Unconventional Ginzburg-Landau energy: a common pathway to novel physics

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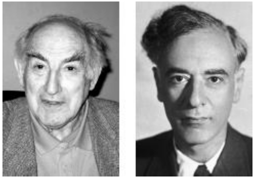
CAP 2026 - Theory Canada 18

June 19, 2026

Outline

- Noncentrosymmetric superconductors: Lifshitz invariants and nonuniform states
- “Superconducting diode” effect in FFLO superconductors
- Multiband superconductors: TRSB states stabilized by interband pairing

NSERC CRSNG



(from www.nobelprize.org)

Vitaly Ginzburg (2003):
"*for pioneering contributions to the theory of superconductors and superfluids*"

Lev Landau (1962):
"*for his pioneering theories for condensed matter, especially liquid helium*"

Superconducting order parameter: $\psi(\boldsymbol{r}) = |\psi(\boldsymbol{r})|e^{i\varphi(\boldsymbol{r})}$ – wave function of Cooper pairs

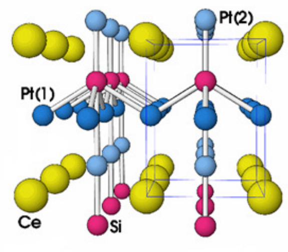
Free energy expansion near T_c:

$$F_s = F_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + K|\boldsymbol{\nabla}\psi|^2, \quad \alpha = a(T - T_c), \quad a, \beta, K > 0$$

New features in unconventional SCs?

Noncentrosymmetric superconductors

O	Li_2Pt_3B (2K), Li_2Pd_3B (8K)
T	LaRhSi (4K), LaIrSi (2K)
$\mathbf{C}_{4v}$	$CePt_3Si$ (0.5K), $CeRhSi_3$ (1K)
$\mathbf{C}_{6v}$	MoN (15K), GaN (6K)
$\mathbf{D}_{3h}$	MoC (9K), NbSe (6K)
$\mathbf{C}_{3v}$	MoS_2 (1K)
...	...

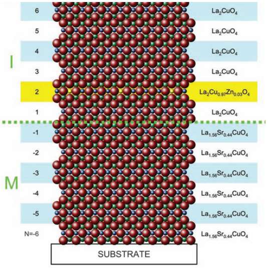


(from Bauer *et al*, PRL **92**, 027003 (2004))

Insulator/insulator interface
(LAO/STO, LTO/STO)

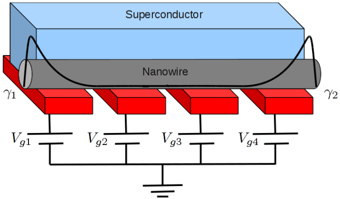
Metal/insulator interface (LSCO/LCO)

Doped insulator surface (STO, WO_3)



(from Pereiro *et al*, Phys. Express **1**, 208 (2011))

Proximity-induced SC
in semiconducting wires



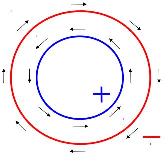
(from Leijnse, Flensberg, SST **27**, 124003 (2012))

Superconducting pairing in nondegenerate bands

No inversion + electron-lattice SO coupling $\rightarrow$ nondegenerate electron bands

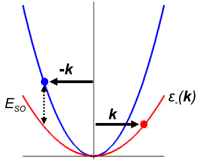
Example: Rashba model

$\hat{H} = \epsilon(\boldsymbol{k}) + \alpha(k_y\hat{\sigma}_x - k_x\hat{\sigma}_y)$

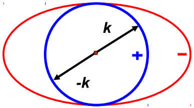


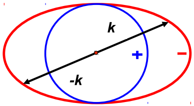
band splitting $E_{\rm SOC} \gg$ SC energy scales

no interband pairing



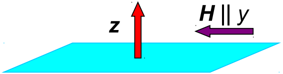
Intraband pairing: $\Delta_+(\boldsymbol{k},\boldsymbol{r}) = \eta_+(\boldsymbol{r})\phi(\boldsymbol{k})$, $\Delta_-(\boldsymbol{k},\boldsymbol{r}) = \eta_-(\boldsymbol{r})\phi(\boldsymbol{k})$, $\phi(\boldsymbol{k}) = \phi(-\boldsymbol{k})$





GL free energy with Lifshitz invariants

2D superconductor
in a parallel field



no orbital effects

GL free energy density: $F = F_+ + F_- + \tilde{F}$

Intraband terms: $F_\lambda = \alpha_\lambda |\eta_\lambda|^2 + \frac{\beta_\lambda}{2}|\eta_\lambda|^4 + K_\lambda |\boldsymbol{\nabla}\eta_\lambda|^2 + \underbrace{\tilde{K}_\lambda \,\mathrm{Im}\,[\eta_\lambda^*(\boldsymbol{H}\times\boldsymbol{\nabla})_z\eta_\lambda]}_{\text{Lifshitz invariant}} + \ldots$

Interband ("Josephson") pair tunneling: $\tilde{F} = \gamma(\eta_+^*\eta_- + \eta_-^*\eta_+)$

Lifshitz invariants $\rightarrow$ nonuniform SC states

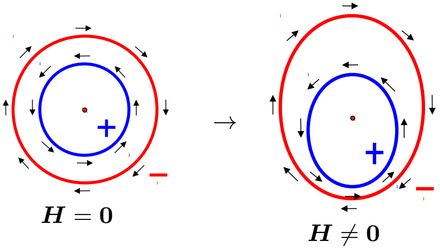
Nonuniform superconducting states

Weak field: helical state $\eta_\lambda(\boldsymbol{r}) = \eta_\lambda e^{iqx}$

$$q \propto H, \quad T_{c0} - T_c(H) \propto H^2, \quad \text{no supercurrent:} \quad j_x \propto \frac{\partial \mathcal{F}}{\partial A_x} \propto \frac{\partial \mathcal{F}}{\partial q} = 0$$

Origin of the helical instability:
band displacement and deformation by $\boldsymbol{H}$

$$\xi_\lambda(\boldsymbol{k}) = \epsilon(\boldsymbol{k}) + \lambda|\boldsymbol{\gamma}(\boldsymbol{k}) + \mu_B \boldsymbol{H}| \neq \xi_\lambda(-\boldsymbol{k})$$



Strong field: interband phase soliton lattice, stripe states

Zero-field nonuniform superconducting states

3D tetragonal SC (point group $\mathbf{C}_{4v}$): $F = F_+ + F_- + \tilde{F} + F_L$

Zero-field Lifshitz invariant: $F_L = K_L\,\mathrm{Re}(\eta_+^*\nabla_z\eta_- - \eta_-^*\nabla_z\eta_+)$

nonuniform instability: $\eta_\lambda = \eta_{\lambda,0}e^{iqz}$ if $K_L > K_{L,c}$

Microscopic origin:

$$\hat{H}_{\mathrm{int}} = \frac{1}{2\mathcal{V}}\sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}}\sum_{\lambda,\lambda'=\pm} V_{\lambda\lambda'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q})\,\hat{c}^\dagger_{\boldsymbol{k}+\boldsymbol{q},\lambda}\hat{\tilde{c}}^\dagger_{\boldsymbol{k},\lambda}\hat{\tilde{c}}_{\boldsymbol{k}',\lambda'}\hat{c}_{\boldsymbol{k}'+\boldsymbol{q},\lambda'} \quad (\hat{\tilde{c}}^\dagger_{\boldsymbol{k},\lambda} = K\hat{c}^\dagger_{\boldsymbol{k},\lambda}K^{-1})$$

$$V_{\lambda\lambda'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}) = v_{\lambda\lambda'}(\boldsymbol{k},\boldsymbol{k}') + i\boldsymbol{b}_{\lambda\lambda'}(\boldsymbol{k},\boldsymbol{k}')\boldsymbol{q} + \mathcal{O}(q^2) \quad \rightarrow \quad K_L \propto |\langle\, \boldsymbol{b}_{+-}\,\rangle_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'}|$$

invariant polar vector
$\neq 0$ in pyroelectric crystals

Zero-field nonuniform superconducting states

Other types of zero-field Lifshitz invariants:

Weak antisymmetric SOC + spin-triplet pairing: $\hat{\Delta}(\boldsymbol{k},\boldsymbol{r}) = \boldsymbol{d}(\boldsymbol{k},\boldsymbol{r})(i\hat{\boldsymbol{\sigma}}\hat{\sigma}_2)$

Example: 3-component order parameter $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ in a cubic crystal (point group $\mathbf{O}$)

$$\boldsymbol{d}(\boldsymbol{k},\boldsymbol{r}) = \eta_1(\boldsymbol{r})\boldsymbol{\varphi}_1(\boldsymbol{k}) + \eta_2(\boldsymbol{r})\boldsymbol{\varphi}_2(\boldsymbol{k}) + \eta_3(\boldsymbol{r})\boldsymbol{\varphi}_3(\boldsymbol{k}), \qquad \varphi_{a,i}(\boldsymbol{k}) \propto e_{aij}\hat{k}_j$$

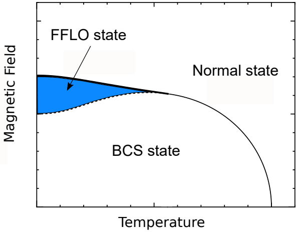
Lifshitz invariant: $F_L = K_L(\eta_1^*\nabla_y\eta_3 + \eta_2^*\nabla_z\eta_1 + \eta_3^*\nabla_x\eta_2 + \mathrm{c.c.})$

Origin: $\boldsymbol{k}$-odd terms in the triplet Cooper kernel

$$K_{ab}(\boldsymbol{q}) = \frac{1}{2}T\sum_{\omega_n,\boldsymbol{k}}\sum_{\lambda=\pm}\varphi^*_{a,i}\varphi_{b,j}\left[\hat{\gamma}_i\hat{\gamma}_j G_\lambda(\boldsymbol{k}+\boldsymbol{q},\omega_n)G_\lambda(-\boldsymbol{k},-\omega_n)\right.$$
$$\left.+(\delta_{ij} - \hat{\gamma}_i\hat{\gamma}_j - i\lambda e_{ijl}\hat{\gamma}_l)G_\lambda(\boldsymbol{k}+\boldsymbol{q},\omega_n)G_{-\lambda}(-\boldsymbol{k},-\omega_n)\right]$$

FFLO superconductors

Fulde-Ferrell-Larkin-Ovchinnikov states can be realized in a clean paramagnetically-limited SC



FF state: $\psi(\boldsymbol{r}) = \Delta_0 e^{i\boldsymbol{qr}}$

LO state: $\psi(\boldsymbol{r}) = \Delta_0 \cos(\boldsymbol{qr})$

Possible experimental realizations:
layered organic SCs, heavy-fermion SCs, population-imbalanced “cold” Fermi gases, ...

Phenomenological view

Modified Ginzburg-Landau free energy:

$$F = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + K|\boldsymbol{\nabla}\psi|^2 + \tilde{K}|\boldsymbol{\nabla}^2\psi|^2$$

$K < 0, \quad \tilde{K} > 0 \quad \rightarrow$ nonuniform SC instability

with disorder, pairing anisotropy: $K < 0$ but $\beta > 0$ $\Rightarrow$ FF state is stable

Modified supercurrent:

$$\boldsymbol{j}_s = -\frac{4e}{\hbar}\,\mathrm{Im}\left\{K(\psi^*\boldsymbol{\nabla}\psi) + \tilde{K}\left[(\boldsymbol{\nabla}\psi)^*\nabla^2\psi - \psi^*\boldsymbol{\nabla}\nabla^2\psi\right]\right\}$$

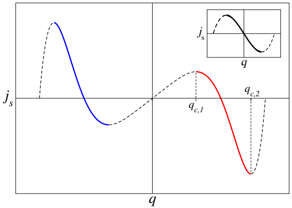
Modified GL free energy:

$$F = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + K|\nabla_x\psi|^2 + \tilde{K}|\nabla_x^2\psi|^2 \quad (K < 0,\ \tilde{K} > 0)$$

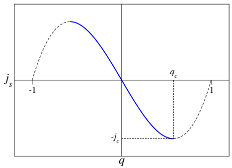
$$j_s = -\frac{4e}{\hbar}\,\mathrm{Im}\left\{K(\psi^*\nabla_x\psi) + \tilde{K}\left[(\nabla_x\psi)^*\nabla_x^2\psi - \psi^*\nabla_x^3\psi\right]\right\}$$

Current-carrying FF state: $\psi(x) = \Delta(q)e^{iqx}$

stable if
$q_{c,1} \le |q| \le q_{c,2}$



compare to usual BCS



Two stable states at given current I: $\psi_+ = \Delta(q_+)e^{iq_+x}$ and $\psi_- = \Delta(q_-)e^{iq_-x}$

BCS *vs* FF

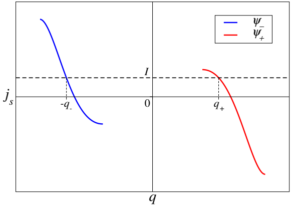
Usual BCS superconductor: one critical current j_c, one current-carrying state at any $|I| \leq j_c$

FF superconductor: two critical currents j_{c1} and j_{c2}, two current-carrying states

$-j_{c,2} < I < -j_{c,1}$
current carried by ψ_+

$j_{c,1} < I < j_{c,2}$
current carried by ψ_-

$|I| < j_{c,1}$
current carried by ψ_+ or ψ_-



ψ_+ is metastable at
$0 < I \leq j_{c,1}$

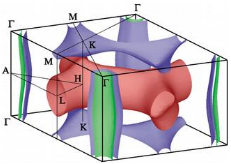
ψ_- is metastable at
$-j_{c,1} \leq I < 0$

Branch switching via domain wall formation?

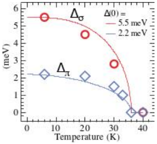
$N > 1$ bands participating in superconductivity:

MgB_2, $NbSe_2$, Nb_3Sn, heavy-fermion SCs (UPt_3, $CeCoIn_5$, URu_2Si_2), Fe-based, Sr_2RuO_4, TMDs, oxide interfaces (e.g. LAO/STO), ...

MgB_2

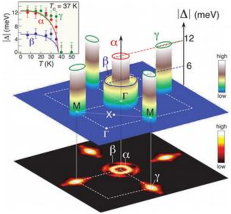


(from J. Kortus et al, PRL **86**, 4656 (2001))



(from S. Tsuda et al, PRL **91**, 127001 (2003))

$Ba_{0.6}K_{0.4}Fe_2As_2$



(from H. Ding et al, EPL **83**, 47001 (2008))

Interband pairing

All pairing interactions are possible:

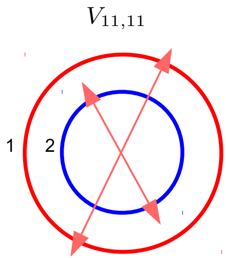
$$\hat{H}_{\rm int} = \frac{1}{2\mathcal{V}} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} \sum_{n_i s_i} V^{s_1 s_2 s_3 s_4}_{n_1 n_2 n_3 n_4}(\boldsymbol{k},\boldsymbol{k}')\; c^\dagger_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2},n_1,s_1} \tilde{c}^\dagger_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2},n_2,s_2} \tilde{c}_{\boldsymbol{k}'-\frac{\boldsymbol{q}}{2},n_3,s_3} c_{\boldsymbol{k}'+\frac{\boldsymbol{q}}{2},n_4,s_4}$$

intraband pairing : $V_{11,11}\; c_1^\dagger \tilde{c}_1^\dagger \tilde{c}_1 c_1, \quad V_{22,22}\; c_2^\dagger \tilde{c}_2^\dagger \tilde{c}_2 c_2$

“Josephson” pair tunneling : $V_{11,22}\; c_1^\dagger \tilde{c}_1^\dagger \tilde{c}_2 c_2$

purely interband pairing : $V_{12,12}\; c_1^\dagger \tilde{c}_2^\dagger \tilde{c}_1 c_2$

mixed band pairing : $V_{11,12}\; c_1^\dagger \tilde{c}_1^\dagger \tilde{c}_1 c_2, \quad V_{22,12}\; c_2^\dagger \tilde{c}_2^\dagger \tilde{c}_1 c_2$



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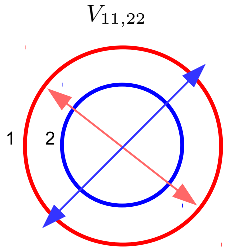

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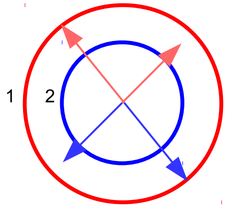
$V_{12,12}$



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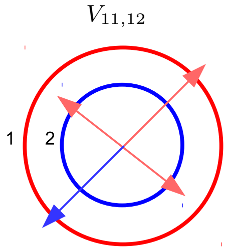
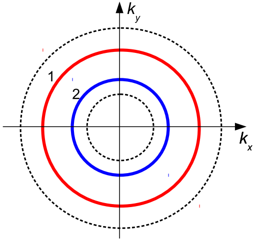


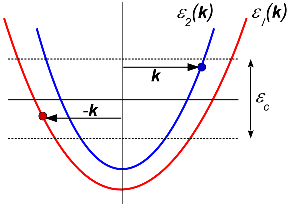
Interband Cooper pairs in two-band superconductors

Mean-field Hamiltonian: $$\hat{H} = \sum_{\boldsymbol{k},s} \left[\varepsilon_1(\boldsymbol{k}) c^\dagger_{\boldsymbol{k},1,s} c_{\boldsymbol{k},1,s} + \varepsilon_2(\boldsymbol{k}) c^\dagger_{\boldsymbol{k},2,s} c_{\boldsymbol{k},2,s} \right] + \hat{H}_{\rm pair}$$

2 Fermi surfaces inside
"thick" pairing shell

"BCS" energy cutoff
$\varepsilon_c >$ band splitting





$$\hat{H}_{\rm pair} = \frac{1}{2} \sum_{\boldsymbol{k},ss'} \Big[\Delta_{11,ss'}(\boldsymbol{k}) c^\dagger_{\boldsymbol{k},1,s} \tilde{c}^\dagger_{\boldsymbol{k},1,s'} + \Delta_{22,ss'}(\boldsymbol{k}) c^\dagger_{\boldsymbol{k},2,s} \tilde{c}^\dagger_{\boldsymbol{k},2,s'}$$
$$+ \Delta_{12,ss'}(\boldsymbol{k}) c^\dagger_{\boldsymbol{k},1,s} \tilde{c}^\dagger_{\boldsymbol{k},2,s'} + \Delta_{21,ss'}(\boldsymbol{k}) c^\dagger_{\boldsymbol{k},2,s} \tilde{c}^\dagger_{\boldsymbol{k},1,s'} + \text{H.c.} \Big]$$

Ginzburg-Landau free energy with interband pairing

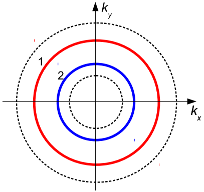
3 order parameter components: $\eta_1 = |\eta_1| e^{i\varphi_1}$, $\eta_2 = |\eta_2| e^{i\varphi_2}$, $\tilde{\eta} = |\tilde{\eta}|$

TRSB states: $\varphi_{1,2} \neq 0, \pi$

$$F_{GL} = F_2 + F_4$$

$$F_2 = \alpha_1(T)|\eta_1|^2 + \alpha_2(T)|\eta_2|^2 + \gamma(\eta_1^*\eta_2 + \text{c.c.}) + K_1|\nabla\eta_1|^2 + K_2|\nabla\eta_2|^2 + \tilde{\alpha}|\tilde{\eta}|^2 + \tilde{\gamma}_1(\eta_1^*\tilde{\eta} + \text{c.c.}) + \tilde{\gamma}_2(\eta_2^*\tilde{\eta} + \text{c.c.}) + \tilde{K}|\nabla\tilde{\eta}|^2$$

$$F_4 = \beta_1|\eta_1|^4 + \beta_2|\eta_2|^4 + \tilde{\beta}_1|\eta_1|^2|\tilde{\eta}|^2 + \tilde{\beta}_2|\eta_2|^2|\tilde{\eta}|^2 + \tilde{\beta}_3|\tilde{\eta}|^4 + \tilde{\beta}_4(\eta_1\eta_2\tilde{\eta}^{*,2} + \text{c.c.})$$



$\tilde{K} < 0$, $\tilde{\beta}_3 < 0$ – pairing $(|\boldsymbol{k}, 1\rangle, |-\boldsymbol{k} - \boldsymbol{q}, 2\rangle)$
interband FFLO-like instability?

Stable superconducting states

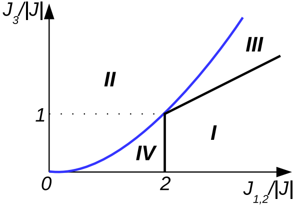
Uniform SC state: $$F_{\text{phase}} = \underbrace{J\cos(\varphi_1-\varphi_2)}_{\text{intraband only}} + \underbrace{J_1\cos\varphi_1 + J_2\cos\varphi_2 + J_3\cos(\varphi_1+\varphi_2)}_{\text{due to interband pairs}}$$

two-band SC, no interband pairing: $J_1 = J_2 = J_3 = 0 \quad \rightarrow$ no TRSB states

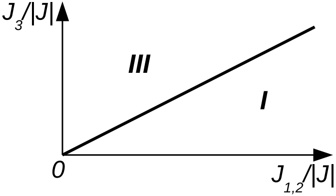
Stable superconducting states

Uniform SC state: $F_{\text{phase}} = \underbrace{J\cos(\varphi_1-\varphi_2)}_{\text{intraband only}} + \underbrace{J_1\cos\varphi_1 + J_2\cos\varphi_2 + J_3\cos(\varphi_1+\varphi_2)}_{\text{due to interband pairs}}$

$J > 0$



$J < 0$



TRI states: I ($\varphi_1=\varphi_2=\pi$), II ($\varphi_1=0,\ \varphi_2=\pi$)

TRSB states: III ($\varphi_1=\varphi_2\neq 0,\pi$), IV ($\varphi_1=-\varphi_2\neq 0,\pi$)

Conclusions

- Linear gradient terms (Lifshitz invariants) lead to a variety of nonuniform SC states, both with and without magnetic field
- Novel features in current-carrying FF states: “SC diode” effect, domain walls
- Interband pairing stabilizes TR symmetry-breaking SC states