_\alpha$$

Noether Current:

$$J^{\rho} = \eta^{\rho\lambda}\mathcal{L}\delta x_{\lambda} + \frac{\partial\mathcal{L}}{\partial(\partial_{\rho}h_{\omega\sigma})}\delta h_{\omega\sigma} + \frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\partial_{\zeta}h_{\omega\sigma})}\partial_{\zeta}\delta h_{\omega\sigma} - [\partial_{\zeta}\frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\partial_{\zeta}h_{\omega\sigma})}]\delta h_{\omega\sigma}$$

Each term's coefficient is a function of uppercase coefficients {B, C, D}.

We are left with a system of equations of these coefficients for each of the 42 unique terms of the form (dhdh) and (hddh).

$$T^{\rho\lambda} = b_1\partial_{\alpha}h^{\rho\lambda}\partial_{\beta}h^{\alpha\beta} + b_2\partial_{\alpha}h^{\rho\lambda}\partial^{\alpha}h + b_3\partial_{\alpha}h^{\rho\alpha}\partial_{\beta}h^{\lambda\beta} + \cdots$$

Noether Identity:

We Have:

- The most general field transformations (symmetries).
- The most general Lagrangian density which yield the spin-2 EOM.
- General system of coefficients that relates each term to the symmetries and Lagrangian density.

We Want:

- A set of restrictions on the general field transformations.
- Noether identity to be satisfied for transformations that correspond to the energy-momentum tensors.
- Remaining energy-momentum tensors with these restrictions applied.

Noether Identity Satisfied:

We applied a direct naïve approach to the Noether identity:

$$E^{\omega\sigma}\delta h_{\omega\sigma} + \partial_\rho J^\rho = 0$$

From this we see the only symmetry is the canonical:

$$\delta h_{\omega\sigma} = -\partial^\alpha h_{\omega\sigma} a_\alpha$$

Further questions that we can ask:

Can we apply a less naïve approach where terms can split contribution to EOM, energy-momentum tensors, or total divergences of the EOM, (On-Shell equivalence)?

Bessel-Hagen currents for the Fierz-Pauli action:

Recent work uses the Bessel-Hagen method for the the Fierz-Pauli action [4]:

- For electromagnetism, this method provides a direct Noetherian derivation of the gauge invariant energy-momentum tensor
- They discuss analogue choices for Fierz-Pauli and limitations

For Fierz-Pauli action there is no single gauge invariant local tensor [5]

This work illustrates the effects of a lack of gauge invariance in the larger question of uniqueness.

[4] Hobson, M., Barker, W., & Lasenby, A. (2026). Bessel-Hagen currents for the Fierz-Pauli action. *arXiv preprint arXiv:2605.19719*.
[5] Magnano, G., & Sokolowski, L. M. (2002). Symmetry properties under arbitrary field redefinitions of the metric energy–momentum tensor in classical field theories and gravity. *Classical and Quantum Gravity*, *19*(2), 223-236.

Conclusion:

- There exists a non-uniqueness problem for energy-momentum tensors in linearized gravity.

- We work in linearized gravity, to address non-uniqueness with Noether's theorem.

- There are open questions about the Noetherian energy-momentum tensors from how the Noether identity must be satisfied, and the Bessel-Hagen method.