Confronting neutrino theories with correlations of oscillation data

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Bringing theory and experiments together for precision neutrino physics

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THEORY ⇨ COMBINE ⇦ EXPERIMENT

basic concept

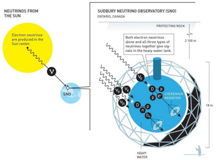
$$f_1(x_i) = g(x_k, \ldots).$$
$$\downarrow$$
$$\chi^2(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$$
$$\uparrow$$
$$f_2(x_j) = (x_m, \ldots).$$

for neutrinos: eg. mixing scheme relations + "full range" χ^2 correlated data

a minimal dataset: $\chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\rm CP})$

or better, a dataset with masses:

$$\begin{array}{c} \sin^2\theta_{12} = ... \\ \downarrow \\ \chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\rm CP}, \Delta m^2_{21}, \Delta m^2_{31}) \\ \uparrow \\ \cos\delta_{\rm CP} = ... \end{array}$$



Sudbury Neutrino Observatory detects neutrinos from the Sun, where only electron-neutrinos are produced. The reactions between neutrinos and the heavy water in the tank yielded the possibility to measure both electron-neutrinos and all three types of neutrinos combined. It was discovered that the electron-neutrinos were fewer than expected, while the total number of all three types of neutrinos combined still corresponded to expectations. The conclusion was that some of the electron-neutrinos had changed into another identity.

Illustration: © Johan Jarnestad/The Royal Swedish Academy of Sciences

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MLA style: Popular information. NobelPrize.org. Nobel Prize Outreach AB 2024. Sun. 15 Sep 2024. <https://www.nobelprize.org/prizes/physics/2015/popular-information/>

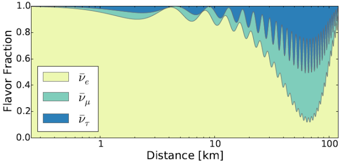


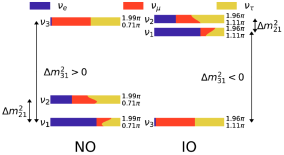
FIG. 1. **Illustration of neutrino oscillations.** The expected flavor composition of the reactor neutrino flux, for neutrinos of 4 MeV energy used as an example, is plotted as a function of distance to the reactor cores. arXiv:1503.01059

Normal mass ordering (NO)

$$m_1 = m_0\,,$$
$$m_2 = \sqrt{m_0^2 + \Delta m_{21}^2}\,,$$
$$m_3 = \sqrt{m_0^2 + \Delta m_{31}^2}\,,$$

Inverted mass ordering (IO)

$$m_1 = \sqrt{m_0^2 - \Delta m_{21}^2 - \Delta m_{32}^2}\,,$$
$$m_2 = \sqrt{m_0^2 - \Delta m_{32}^2}\,,$$
$$m_3 = m_0\,,$$



Taken from https://globalfit.astroparticles.es, updated arXiv:1806.11051, Figure 1

$$\Delta m_{21}^2 > 0 \qquad |\Delta m_{31(32)}^2| > 0 \qquad \text{NO or IO ?}$$

$$\Sigma m_i < 0.12 \text{ eV at } 95\% \text{ CL from Planck (CMB)}$$

3ν mixing

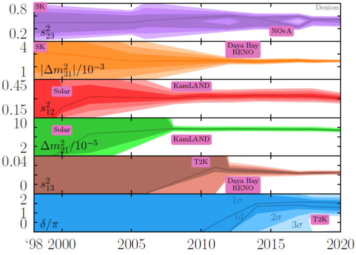
Neutrino flavor and mass eigenstates are related by

$$|\nu_\alpha\rangle = \mathrm{U}_{\alpha i}\,|\nu_i\rangle$$

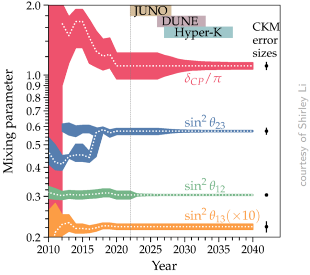
Pontecorvo–Maki–Nakagawa–Sakata parametrization of mixing matrix

$$\mathrm{U}_{\mathrm{PMNS}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix} \begin{bmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$, $\delta \equiv \delta_{\mathrm{CP}}$.



Peter B. Denton talk, link here.



arXiv:2204.08668, Figure 2.1

Dirac or Majorana ? $(3+n)\nu$, n=0 ?

NO or IO ? $m_{lightest} \neq 0$?

flavor symmetry ? mixing scheme ?

θ_{23} octant ?

$\delta_{\mathrm{CP}} \neq 0$?

$\Sigma m_i < 0.12$ eV $\leftarrow$ this we know from Planck at 95% CL.

$\theta_{13} \neq 0$ $\leftarrow$ this we know from reactor experiments, e.g. Daya Bay.

Experimental data indicate that $\delta_{\mathrm{CP}} \neq 0$ and NO are favored.

Closer look at NuFIT data

NuFIT 5.3 (2024)

without SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.3$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$
$\theta_{12}/^\circ$	$33.66^{+0.73}_{-0.70}$	$31.60 \to 35.94$	$33.67^{+0.73}_{-0.71}$	$31.61 \to 35.94$
$\sin^2\theta_{23}$	$0.572^{+0.018}_{-0.023}$	$0.407 \to 0.620$	$0.578^{+0.016}_{-0.021}$	$0.412 \to 0.623$
$\theta_{23}/^\circ$	$49.1^{+1.0}_{-1.3}$	$39.6 \to 51.9$	$49.5^{+0.9}_{-1.2}$	$39.9 \to 52.1$
$\sin^2\theta_{13}$	$0.02203^{+0.00056}_{-0.00058}$	$0.02029 \to 0.02391$	$0.02219^{+0.00059}_{-0.00057}$	$0.02047 \to 0.02396$
$\theta_{13}/^\circ$	$8.54^{+0.11}_{-0.11}$	$8.19 \to 8.89$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.90$
$\delta_{CP}/^\circ$	197^{+41}_{-25}	$108 \to 404$	286^{+27}_{-32}	$192 \to 360$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.511^{+0.027}_{-0.027}$	$+2.428 \to +2.597$	$-2.498^{+0.032}_{-0.024}$	$-2.581 \to -2.409$

with SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 9.1$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$	$0.307^{+0.012}_{-0.011}$	$0.275 \to 0.344$
$\theta_{12}/^\circ$	$33.67^{+0.73}_{-0.71}$	$31.61 \to 35.94$	$33.67^{+0.73}_{-0.71}$	$31.61 \to 35.94$
$\sin^2\theta_{23}$	$0.454^{+0.019}_{-0.016}$	$0.411 \to 0.606$	$0.568^{+0.016}_{-0.021}$	$0.412 \to 0.611$
$\theta_{23}/^\circ$	$42.3^{+1.1}_{-0.9}$	$39.9 \to 51.1$	$48.9^{+0.9}_{-1.2}$	$39.9 \to 51.4$
$\sin^2\theta_{13}$	$0.02224^{+0.00056}_{-0.00057}$	$0.02047 \to 0.02397$	$0.02222^{+0.00069}_{-0.00057}$	$0.02049 \to 0.02420$
$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.13}_{-0.11}$	$8.23 \to 8.95$
$\delta_{CP}/^\circ$	232^{+39}_{-25}	$139 \to 350$	273^{+24}_{-26}	$195 \to 342$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.81 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.505^{+0.024}_{-0.026}$	$+2.426 \to +2.586$	$-2.487^{+0.027}_{-0.024}$	$-2.566 \to -2.407$

NuFIT 6.1 (2025)

IC23 without SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 1.5$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.3088^{+0.0067}_{-0.0066}$	$0.2893 \to 0.3295$	$0.3088^{+0.0067}_{-0.0066}$	$0.2893 \to 0.3295$
$\theta_{12}/^\circ$	$33.76^{+0.42}_{-0.41}$	$32.54 \to 35.03$	$33.76^{+0.42}_{-0.41}$	$32.54 \to 35.03$
$\sin^2\theta_{23}$	$0.470^{+0.017}_{-0.014}$	$0.432 \to 0.587$	$0.555^{+0.013}_{-0.016}$	$0.437 \to 0.590$
$\theta_{23}/^\circ$	$43.27^{+1.0}_{-0.82}$	$41.11 \to 50.02$	$48.15^{+0.75}_{-0.92}$	$41.40 \to 50.21$
$\sin^2\theta_{13}$	$0.02249^{+0.00057}_{-0.00057}$	$0.02070 \to 0.02420$	$0.02261^{+0.00056}_{-0.00056}$	$0.02091 \to 0.02433$
$\theta_{13}/^\circ$	$8.62^{+0.11}_{-0.11}$	$8.27 \to 8.95$	$8.65^{+0.11}_{-0.11}$	$8.31 \to 8.97$
$\delta_{CP}/^\circ$	207^{+23}_{-20}	$114 \to 405$	283^{+24}_{-28}	$202 \to 347$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.537^{+0.094}_{-0.10}$	$7.236 \to 7.823$	$7.537^{+0.094}_{-0.10}$	$7.236 \to 7.822$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.521^{+0.026}_{-0.018}$	$+2.454 \to +2.592$	$-2.500^{+0.024}_{-0.023}$	$-2.569 \to -2.430$

IC24 with SK atmospheric data	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 5.9$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.3088^{+0.0067}_{-0.0066}$	$0.2893 \to 0.3295$	$0.3088^{+0.0067}_{-0.0066}$	$0.2893 \to 0.3295$
$\theta_{12}/^\circ$	$33.76^{+0.42}_{-0.41}$	$32.54 \to 35.03$	$33.76^{+0.42}_{-0.41}$	$32.54 \to 35.03$
$\sin^2\theta_{23}$	$0.470^{+0.017}_{-0.014}$	$0.435 \to 0.584$	$0.550^{+0.013}_{-0.016}$	$0.439 \to 0.584$
$\theta_{23}/^\circ$	$43.29^{+0.96}_{-0.79}$	$41.27 \to 49.86$	$47.90^{+0.73}_{-0.92}$	$41.51 \to 49.83$
$\sin^2\theta_{13}$	$0.02248^{+0.00055}_{-0.00059}$	$0.02064 \to 0.02418$	$0.02262^{+0.00057}_{-0.00056}$	$0.02093 \to 0.02441$
$\theta_{13}/^\circ$	$8.62^{+0.11}_{-0.11}$	$8.26 \to 8.95$	$8.65^{+0.11}_{-0.11}$	$8.32 \to 8.99$
$\delta_{CP}/^\circ$	212^{+26}_{-36}	$125 \to 365$	274^{+22}_{-25}	$203 \to 335$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.537^{+0.094}_{-0.10}$	$7.236 \to 7.823$	$7.537^{+0.094}_{-0.10}$	$7.236 \to 7.822$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.511^{+0.021}_{-0.020}$	$+2.450 \to +2.576$	$-2.483^{+0.020}_{-0.020}$	$-2.547 \to -2.421$

1. NuFIT (1D, 2D + single 3D data set)
http://www.nu-fit.org

www.nu-fit.org

Home

v6.1: Three-neutrino fit based on data available in November 2025

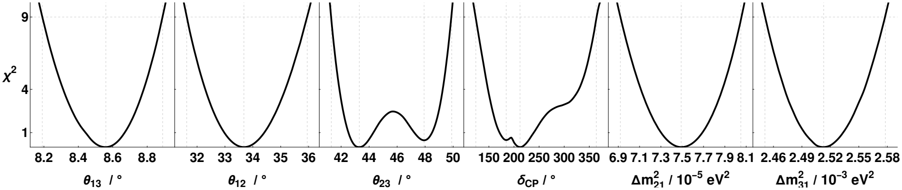
2. χ^2 profiles from Valencia neutrino global fit (1D, 2D data sets)
https://globalfit.astroparticles.es

NEUTRINO GLOBAL FIT
RESULTS FROM THE NEUTRINO OSCILLATIONS GLOBAL FIT

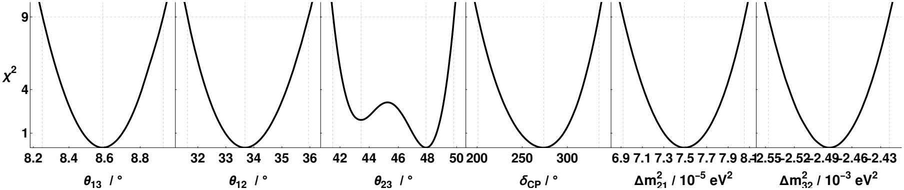
χ^2 profiles

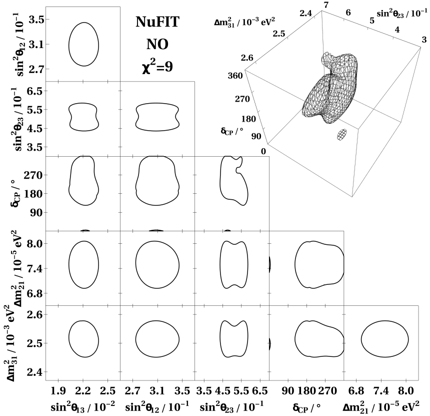
1D profiles of oscillation parameters, 3σ ranges, b-fs, NuFIT v6.0

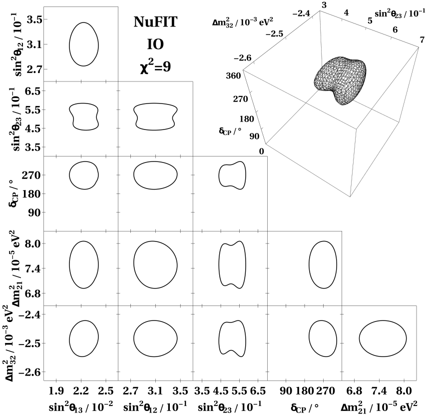
NO



IO







Currently, up to 3D datasets are available.

Mathematica output based on NuFIT v6.0 datasets.

```
In[686]:= {{"", "sin²θ23=0.57", "Δm²31=0.00255 eV²", "δCP=290°"},
  {"χ²", deltachisq["NO"][{"T23"}][5.7], deltachisq["NO"][{"DMA"}][2.55], deltachisq["NO"][{"DCP"}][290]}} // TableForm
```

Out[686]//TableForm=

	$\sin^2\theta_{23}=0.57$	$\Delta m^2_{31}=0.00255$ eV^2	$\delta_{CP}=290°$
χ^2	2.71635	3.07873	2.82341

← **treated independently, each $\chi^2 \sim 3$**

```
In[687]:= {{"", "sin²θ23=0.57 ∧ Δm²31=0.00255 eV²"}, {"χ²", deltachisq["NO"][{"T23", "DMA"}][5.7, 2.55]}} // TableForm
```

Out[687]//TableForm=

	$\sin^2\theta_{23}=0.57 \wedge \Delta m^2_{31}=0.00255$ eV^2
χ^2	4.68546

←

```
In[688]:= {{"", "Δm²31=0.00255 eV² ∧ δCP=290°"}, {"χ²", deltachisq["NO"][{"DMA", "DCP"}][2.55, 290]}} // TableForm
```

Out[688]//TableForm=

	$\Delta m^2_{31}=0.00255$ eV^2 $\wedge\ \delta_{CP}=290°$
χ^2	7.56033

← **composition of two, $\chi^2 \sim 5$ to 13**

```
In[689]:= {{"", "sin²θ23=0.57 ∧ δCP=290°"}, {"χ²", deltachisq["NO"][{"T23", "DCP"}][5.7, 290]}} // TableForm
```

Out[689]//TableForm=

	$\sin^2\theta_{23}=0.57 \wedge \delta_{CP}=290°$
χ^2	13.454

←

```
In[690]:= {{"", "sin²θ23=0.57 ∧ Δm²31=0.00255 eV² ∧ δCP=290°"}, {"χ²", deltachisq["NO"][{"T23", "DMA", "DCP"}][5.7, 2.55, 290]}} // TableForm
```

Out[690]//TableForm=

	$\sin^2\theta_{23}=0.57 \wedge \Delta m^2_{31}=0.00255$ eV^2 $\wedge\ \delta_{CP}=290°$
χ^2	17.4936

← **composition of three, $\chi^2 \sim 17$**

Insight into the χ^2 value is crucial to confront theory with exp.

To extend the number of random variables we introduce conservative estimations:

$$\chi^2_{\mathrm{ext}}(X_1, ..., X_i) \equiv \max\left\{\chi^2(X_1, ..., X_{k-1}, X_{k+1}, ..., X_i), \quad k \in \{1...i\}\right\}.$$

and in the context of neutrino parameters e.g.

$$\chi^2_{\mathrm{ext}}(\theta_{13}, \Delta m^2_{31}, \delta_{\mathrm{CP}}) \equiv \max\left\{\chi^2_{\mathrm{NuFIT}}(\Delta m^2_{31}, \delta_{\mathrm{CP}}), \chi^2_{\mathrm{NuFIT}}(\theta_{13}, \delta_{\mathrm{CP}}), \chi^2_{\mathrm{NuFIT}}(\theta_{13}, \Delta m^2_{31})\right\}$$

Tested against complete minimized NuFIT data sets, both mass orderings, that

$$\chi^2(X_1, ..., X_i) \geqslant \max\left\{\chi^2(X_1, ..., X_{k-1}, X_{k+1}, ..., X_i), \quad k \in \{1...i\}\right\}.$$

Example NuFIT data test:

$$\chi^2(\theta_{23}, \Delta m^2_{31}, \delta_{\mathrm{CP}}) \geqslant \max\left\{\chi^2(\Delta m^2_{31}, \delta_{\mathrm{CP}}), \chi^2(\theta_{23}, \delta_{\mathrm{CP}}), \chi^2(\theta_{23}, \Delta m^2_{31})\right\}$$

$17.5 \geqslant \max\{7.6, 13.5, 4.7\}$ $\leftarrow$ numbers taken from slide 11

due to "$\geqslant$" underestimation ($13.5 < 17.5$) $\Rightarrow$ no risk of excluding allowed region (volume).

Reduce.

To reduce the number of random variables, we carry out projections:

$$\chi^2_{\rm red}(X_1,\dots,X_{k-1},X_{k+1},\dots,X_n) \equiv \text{min}\left\{\chi^2(X_1,\dots,X_k=x_k,\dots,X_n) \text{ for any } x_k\right\}.$$

and in the context of neutrino parameters e.g.

$$\chi^2_{\rm red}(\theta_{23},\theta_{13},\delta_{\rm CP}) = \text{min}\left\{\chi^2(\theta_{23},\theta_{13},\theta_{12},\delta_{\rm CP}) \text{ for any } \theta_{12}\right\}.$$

Tested against complete minimized NuFIT data sets, both mass orderings, that

$$\chi^2(X_1,\dots,X_{k-1},X_{k+1},\dots,X_n) = \text{min}\left\{\chi^2(X_1,\dots,X_k=x_k,\dots,X_n) \text{ for any } x_k\right\}.$$

Example:

$$\chi^2_{\rm NuFIT}(\delta_{\rm CP}) = \text{min}\left\{\chi^2_{\rm NuFIT}(\theta_{23},\delta_{\rm CP}) \text{ for any } \theta_{23}\right\}.$$

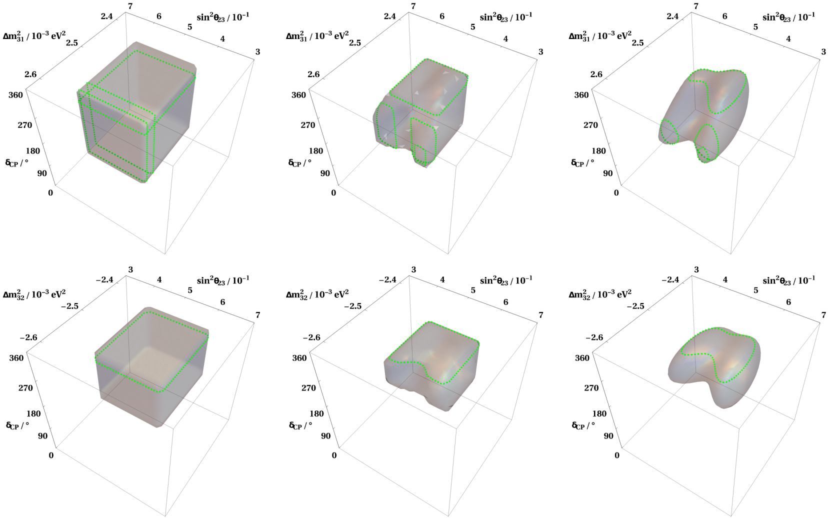
A full (6D) correlated data from global fit (experiment) would further constrain the results.

The extension formula may still be useful due to numerical complexity and memory requirements of 4D-6D data sets.

Reconstructed χ^2 volumes from NuFIT (v5.3) 1D and 2D data *vs* 3D data

Here presented for $\chi^2 \leqslant 9$, NO (upper panel) and IO (lower panel).
max(...) extension formula (slide 12) applied to 1D and 2D data to reconstruct the 3D valume

1D data only (3σ ranges) | 2D correlated data only | 3D correlated data

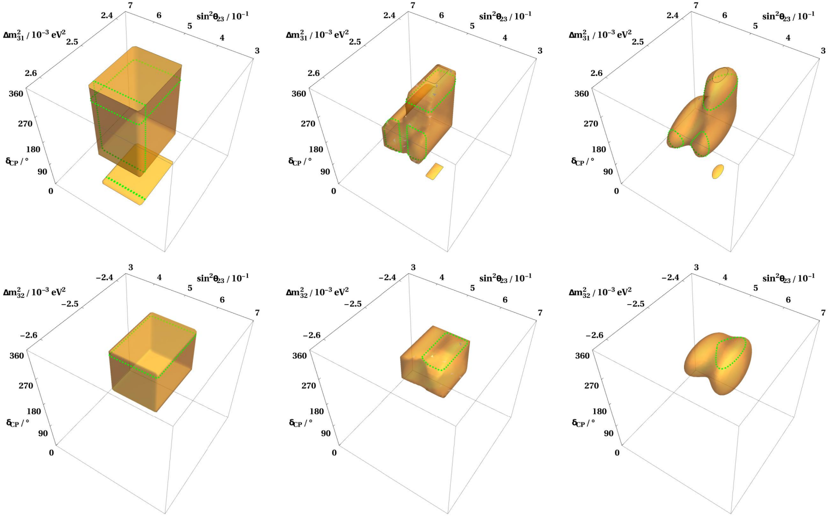


Reconstructed χ^2 volumes from NuFIT (v6.1) 1D and 2D data *vs* 3D data

Here presented for $\chi^2 \leqslant 9$, NO (upper panel) and IO (lower panel).

max(...) extension formula (slide 12) applied to 1D and 2D data to reconstruct the 3D volume

1D data only (3σ ranges) | 2D correlated data only | 3D correlated data



$\chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\mathrm{CP}}, \Delta m^2_{21}, \Delta m^2_{31})$ from global fit (could be without masses)

The NuFIT collaboration performing a global analysis of neutrino oscillation data provides one-dimensional χ^2 projections for $\sin^2\theta_{ij}$ and δ [49]. We denote them as $\chi^2_i(x_i)$, $i = 1, 2, 3, 4$, where x_i are components of $\vec{x} = (\sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}, \delta)$. Using these projections, we construct an approximate global χ^2 function as

arXiv:1804.00182

$$\chi^2(\vec{x}) = \sum_{i=1}^{4} \chi^2_i(x_i)\,. \tag{4.1}$$

Given the distinct experimental sensitivities, we construct the total χ^2 function as the sum of individual contributions from the solar, reactor, and atmospheric sectors, namely,

$$\chi^2 = \chi^2_{\mathrm{Solar}}\left[\theta_{12}\left(\boldsymbol{p}\right), \Delta m^2_{21}\left(\boldsymbol{p}\right)\right] + \chi^2_{\mathrm{Atmos}}\left[\theta_{23}\left(\boldsymbol{p}\right), \Delta m^2_{3\ell}\left(\boldsymbol{p}\right)\right] + \chi^2_{\mathrm{Reactor}}\left[\theta_{13}\left(\boldsymbol{p}\right)\right] \,, \tag{19}$$

arXiv:2511.15654

max(...) formula at the end always return unchanged NuFIT χ^2 value (except needed interpolation to join the discrete data points - no extrapolation).

$$\chi^2_{\mathrm{ext}}(\theta_{13}, \Delta m^2_{31}, \delta_{\mathrm{CP}})$$

is estimated by

$$\max \begin{cases} \chi^2_{\mathrm{NuFIT}}(\Delta m^2_{31}, \delta_{\mathrm{CP}}) \\ \chi^2_{\mathrm{NuFIT}}(\theta_{13}, \delta_{\mathrm{CP}}) \\ \chi^2_{\mathrm{NuFIT}}(\theta_{13}, \Delta m^2_{31}) \end{cases}$$

The maximum is taken after checking each χ^2 value of global fit available subsets of parameters.

$\theta_{13} \neq 0$, Daya Bay, RENO (2012)

BM, TB, GR, HG disfavored by non-zero θ_{13}.

$$\mathrm{U_{PMNS}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix}$$

$$\theta_{13} \neq 0^\circ \qquad \Downarrow \qquad \theta_{23} = 45^\circ$$

$$\mathrm{U_0} = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Downarrow$$

$\theta_{12} = 45^\circ$, $s_{12} = 1/\sqrt{2}$ | $\theta_{12} = 35,26^\circ$, $s_{12} = 1/\sqrt{3}$ | $\theta_{12} = 31,7^\circ$ | $\theta_{12} = 30^\circ$, $s_{12} = 1/2$

Bimaximal Mixing

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Tribimaximal Mixing

$$\begin{bmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Golden Ratio Mixing

$$\begin{bmatrix} \frac{\varphi}{\sqrt{2+\varphi}} & \frac{1}{\sqrt{2+\varphi}} & 0 \\ \frac{-1}{\sqrt{4+2\varphi}} & \frac{\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{4+2\varphi}} & \frac{-\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hexagonal Mixing

$$\begin{bmatrix} \sqrt{\frac{3}{4}} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Golden Ratio Mixing: tg $\theta_{12} = 1/\varphi$, $\varphi = (1+\sqrt{5})/2$ being the golden ratio.

$$\mathrm{U_{TBM}} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathrm{U_{PMNS}} \simeq \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0.15 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

```
In[688]:= Sin[8.6*Pi/180]
          Sin[8.6*Pi/180]^2
Out[688]= 0.149535
Out[689]= 0.0223608
```

$$|\mathrm{U_{TM_1}}| = \begin{bmatrix} \frac{2}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \end{bmatrix}, \qquad |\mathrm{U_{TM_2}}| = \begin{bmatrix} * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \end{bmatrix},$$

$$\mathrm{U_{TM_1}} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} & \frac{s_\theta}{\sqrt{3}} e^{-i\gamma} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} - \frac{s_\theta}{\sqrt{2}} e^{i\gamma} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} - \frac{c_\theta}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} - \frac{s}{\sqrt{2}} e^{i\gamma} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} + \frac{c_\theta}{\sqrt{2}} \end{bmatrix},$$

$$\mathrm{U_{TM_2}} = \begin{bmatrix} \frac{2c_\theta}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{2s_\theta}{\sqrt{6}} e^{-i\gamma} \\ -\frac{c_\theta}{\sqrt{6}} + \frac{s}{\sqrt{2}} e^{i\gamma} & \frac{1}{\sqrt{3}} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} - \frac{c_\theta}{\sqrt{2}} \\ -\frac{c_\theta}{\sqrt{6}} + \frac{s}{\sqrt{2}} e^{i\gamma} & \frac{1}{\sqrt{3}} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} + \frac{c_\theta}{\sqrt{2}} \end{bmatrix}.$$

Flavor and mass mixing:

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}.$$

TM_1 and TM_2 mixing-schemes structure

$$|\mathrm{U}_{\mathrm{TM}_1}| = \begin{bmatrix} \frac{2}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \end{bmatrix}, \quad |\mathrm{U}_{\mathrm{TM}_2}| = \begin{bmatrix} * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \end{bmatrix}.$$

Partial μ-τ ($\mu 1$-$\tau 1$, $\mu 2$-$\tau 2$) reflection symmetry mixing structure:

$$|U_{\mu 1}| = |U_{\tau 1}|, \quad |U_{\mu 2}| = |U_{\tau 2}|.$$

TM_1 variant of more general $\mu 1$-$\tau 1$ $\quad$ TM_2 variant of more general $\mu 2$-$\tau 2$

TM_1 and TM_2 mixing schemes, partial $\mu-\tau$ symmetry

Comparing the corresponding elements of the first column of $\mathrm{U}_{\mathrm{PMNS}}$ and $\mathrm{U}_{\mathrm{TM}_1}$.

$$|\mathrm{U}_{\mathrm{e}1}|^2 = c_{12}^2 c_{13}^2 = 2/3 \quad : \quad \theta_{12}^{\mathrm{TM}_1}(\theta_{13}) \quad : \quad s_{12}^2 = \frac{1-3s_{13}^2}{3-3s_{13}^2},$$

$$|\mathrm{U}_{\mu 1}|^2 = |\mathrm{U}_{\tau 1}|^2 = 1/6 \quad : \quad \delta_{\mathrm{CP}}^{\mathrm{TM}_1}(\theta_{13}, \theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = \frac{(1-5s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{2(1-3s_{13}^2)(1-s_{23}^2)}}.$$

Comparing the corresponding elements of the second column of $\mathrm{U}_{\mathrm{PMNS}}$ and $\mathrm{U}_{\mathrm{TM}_2}$.

$$|\mathrm{U}_{\mathrm{e}2}|^2 = s_{12}^2 c_{13}^2 = 1/3 \quad : \quad \theta_{12}^{\mathrm{TM}_2}(\theta_{13}) \quad : \quad s_{12}^2 = \frac{1}{3-3s_{13}^2},$$

$$|\mathrm{U}_{\mu 2}|^2 = |\mathrm{U}_{\tau 2}|^2 = 1/3 \quad : \quad \delta_{\mathrm{CP}}^{\mathrm{TM}_2}(\theta_{13}, \theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = -\frac{(2-4s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{(2-3s_{13}^2)(1-s_{23}^2)}}.$$

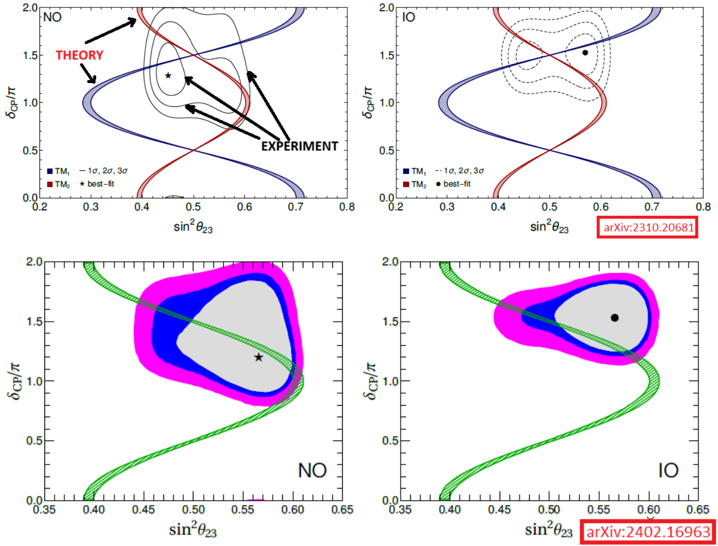
In partial μ-τ reflection symmetry, the mixing matrix symmetry is given by:

$$|U_{\mu 1}| = |U_{\tau 1}| \quad : \quad \delta_{\mathrm{CP}}^{\mu 1\text{-}\tau 1}(\theta_{13}, \theta_{12}, \theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = \frac{(c_{23}^2 - s_{23}^2)(c_{12}^2 s_{13}^2 - s_{12}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}},$$

$$|U_{\mu 2}| = |U_{\tau 2}| \quad : \quad \delta_{\mathrm{CP}}^{\mu 2\text{-}\tau 2}(\theta_{13}, \theta_{12}, \theta_{23}) \quad : \quad \cos\delta_{\mathrm{CP}} = \frac{(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2 s_{13}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}}.$$

TM$_1$	TM$_2$
$s_{12}^2 = \frac{1-3s_{13}^2}{3-3s_{13}^2}$	$s_{12}^2 = \frac{1}{3-3s_{13}^2}$
$\cos\delta_{\rm CP} = \frac{(1-5s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{2(1-3s_{13}^2)(1-s_{23}^2)}}$	$\cos\delta_{\rm CP} = -\frac{(2-4s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{(2-3s_{13}^2)(1-s_{23}^2)}}$
$\mu 1 - \tau 1$	$\mu 2 - \tau 2$
$\cos\delta_{\rm CP} = \frac{(c_{23}^2-s_{23}^2)(c_{12}^2 s_{13}^2 - s_{12}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}}$	$\cos\delta_{\rm CP} = \frac{(c_{23}^2-s_{23}^2)(c_{12}^2 - s_{12}^2 s_{13}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}}.$

How to confront the mixing schemes with experimental data ?



⇒ **Next, we will combine theory with experiment which are flashed independently on the plots.**

COMBINE correlated data and mixing-scheme relations

THEORY

⇩

MIXING SCHEME RELATIONS

⇩

COMBINE ⇦ **EXTEND TO FULL RANGE**

EXPERIMENT

⇩

OSCILLATION PARAMTERS CORRELATIONS

⇩

EXTEND TO FULL RANGE

TM_1 mixing relations + extended to full range correlated data, example:

$$\sin^2\theta_{12} = \frac{1-3\sin^2\theta_{13}}{3-3\sin^2\theta_{13}}$$

$$\downarrow$$

$$\chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\mathrm{CP}}, \Delta m^2_{21}, \Delta m^2_{31})$$

$$\uparrow$$

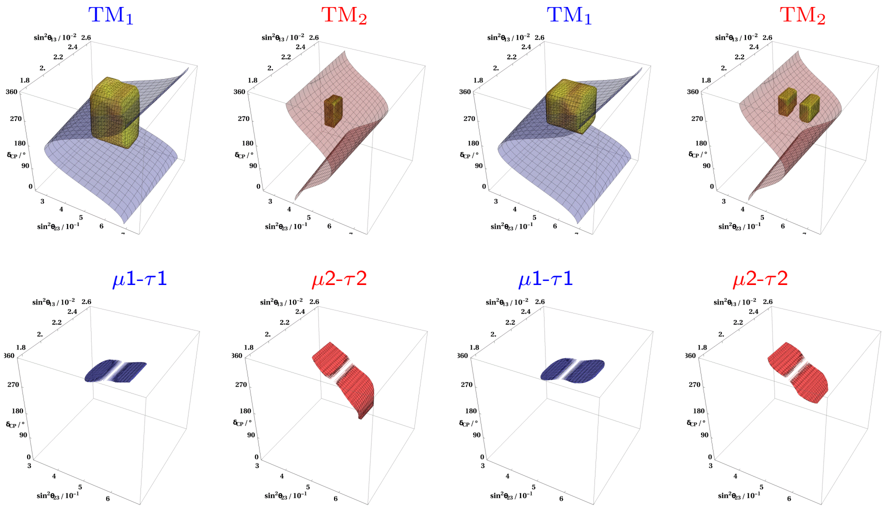
$$\cos\delta_{\mathrm{CP}} = \frac{(1-5\sin^2\theta_{13})(2\sin^2_{23}-1)}{4\sin_{13}\sin\theta_{23}\sqrt{2(1-3\sin^2\theta_{13})(1-\sin^2\theta_{23})}}$$

In mixing schemes some parameter values are determined by other parameters values

For any choice of parameters, the equations for these parameters must be solved first.

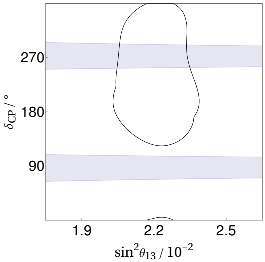
NO and IO need to be treated separately.

$\Delta\chi^2 \leqslant 9$, NO (left panel) and IO (right panel) (volumes NuFIT v5.3 + $\theta_{12}(\theta_{13})$ relation):



3D projections of 4D intersections for partial μ-τ reflection symmetry.

Impact of correlated data on model's predictions (NuFIT v6.0)

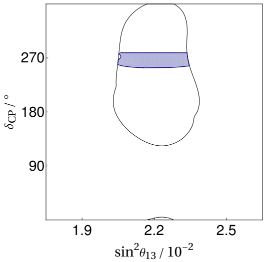


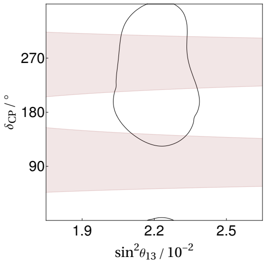
THEORY AND EXPERIMENT

FLASHED INDEPEDENTLY

TM₁

COMBINED



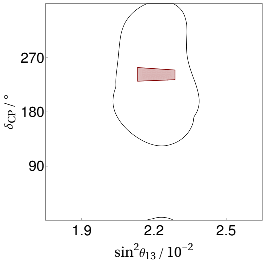


THEORY AND EXPERIMENT

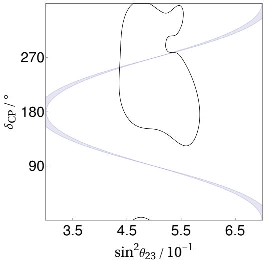
FLASHED INDEPEDENTLY

TM₂

COMBINED



NO, $\chi^2 < 9$

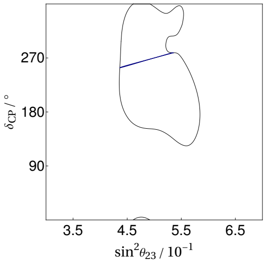


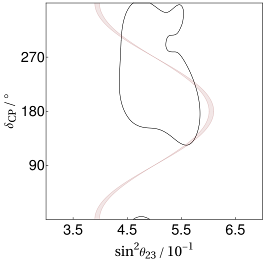
THEORY AND EXPERIMENT

FLASHED INDEPEDENTLY

TM1

COMBINED



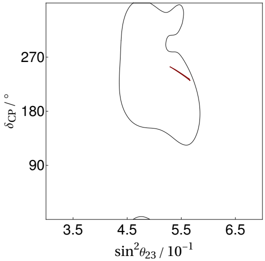


THEORY AND EXPERIMENT

FLASHED INDEPEDENTLY

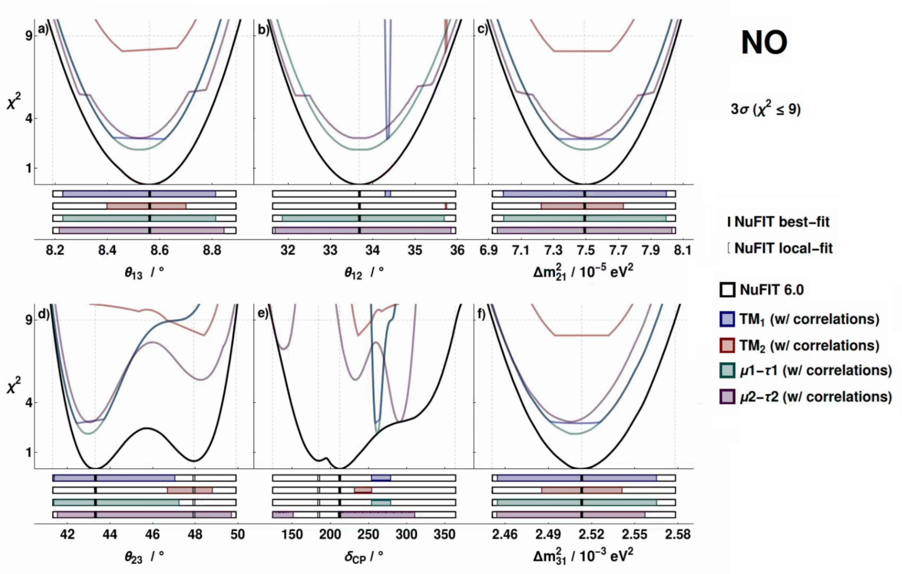
TM2

COMBINED

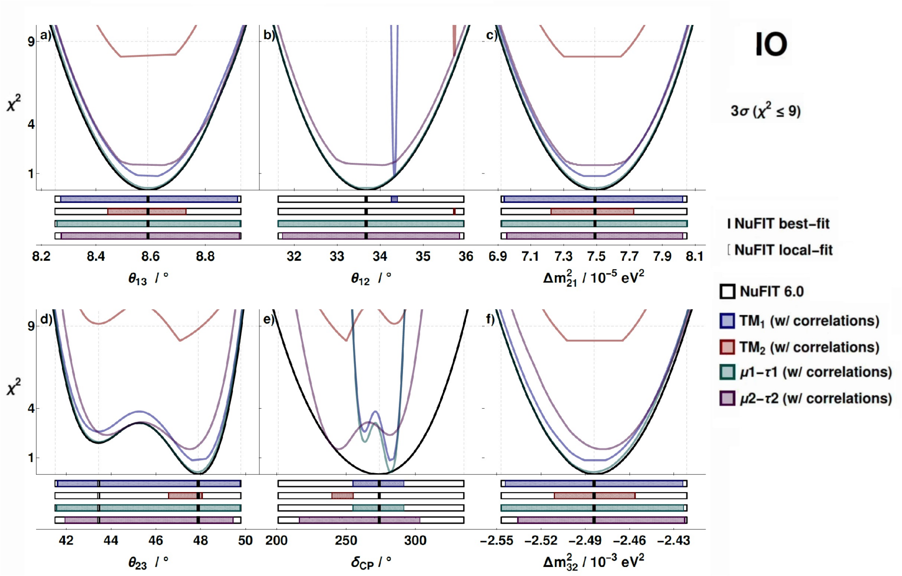


NO, $\chi^2 < 9$

Extracted ranges of model's parameters from correlated data (NuFIT v6.0)



χ^2_{min}: TM$_1 = 2.8$, TM$_2 = 8.1$, $\mu1-\tau1 = 2.1$, $\mu2-\tau2 = 2.8$



$$\chi^2_{min}: \mathrm{TM}_1 = 0.9,\ \mathrm{TM}_2 = 8.1,\ \mu1-\tau1 = 0.1,\ \mu2-\tau2 = 1.5$$

Results for NuFIT 6.0 data sets (w/ atmospheric ν)					
		TM_1	TM_2	$\mu1$-$\tau1$	$\mu2$-$\tau2$
χ^2_{min}	NO	2.8	8.1	2.1	2.8
	IO	0.9	8.1	0.1	1.5

Results for NuFIT 6.1 data sets (w/ atmospheric ν)					
		TM_1	TM_2	$\mu1$-$\tau1$	$\mu2$-$\tau2$
χ^2_{min}	NO	3.9	21.5	2.4	3.2
	IO	1.9	21.5	0.1	1.6

JUNO results change numbers, not preferences.

Can we minimize χ^2?

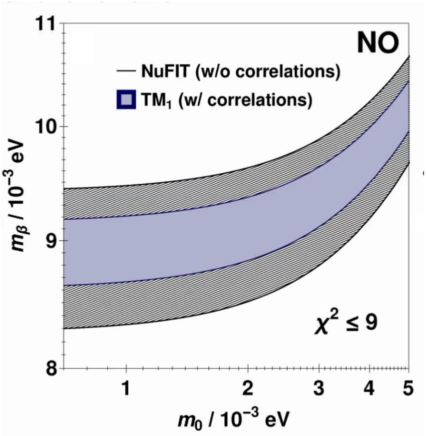
We can confront the constrains coming from mixing schemes to global-fit data.

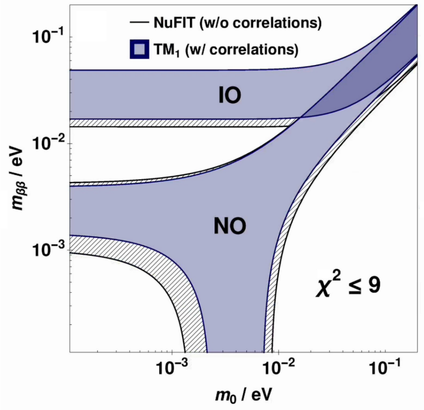
Extra parameter example $m_\beta(m_0)$, TM_1, NO, $\chi^2 < 9$

$$
\begin{cases}
\chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\mathrm{CP}}, \Delta m^2_{21}, \Delta m^2_{31}) \leqslant 9\,, \\
m_\beta = \sqrt{m_0^2 + \Delta m^2_{21} c^2_{13} s^2_{12} + \Delta m^2_{31} s^2_{13}}\,, \\
s^2_{12} = \frac{1-3s^2_{13}}{3-3s^2_{13}}\,, \\
\cos\delta_{\mathrm{CP}} = \frac{(1-5s^2_{13})(2s^2_{23}-1)}{4 s_{13} s_{23}\sqrt{2(1-3s^2_{13})(1-s^2_{23})}}\,.
\end{cases}
$$

Extra parameters example $m_{\beta\beta}(m_0)$, TM$_1$, NO, $\chi^2 < 9$

$$
\begin{cases}
\chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\mathrm{CP}}, \Delta m^2_{21}, \Delta m^2_{31}) \leqslant 9 \,, \\[2ex]
m_{\beta\beta} = \Big| m_0 c^2_{13} c^2_{12} e^{i2(\alpha_1 - \delta_{\mathrm{CP}})} + \sqrt{m_0^2 + \Delta m^2_{21}}\, c^2_{13} s^2_{12} e^{i2(\alpha_2 - \delta_{\mathrm{CP}})} \\
\qquad\qquad\qquad\qquad\qquad + \sqrt{m_0^2 + \Delta m^2_{31}}\, s^2_{13} \Big| \,, \\[2ex]
s^2_{12} = \frac{1 - 3 s^2_{13}}{3 - 3 s^2_{13}} \,, \\[2ex]
\cos\delta_{\mathrm{CP}} = \frac{(1 - 5 s^2_{13})(2 s^2_{23} - 1)}{4 s_{13} s_{23} \sqrt{2(1 - 3 s^2_{13})(1 - s^2_{23})}} \,, \\[2ex]
0 \leqslant \alpha_1, \alpha_2 \leqslant 2\pi \,.
\end{cases}
$$





Solution of eqs from slides 32 and 33

Summary

Multidimensional correlated datasets on neutrino oscillation parameters allows for confronting the theoretical models with experimental data

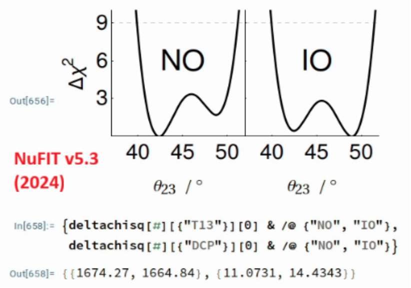
With the presented approach, we

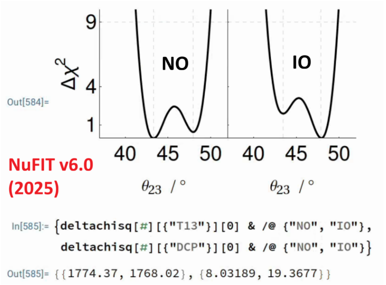
- find χ^2_{min} "floor" for the model
- test TM_1, TM_2 and partial μ-τ mixing-schemes over the full range of parameters at a given χ^2 level
- some parameters are exceptionally constrained (for NuFIT v6.0)
  - θ_{12} for TM_1 and TM_2,
  - θ_{23} for TM_2,
  - δ_{CP} for all considered mixing-schemes,
  - Δm^2_{21} and $\Delta m^2_{31(32)}$ for TM_2
- TM_2 is outside the reasonable χ^2_{min} value (for NuFIT v6.1)
- find tighter, model-dependent constraints on the effective neutrino masses
- potentially could be used for χ^2 estimation in SO(3) parametrization

The method is general and can be applied to any model providing mixing-scheme relations in the form analytical expressions involving the neutrino oscillation parameters.

Thank you
for your attention.

3ν mixing





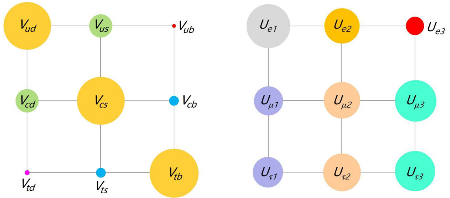
Left: Mathematica output based on NuFIT v5.3 data (2024). **Right:** Mathematica output based on NuFIT v6.0 data (2025).

NO

$$\begin{pmatrix} 0.800 \to 0.843 & 0.519 \to 0.580 & 0.143 \to 0.155 \\ 0.354 \to 0.461 & 0.533 \to 0.620 & 0.635 \to 0.769 \\ 0.277 \to 0.485 & 0.589 \to 0.616 & 0.620 \to 0.759 \end{pmatrix}$$

IO

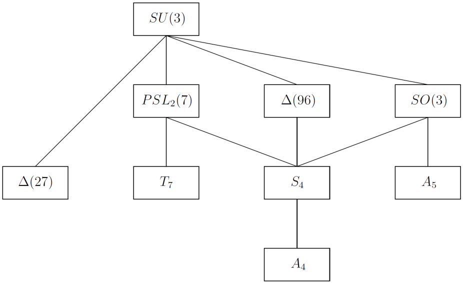
$$\begin{pmatrix} 0.800 \to 0.843 & 0.519 \to 0.580 & 0.143 \to 0.156 \\ 0.288 \to 0.479 & 0.567 \to 0.606 & 0.635 \to 0.772 \\ 0.246 \to 0.527 & 0.585 \to 0.603 & 0.616 \to 0.759 \end{pmatrix}$$



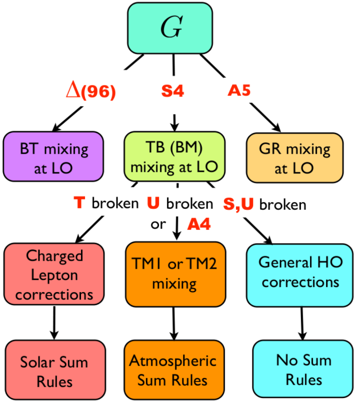
(a) CKM quark flavor mixing (b) PMNS lepton flavor mixing

Left: Mixing matrix ($|\mathrm{U}_{\mathrm{PMNS}}|$) ranges in 3σ, NuFIT v5.3. **Right:** Taken from arXiv:2210.11922, Figure 2.

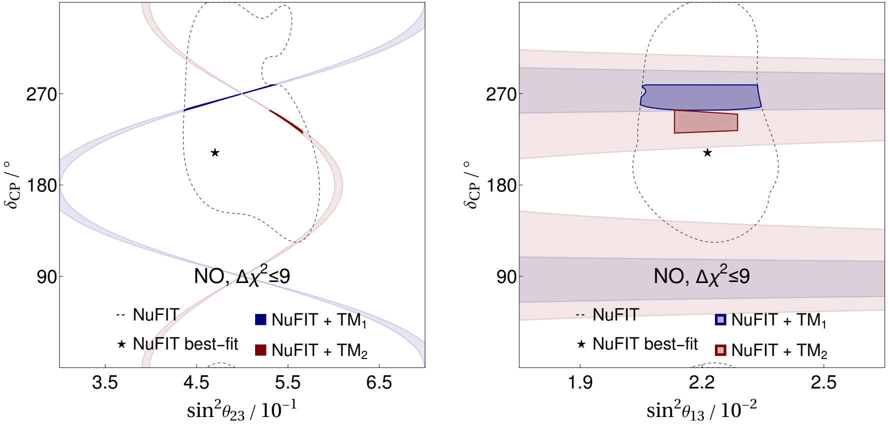
Flavor symmetries and mixing-schemes



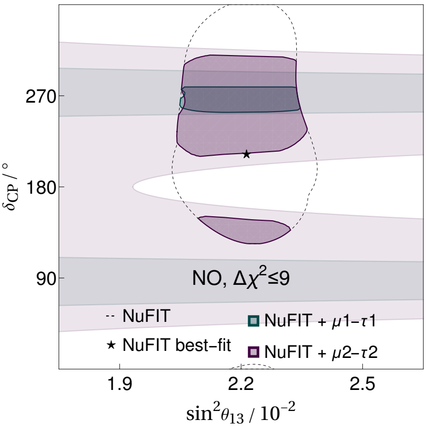
arXiv:1301.1340 S.F. King, C. Luhn

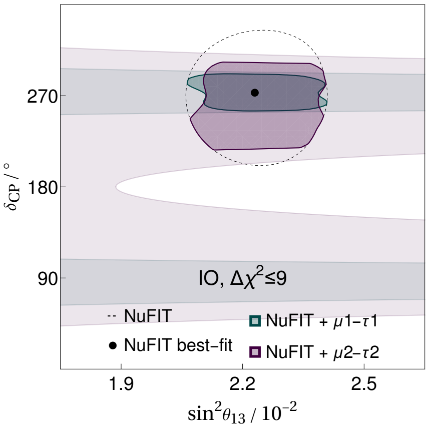


Experiment / Theory / Combined at $\chi^2 < 9$ (NuFIT v6.0)



Experiment / Theory / Combined at $\chi^2 < 9$ (NuFIT v6.0)





Effective neutrino mass

Effective electron neutrino mass:

$$m_\beta^2 = c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2 = \begin{cases} \text{NO:} \quad m_0^2 + \Delta m_{21}^2 c_{13}^2 s_{12}^2 + \Delta m_{3\ell}^2 s_{13}^2\,, \\ \text{IO:} \quad m_0^2 - \Delta m_{21}^2 c_{13}^2 c_{12}^2 - \Delta m_{3\ell}^2 c_{13}^2. \end{cases}$$

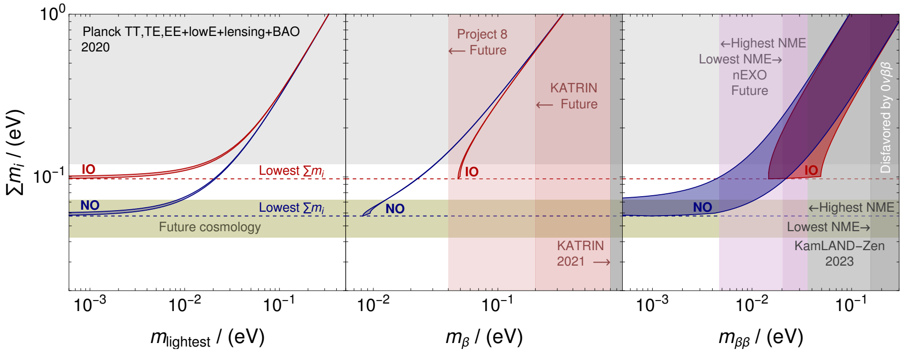
Effective Majorana mass:

$$m_{\beta\beta} = \left| \sum_i m_i U_{ei}^2 \right| = \left| m_1 c_{13}^2 c_{12}^2 e^{i2\alpha_1} + m_2 c_{13}^2 s_{12}^2 e^{i2\alpha_2} + m_3 s_{13}^2 e^{-i2\delta_{\rm CP}} \right|$$

$$= \begin{cases} \text{NO:} \quad m_0 \left| c_{13}^2 c_{12}^2 e^{i2(\alpha_1 - \delta_{\rm CP})} + \sqrt{1 + \frac{\Delta m_{21}^2}{m_0^2}}\, c_{13}^2 s_{12}^2 e^{i2(\alpha_2 - \delta_{\rm CP})} + \sqrt{1 + \frac{\Delta m_{3\ell}^2}{m_0^2}}\, s_{13}^2 \right| \\ \text{IO:} \quad m_0 \left| \sqrt{1 - \frac{\Delta m_{3\ell}^2 + \Delta m_{21}^2}{m_0^2}}\, c_{13}^2 c_{12}^2 e^{i2(\alpha_1 - \delta_{\rm CP})} + \sqrt{1 - \frac{\Delta m_{3\ell}^2}{m_0^2}}\, c_{13}^2 s_{12}^2 e^{i2(\alpha_2 - \delta_{\rm CP})} + s_{13}^2 \right|. \end{cases}$$

Here $m_0 = m_1$ (m_3) and $\ell = 1$ (2) in $\Delta m_{3\ell}^2$ for NO (IO).

Neutrino mass



Planck: $\sum_i m_i < 0.12$ eV, 95% CL, arXiv:1807.06209

KATRIN: $m_\beta < 0.8$ eV, 90% CL, arXiv:2105.08533

KamLAND-Zen: $m_{\beta\beta} <$ (0.036 to) 0.156 eV, 90% CL, arXiv:2203.02139