Life on the far side of the U_{PMNS}
SO(3) parameterization of neutrino mixing matrix

U_{SO3} U_{PMNS}

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2509.25328, PLB 874 (2026) 140219 with Jaroslaw Duda & Janusz Gluza

1st Annual MAPS Meeting
Ruđer Bošković Institute
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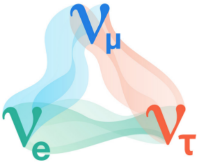
Neutrino mixing



OSCILLATING

Neutrinos come in three types, called flavors. There are electron neutrinos, muon neutrinos and tau neutrinos. One of the strangest aspects of neutrinos is that they don't pick just one flavor and stick to it. They oscillate between all three.

LIGHTWEIGHT

Neutrinos weigh almost nothing, and they travel close to the speed of light. Neutrino masses are so small that so far no experiment has succeeded in measuring them. The masses of other fundamental particles come from the Higgs field, but neutrinos might get their masses another way.

Neutrino parameters and the known unknowns

- Neutrinos are special!! It's flavor and mass eigenstates are related by :

$$|\nu_e, \nu_\mu, \nu_\tau\rangle^T_{\text{flavor}} = U_{\alpha i}|\nu_1, \nu_2, \nu_3\rangle^T_{\text{mass}}$$

- Pontecorvo-Maki-Nakagawa-Sakata parametrization: $\boxed{U_{\alpha i} = U_{\text{PMNS}}}$

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- Elements of $U_{\alpha i}$ are not necessarily angles, however such 'angle' parametrization has its advantages.
- Inherited from the quark sector.
- We choose this parametrization because of the smallness of U_{e3}

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- Elements of $U_{\alpha i}$ are not necessarily angles, however such 'angle' parametrization has its advantages.
- Inherited from the quark sector.
- We choose this parametrization because of the smallness of U_{e3}
- We are not fitting any fundamental objects, it's just elements of some parametrization that we have chosen.

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- Large Lepton Mixings

$$|U_{\alpha i}| \sim \begin{pmatrix} 0.79 - 0.86 & 0.50 - 0.61 & 0.14 - 0.16 \\ 0.24 - 0.52 & 0.44 - 0.69 & 0.63 - 0.79 \\ 0.26 - 0.52 & 0.47 - 0.71 & 0.60 - 0.77 \end{pmatrix}$$

- Small Quark Mixings

$$|V_{CKM}| \sim \begin{pmatrix} 0.9745 - 0.9757 & 0.219 - 0.224 & 0.002 - 0.005 \\ 0.218 - 0.224 & 0.9736 - 0.9750 & 0.036 - 0.046 \\ 0.004 - 0.014 & 0.034 - 0.046 & 0.9989 - 0.9993 \end{pmatrix}$$

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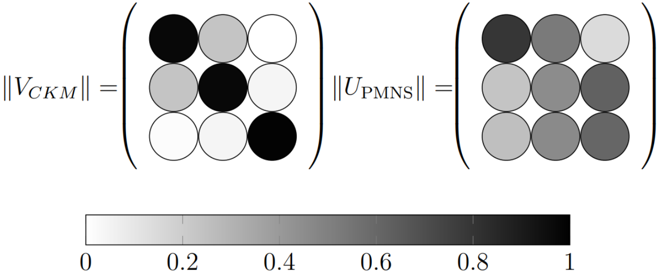
here $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. Figure : D. Meloni, 1709.02662



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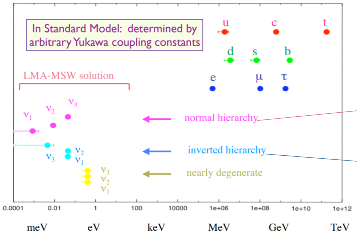
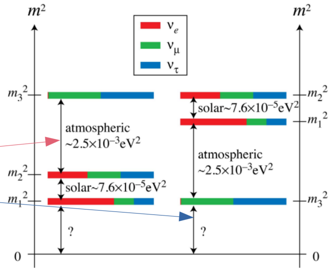
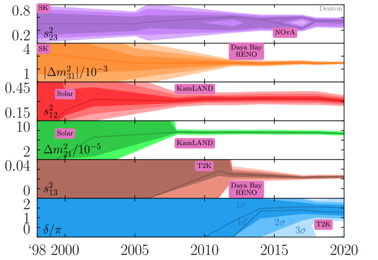
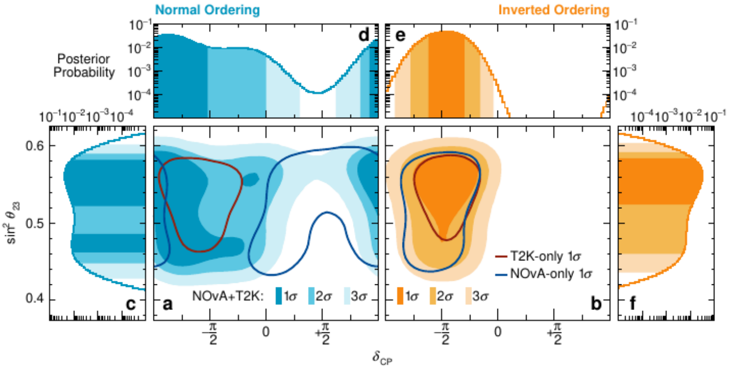






	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.6$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.012}_{-0.012}$	$0.269 \to 0.343$
$\theta_{12}/^\circ$	$33.44^{+0.77}_{-0.74}$	$31.27 \to 35.86$	$33.45^{+0.77}_{-0.74}$	$31.27 \to 35.87$
$\sin^2\theta_{23}$	$0.573^{+0.018}_{-0.023}$	$0.405 \to 0.620$	$0.578^{+0.017}_{-0.021}$	$0.410 \to 0.623$
$\theta_{23}/^\circ$	$49.2^{+1.0}_{-1.3}$	$39.5 \to 52.0$	$49.5^{+1.0}_{-1.2}$	$39.8 \to 52.1$
$\sin^2\theta_{13}$	$0.02220^{+0.00068}_{-0.00062}$	$0.02034 \to 0.02430$	$0.02238^{+0.00064}_{-0.00062}$	$0.02053 \to 0.02434$
$\theta_{13}/^\circ$	$8.57^{+0.13}_{-0.12}$	$8.20 \to 8.97$	$8.60^{+0.12}_{-0.12}$	$8.24 \to 8.98$
$\delta_{CP}/^\circ$	194^{+52}_{-25}	$105 \to 405$	287^{+27}_{-32}	$192 \to 361$
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.515^{+0.028}_{-0.028}$	$+2.431 \to +2.599$	$-2.498^{+0.028}_{-0.029}$	$-2.584 \to -2.413$

Neutrino parameters and the known unknowns



$\boxed{\text{CP-conserving } \delta_{\mathrm{CP}} = 0/180^\circ \text{ is still possible!!}}$

See evening talks today by Monojit and Dinesh.

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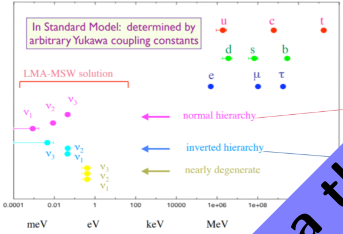
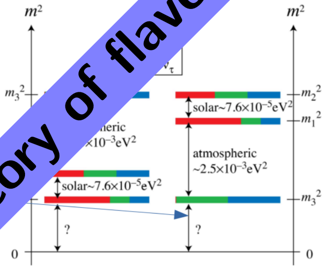
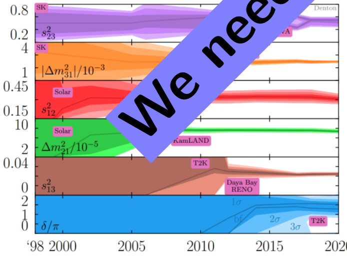




We need a theory of flavor!!



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Common wisdom to explain lepton flavor mixing

Anarchy

- Neutrino mixing anarchy is the hypothesis that the leptonic mixing matrix can be described as the result of a random draw from an unbiased distribution of unitary 3×3 matrices.
- Random analysis without imposing prior theories or symmetries on the mass and mixing matrices.
- This hypothesis does not make any correlation among the neutrino masses and mixing parameters

de Gouvea, Haba, Hall, Murayama : 9911341, 0009174, 1204.1249

Pheno Review: Prog.Part.Nucl.Phys. 2024; Chauhan, Dev, Dubovyk, Dziewit, Flieger, Grzanka, Gluza, Karmakar, Zieba

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Texture

- More specific studies with imposed mass or mixing textures for which models with underlying symmetries can be sought.
- It's an intermediate approach
- Some texture zeros of neutrino mass matrices can be eliminated.

Talk : Marek Michalczyk on Wednesday

Alejandro Ibarra, Graham Ross: Phys.Lett.B 2003

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Symmetry

- Theoretical studies where some explicit symmetries at the Yukawa Lagrangian level are assumed and corresponding extended particle sector is defined.
- The symmetry-based approach to explain the non-trivial mixing in the lepton sector known as family symmetry or horizontal symmetry

Talk : Janusz Gluza on Wednesday

Review: Altarelli, Feruglio (Rev. Mod. Phys.) 1002.0211

Pheno Review: Prog.Part.Nucl.Phys. 2024; Chauhan, Dev, Dubovyk, Dziewit, Flieger, Grzanka, Gluza, Karmakar, Zieba

Symmetries, why?

$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & s_{13}c_{23} \end{pmatrix}$$

$\Downarrow$

(Prior to 2012)

$$\boxed{s_{23} = 1/\sqrt{2}\ (\theta_{23} = 45^\circ) \text{ and } \theta_{13} = 0}$$

$\Downarrow$

$$U_0 = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$\theta_{12} = 45^\circ (s_{12} = 1/\sqrt{2})$	$\theta_{12} = 35.26^\circ (s_{12} = 1/\sqrt{3})$	$\theta_{12} = 31.7^\circ$	$\theta_{12} = 30^\circ (s_{12} = 1/2)$
Bimaximal Mixing	Tribimaximal Mixing	Golden Ratio Mixing	Hexagonal Mixing

$$U_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{\varphi}{\sqrt{2+\varphi}} & \frac{1}{\sqrt{2+\varphi}} & 0 \\ \frac{-1}{\sqrt{4+2\varphi}} & \frac{\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{4+2\varphi}} & \frac{-\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{3}{4}} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Fukugita, Tanimoto, Yanagida PRD98; Harrison Perkins, Scott PLB02; Dutta,Ramond NPB03; Rodejohann et. al. EPJC10

(GR: $\tan\theta_{12} = 1/\phi$ where $\phi = (1+\sqrt{5})/2$)

Symmetries, why? Theory Perspective

$\boxed{\text{Lagrangian/neutrino mass mechanism}} \otimes \boxed{\text{Symmetry}} \Rightarrow \boxed{\text{Observed Mixing}}$

Example: $\mu - \tau$ permutation symmetry and tribimaximal mixing

$$m_\nu = U_0^\star \, \mathrm{diag}(m_1, m_2, m_3) U_0^\dagger \,,$$

such a mixing matrices can easily diagonalize a $\mu - \tau$ symmetric (transformations $\nu_e \to \nu_e$, $\nu_\mu \to \nu_\tau$, $\nu_\tau \to \nu_\mu$ under which the neutrino mass term remains unchanged) neutrino mass matrix of the form

$$m_\nu = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix},$$

With $A + B = C + D$ this matrix yields tribimaximal mixing pattern where $s_{12} = 1/\sqrt{3}$ *i.e.*, $\theta_{12} = 35.26^\circ$

- Compatible Mixing Matrix :

$$U_{\mathrm{TB}} \simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Neutrino Oscillation: as we measure it

$$|U_{\alpha i}| \sim \begin{pmatrix} 0.79 - 0.86 & 0.50 - 0.61 & 0.14 - 0.16 \\ 0.24 - 0.52 & 0.44 - 0.69 & 0.63 - 0.79 \\ 0.26 - 0.52 & 0.47 - 0.71 & 0.60 - 0.77 \end{pmatrix}$$

Neutrino Oscillation

- Order dependent mixing matrix

$$U_{PMNS} = R_x(\theta_{23})R_y(\theta_{13},\delta_{\mathrm{CP}})R_z(\theta_{12})U_M(\alpha_1,\alpha_2) = U$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\mathrm{CP}}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\mathrm{CP}}} & c_{23}c_{13} \end{pmatrix}$$

- Probability:

$$P_{\nu_\alpha\to\nu_\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\mathrm{Re}\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin^2\left(\frac{\Delta m^2_{ij}L}{4E}\right)$$
$$+2\sum_{i>j}\mathrm{Im}\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)$$

Neutrino Oscillation and U_{PMNS}

- Order-dependent mixing matrix : **Why this particular order?**

$$U_{PMNS} = R_x(\theta_{23}) R_y(\theta_{13}, \delta_{\rm CP}) R_z(\theta_{12}) U_M(\alpha_1, \alpha_2) = U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{\rm CP}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{\rm CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{\rm CP}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{\rm CP}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{\rm CP}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{\rm CP}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{\rm CP}} & c_{23} c_{13} \end{pmatrix}$$

'Secrets' of the Standard Parametrization

- Placing $R_x(\theta_{23})$ (atmospheric) on the left and $R_z(\theta_{12})$ **(solar) on the right** $\Rightarrow$ extreme physical limit $\Rightarrow$ clean 2×2 framework.
- The mass hierarchy ($\Delta m^2_{21} \ll |\Delta m^2_{31}|$, one-mass-scale dominance approximation) and the smallness of θ_{13}

Neutrino Oscillation and U_{PMNS}

Tuning the experimental baseline scale $\Delta_{ij} = \frac{\Delta m^2_{ij} L}{4E}$, specific frequencies decouple:

Experiment	**Physical Limit / Regime**	**Structural Reduction**	**Parameters**
Atmospheric	Long base-line $\Delta m^2_{21} \to 0$ $(\Delta_{31} \sim \pi/2, \Delta_{21} \to 0)$ ν_1, ν_2 mass-degenerate	$P_{\mu\mu} \approx$ $1 - \sin^2(2\theta_{23}) \cos^4(\theta_{13}) \sin^2\left(\frac{\Delta m^2_{31} L}{4E}\right)$ $R_z(\theta_{12})$ physically drops out $\theta_{13} \to 0 \implies \nu_e$ decouples	$\theta_{23}, \lvert\Delta m^2_{31}\rvert$
Solar	Very-long baseline $\Delta m^2_{31} \to \infty$ $(\Delta_{21} \sim \frac{\pi}{2}, \Delta_{31} >> 1)$ Rapid atmospheric oscillations average out $(\sin^2 \Delta_{31} \to \frac{1}{2})$; ν_3 kinematically decouples	$P_{ee} \approx \cos^4(\theta_{13}) \Big[1 - \sin^2(2\theta_{12}) \times$ $\sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right) \Big] + \sin^4(\theta_{13})$ $R_Y(\theta_{13}) \to$ const. shift 2×2 solar block isolates	$\theta_{12}, \Delta m^2_{21}$

Neutrino Oscillation and U_{PMNS}

- Elements of U_{PMNS}

SNO/KamLAND $|U_{e2}|^2$

Reactor/LBL $|U_{e3}|^2(1-|U_{e3}|^2)$

Atm Nus/LBL $|U_{\mu 3}|^2(1-|U_{\mu 3}|^2)$

$$\begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12}-s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12}-s_{13}s_{23}s_{12}e^{i\delta} & c_{13}s_{23} \\ s_{23}s_{12}-s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12}-s_{13}c_{23}s_{12}e^{i\delta} & c_{13}c_{23} \end{pmatrix}$$

- Experimentally, with $\delta_{\rm CP} = 180^\circ$

$$U \to V_{\rm osc} = \begin{pmatrix} 0.817 \div 0.831 & 0.535 \div 0.558 & -0.150 \div -0.146 \\ -0.313 \div -0.273 & 0.615 \div 0.657 & 0.686 \div 0.738 \\ 0.471 \div 0.498 & -0.569 \div -0.519 & 0.658 \div 0.713 \end{pmatrix}$$

- Even with other CP conserving scenario, $\delta_{\rm CP} = 0^\circ$, measured values of $\theta_{23}, \theta_{13}, \theta_{12}$ remains same. U_{PMNS} does not distinguish the CP-conserving scenarios.

More on U_{PMNS}

- The standard parameterization of the U matrix is one of six possible products of three-dimensional rotations for 3 flavors
- We adopted : Small-angle approximations $\Rightarrow$ usual oscillation paradigm

$$
\begin{aligned}
U_{PMNS} &= R_x(\theta_{23},0)R_y(\theta_{13},\delta_{\rm CP})R_z(\theta_{12},0) \\
&= \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta_{\rm CP}} \\ & 1 & \\ -s_{13}e^{i\delta_{\rm CP}} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & c_{23}c_{13} \end{pmatrix}
\end{aligned}
$$

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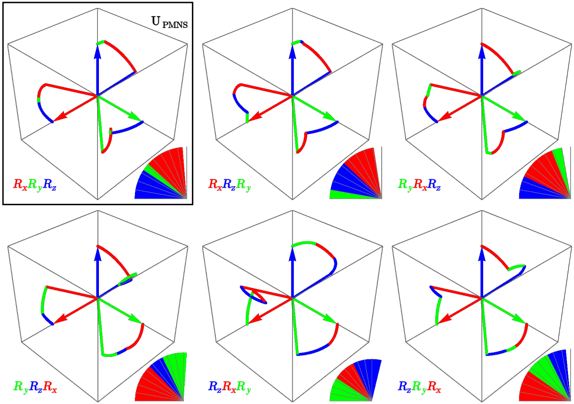
\end{aligned}
$$

- Other possibilities (for Tait–Bryan angles):

$$R_xR_zR_y, \quad R_yR_xR_z, \quad R_yR_zR_x, \quad R_zR_xR_y, \quad R_zR_yR_x$$

"Who ordered that?"-Isidor Rabi (90 years ago after muon discovery)

- Six possibilities :



More on U_{PMNS}

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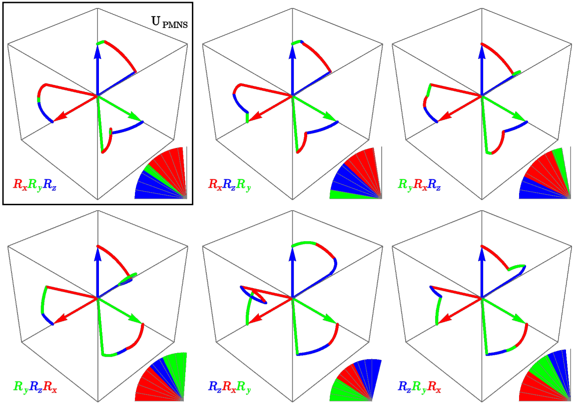


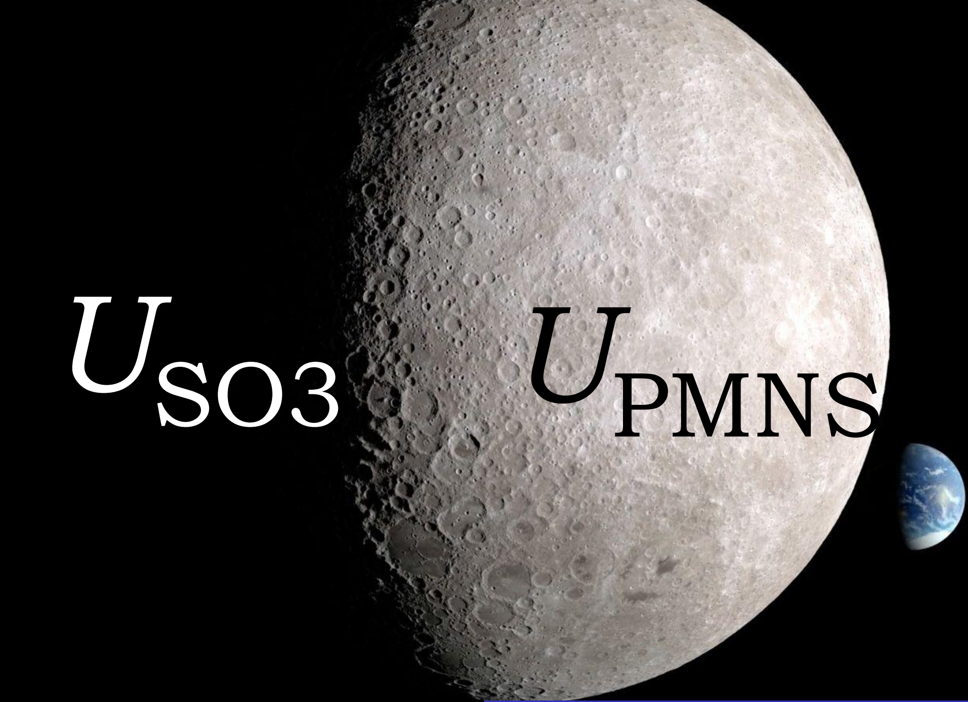
$$\begin{aligned} U &= R_x(43.3^\circ)R_y(8.56^\circ)R_z(33.68^\circ) = R_x(37.64^\circ)R_z(10.25^\circ)R_y(33.26^\circ) \\ &= R_y(42.70^\circ)R_x(11.69^\circ)R_z(25.70^\circ) = R_y(18.58^\circ)R_z(45.68^\circ)R_x(29.76^\circ) \\ &= R_z(33.17^\circ)R_x(39.63^\circ)R_y(30.71^\circ) = R_z(35.36^\circ)R_y(28.06^\circ)R_x(21.17^\circ). \end{aligned}$$

- After decades of observation :

$$V_{\rm osc} = \begin{pmatrix} 0.817 \div 0.831 & 0.535 \div 0.558 & -0.150 \div -0.146 \\ -0.313 \div -0.273 & 0.615 \div 0.657 & 0.686 \div 0.738 \\ 0.471 \div 0.498 & -0.569 \div -0.519 & 0.658 \div 0.713 \end{pmatrix}$$

$$\boxed{|(V_{\rm osc})_{e1}|^2 > |(V_{\rm osc})_{\mu 1}|^2 > |(V_{\rm osc})_{\tau 1}|^2}$$

- Permutations of Euler matrices are qualitatively equivalent
- Mixing angles are parameters of the U matrix, not physical observables, fitted experimentally
- There is nothing special in choosing U_{PMNS} as far as neutrino mixing parametrization is concerned
- Current choice of U_{PMNS} is order dependent
- Specific representation matters as theoretical considerations and mass/mixing models are based on this choice.


U_{SO3}
U_{PMNS}

Neutrino Mixing

Q: Can neutrino mixing originate directly from any symmetry principle?

$\boxed{\text{Lagrangian/neutrino mass mechanism}} \otimes \boxed{\boxed{\text{Symmetry}} \Rightarrow \boxed{\text{Observed Mixing}}}$

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Ans: Yes, with Special Orthogonal group in 3 dimensions, denoted as SO(3) but with some caveats

SO(3) Group Theory

- The continuous SO(3) family symmetry is one of the most widely used Lie groups in mathematics and physics. Mathematically, SO(3) is the set of all (3×3) real matrices R that satisfy: $\boxed{R^T R = \mathbf{I} \text{ and } \det(\mathrm{R}) = +1}$.

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- Fundamental three-dimensional representaion of the standard antisymmetric SO(3) generators :

$$G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, G_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, G_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $G_i^T = -G_i$ and the commutator of the generators can be written as

$$[G_i, G_j] = \epsilon_{ijk} G_k, \quad i, j, k = x, y, z.$$

SO(3) and Euler rotaions

- Euler rotations are elements of SO(3):
- The matrix $R_x(\theta_{23}), R_y(\theta_{13}, \delta_{\rm CP} = 180^\circ), R_z(\theta_{12})$ can be obtained via SO(3) generators G_x, G_y, G_z

$$\begin{aligned}
U &= e^{-\theta_{23} G_x} e^{-\theta_{13} G_y} e^{-\theta_{12} G_z}, \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & -s_{13} \\ 0 & 1 & 0 \\ s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & -s_{13} \\ -s_{12}c_{23} + c_{12}s_{23}s_{13} & c_{12}c_{23} + s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} + c_{12}c_{23}s_{13} & -c_{12}s_{23} + s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix},
\end{aligned}$$

where $\delta_{\rm CP} = 180^\circ$. For $\delta_{\rm CP} = 0^\circ$, replace with $e^{+\theta_{13} G_y}$

- Other combinations, such as $e^{-\theta_{23} G_x} e^{-\theta_{12} G_z} e^{-\theta_{13} G_y}$ or $e^{-\theta_{12} G_z} e^{-\theta_{23} G_x} e^{-\theta_{13} G_y}$ will imply different values of the mixing angles, not reflecting nearly maximal mixings in U_{PMNS}

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&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & -s_{13} \\ -s_{12}c_{23} + c_{12}s_{23}s_{13} & c_{12}c_{23} + s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} + c_{12}c_{23}s_{13} & -c_{12}s_{23} + s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix},
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SO(3) order independent parameterization

- How to avoid the order dependence for the multiplication of group matrices ?
- We propose, an *order independent parameterization* of rotations in a plane perpendicular to $\vec{\Theta} = (\theta_x, \theta_y, \theta_z)$ through the matrix exponent.
- Using the three SO(3) generators, and the three mixing angles $(\theta_x, \theta_y, \theta_z)$:

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$$U_{\rm SO3} = \exp(-\vec{\Theta}.\vec{G}) = \exp(-\theta_x G_x - \theta_y G_y - \theta_z G_z) = \exp\begin{pmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\theta_x^2+\cos\theta\left(\theta_y^2+\theta_z^2\right)}{\theta^2} & \frac{\theta\theta_z\sin\theta-\theta_x\theta_y(\cos\theta-1)}{\theta^2} & -\frac{\theta_x\theta_z(\cos\theta-1)+\theta\theta_y\sin\theta}{\theta^2} \\ -\frac{\theta_x\theta_y(\cos\theta-1)+\theta\theta_z\sin\theta}{\theta^2} & \frac{\cos\theta\left(\theta_x^2+\theta_z^2\right)+\theta_y^2}{\theta^2} & \frac{\theta\theta_x\sin\theta-\theta_y\theta_z(\cos\theta-1)}{\theta^2} \\ \frac{\theta\theta_y\sin\theta-\theta_x\theta_z(\cos\theta-1)}{\theta^2} & -\frac{\theta\theta_x\sin\theta+\theta_y\theta_z(\cos\theta-1)}{\theta^2} & \frac{\cos\theta\left(\theta_x^2+\theta_y^2\right)+\theta_z^2}{\theta^2} \end{pmatrix}.$$

where $\theta = \sqrt{\theta_x^2 + \theta_y^2 + \theta_z^2}$.

$\boxed{U_{\rm SO3} \text{ is also orthogonal and unimodular} \Rightarrow \text{Rotation}}$

SO(3) order independent parameterization

Warning !!

- The exponential form $U_{\mathrm{SO3}} = \exp(-\vec{\Theta}.\vec{G}) = \exp(-\theta_x G_x - \theta_y G_y - \theta_z G_z)$, is in general basis-dependent.
- Under a basis transformation, the magnitude of θ_x, θ_y and θ_y will change.

Good news !!

- The rotation itself, however, will remain the same, and all such forms are related by similarity transformation.
- Since similarity transformations preserve intrinsic quantities such as the rotation angle, the physical content of the transformation remains unchanged.
- Here we work with the fundamental three-dimensional representaion of the standard antisymmetric SO(3) generators (and their commutator relations)

SO(3) order independent parameterization

- U_{SO3} explains V_{osc}, with $\delta_{\mathrm{CP}} = 180°$

$$\boxed{\theta_x = 43.8° \pm 2.1°, \quad \theta_y = 21.7° \pm 0.6°, \quad \theta_z = 28.3° \pm 0.8°}$$

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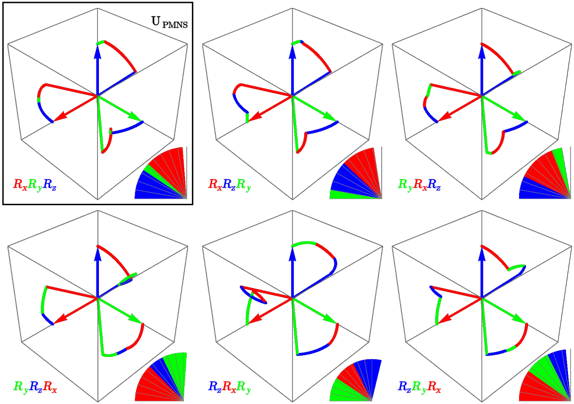
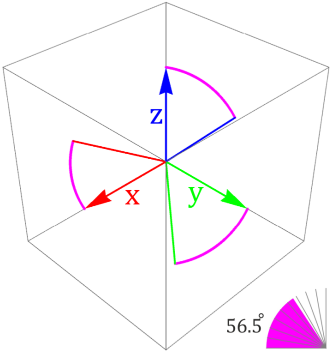


SO(3) order independent parameterization

- U_{SO3} explains V_{osc}, with $\delta_{\mathrm{CP}} = 180^\circ$

$$\boxed{\theta_x = 43.8^\circ \pm 2.1^\circ, \quad \theta_y = 21.7^\circ \pm 0.6^\circ, \quad \theta_z = 28.3^\circ \pm 0.8^\circ}$$



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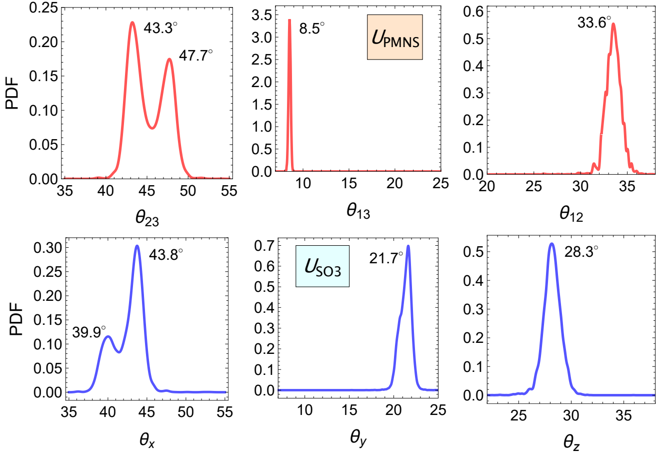

NuFIT χ^2 $\rightarrow$ probability distributions of mixing angles $\rightarrow$ Monte Carlo sampling $\rightarrow$ PMNS matrices $\rightarrow$ nearest SO(3) matrices $\rightarrow$ distributions of new angles

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CP-Conserving cases are different

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- $\delta_{\text{CP}} = 0^\circ$

$$\boxed{\theta_x = 46.5^\circ \pm 2.4^\circ, \quad \theta_y = 5.4^\circ \pm 0.8^\circ, \quad \theta_z = 35.1^\circ \pm 0.7^\circ}$$

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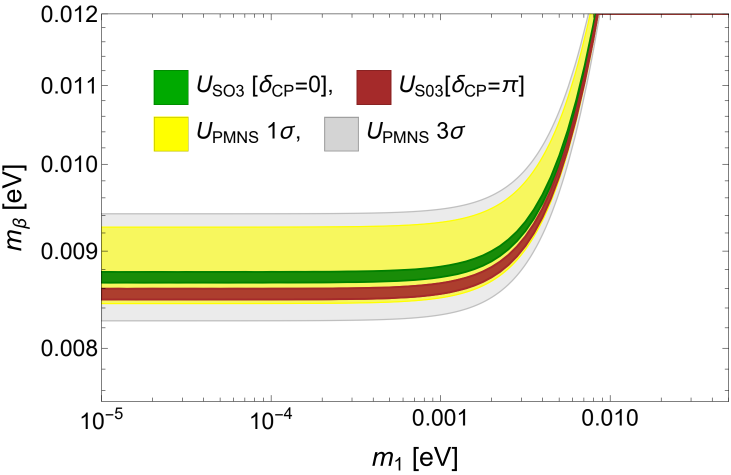
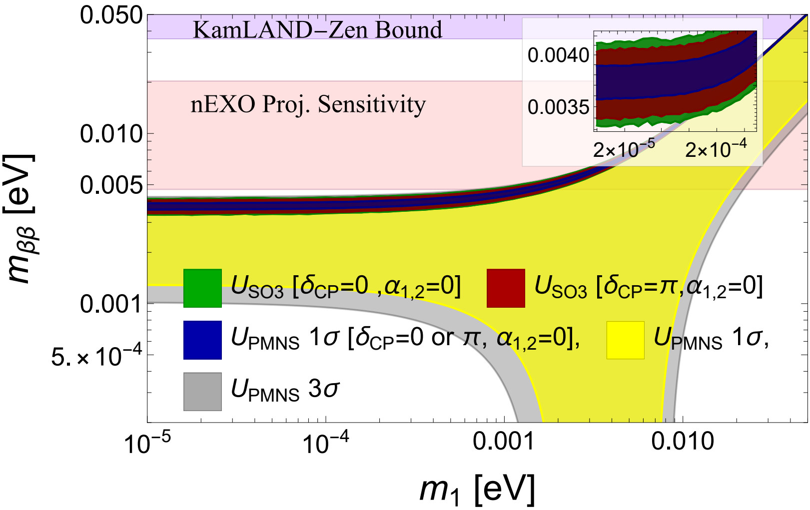
What do these results mean? Next talk by Elena (studied many phenomenological aspects within a short period of time)

SO(3) parameterization: phenomenology



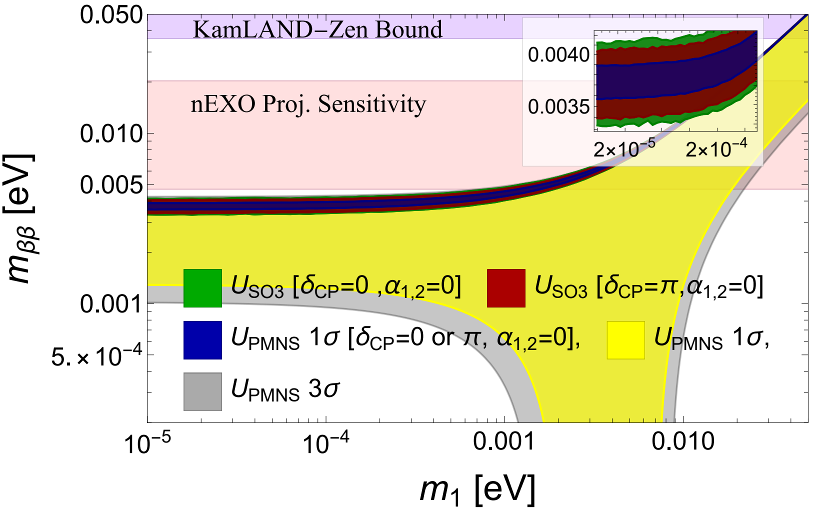
$$m_\beta^2 = \frac{\sum_i m_i^2 |U_{ei}|^2}{\sum_i |U_{ei}|^2} = \sum_i m_i^2 |U_{ei}|^2$$

SO(3) parameterization: phenomenology



$m_{\beta\beta} = \left| \sum_i m_i U_{ei}^2 \right|$, for NO with vanishing Majorana phases

SO(3) parameterization: phenomenology



Inverted Ordering with $\delta_{\rm CP} = 180^\circ$ is excluded by T2k/NOνA.

SO(3) Parametrization : Outlook

- We propose a parameterization of the 3-dimensional U matrix based on the SO(3) group, which corresponds to one general rotation in 3-dimensional space.
- To harmonize the signs of elements of the SO(3) generators and related commutator relation with the neutrino mixing matrix that follow, we infer $\delta_{\rm CP} = 0/180^\circ$
- Both T2K and NOνA disfavours inverted ordering with $\delta_{\rm CP} = 180^\circ$. In the framework of the SO(3) parameterization, we remain with the normal mass ordering.

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- Yields near maximal mixing for $\delta_{\rm CP} = 0°$, whereas relatively large, 'democratic' values of the mixing angles $\theta_x, \theta_y, \theta_z$ for $\delta_{\rm CP} = 180°$

Future Works

- Implications in neutrino oscillation pheno (next by Elena)
- Constructions of models that naturally generate the SO(3) mixing.
- Consequences beyond neutrino mixing,
  e.g., The fixed $\delta_{\rm CP} = 0/180°$ prediction eliminates CP violation in the lepton sector, which could have implications for a wide variety of leptogenesis scenarios. With hierarchical right-handed neutrinos, the lepton asymmetry vanishes for vanilla leptogenesis
- How to include CP-conserving cases?

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Thank you!!