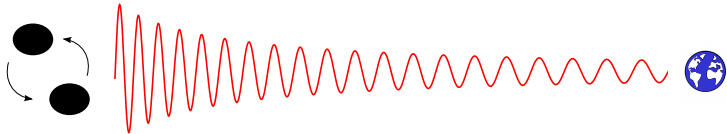


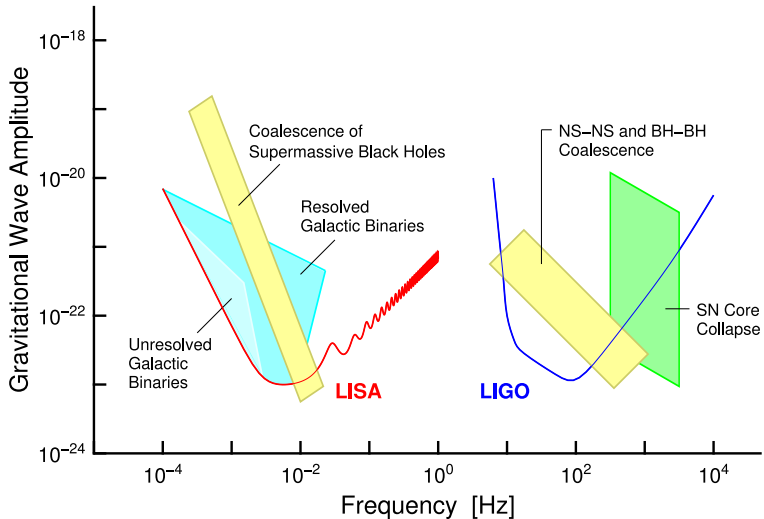
The two-body problem in General Relativity

Alexandre Le Tiec

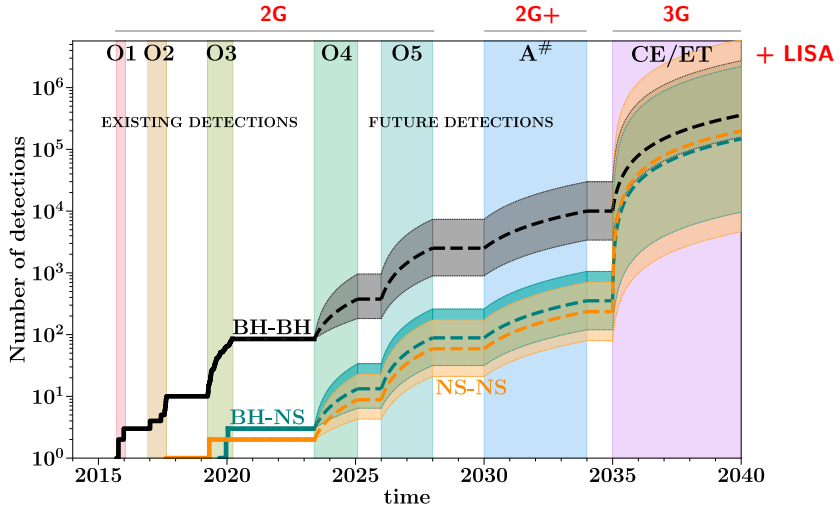
ICTP–SAIFR | UCLA Bhaumik Institute
CNRS | LUX, Observatoire de Paris



Promising sources of gravitational waves



From detections to precision physics



Gravitational-wave astronomy is becoming a **precision science**

Gravitational-wave science

Fundamental physics

- Strong-field tests of GR
- Black hole no-hair theorem
- Cosmic censorship conjecture
- Dark energy equation of state
- Alternatives to general relativity

Astrophysics

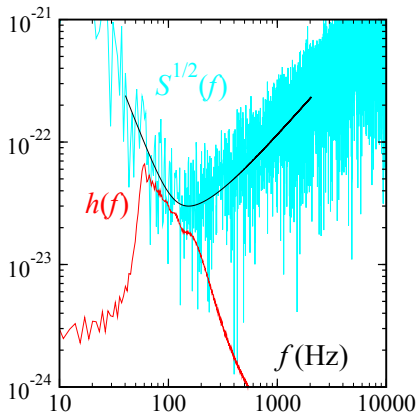
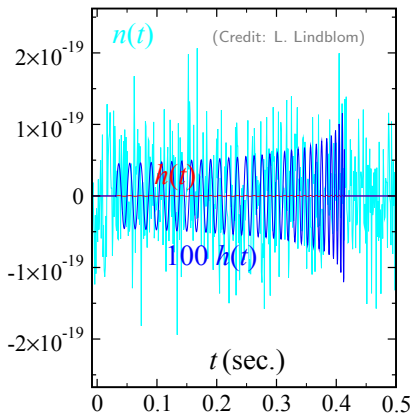
- Formation and evolution of compact binaries
- Origin and mechanisms of γ -ray bursts
- Internal structure of neutron stars

Cosmology

- Cosmography and measure of Hubble's constant
- Origin and growth of supermassive black holes
- Phase transitions during primordial Universe

→ **Lectures by Konstantin and Guilherme**

Need for accurate template waveforms



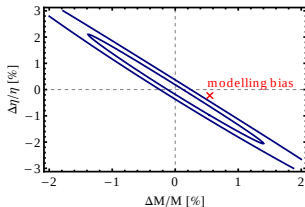
If the expected signal is *known in advance* then $n(t)$ can be filtered and $h(t)$ recovered by **matched filtering** → **template waveforms**

The limits of current waveform models

Accuracy problem → Lecture I

For 3G detectors we find that the mismatch error for semi-analytical models needs to be reduced by at least **three orders of magnitude** and for NR waveforms by **one order of magnitude**.

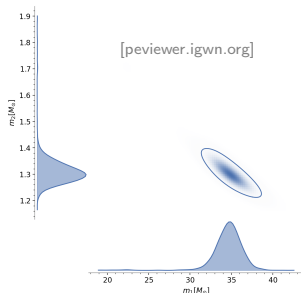
[Pürrer & Haster, PRR 2020]



Coverage problem → Lecture II

The mass ratio of event GW191219_163120's source is inferred to be $q = 0.038^{+0.005}_{-0.004}$, which is **extremely challenging** for waveform modeling, and thus there may be **systematic uncertainties** in results for this candidate.

[LVK Collaboration, PRX 2023]



Outline

Lecture I — From the Kepler problem to relativistic inspiral

Lecture II — From geodesic motion to gravitational self-force

Lecture III — Interfaces, mappings, and invariant dynamics

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Some historical context



Johannes Kepler
(1571-1630)



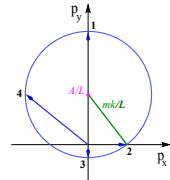
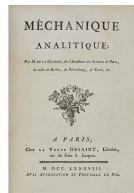
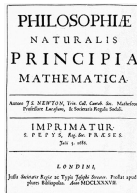
Isaac Newton
(1642-1727)



Joseph-Louis Lagrange
(1736-1813)



William R. Hamilton
(1805-1865)

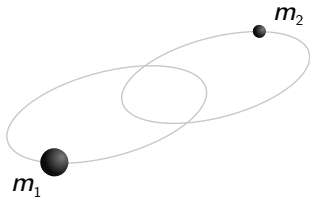


But also: Bernoulli (1710), Clairaut (1747–52), Euler (1740s), d'Alembert (1743–50), Laplace (1799–1825), Legendre (1780s–1820s), Poisson (1800s–30s), Gauss (1800s–40s), Jacobi (1830s–40s), Dirichlet (1830s–50s), Bertrand (1873), Levi-Civita (1904–24), Runge (1919), Lenz (1924), ...

Newtonian two-body Hamiltonian

Real two-body problem

$$H(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{Gm_1m_2}{r_{12}}$$

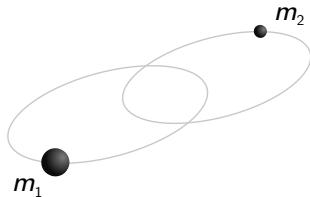


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$$\downarrow \quad \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}$$



Newtonian two-body Hamiltonian

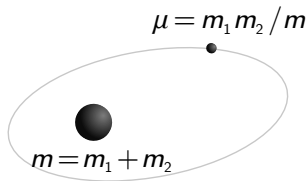
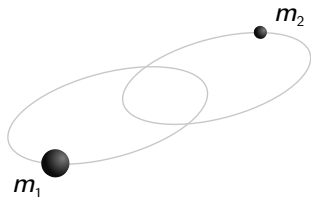
Real two-body problem

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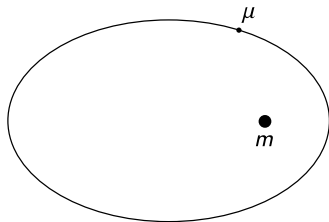
$$\downarrow \quad \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}$$

Effective one-body problem

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2\mu} - \frac{Gm\mu}{r}$$



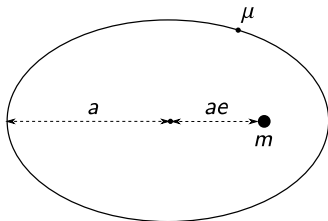
Integrability and Keplerian solution



Integrability and Keplerian solution

Keplerian elements

- eccentricity e
- semi-major axis a



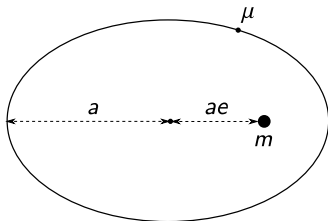
Integrability and Keplerian solution

Keplerian elements

- eccentricity e
- semi-major axis a

Constants of motion

- energy $E = -Gm\mu/(2a)$
- ang. mom. $L = \mu\sqrt{Gma(1 - e^2)}$



Integrability and Keplerian solution

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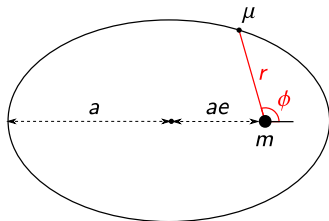
- energy $E = -Gm\mu/(2a)$
- ang. mom. $L = \mu\sqrt{Gma(1 - e^2)}$

Parametric solution

$$r(u) = a(1 - e \cos u)$$

$$\phi(u) = 2 \arctan \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \right)$$

$$\nu(u) = u - e \sin u = n(t - t_0)$$



Integrability and Keplerian solution

Keplerian elements

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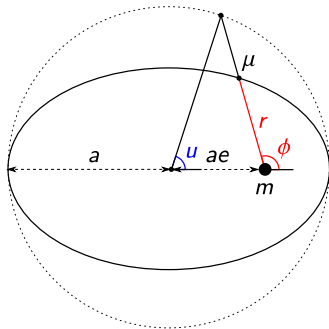
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Integrability and Keplerian solution

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Constants of motion

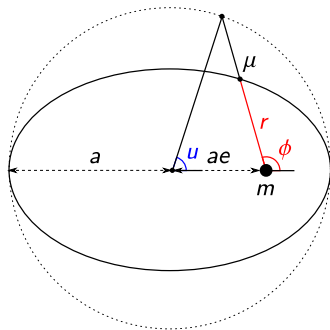
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Kepler's third law: $n^2 a^3 = Gm$

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Classical vs relativistic

Newtonian Gravity

Fixed Euclidean space + time

Natural center-of-mass reduction

Instantaneous interaction

Linear superposition of potentials

Conservative closed system

General Relativity

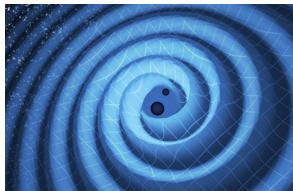
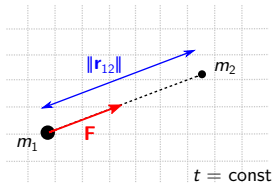
Dynamical spacetime

No exact analogous reduction

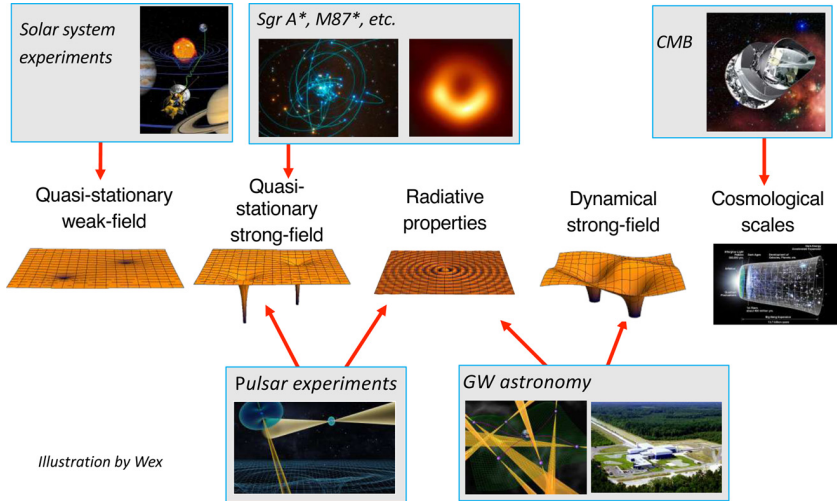
Retarded gravitational field

Nonlinear field equations

System radiates GWs



Different regimes of gravity



What is the relativistic two-body problem?

Problem of motion

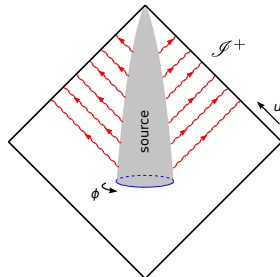


- equations of motion
- orbital dynamics
- strong-field GR

radiation reaction



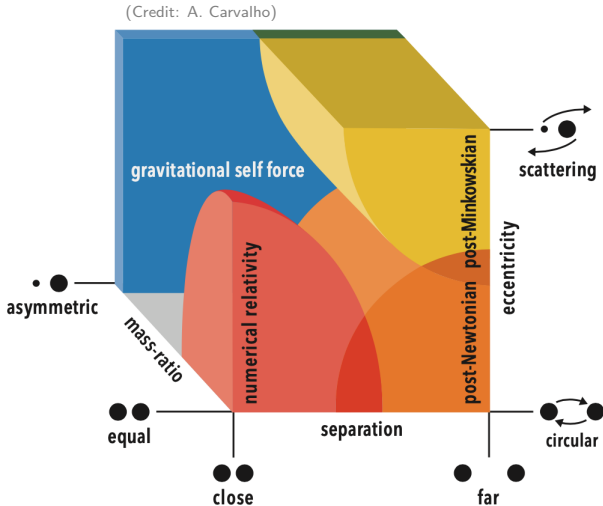
Problem of radiation



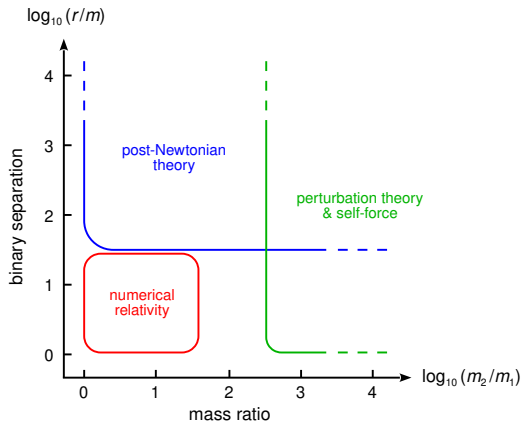
- wave generation
- wave propagation
- asymptotic structure

No known exact analytic solution describing two generic self-gravitating compact objects in GR

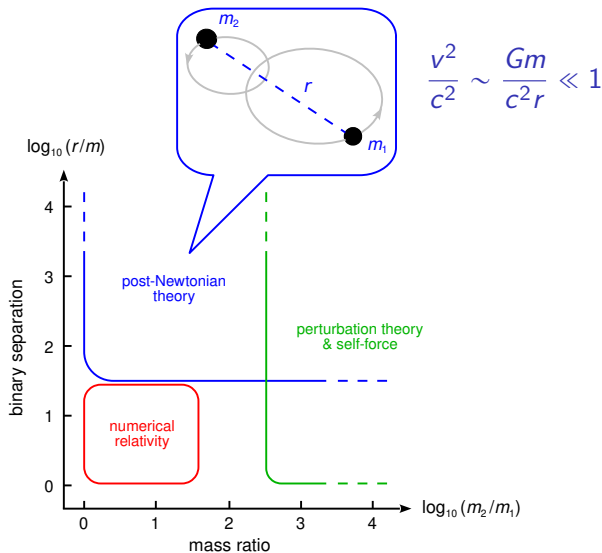
Modelling compact-binary coalescences



Modelling compact-binary coalescences

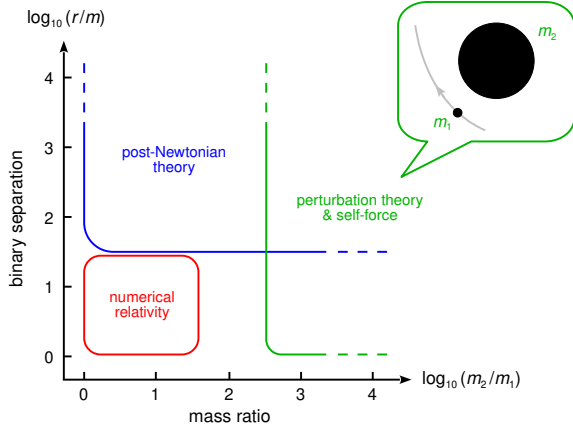


Modelling compact-binary coalescences

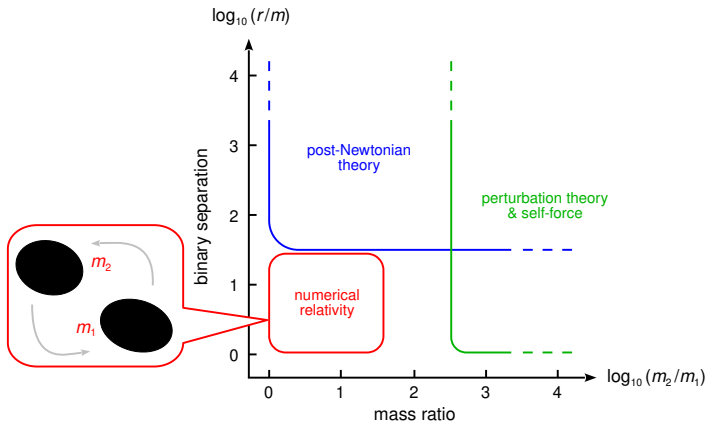


Modelling compact-binary coalescences

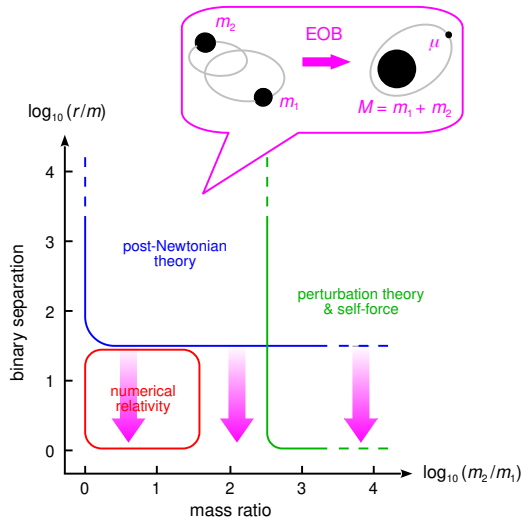
$$\epsilon \equiv \frac{m_1}{m_2} \ll 1$$



Modelling compact-binary coalescences



Modelling compact-binary coalescences



Outline

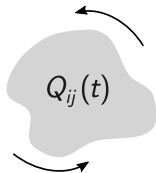
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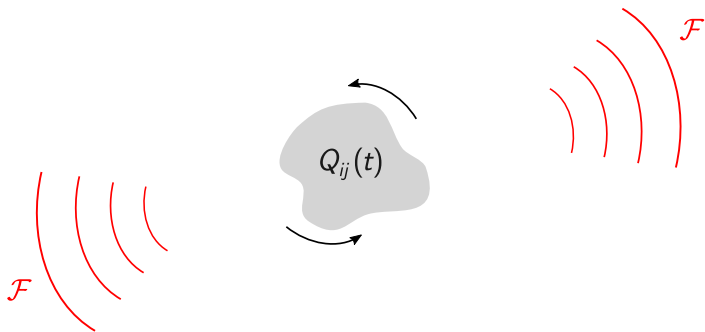
Lecture II — From geodesic motion to gravitational self-force

Lecture III — Interfaces, mappings, and invariant dynamics

Leading-order gravitational-wave emission



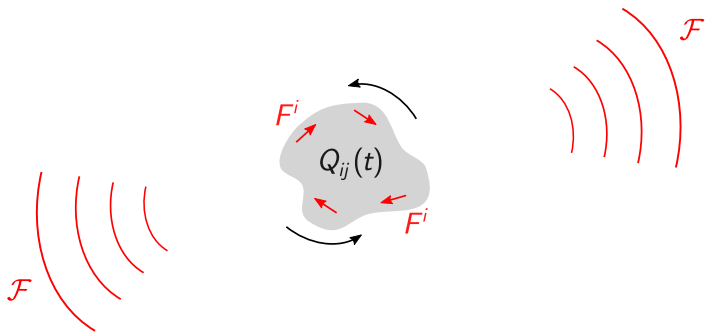
Leading-order gravitational-wave emission



GW energy flux

$$\mathcal{F}(t) = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle$$

Leading-order gravitational-wave emission



GW energy flux

$$\mathcal{F}(t) = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle \quad \longleftrightarrow$$

Radiation reaction force

$$F^i(t, \mathbf{x}) = -\frac{2G}{5c^5} \rho Q_{ij}^{(5)} x^j$$

Radiation reaction and secular evolution

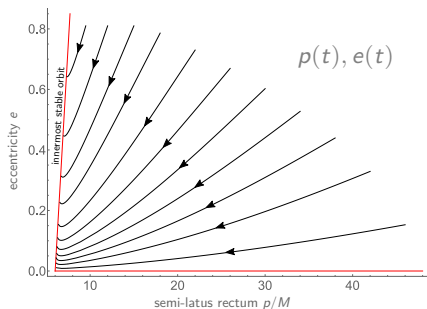
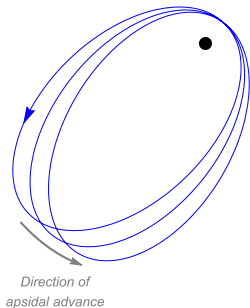
Conservative dynamics

$$E = E_0, \quad L = L_0$$

GW emission
+ rad. reac.

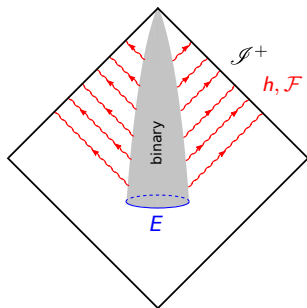
Secular evolution

$$\dot{E} = -\mathcal{F}, \quad \dot{L} = -\mathcal{G}$$



In an isolated binary, **gravitational radiation** drives **secular evolution** of the conservative orbital dynamics

From dynamics to waveform

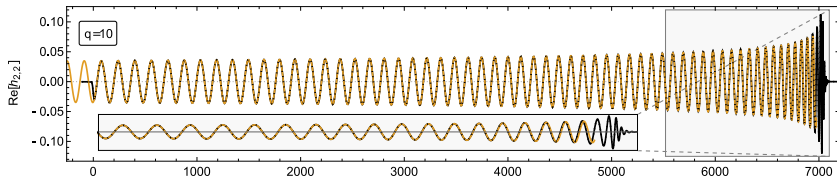


Adiabatic approximation: $T_{\text{orb}} \ll T_{\text{RR}}$

$$\dot{E} = -\mathcal{F} \implies \dot{\Omega} = -\mathcal{F}(\Omega)/E'(\Omega)$$

$\Downarrow$

$$h[\phi(t)] \longleftarrow \phi(t) = \int^t \Omega(t') dt'$$



Example: perturbative inspiral waveform [Wardell *et al.*, PRL 2023]

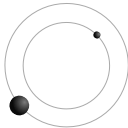
Beyond quasi-circular inspirals

Dynamics

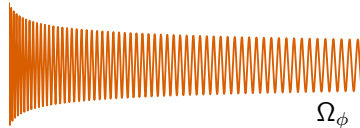
Balance laws

Waveform

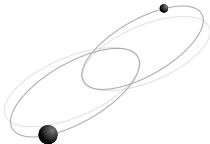
Quasi-circular



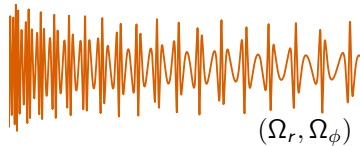
$$\dot{E} = -\mathcal{F}$$



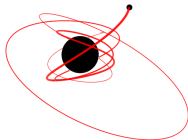
Quasi-eccentric



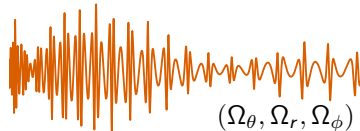
$$\begin{aligned}\dot{E} &= -\mathcal{F} \\ \dot{L} &= -\mathcal{G}\end{aligned}$$



Generic orbit



$$\begin{aligned}\dot{E} &= -\mathcal{F} \\ \dot{L} &= -\mathcal{G} \\ \dot{Q} &= ?\end{aligned}$$



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Small parameter

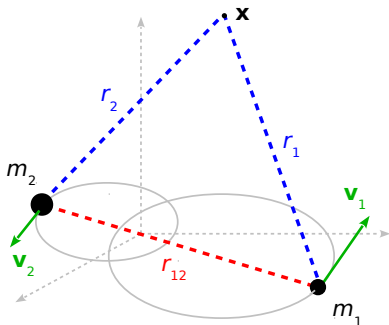
$$\varepsilon \sim \frac{\mathbf{v}_{12}^2}{c^2} \sim \frac{Gm}{r_{12}c^2} \ll 1$$

Example

$$g_{00}(t, \mathbf{x}) = -1 + \underbrace{\frac{2Gm_1}{r_1 c^2}}_{\text{Newtonian}} + \underbrace{\frac{4Gm_2 \mathbf{v}_2^2}{r_2 c^4}}_{\text{1PN term}} + \dots + (1 \leftrightarrow 2)$$

Notation

n PN order refers to effects $O(c^{-2n})$ with respect to “Newtonian” solution



LL formulation and harmonic gauge

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta} - \eta^{\alpha\beta}$$

LL formulation and harmonic gauge

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

LL formulation and harmonic gauge

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Longleftrightarrow \quad \begin{cases} \partial_\alpha h^{\alpha\beta} = 0 \\ \square_\eta h^{\alpha\beta} = 16\pi |g| T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}[h]}_{\substack{\text{nonlinearities} \\ \partial h \partial h + \dots}} \end{cases}$$

LL formulation and harmonic gauge

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$$

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LL formulation and harmonic gauge

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LL formulation and harmonic gauge

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LL formulation and harmonic gauge

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Weak-field approximation

$$|h^{\alpha\beta}| \ll 1$$

LL formulation and harmonic gauge

Metric potential

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Einstein field equations

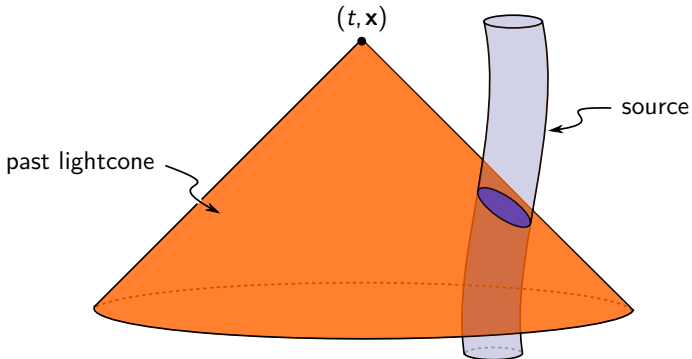
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Weak-field approximation

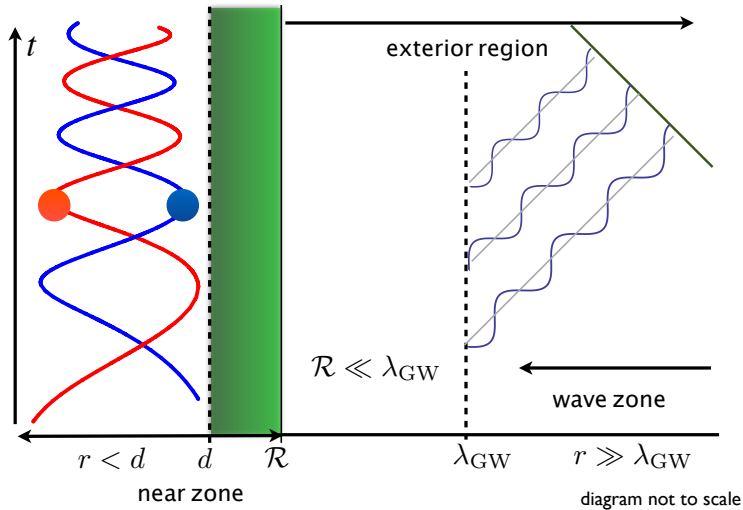
$$|h^{\alpha\beta}| \ll 1 \quad \Longrightarrow \quad \text{perturbative nonlinear treatment}$$

Flat space retarded propagator

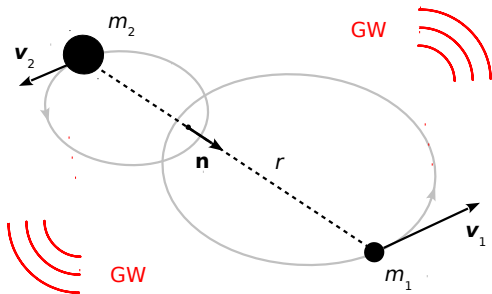
$$h^{\alpha\beta}(t, \mathbf{x}) = -4 \int_{\mathbb{R}^3} \frac{d^3\mathbf{y}}{|\mathbf{x} - \mathbf{y}|} \tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{y}|/c, \mathbf{y})$$



A wave generation formalism



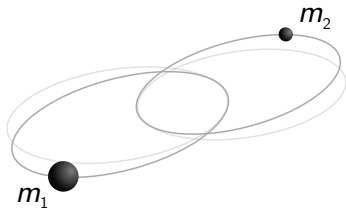
Two-body equations of motion



$$\frac{d\mathbf{v}_1}{dt} = \underbrace{-\frac{Gm_2}{r^2} \mathbf{n} + \frac{\mathbf{A}_{1PN}}{c^2} + \frac{\mathbf{A}_{2PN}}{c^4}}_{\text{conservative terms}} + \underbrace{\frac{\mathbf{A}_{2.5PN}}{c^5}}_{\text{radiation reaction}} + \underbrace{\frac{\mathbf{A}_{3PN}}{c^6}}_{\text{conserv.}} + \underbrace{\frac{\mathbf{A}_{3.5PN}}{c^7}}_{\text{radiation reaction}} + \dots$$

Two-body Hamiltonian at 1PN order

$$H = H_N + \frac{1}{c^2} H_{1\text{PN}} + O(c^{-4})$$



$$\begin{aligned} H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{4} \frac{G^2 m_1 m_2 m}{r_{12}^2} \\ & + \frac{1}{8} \frac{G m_1 m_2}{r_{12}} \left[-12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{m_1 m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right] \\ & + (1 \leftrightarrow 2) \end{aligned}$$

Gravitational-wave tail effect at 4PN order

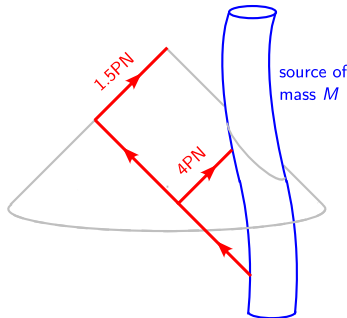
- Starting at **4PN order**, the near-zone metric depends on the entire past history of the **source**:

$$g_{00}^{\text{tail}}(t, \mathbf{x}) = -\frac{8G^2 M}{5c^{10}} x^i x^j \int_{-\infty}^t dt' Q_{ij}^{(7)}(t') \ln \left(\frac{c(t-t')}{2r} \right)$$

- This leads to a **4PN non-local-in-time** contribution to the Fokker action:

$$S_F^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{dt dt'}{|t-t'|} \ddot{Q}_{ij}(t) \ddot{Q}_{ij}(t')$$

- And to a **1.5PN** relative correction to the leading radiation-reaction force



State-of-the-art: 5PN equations of motion

$$\begin{aligned}
 \frac{d\mathbf{v}_1}{dt} = & \underbrace{-\frac{Gm_2}{r^2} \mathbf{n} + \frac{\mathbf{A}_{1PN}}{c^2} + \frac{\mathbf{A}_{2PN}}{c^4}}_{\text{conservative terms}} + \underbrace{\frac{\mathbf{A}_{2.5PN}}{c^5}}_{\text{RR}} + \underbrace{\frac{\mathbf{A}_{3PN}}{c^6}}_{\text{cons.}} + \underbrace{\frac{\mathbf{A}_{3.5PN}}{c^7}}_{\text{RR}} + \\
 & + \underbrace{\frac{\mathbf{A}_{4PN}}{c^8}}_{\text{local cons. + nonlocal tail}} + \underbrace{\frac{\mathbf{A}_{4.5PN}}{c^9}}_{\text{RR}} + \underbrace{\frac{\mathbf{A}_{5PN}}{c^{10}}}_{\text{local cons. + nonlocal tail}} + \dots
 \end{aligned}$$

4PN { [Jaranowski & Schäfer 2012, 2013; Damour, Jaranowski & Schäfer 2014] ADM Hamiltonian
 [Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015–2017] Fokker Lagrangian
 [Foffa, Mastrolia, Sturani & Sturm 2012–2017] Effective Field Theory

5PN { [Bini, Damour & Geralico 2020] PN/PM/SF/EOB synthesis
 [Blümlein, Maier, Marquard & Schäfer 2022] Effective Field Theory
 [Porto, Riva & Yang 2025] nonlinear radiation reaction
 [Porto & Riva 2026] hereditary + RR contributions

Phasing for inspiralling compact binaries

- Conservative orbital dynamics → **5PN** binding energy

$$E(\Omega) = \underbrace{-\frac{\mu}{2} (m\Omega)^{2/3}}_{\text{Newtonian binding energy}} \underbrace{\left(1 + \dots\right)}_{\text{5PN relative correction}}$$

Phasing for inspiralling compact binaries

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- Wave generation formalism → **4.5PN** GW energy flux

$$\mathcal{F}(\Omega) = \underbrace{\frac{32}{5} \nu^2 (m\Omega)^5}_{\text{Einstein's quad. formula}} \underbrace{\left(1 + \dots\right)}_{\text{4.5PN relative correction}}$$

Phasing for inspiralling compact binaries

- Conservative orbital dynamics $\rightarrow$ **5PN** binding energy

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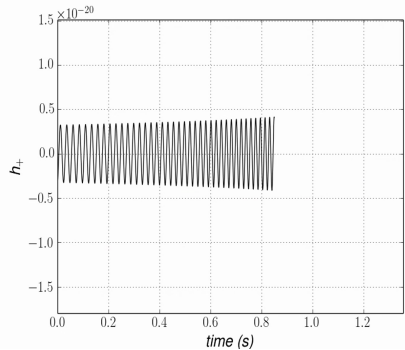
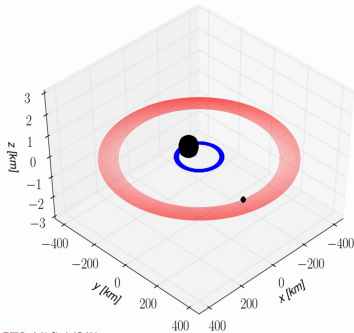
- Wave generation formalism $\rightarrow$ **4.5PN** GW energy flux

$$\mathcal{F}(\Omega) = \underbrace{\frac{32}{5} \nu^2 (m\Omega)^5}_{\text{Einstein's quad. formula}} \underbrace{\left(1 + \dots\right)}_{\text{4.5PN relative correction}}$$

- Energy balance $\rightarrow$ **4.5PN** orbital phase and GW phase

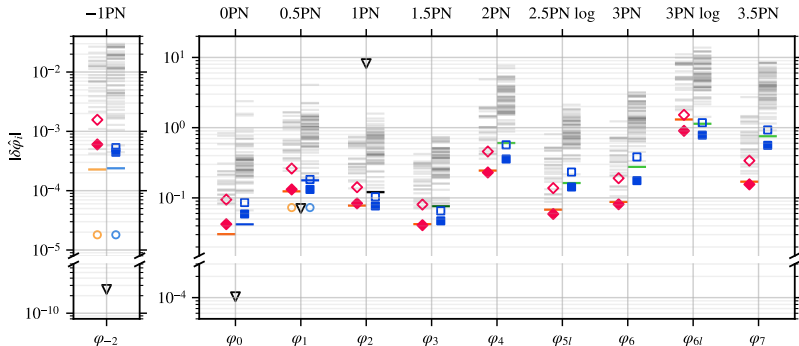
$$\frac{dE}{dt} = -\mathcal{F} \implies \frac{d\Omega}{dt} = -\frac{\mathcal{F}(\Omega)}{E'(\Omega)} \implies \phi(t) = \int^t \Omega(t') dt'$$

Waveform for inspiralling compact binaries



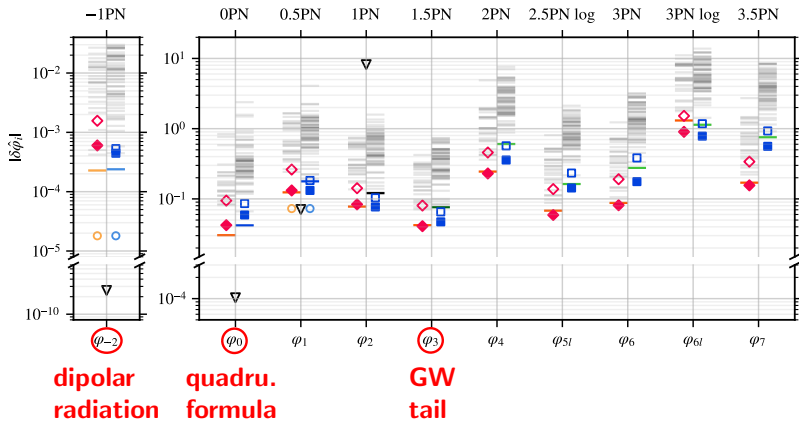
Constraining post-Newtonian parameters

- | | | |
|-----------------------|-------------------------|-------------------------|
| ◆ O4a+O4b FTI | ▼ PSR J0737-3039A/B | — GW250114_082203 TIGER |
| ◇ O4a FTI | ■ GWTC-5.0 TIGER | — GW230627_015337 TIGER |
| ○ GW170817 FTI | □ GWTC-4.0 TIGER | — GW190814 TIGER |
| — GW230518_125908 FTI | ○ GW170817 TIGER | — GW241011_233834 TIGER |
| — GW250114_082203 FTI | — GW230518_125908 TIGER | |

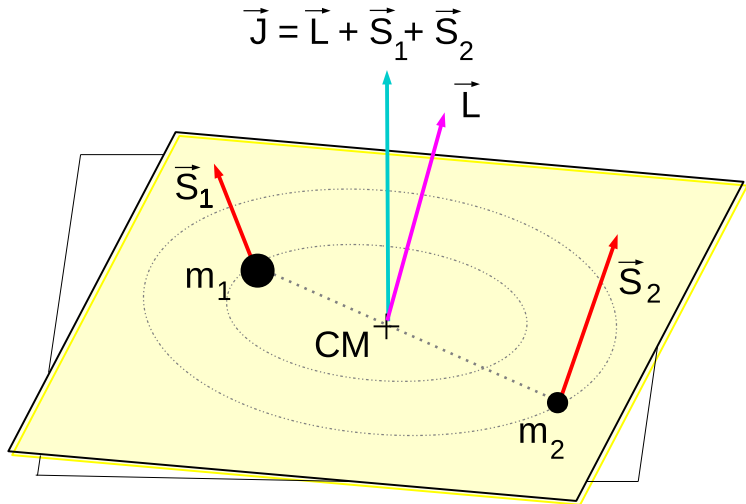


Constraining post-Newtonian parameters

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Binary systems of spinning compact bodies



(Credit: L. Blanchet)

Spin-orbit coupling at leading order

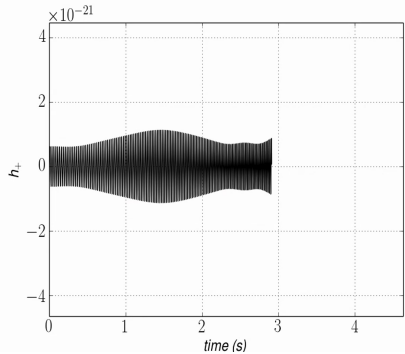
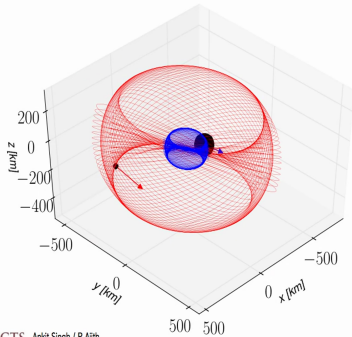


$$\frac{d\mathbf{S}_a}{dt} = \boldsymbol{\Omega}_a \times \mathbf{S}_a$$

$$H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = H_{\text{orb}}(\mathbf{x}_a, \mathbf{p}_a) + \overbrace{\sum_b \boldsymbol{\Omega}_b(\mathbf{x}_a, \mathbf{p}_a) \cdot \mathbf{S}_b}^{\text{spin-orbit coupling}}$$

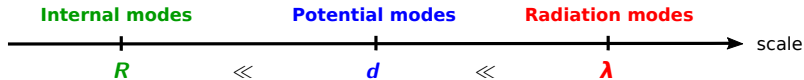
$$\boldsymbol{\Omega}_1(\mathbf{x}_a, \mathbf{p}_a) = \frac{G}{c^2 r_{12}^2} \left(\frac{3m_2}{2m_1} \mathbf{n}_{12} \times \mathbf{p}_1 - 2\mathbf{n}_{12} \times \mathbf{p}_2 \right) \propto \mathbf{L}$$

Spin effects on the waveform



EFT formulation of the PN approximation

Hierarchy of scales



Effective action

$$S_{\text{eff}} = \underbrace{S_{\text{EH}}[g] - \sum_a m_a \int d\tau_a}_{\text{point-particle EFT}} + \underbrace{\sum_a c_{E,a} \int d\tau_a \left(E_{\alpha\beta} E^{\alpha\beta} + \dots \right)}_{\text{finite-size operators}}$$

Power counting

$$\epsilon \equiv v^2 \sim R/d \quad \text{Newtonian} = O(\epsilon^0), \quad \text{Tidal quadrupole} = O(\epsilon^5)$$

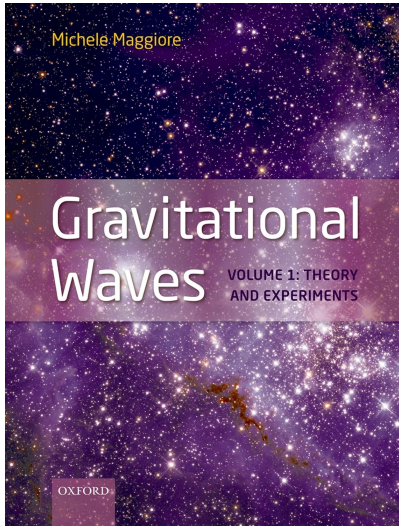
State-of-the-art of known PN results

PN order	Dynamics				Dissipative flux			
	Non-spinning	Spinning			Non-spinning	Spinning		
		SO	SS	Higher spins		SO	SS	Higher spins
0	✓	—	—	—	✓	—	—	—
1	✓	—	—	—	✓	—	—	—
1.5	—	✓	—	—	✓	✓	—	—
2	✓	—	✓	—	✓	—	✓	—
2.5	✓	✓	—	—	✓	✓	—	—
3	✓	—	✓	—	✓	✓	✓	—
3.5	✓	✓	—	✓ (S^3)	✓	✓	✓	✓ (S^3)
4	✓	—	✓	✓ (S^4)	✓	✓	✓	
4.5	✓	✓	—	✓ (S^3)	✓			
5	✓	—	✓	✓ (S^4)				

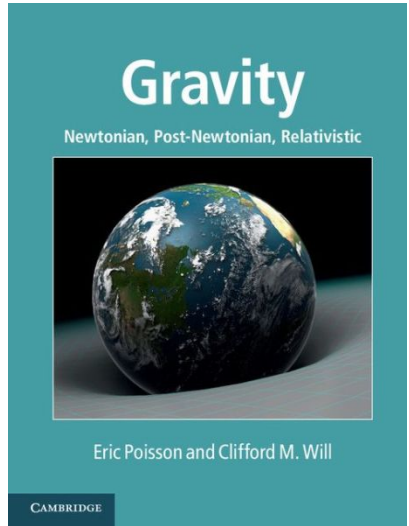
Further reading

- L. Blanchet
Post-Newtonian theory for gravitational waves
Living Rev. Rel. **27**, 4 (2024)
- G. Schäfer & P. Jaranowski
Hamiltonian formulation of general relativity and post-Newtonian dynamics of compact binaries
Living Rev. Rel. **27**, 2 (2024)
- M. Levi
Effective field theories of post-Newtonian gravity: a comprehensive review
Rep. Prog. Phys. **83**, 075901 (2020)
- R. A. Porto
The effective field theorist's approach to gravitational dynamics
Phys. Rept. **633**, 1 (2016)
- S. Foffa & R. Sturani
Effective field theory methods to model compact binaries
Class. Quantum Grav. **31**, 043001 (2014)
- T. Futamase & Y. Itoh
The post-Newtonian approximation for relativistic compact binaries
Living Rev. Rel. **10**, 2 (2007)

Further reading



2007



2014

Outline

Lecture I — From the Kepler problem to relativistic inspiral

Lecture II — From geodesic motion to gravitational self-force

- ① The extreme-mass-ratio limit
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- ③ Black-hole perturbation theory and self-force
- ④ Adiabatic, post-adiabatic, and second-order waveforms

Lecture III — Interfaces, mappings, and invariant dynamics

Outline

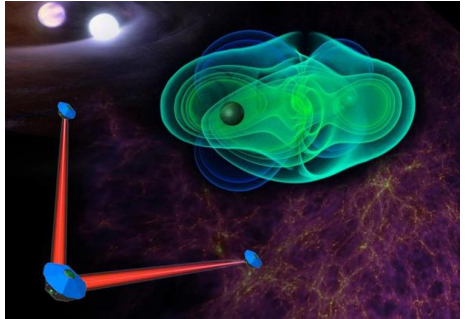
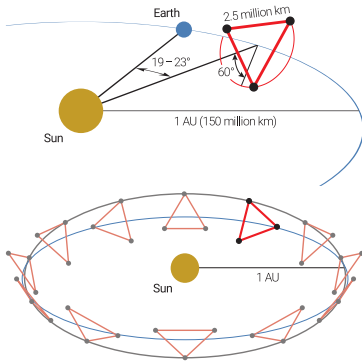
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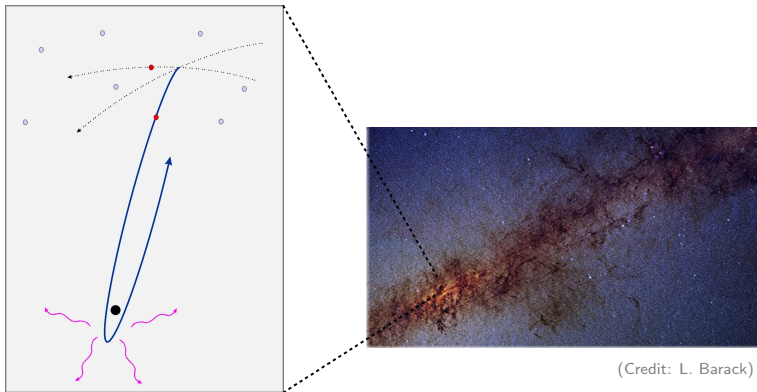
Lecture III — Interfaces, mappings, and invariant dynamics

LISA: a gravitational antenna in space



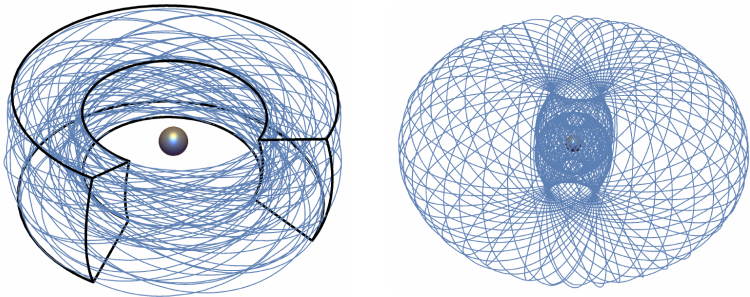
The **LISA mission** was officially adopted/approved by ESA in 2024, with a launch planned for **2035**

Extreme Mass Ratio Inspirals (EMRIs)



- LISA sensitive to $M \sim 10^5 - 10^7 M_\odot \rightarrow q \equiv M/m \sim 10^3 - 10^6$
- $T_{\text{orb}} \propto M \sim \text{hr}$ and $T_{\text{insp}} \propto M^2/m \sim \text{yrs} \rightarrow N_{\text{band}} \sim 10^5 \text{ cycles}$

EMRIs as probes of strong-field geometry

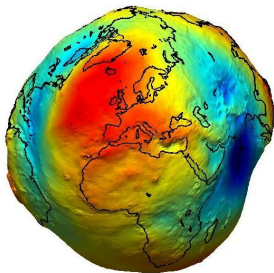


Assuming the central object is a **Kerr black hole**:

- Orbits are **tri-periodic** (1 rotation + 2 librations)
- Orbits are **ergodic** (phase space-filling) in general
- Principal elements undergo conservative secular **precession**
- Principal elements drift under gravitational **radiation-reaction**

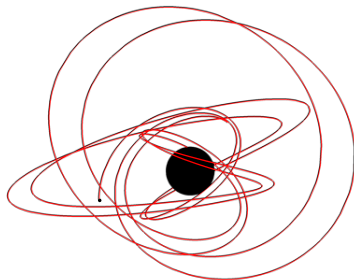
EMRIs as probes of strong-field geometry

Geodesy



M_{lm} arbitrary

Botromeladesy



$$M_{l0} + iS_{l0} = M(ia)^l$$

Objective: test the black hole **no-hair theorem** of GR

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Lecture III — Interfaces, mappings, and invariant dynamics

Hamiltonian formulation of geodesic motion

Canonical Hamiltonian

$$H(x, p) = \frac{1}{2m} g^{\alpha\beta}(x) p_\alpha p_\beta$$

Hamiltonian formulation of geodesic motion

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Constants of the motion

- On-shell value $H = -\frac{1}{2}m$
- Energy $E = -t^\alpha p_\alpha = -p_t$
- Ang. mom. $L_z = \phi^\alpha p_\alpha = p_\phi$
- Carter constant $Q = K^{\alpha\beta} p_\alpha p_\beta$

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Completely integrable

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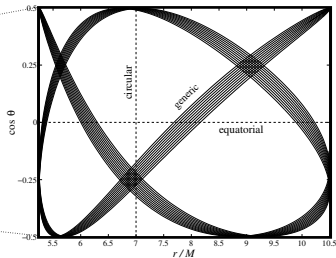
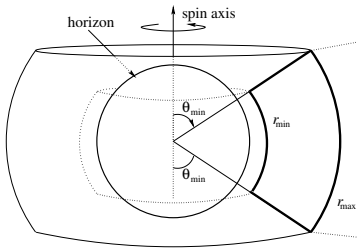
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Generalized action-angle variables

- Canonical transformation $(x^\alpha, p_\alpha) \rightarrow (\theta^\alpha, I_\alpha)$ to generalized **action-angle** variables, with

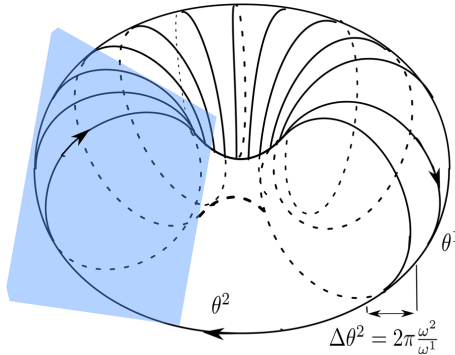
$$I_\alpha \equiv \frac{1}{2\pi} \int_{\gamma_\alpha} \mathbf{p} \cdot d\mathbf{x} \quad \text{and} \quad \begin{cases} x^\alpha(\theta + 2\pi, I) = x^\alpha(\theta, I) \\ p_\alpha(\theta + 2\pi, I) = p_\alpha(\theta, I) \end{cases}$$

- Hamilton's equations of motion for $\mathcal{H}(\theta, I) = H(x, p)$:

$$\dot{\theta}^\alpha = \left(\frac{\partial \mathcal{H}}{\partial I_\alpha} \right)_\theta \equiv \omega^\alpha(I), \quad \dot{I}_\alpha = - \left(\frac{\partial \mathcal{H}}{\partial \theta^\alpha} \right)_I = 0$$

- The **coordinate-invariant** information is contained in the four fundamental frequencies $\omega^\alpha(I) \rightarrow$ [Lecture III](#)

Resonant orbits in Kerr spacetime



Orbital resonances are a **challenge** for data analysis of EMRI signals

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Lecture III — Interfaces, mappings, and invariant dynamics

Linearized Einstein equation

Linearization about a vacuum background

$$g_{\alpha\beta}^{\text{exact}} = g_{\alpha\beta} + h_{\alpha\beta}, \quad |h_{\alpha\beta}| \ll 1$$
$$G_{\alpha\beta}[g + h] = \cancel{G_{\alpha\beta}[g]} + \delta G_{\alpha\beta}[h] + O(h^2)$$

Lorenz gauge condition and wave equation

$$\nabla^\alpha \bar{h}_{\alpha\beta} = 0, \quad \bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h$$

$$\boxed{\square_g \bar{h}_{\alpha\beta} + 2R_{\rho\alpha\sigma\beta} \bar{h}^{\rho\sigma} = -16\pi T_{\alpha\beta}}$$

Linear PDE on a curved background $\rightarrow$ exploit the **symmetries** of $g_{\alpha\beta}$

Black-hole perturbation theory: Schwarzschild

Symmetry decomposition

- Stationary $\rightarrow e^{-i\omega t}$
- Spherical symmetry $\rightarrow$ tensor spherical harmonics
- Even (+) and odd (−) parity sectors **decouple**

Regge–Wheeler–Zerilli–Moncrief

$$h_{\alpha\beta} = \sum_{lm,\pm} h_{\alpha\beta}^{lm,\pm}$$

$$\frac{d^2 \psi_{lm}^{\pm}}{dr_*^2} + [\omega^2 - V_l^{\pm}(r)] \psi_{lm}^{\pm} = S_{lm}^{\pm}$$

Waveform + fluxes

Black-hole perturbation theory: Kerr

Symmetry decomposition

- Stationary + axial symmetry $\rightarrow e^{-i\omega t} e^{im\phi}$
- Spin-weighted spheroidal harmonics $_{-2}S_{lm}^{a\omega}(\theta)$
- Teukolsky equation for ψ_4 is **separable**

Separated perturbations

$$\psi_4 \sim \sum_{lm} \int d\omega R_{lm\omega}(r) {}_{-2}S_{lm}^{a\omega}(\theta) e^{i(m\phi - \omega t)}$$

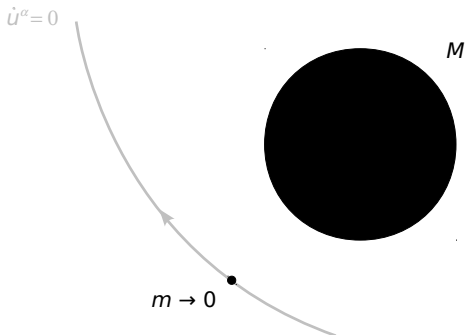
$$\omega_{mkn} = m\Omega_\phi + k\Omega_\theta + n\Omega_r$$

Waveform + fluxes

Gravitational self-force: A primer

Spacetime metric

$$g_{\alpha\beta}^{\text{exact}} = g_{\alpha\beta}$$



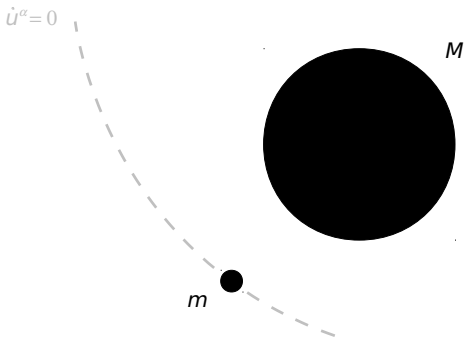
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Spacetime metric

$$g_{\alpha\beta}^{\text{exact}} = g_{\alpha\beta}$$

Small parameter

$$\epsilon \equiv \frac{m}{M} \ll 1$$



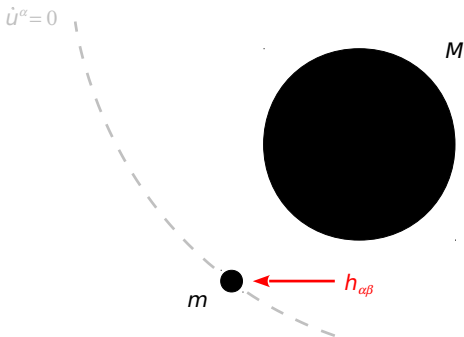
Gravitational self-force: A primer

Spacetime metric

$$g_{\alpha\beta}^{\text{exact}} = g_{\alpha\beta} + h_{\alpha\beta}$$

Small parameter

$$\epsilon \equiv \frac{m}{M} \ll 1$$



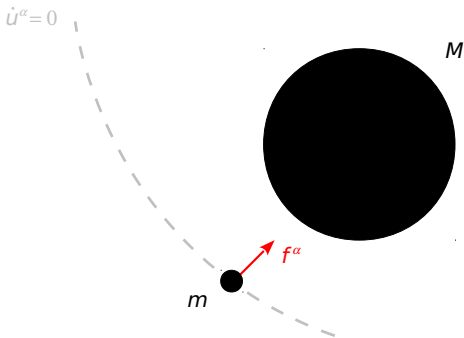
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Gravitational self-force: A primer

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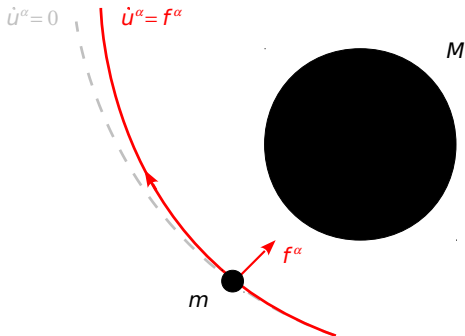
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Small parameter

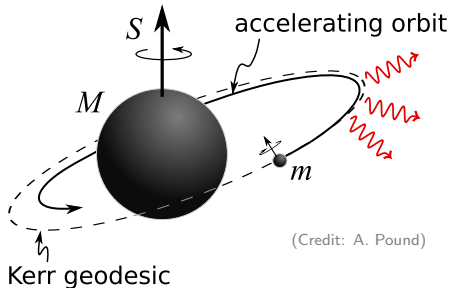
$$\epsilon \equiv \frac{m}{M} \ll 1$$

Gravitational self-force

$$\dot{u}^\alpha \equiv u^\beta \nabla_\beta u^\alpha = f^\alpha[h]$$

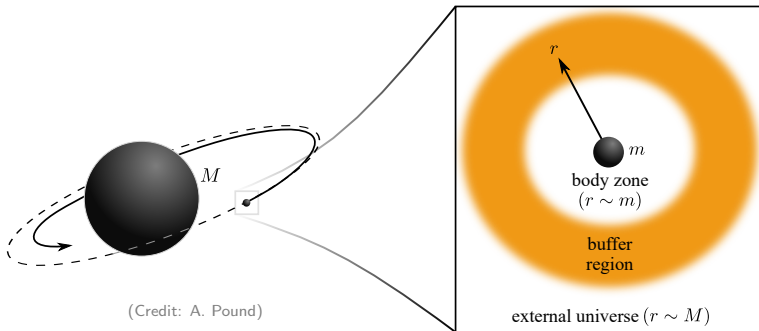


Gravitational self-force: A primer



- Dissipative component $\longleftrightarrow$ **gravitational-wave** emission
- Conservative component $\longleftrightarrow$ secular precession effects

Matched asymptotic expansions

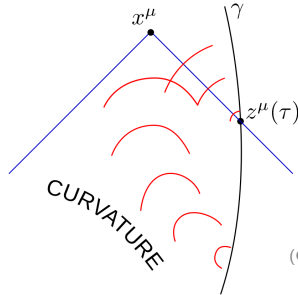


- Trajectory defined on background spacetime using a suitable far-zone limit; constrained by **matching near & far expansions**
- **No resort to point particles:** notion *derived* rather than assumed
→ one can associate a worldline γ , four-velocity u^α and mass m

Post-geodesic equation of motion

Metric perturbation

$$h_{\alpha\beta} = h_{\alpha\beta}^{\text{direct}} + h_{\alpha\beta}^{\text{tail}}$$



(Credit: A. Pound)

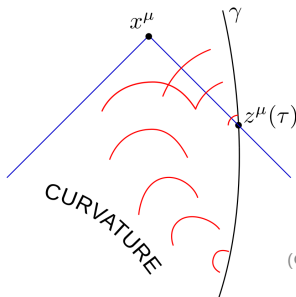
Post-geodesic equation of motion

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$$h_{\alpha\beta} = h_{\alpha\beta}^{\text{direct}} + h_{\alpha\beta}^{\text{tail}}$$

Tail contribution

$$h_{\alpha\beta}^{\text{tail}}(x) = m \int_{-\infty}^{\tau^-} \left(G_{\alpha\beta\alpha'\beta'} - \frac{1}{2} g_{\alpha\beta} G_{\lambda}{}^{\lambda}{}_{\alpha'\beta'} \right) (x, z(\tau')) u^{\alpha'} u^{\beta'} d\tau'$$

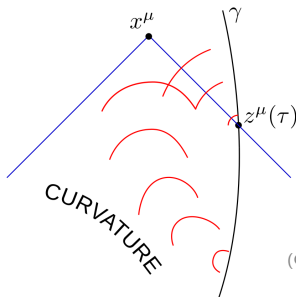


(Credit: A. Pound)

Post-geodesic equation of motion

Metric perturbation

$$h_{\alpha\beta} = h_{\alpha\beta}^{\text{direct}} + h_{\alpha\beta}^{\text{tail}}$$



(Credit: A. Pound)

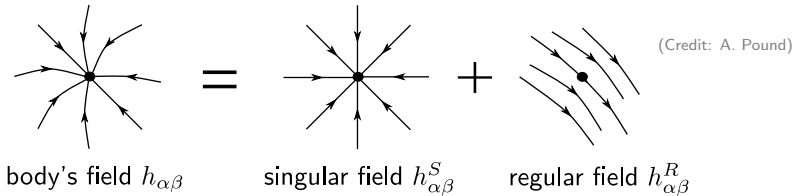
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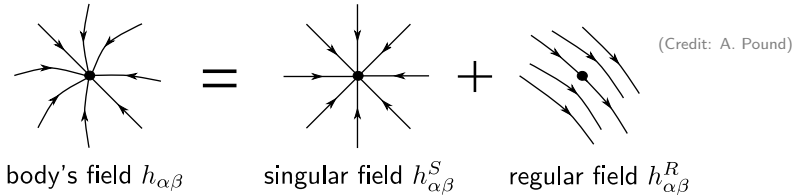
MiSaTaQuWa equation

$$\dot{u}^{\alpha} = (g^{\alpha\beta} + u^{\alpha} u^{\beta}) \left(\frac{1}{2} \nabla_{\beta} h_{\rho\sigma}^{\text{tail}} - \nabla_{\rho} h_{\beta\sigma}^{\text{tail}} \right) |_{z(\tau)} u^{\rho} u^{\sigma} \equiv f^{\alpha} [h^{\text{tail}}]$$

Generalized equivalence principle



Generalized equivalence principle



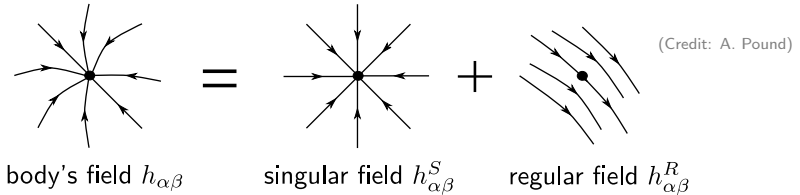
singular/self field

$$h^S \sim m/r$$

$$\square h^S \sim -16\pi T$$

$$f^\alpha[h^S] = 0$$

Generalized equivalence principle



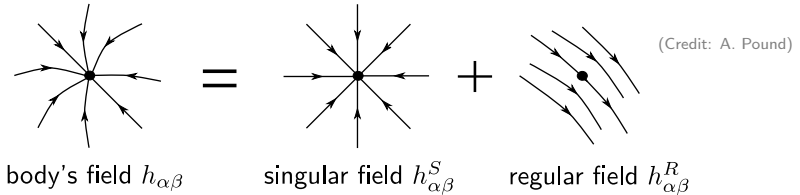
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regular/residual field

$$\begin{aligned}h^R &\sim h^{\text{tail}} + \text{local terms} \\ \square h^R &\sim 0 \\ \dot{u}^\alpha &= f^\alpha[h^R]\end{aligned}$$

Generalized equivalence principle



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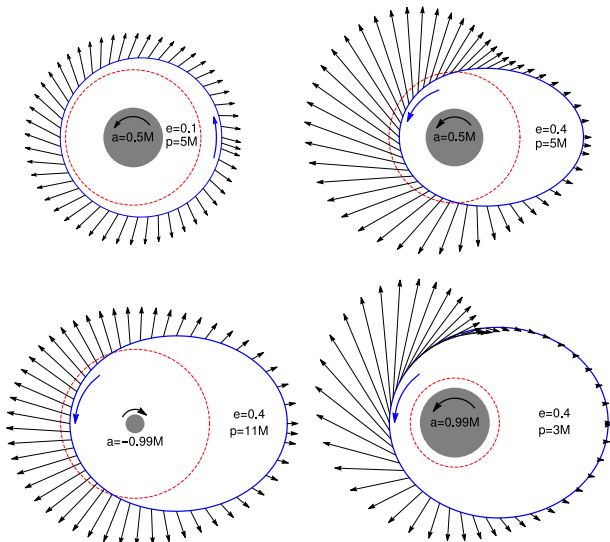
$$\square h^R \sim 0$$

$$\dot{u}^\alpha = f^\alpha[h^R]$$

self-accelerated motion in $g_{\alpha\beta} \iff$ **geodesic motion** in $g_{\alpha\beta} + h_{\alpha\beta}^R$

Self-force along fixed geodesic orbits

Sample results for equatorial orbits in Kerr



Outline

Lecture I — From the Kepler problem to relativistic inspiral

Lecture II — From geodesic motion to gravitational self-force

- ① The extreme-mass-ratio limit
- ② Geodesic motion in Schwarzschild/Kerr
- ③ Black-hole perturbation theory and self-force
- ④ Adiabatic, post-adiabatic, and second-order waveforms

Lecture III — Interfaces, mappings, and invariant dynamics

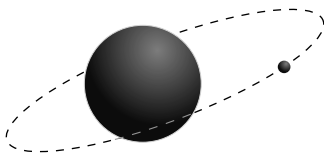
Adiabatic approximation

$$\frac{T_{\text{RR}}}{T_{\text{orb}}} \propto \frac{M}{m} \gg 1$$



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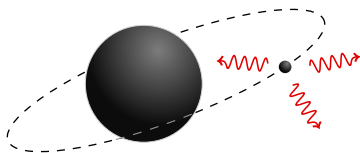
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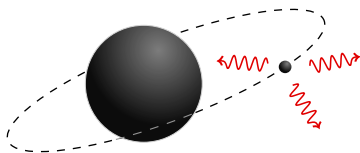
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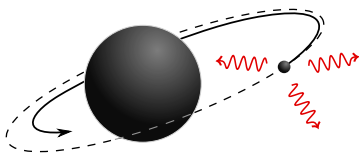


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$$\langle \dot{E} \rangle = -\mathcal{F}_E, \quad \langle \dot{L}_z \rangle = -\mathcal{F}_{L_z}, \quad \langle \dot{Q} \rangle = ?$$

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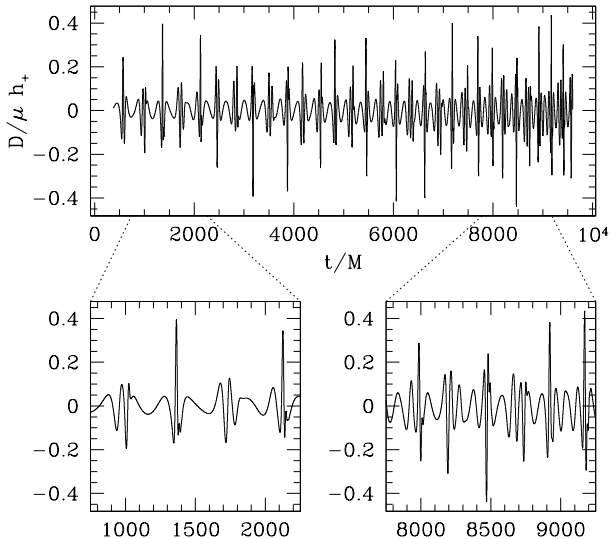


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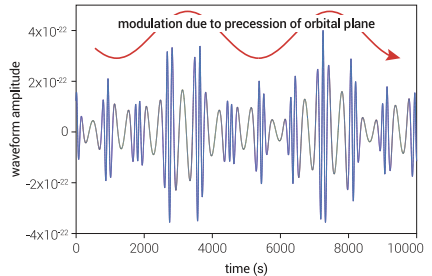
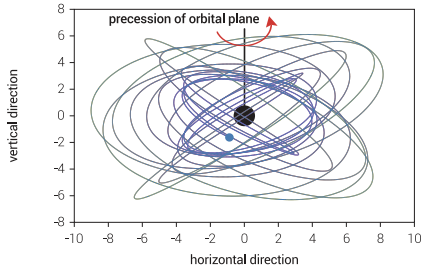
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- **Update** the orbit and play again!

Waveform in the adiabatic approximation



Waveform in the adiabatic approximation



(Credit: Gair et al. 2021)

Accuracy requirement for the phasing

- Over an inspiral timescale $T_{\text{RR}} \sim M/\epsilon$, the GW phase is given by the expansion

$$\phi = \frac{1}{\epsilon} [\phi_0 + \epsilon \phi_1 + O(\epsilon^2)]$$

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We need to account for the local effects of the metric perturbation on the body's orbital motion

Einstein field equations

- Expand the exact metric about a given Kerr background $g_{\alpha\beta}$:

$$g_{\alpha\beta}^{\text{exact}} = g_{\alpha\beta} + \epsilon h_{\alpha\beta}^{(1)} + \epsilon^2 h_{\alpha\beta}^{(2)} + O(\epsilon^3)$$

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- Expand the Einstein tensor $G_{\alpha\beta}$ and the energy-momentum tensor $T_{\alpha\beta}$ accordingly, and equate order by order in ϵ :

$$O(\epsilon^0) : \quad G_{\alpha\beta}[g] = 0$$

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- Naive point-particle source becomes insufficient at $O(\epsilon^2)$
→ decompose into puncture/residual fields: *effective source*

Two-timescale expansion

- Separation of timescales during **inspiral**: $T_{\text{orb}}/T_{\text{RR}} \sim \epsilon \ll 1$

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- The two-timescale expansion of the metric then reads

$$g_{\alpha\beta}^{\text{exact}}(t, \mathbf{x}, \epsilon) = g_{\alpha\beta}(\mathbf{x}) + \epsilon h_{\alpha\beta}^{(1)}(\mathbf{x}, \mathbf{I}, \psi) + \epsilon^2 h_{\alpha\beta}^{(2)}(\mathbf{x}, \mathbf{I}, \psi) + O(\epsilon^3)$$

$$\text{where } h_{\alpha\beta}^{(n)} = \sum_{\mathbf{k}} \underbrace{h_{\alpha\beta}^{(n,\mathbf{k})}(\mathbf{x}, \mathbf{I})}_{\text{slowly evolving amplitude}} \times \underbrace{e^{-i\mathbf{k} \cdot \psi}}_{\text{rapidly evolving phase}}$$

Post-adiabatic orbit evolution

- The actions $\mathbf{I}$ and angles/phases ψ evolve according to

$$\dot{\mathbf{I}} = \sum_{\mathbf{k}} [\epsilon \mathbf{G}_{(1,\mathbf{k})}(\tilde{t}) + \epsilon^2 \mathbf{G}_{(2,\mathbf{k})}(\tilde{t}) + O(\epsilon^3)] e^{-i\mathbf{k} \cdot \psi}$$
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Adiabatic (0PA)

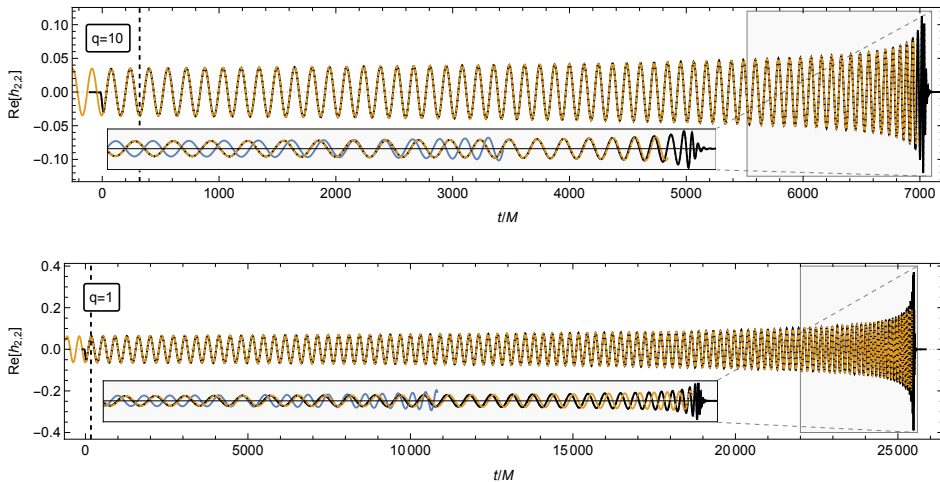
- averaged first-order dissipative self-force

Post-Adiabatic (1PA)

- oscillatory first-order dissipative
- complete first-order conservative
- averaged second-order dissipative

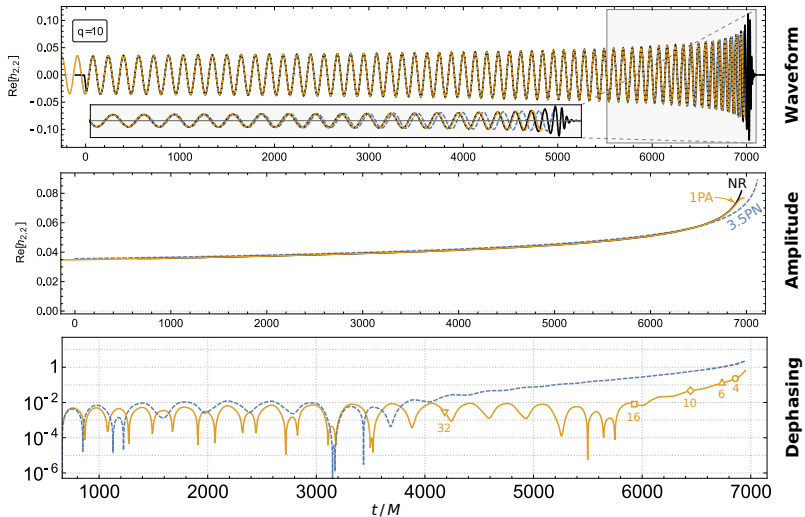
Post-adiabatic gravitational waveforms

Nonspinning binaries on quasicircular orbits



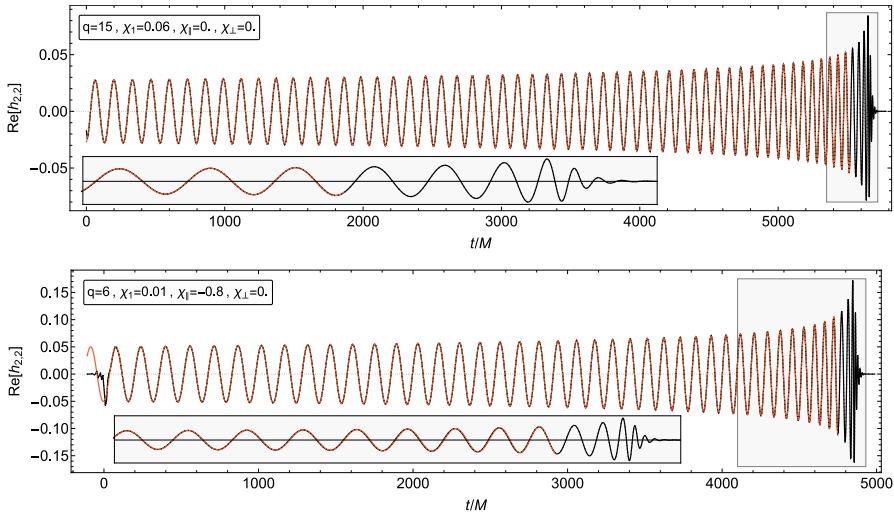
Post-adiabatic gravitational waveforms

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Post-adiabatic gravitational waveforms

Nonprecessing, spinning binaries on quasicircular orbits



Progress and challenges

Background Spacetime	Orbital Configuration	Adiabatic	Post-1-adiabatic			
		1SF (Dissipative)	1SF (Conservative)	2SF (Dissipative)	Spin Effects (Conservative)	Spin Effects (Dissipative)
Schwarzschild	Circular	✓✓✓	✓✓✓	✓✓✓	✓✓✓	✓✓✓
	Eccentric	✓✓✓	✓✓✓	X	✓✓, ✓✓✓*	✓, ✓✓*
Kerr	Circular	✓✓✓	✓✓	X	✓, ✓✓*	✓✓✓*
	Eccentric Equatorial	✓✓✓	✓✓	X	✓, ✓✓*	✓✓*
	Generic	✓✓✓	✓	X	✓	✓*
	Resonances	✓✓✓	✓	X	X	X
✓✓✓ Evolving Waveform		✓✓ Driven Inspiral	✓ Snapshot Calculation		*(Anti-)Aligned Spin Only	

- Generic bound orbits in Kerr
- Transition accross resonances
- Inclusion of the final plunge
- Scattering orbits
- Environmental effects
- Beyond general relativity

Capra Meetings on Radiation Reaction

29th Capra Meeting on Radiation Reaction in General Relativity

Capra in Brussels

The 29th Capra Meeting on Radiation Reaction in General Relativity will be hosted in Brussels from **Monday June 29th to July 3rd 2026**. It will take place at the Université Libre de Bruxelles and will be organised by the International Solvay Institutes.

The conference will be held as an in-person event. The format of the meeting is an amalgamation of a daily invited review talk, short contributed talks and organised discussion sessions.

There is no registration fee. A contribution of 50 euros to the banquet will be requested at registration for the attendees that would like to join the banquet. The registration page can be found [here](#). All participants are expected to adhere to the [Capra Code of Conduct](#).



Organizing Committees

Scientific organizing committee: Leor Barrack, Susanna Barsanti, Béatrice Bonga, Alvin Chua, Lisa Drummond, Scott Hughes, Adam Pound, Vojtěch Witzany.

Local organizing committee: Geoffrey Compère, Marc Henneaux, Loïc Honet, Guillaume Lhost, Gabriel Piovano.

Black Hole Perturbation Toolkit



[Tools](#) [Get started](#) [Docs](#) [Community](#) [Publications](#) 



BLACK HOLE PERTURBATION THEORY

Open tools for modelling gravitational waves

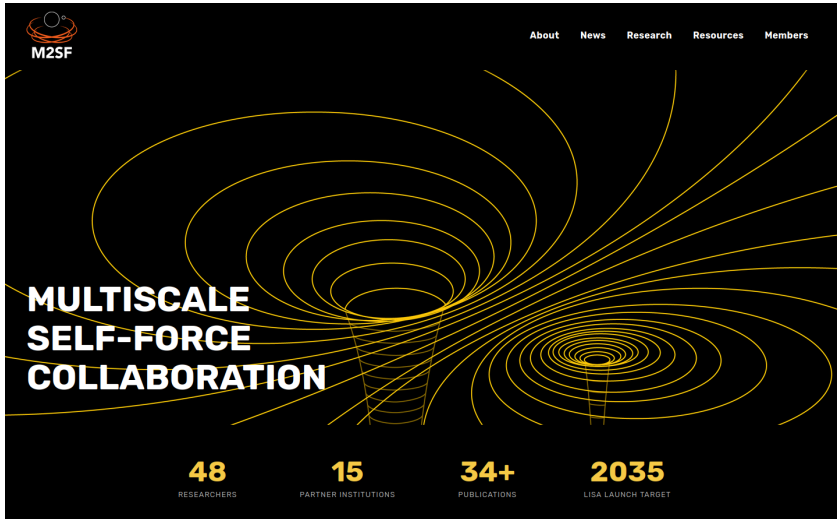
A community-built suite of software and data for black hole perturbation theory, self-force, waveforms and ringdown.

[Browse the tools](#)

[Quick start →](#)



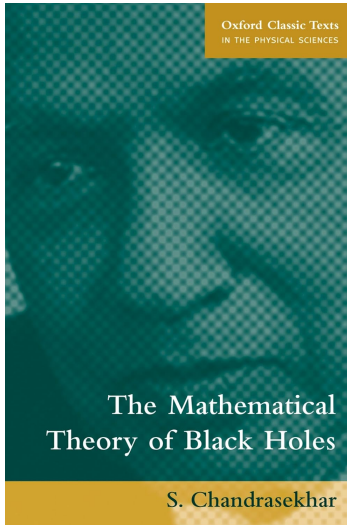
Multiscale self-force collaboration



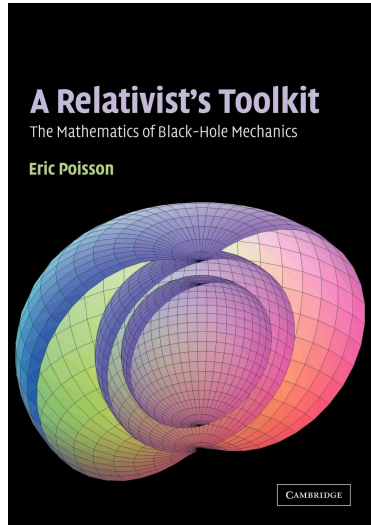
Further reading

- A. Pound & B. Wardell
Black hole perturbation theory and gravitational self-force
In: Handbook of Gravitational Wave Astronomy, Springer (2021)
- L. Barack & A. Pound
Self-force and radiation reaction in general relativity
Rep. Prog. Phys. **82**, 016904 (2019)
- A. I. Harte
Motion in classical field theories and the foundations of the SF problem
Fund. Theor. Phys. **179**, 327 (2015)
- E. Poisson, A. Pound & I. Vega
The motion of point particles in curved spacetimes
Living Rev. Relativ. **14**, 7 (2011)
- L. Barack
Gravitational self-force in extreme mass-ratio inspirals
Class. Quantum Grav. **26**, 213001 (2009)
- M. Sasaki & H. Tagoshi
Analytic black hole perturbation approach to gravitational radiation
Living Rev. Rel. **6**, 5 (2003)

Further reading



1983



2004

Outline

Lecture I — From the Kepler problem to relativistic inspiral

Lecture II — From geodesic motion to gravitational self-force

Lecture III — Interfaces, mappings, and invariant dynamics

- ① The parameter space and the overlap problem
- ② Invariant observables and structural relations
- ③ EOB, NR, and the road to a unified description

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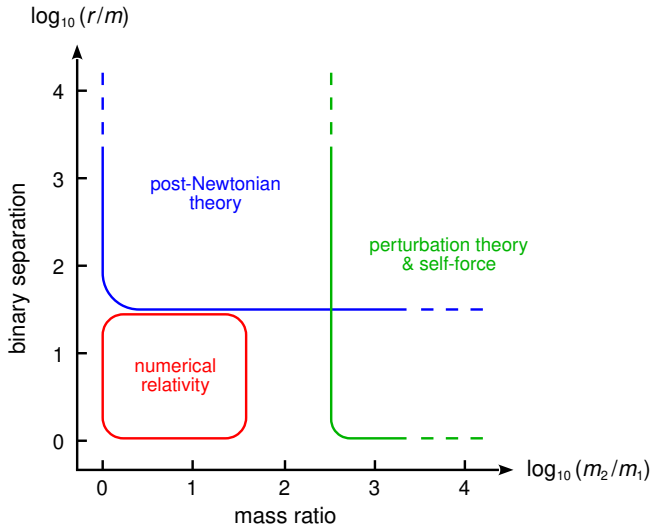
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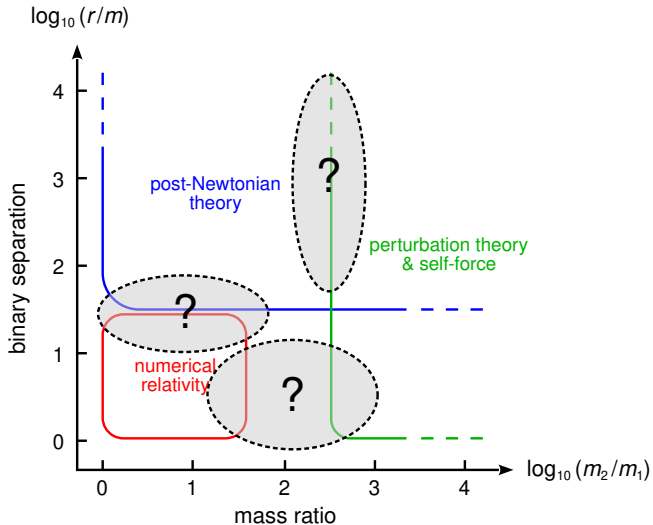
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Compact-binary parameter space



Compact-binary parameter space



Why?

- **Independent checks** of long and complicated calculations
- Identify **domains of validity** of approximation schemes
- **Extract information** inaccessible to other methods
- Develop a **universal model** for compact binaries

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What?

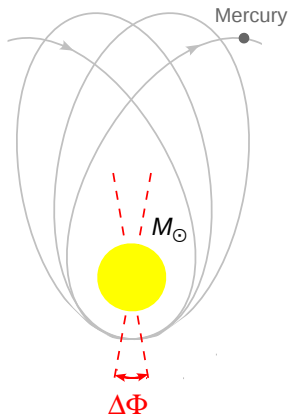
- Conservative effects on the **orbital dynamics**
- Gravitational **waveforms** at future null infinity

Relativistic perihelion advance of Mercury

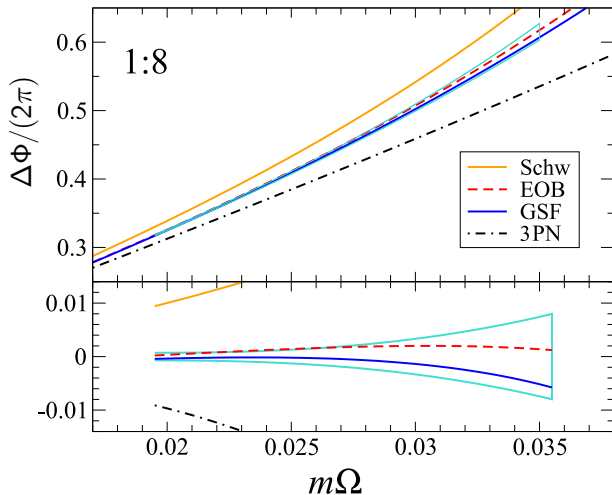
Leading-order relativistic angular advance per orbit:

$$\Delta\phi = \frac{6\pi M_{\odot}}{a(1 - e^2)}$$

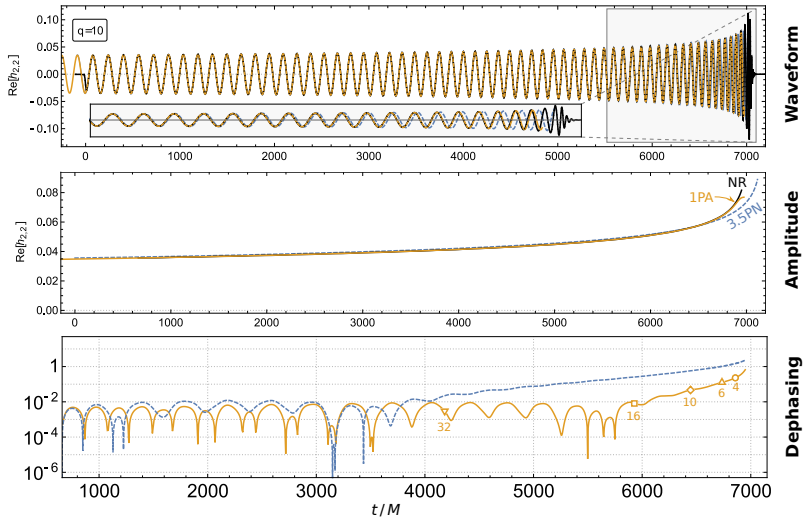
Origin	Amplitude (" /cent.)
Other planets	531.63 ± 0.69
Sun oblateness	0.028 ± 0.001
General relativity	42.98 ± 0.04
Total	574.64 ± 0.69
Observed	574.10 ± 0.65



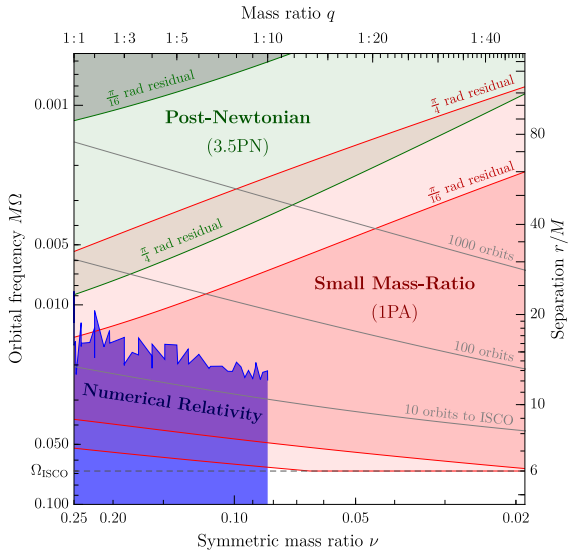
Periastron advance in black-hole binaries



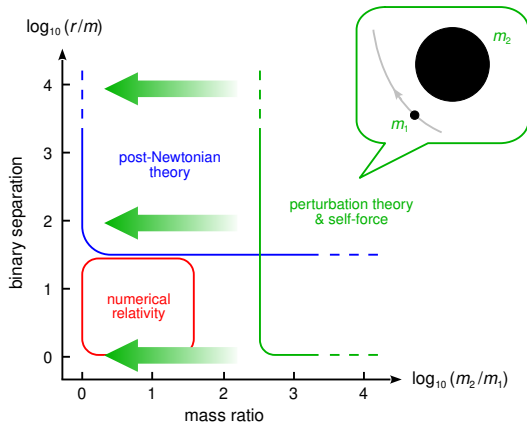
Waveform for quasi-circular inspiral



Perturbation theory for comparable masses



Why does self-force work beyond $\epsilon \ll 1$?



asymmetric

$$\epsilon = m_1/m_2$$

$1 \leftrightarrow 2$ ↓

$$\nu = m_1 m_2 / m^2$$

symmetric

- Extended convergence radius using $0 < \nu \leq \frac{1}{4}$ [Le Tiec, IJMPD 2014]
- Polynomial nonlinearity of the Einstein equation [Harte, PRL 2014]
- Exact EOB energy map to $O(G)$ [Damour, PRD 2016 | Vines, PRD 2017]

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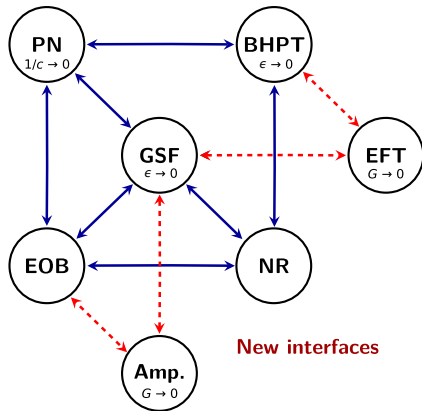
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Towards a *unified* two-body framework

Established synergies



New interfaces

Common structures

- Exchange symmetry
- First-law framework
- B2B dictionary
- EOB mapping

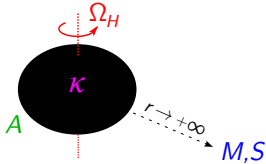
Common invariants

- Frequencies Ω_i
- Particle redshift z
- Binary's energy E
- Angular mom. L
- Radial action I_r
- Scattering angle χ

First law of compact-binary mechanics

Black-hole mechanics

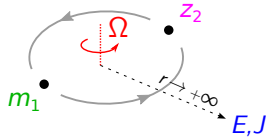
An *isolated*, stationary black hole



$$\delta M - \Omega_H \delta S = \frac{\kappa}{8\pi} \delta A$$

Compact-binary mechanics

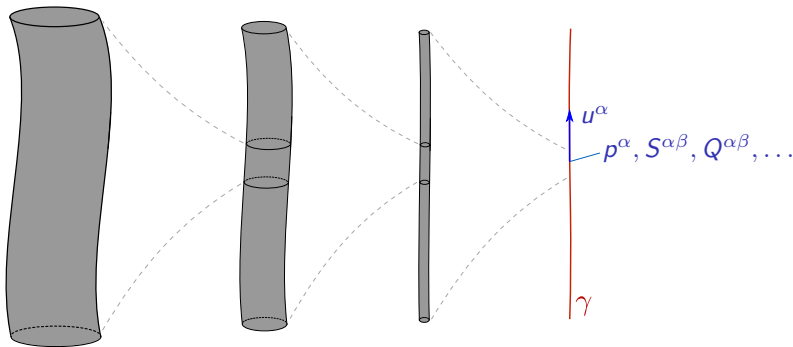
Two *interacting* compact objects



$$\delta E - \Omega \delta J = \sum_a z_a \delta m_a$$

A **framework connecting** black-hole physics, relativistic celestial mechanics and gravitational-wave modelling

Effective description of compact objects



$$T_{\text{eff}}^{\alpha\beta} = \int_{\gamma} d\tau \left[\underbrace{u^{(\alpha} p^{\beta)} \delta_4}_{\text{monopole}} + \underbrace{\nabla_\rho (u^{(\alpha} S^{\beta)\rho} \delta_4)}_{\text{dipole}} + \underbrace{\nabla_\rho \nabla_\sigma (u^{[\rho} Q^{\alpha]} [\beta} u^{\sigma]} \delta_4)}_{\text{quadrupole}} + \dots \right]$$

Quadrupolar particles on a circular orbit

- Helical Killing vector field

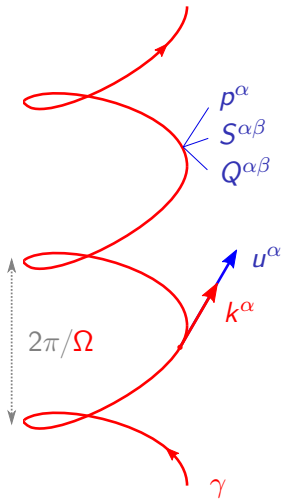
$$k^\alpha = t^\alpha + \Omega \phi^\alpha$$

- Each particle worldline γ is an integral curve of k^α :

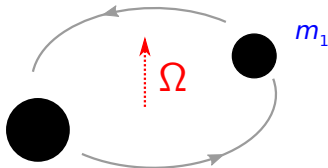
$$k^\alpha|_\gamma = z u^\alpha$$

- The particle multipoles are all Lie-dragged along k^α :

$$\mathcal{L}_k p^\alpha = \mathcal{L}_k S^{\alpha\beta} = \mathcal{L}_k Q^{\alpha\beta} = 0$$

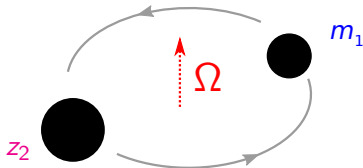


First law with spin and finite-size effects



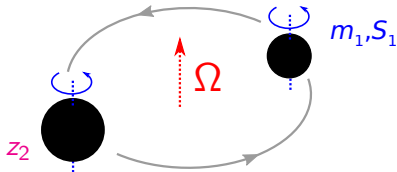
$$\delta E - \Omega \delta J = \sum_a |k|_a \delta m_a$$

First law with spin and finite-size effects



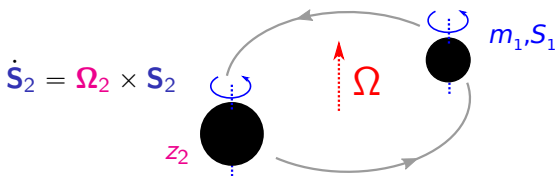
$$\delta E - \Omega \delta J = \sum_a z_a \delta m_a$$

First law with spin and finite-size effects



$$\delta E - \Omega \delta J = \sum_a z_a \delta m_a - \sum_a |\nabla k|_a \delta S_a$$

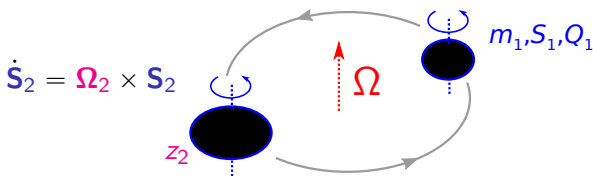
First law with spin and finite-size effects



$$\dot{\mathbf{S}}_2 = \boldsymbol{\Omega}_2 \times \mathbf{S}_2$$

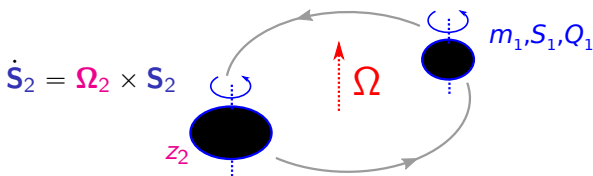
$$\delta E - \Omega \delta J = \sum_a z_a \delta m_a - \sum_a (\Omega - \Omega_a) \delta S_a$$

First law with spin and finite-size effects



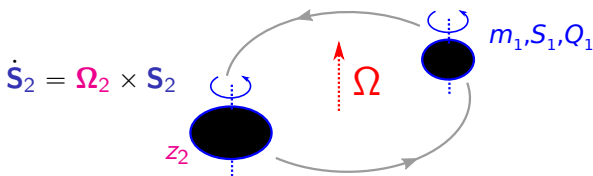
$$\delta E - \Omega \delta J = \sum_a z_a \delta m_a - \sum_a (\Omega - \Omega_a) \delta S_a + \sum_a |\nabla \nabla k|_a \delta Q_a$$

First law with spin and finite-size effects



$$\delta E - \Omega \delta J = \sum_a z_a \delta m_a - \sum_a (\Omega - \Omega_a) \delta S_a + \sum_a \mathcal{E}_a \delta Q_a$$

First law with spin and finite-size effects



$$\delta E - \Omega \delta J = \sum_a z_a \delta m_a - \sum_a (\Omega - \Omega_a) \delta S_a + \sum_a \mathcal{E}_a \delta Q_a$$

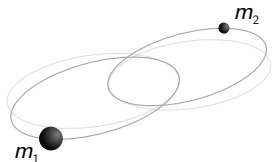
Open question

- Spin-induced quadrupole $Q_{\text{spin}} \sim \kappa S^2$
- Tidally-induced quadrupole $Q_{\text{tidal}} \sim \lambda \mathcal{E}$

First law of mechanics for eccentric orbits

- **Canonical ADM Hamiltonian**

$H(\mathbf{x}_a, \mathbf{p}_a)$ of two point masses



- Variation δH + Hamilton's equation + orbital averaging:

$$\delta E = \Omega_\phi \delta L + \Omega_r \delta I_r + \sum_a \langle z_a \rangle \delta m_a$$

- At 4PN order, GW tails introduce **nonlocality in time**

$$H(\mathbf{x}_a, \mathbf{p}_a; m_a) \longrightarrow H[\mathbf{x}_a(t), \mathbf{p}_a(t); m_a]$$

Averaged redshift for eccentric orbits

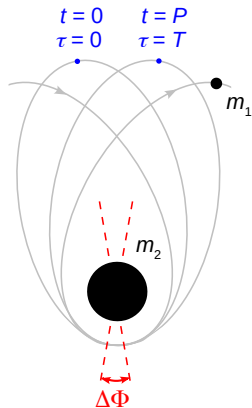
- Generic eccentric orbit parameterized by the two frequencies

$$\Omega_r = \frac{2\pi}{P}, \quad \Omega_\phi = \frac{2\pi + \Delta\Phi}{P}$$

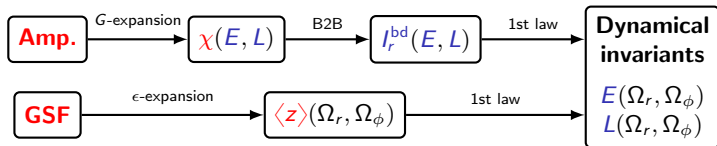
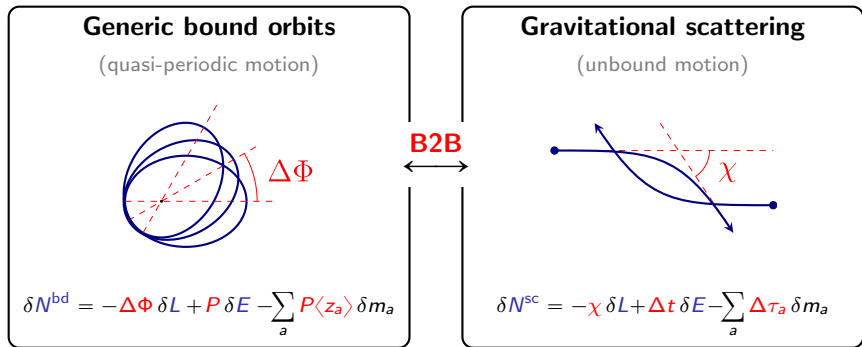
- Time average of redshift $z = d\tau/dt$ over one radial period

$$\langle z \rangle \equiv \frac{1}{P} \int_0^P z(t) dt = \frac{1}{P} \int_0^{\mathcal{T}} d\tau = \frac{\mathcal{T}}{P}$$

- Gauge-invariant function $\langle z \rangle(\Omega_r, \Omega_\phi)$



Connecting bound to unbound orbits



Outline

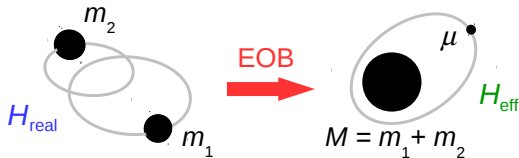
Lecture I — From the Kepler problem to relativistic inspiral

Lecture II — From geodesic motion to gravitational self-force

Lecture III — Interfaces, mappings, and invariant dynamics

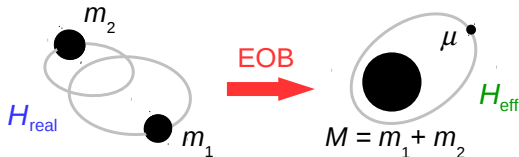
- ① The parameter space and the overlap problem
- ② Invariant observables and structural relations
- ③ EOB, NR, and the road to a unified description

Effective one-body philosophy



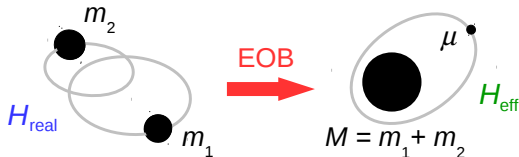
- Motivated by the exact solution in the Newtonian limit

Effective one-body philosophy



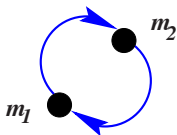
- Motivated by the exact solution in the Newtonian limit
- **By construction**, the EOB framework:
 - Recovers the known PN dynamics as $c^{-1} \rightarrow 0$
 - Recovers the geodesic dynamics when $\epsilon \rightarrow 0$

Effective one-body philosophy

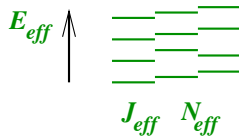
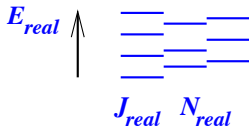
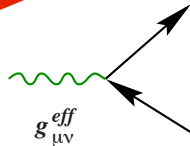
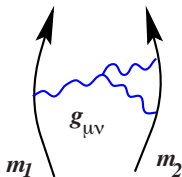


- Motivated by the exact solution in the Newtonian limit
- **By construction**, the EOB framework:
 - Recovers the known PN dynamics as $c^{-1} \rightarrow 0$
 - Recovers the geodesic dynamics when $\epsilon \rightarrow 0$
- The idea was extended to spinning binaries and tidal effects

Real description

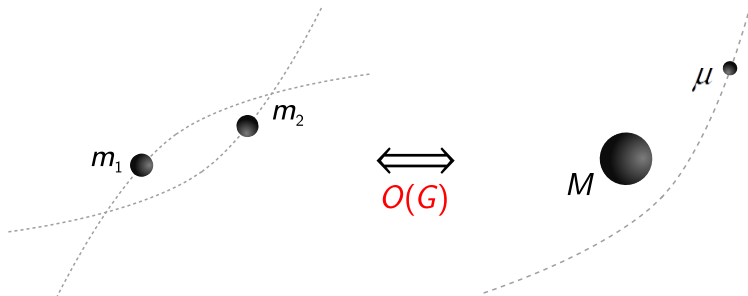


Effective description

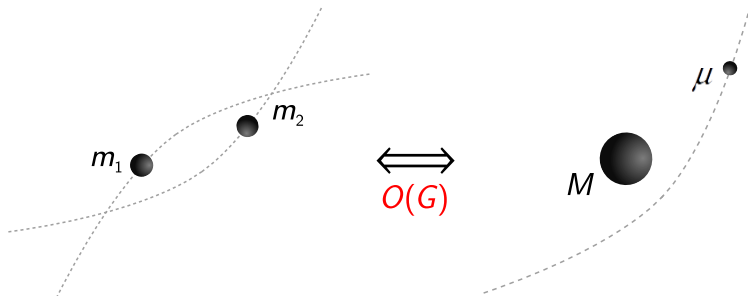


$$E_{eff}(J, N) = f(E_{real}(J, N))$$

Two-body scattering and EOB mappings

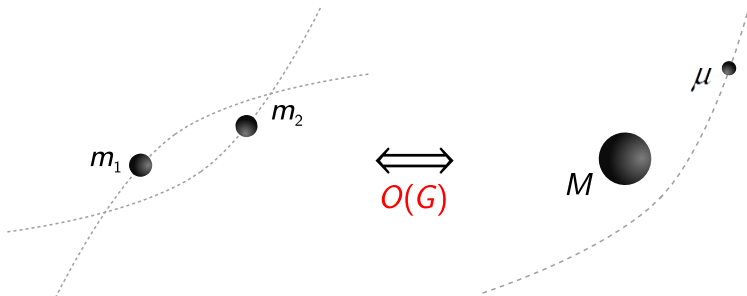


Two-body scattering and EOB mappings



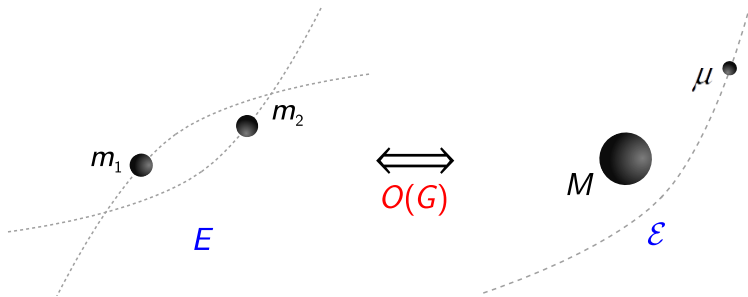
$$M \leftrightarrow m_1 + m_2$$

Two-body scattering and EOB mappings



$$M \leftrightarrow m_1 + m_2 \quad \mu \leftrightarrow \frac{m_1 m_2}{M}$$

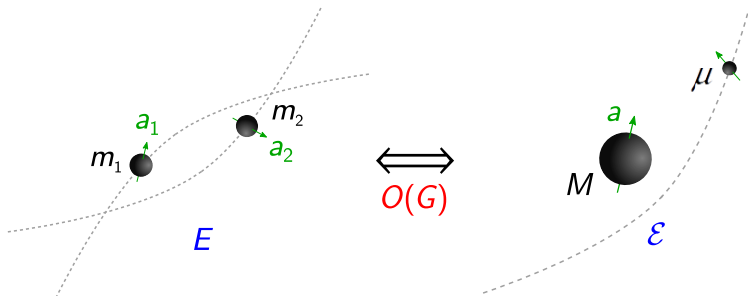
Two-body scattering and EOB mappings



$$M \leftrightarrow m_1 + m_2 \quad \mu \leftrightarrow \frac{m_1 m_2}{M} \quad E_{\text{eff}} \leftrightarrow \mu + \frac{E_{\text{real}}^2 - M^2}{2M}$$

EOB energy map

Two-body scattering and EOB mappings



$$\begin{aligned}
 M &\leftrightarrow m_1 + m_2 & \mu &\leftrightarrow \frac{m_1 m_2}{M} & E_{\text{eff}} &\leftrightarrow \mu + \frac{E_{\text{real}}^2 - M^2}{2M} & a &\leftrightarrow \frac{M}{E_{\text{real}}} (a_1 + a_2) \\
 && && \text{EOB energy map} & & \text{EOB spin map}
 \end{aligned}$$

EOB Hamiltonian dynamics

EOB Hamiltonian

$$H_{\text{real}}^{\text{EOB}} = M \sqrt{1 + 2\nu \left(\frac{H_{\text{eff}}}{\mu} - 1 \right)}, \quad \nu \equiv \frac{\mu}{M} \in [0, 1/4]$$

Effective Hamiltonian

$$H_{\text{eff}} = \mu \sqrt{g_{tt}^{\text{eff}}(r) \left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{g_{rr}^{\text{eff}}(r)} + \dots \right)}$$

Hamilton's equations

$$\dot{r} = \frac{\partial H_{\text{real}}^{\text{EOB}}}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H_{\text{real}}^{\text{EOB}}}{\partial r} + F_r, \quad \dots$$

EOB effective metric

Effective metric

$$ds_{\text{eff}}^2 = -g_{tt}^{\text{eff}}(r; \nu) dt^2 + g_{rr}^{\text{eff}}(r; \nu) dr^2 + r^2 d\Omega^2$$

Effective potentials

$$g_{tt}^{\text{eff}} = \underbrace{1 - \frac{2M}{r}}_{\text{Schwarzschild}} + \nu \underbrace{\left[2 \left(\frac{M}{r} \right)^3 + \left(\frac{94}{3} - \frac{41}{32} \pi^2 \right) \left(\frac{M}{r} \right)^4 + \dots \right]}_{\text{finite mass-ratio "deformation"}}$$

Padé resummation

Motivation: improve convergence of PN series in strong-field regime

EOB waveform generation

Inspiral/plunge

Evolution of Hamiltonian dynamics up to EOB light-ring
Resummations of PN waveform modes and fluxes

$$h^{\text{inspiral}}(t) = \text{“big mess”}$$

Merger/ringdown

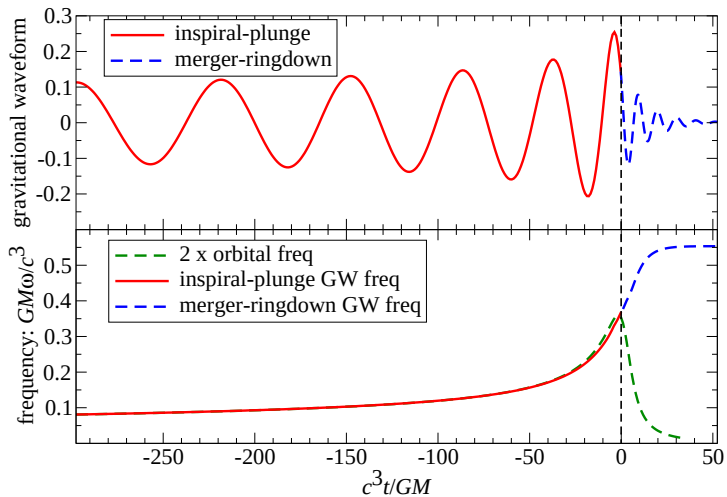
Impose continuity with black hole quasinormal modes ringing

$$h^{\text{ringdown}}(t) = \sum_{nlm} C_{nlm} e^{-t/\tau_{nlm}} \cos(\omega_{nlm}(t - t_m))$$

Final EOB waveform

$$h^{\text{EOB}}(t) = \Theta(t_m - t) h^{\text{inspiral}}(t) + \Theta(t - t_m) h^{\text{ringdown}}(t)$$

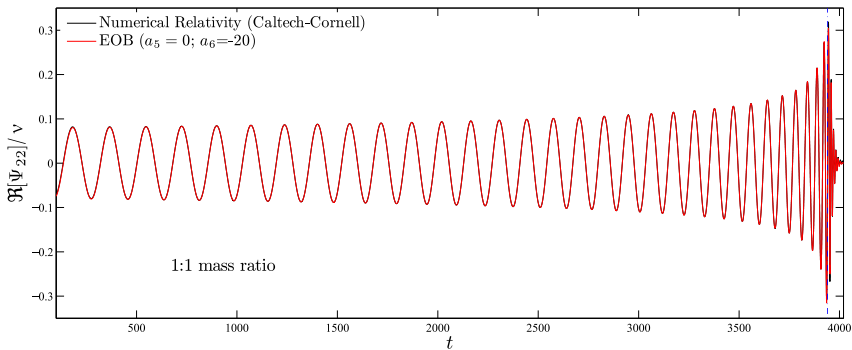
EOB waveform prediction



(Credit: Buonanno & Sathyaprakash 2015)

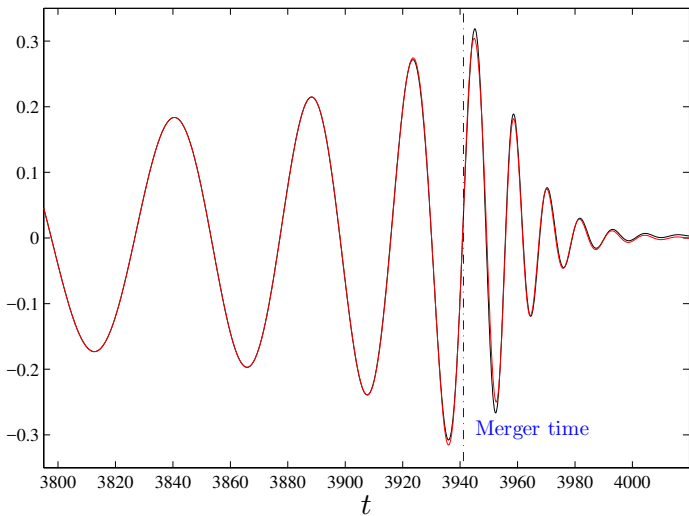
EOB vs NR waveforms

Equal masses and no spins



EOB vs NR waveforms

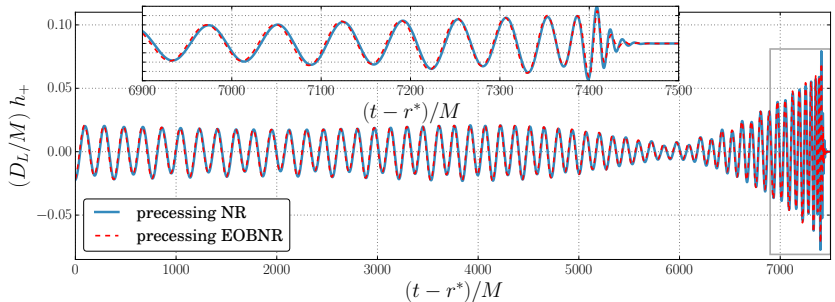
Equal masses and no spins



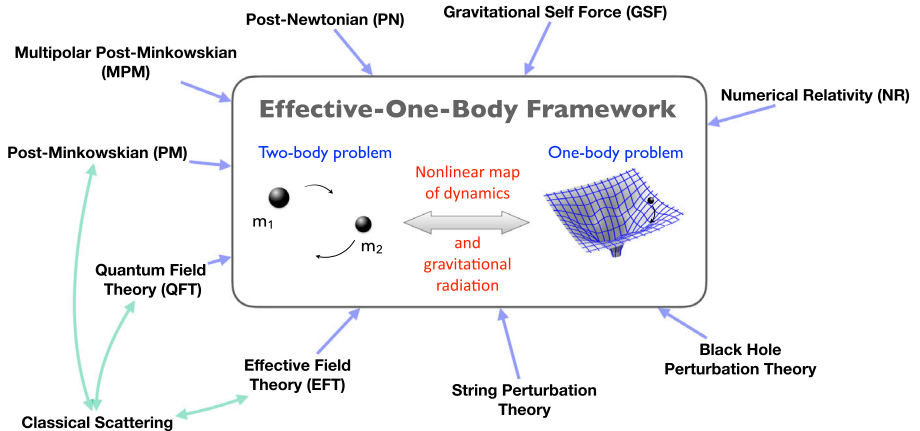
EOB vs NR waveforms

Unequal masses and one precessing spin

$$(q, \chi_1, \chi_2) = (5, +0.5, 0), \iota = \pi/3$$



EOB framework as a synthesis



Further reading

- LISA Consortium Waveform Working Group
Waveform modelling for the Laser Interferometer Space Antenna
Living Rev. Relativ. **28**, 9 (2025)
- N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté & P. Vanhove
Post-Minkowskian expansion from scattering amplitudes
J. Phys. A: Math. Theor **55**, 443014 (2022)
- A. Le Tiec
The overlap of numerical relativity, perturbation theory and post-Newtonian theory in the binary black hole problem
Int. J. Mod. Phys. D **23**, 143022 (2014)
- T. Damour
The general relativistic two body problem and the EOB formalism
In: General Relativity, Cosmology and Astrophysics, Springer (2014)

Obrigado pela sua atenção!

