QFT through the lens of Tree Tensor Networks

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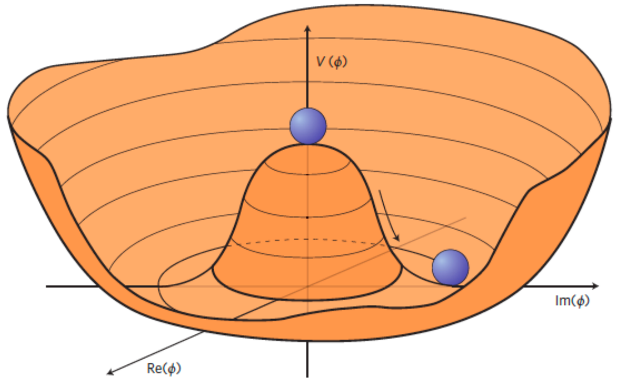
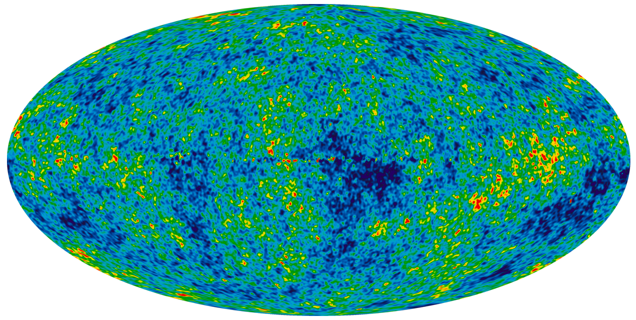
Motivation

Many processes involving nonperturbative QFT we would like to understand better …

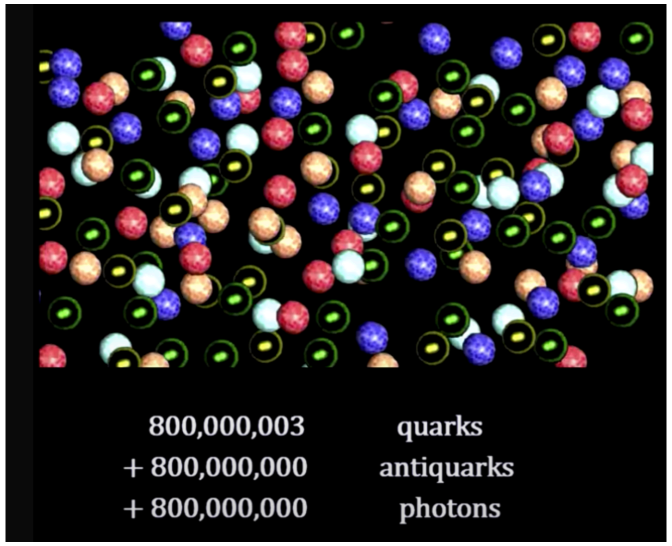
- Electroweak phase transition (Higgs mechanism)



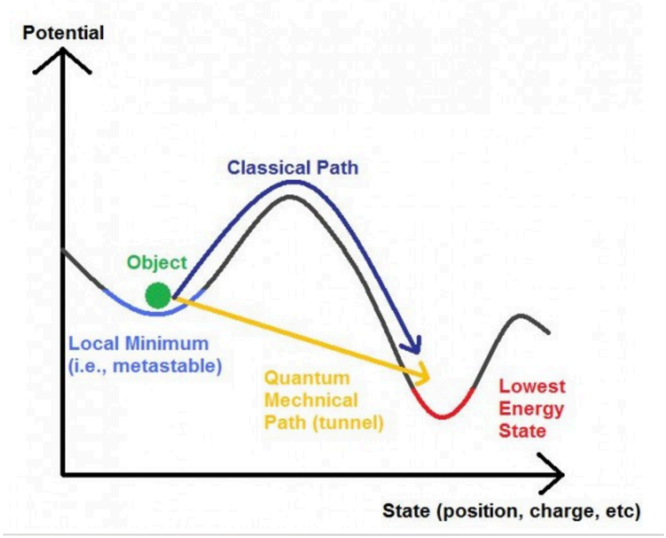
- Inflation



- Baryogenesis (creation of (anti)matter asymmetry)



- Instanton processes:



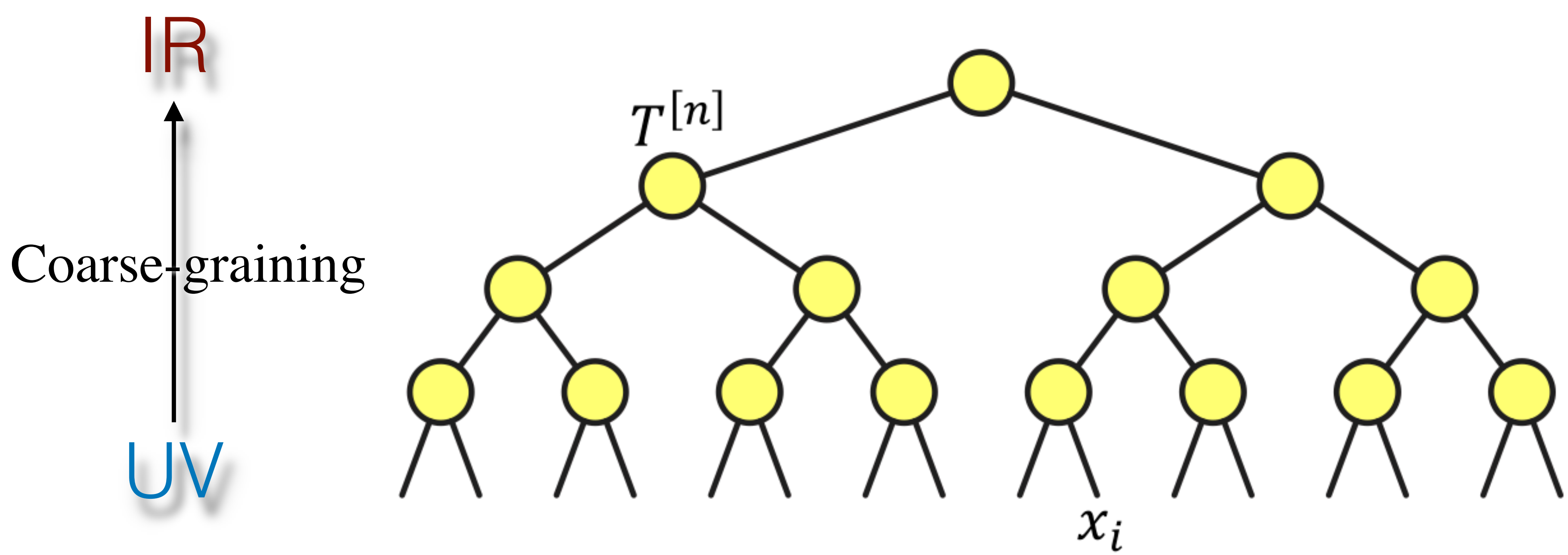
Outline

- Tree Tensor Networks and Renormalisation
- Hamiltonian simulation on TTNs
- Approach for simulating scalar QFTs
- Results

Tree tensor networks and renormalisation

In QFT renormalisation is normally expressed as a vertical arrow going from UV to IR along which degrees of freedom get integrated out.

A Tweet Tensor Network (TTN) instead encodes scale as the loss of information (i.e. entanglement entropy) in the Hilbert space during coarse-graining: **scale ↔ geometry**



White; Ostlund Rommer; Shi–Duan–Vidal, Calabrese–Cardy, Vidal, Swingle, Evenbly–Vidal; figure from Cheng, Wang, Xiang, Zhang

Let’s develop the idea:
Entanglement entropy and renormalisation

Here we are especially interested in this role of TTNs as descriptors of renormalisation. Hence … consider a universe consisting of a bipartite state of two qubits:

$$|\Psi\rangle \;=\; \sqrt{p}\,|0\rangle_A|0\rangle_B + \sqrt{1-p}\,|1\rangle_A|1\rangle_B$$

This state is entangled but *pure*. Density matrix is $\rho = |\Psi\rangle\langle\Psi|$ and hence von Neumann entropy is

$$S_{\rm global} \;=\; -{\rm Tr}\,\rho\log\rho \;=\; 0$$

True for any pure state.

Let's "renormalise" it. Suppose we consider $|i\rangle_B$ to be "redundant" since it is always the same as $|i\rangle_A$

"Integrating out" $|i\rangle_B$ by tracing over it gives effective theory in a "block-spin" basis:

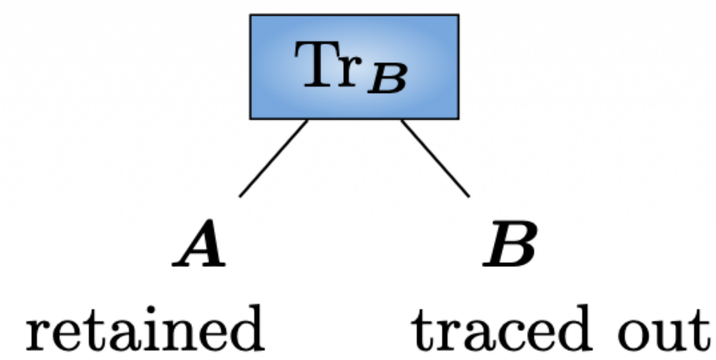
$$\rho_A = p|0\rangle_A\langle 0|_A + (1-p)|1\rangle_A\langle 1|_A$$

This state is *mixed*. Von Neumann entropy is

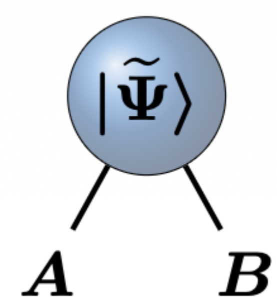
$$S_A = -p\log p - (1-p)\log(1-p) > 0.$$

$\Longrightarrow$ Entropy growth during renormalisation represents the entanglement entropy between the system and what we have integrated out.

There is a useful diagrammatic way to represent this which leads to TTNs:



In TTNs we arrange the state to be in the same ("coarse-graining ready") form but do not actually do the tracing …



But we may decide to discard the small eigenvalues of the density matrix.
e.g. if $p < 10^{-4}$ we might decide to discard it and approximate the global state as

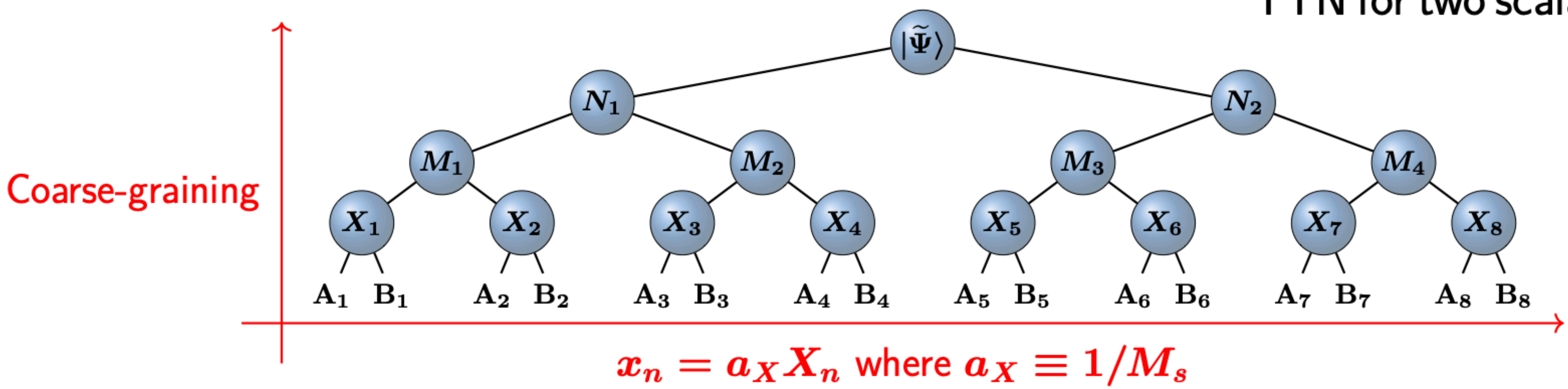
$$|\Psi\rangle \approx |\widetilde{\Psi}\rangle = |1\rangle_A |1\rangle_B$$

This is not really coarse graining, ***in the TTN the state remains pure,*** but can be thought of as ranking the important entanglement in the flow

Generalisation is now obvious

Shi–Duan–Vidal, Calabrese–Cardy, Vidal, Swingle, Evenbly–Vidal

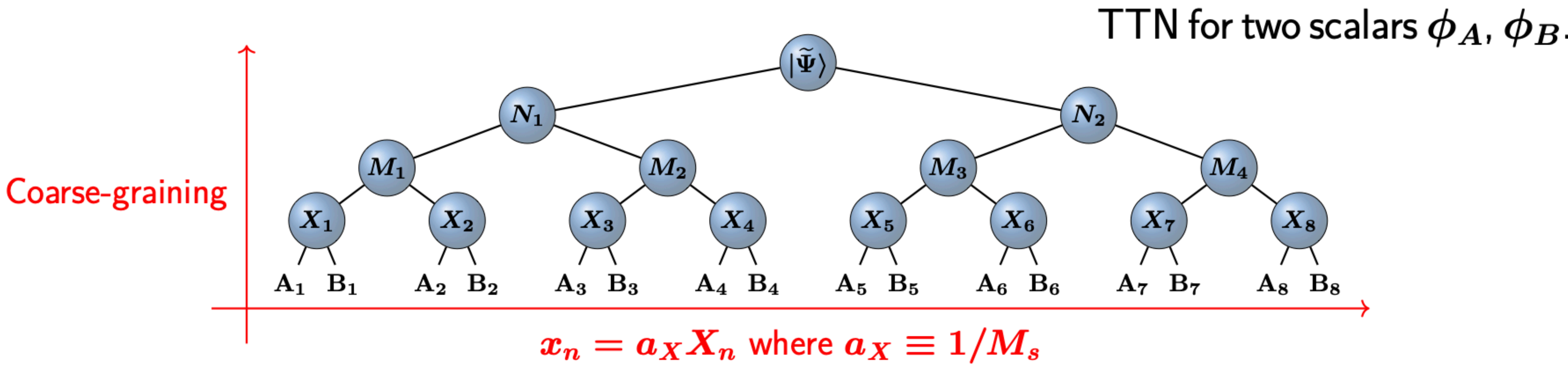
TTN for two scalars ϕ_A, ϕ_B.



Why is this efficient?

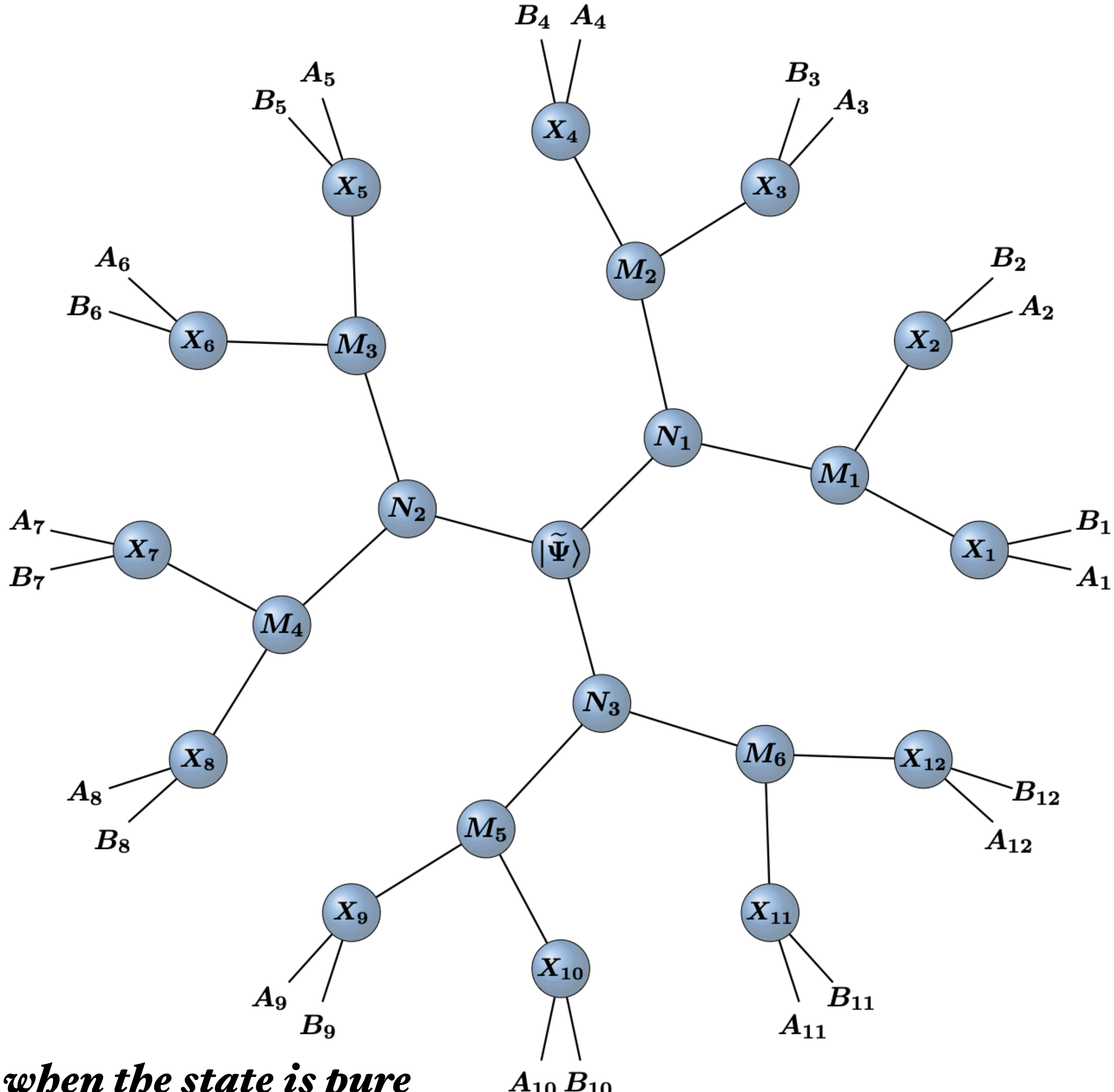
- If we do not truncate, bond dimension grows as $\chi_{\max} \sim d^{2^n} \sim d^N$ where N is the number of nodes. Local Hilbert space of dimension $d = 2$ and $N = 128$ gives 3.10^{38} !
- But the beauty of Tensor Networks is the ability to systematically truncate TTNs at each bond — crucial for making them useful.
- Keeps a pure approximation to the global state

Generalisation is now obvious



- We simply start the network off with trivial bonds and let it evolve in response to physical conditions that we impose truncating as we go — e.g. it is a QFT ground state.
- The point is the entanglement grows through the tree to encode the physics, and the loss of information as we flow to the root.

A word about holography



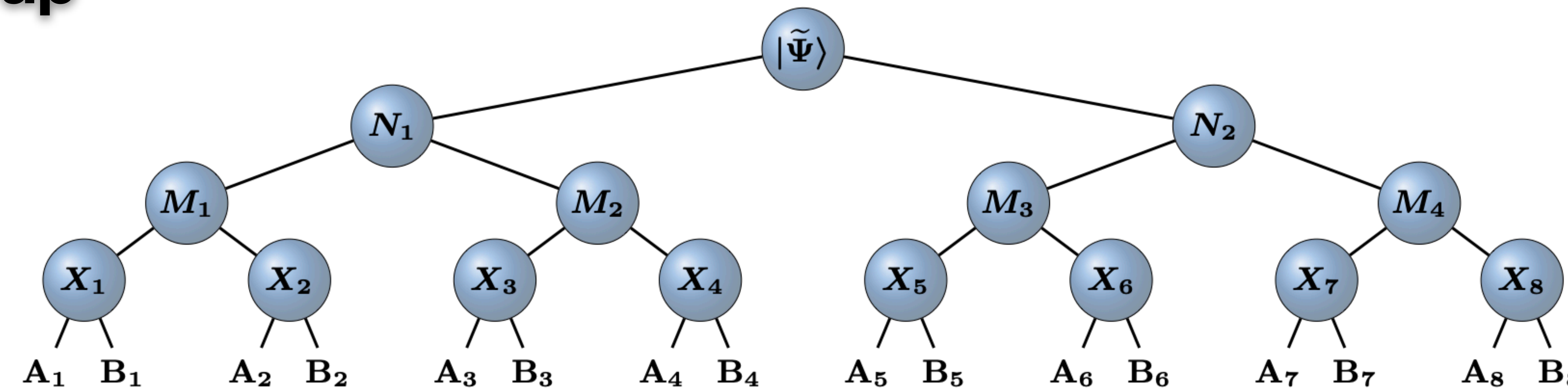
- ***This makes sense when the state is pure***
- In this set-up the root node is less special (also has three indices)
- A truly hyperbolic tiling makes it not distinguished at all (AdS/CFT)

Hamiltonian simulation of QFT

(ITensor using Julia)

Fishman, White and Stoudenmire

Set-up



1D QFT with two scalars ϕ_A and ϕ_B. The N leaf nodes $X_{n=1\ldots N}$ will be our space lattice.

System defined by its H: denoting local quadrature variables $\hat{q}_n$ and $\hat{p}_n$ we have for each of the scalars (dropping the *A* and B indices) that

$$\hat{q}_n(t) \;\leftrightarrow\; \phi(x_n,t)\;, \qquad \hat{p}_n(t) \;\leftrightarrow\; \pi(x_n,t)$$

$$H \;=\; \frac{a}{2}\sum_{n=1}^{N}\left[\hat{p}_n(t)^2 + \left(\frac{\hat{q}_{n+1}(t)-\hat{q}_n(t)}{a}\right)^2 + 2V(\hat{q}_n)\right]$$

NB: what distinguishes QFT from $\prod_{n=1}^{N}$ (local QM)$_n$ is the ***hopping term*** $-\frac{i}{a}\sum_n \hat{q}_n\hat{q}_{n+1}$

Then we will evolve as $|\Psi(t)\rangle = U(t)|\Psi(0)\rangle$ where $U(t) = e^{-iHt}$

How to define $\hat{q}_n \equiv \phi(x_n)$ and $\hat{p}_n \equiv \partial_0\phi(x_n)$??

For example **linear diagonal $\hat{q}$ basis** with

$$
\begin{aligned}
q_n^j &= -L + j \times \delta q \\
\hat{q}_n &= \mathrm{diag}\left(q_n^1 \ldots q_n^d\right) \ , \\
\hat{p}_n^2 &= -\frac{\hbar^2}{\delta q^2}\begin{pmatrix} 2 & -1 & & & \ldots \\ -1 & 2 & -1 & & \ldots \\ & -1 & 2 & & \ldots \\ \vdots & & & \ddots & -1 \\ \vdots & & & -1 & 2 \end{pmatrix}
\end{aligned}
$$

More efficient is **DVR basis** in which $\hat{q}$ is also diagonal:

$$(q_n)_{mm'} = \langle m|\hat{q}_n|m'\rangle$$

$$U^\dagger q_n U = \mathrm{diag}(\tilde{q}_n^1, \ldots, \tilde{q}_n^d)$$

"Diagonalised truncated Fock basis":
e.g. with truncation of 6

$$\hat{q}_n = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix}$$

becomes in DVR basis

$$\hat{q}_n = \frac{1}{\sqrt{\omega_0}}\,\mathrm{diag}\left(-2.35, -1.34, -0.44, 0.44, 1.34, 2.35\right)$$

Overall approach for QFT

- ***Vacuum Preparation***: imaginary Time Evolving Block Decimation (TEBD). This inflates the bonds in the tree
- ***Evolution***: Single site Time Dependent Variational Principle (TDVP)

Preparing the vacuum by cooling

TEBD: given an initial state we can perform evolution by ***Trotterising time,***

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \quad \text{where} \quad U(t) = \left[\prod_i e^{-iH_i\delta t}\right]^{t/\delta t}$$

and apply operators on the state directly and truncating the bonds as we go.

Here we use TEBD to prepare the vacuum and therefore the TTN structure with bonds.
We can do this by evolving through imaginary time $\delta t = -i\delta\tau$ so that

$$|\Psi(\tau + \delta\tau)\rangle = e^{-\delta\tau H}|\Psi(\tau)\rangle$$

and normalising the TTN every step.

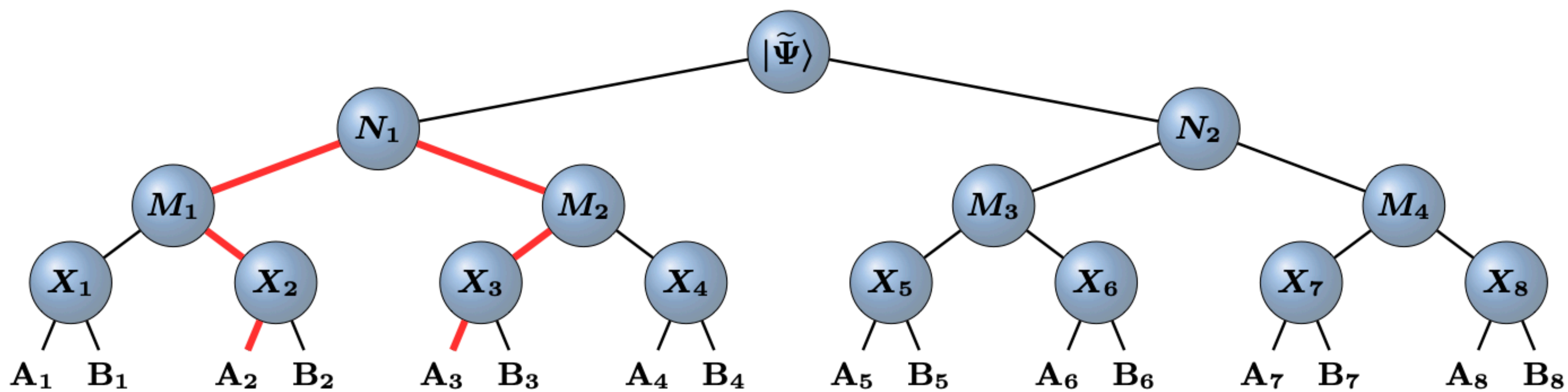
Apply onsite gates locally on the TTN leaves by simple matrix multiplication with

$$\boxed{U_{\mathrm{loc},n}(\delta t) \;=\; e^{-i\delta t\left(\frac{1}{2}\hat{p}_n^2+\hat{q}_n^2+V(\hat{q}_n)\right)}}$$

On the other hand the hopping gates are very hard to do

$$\boxed{U_{\text{hop},n}(\delta t) \;=\; e^{i\delta t\,\hat{q}_n \hat{q}_{n+1}}}$$

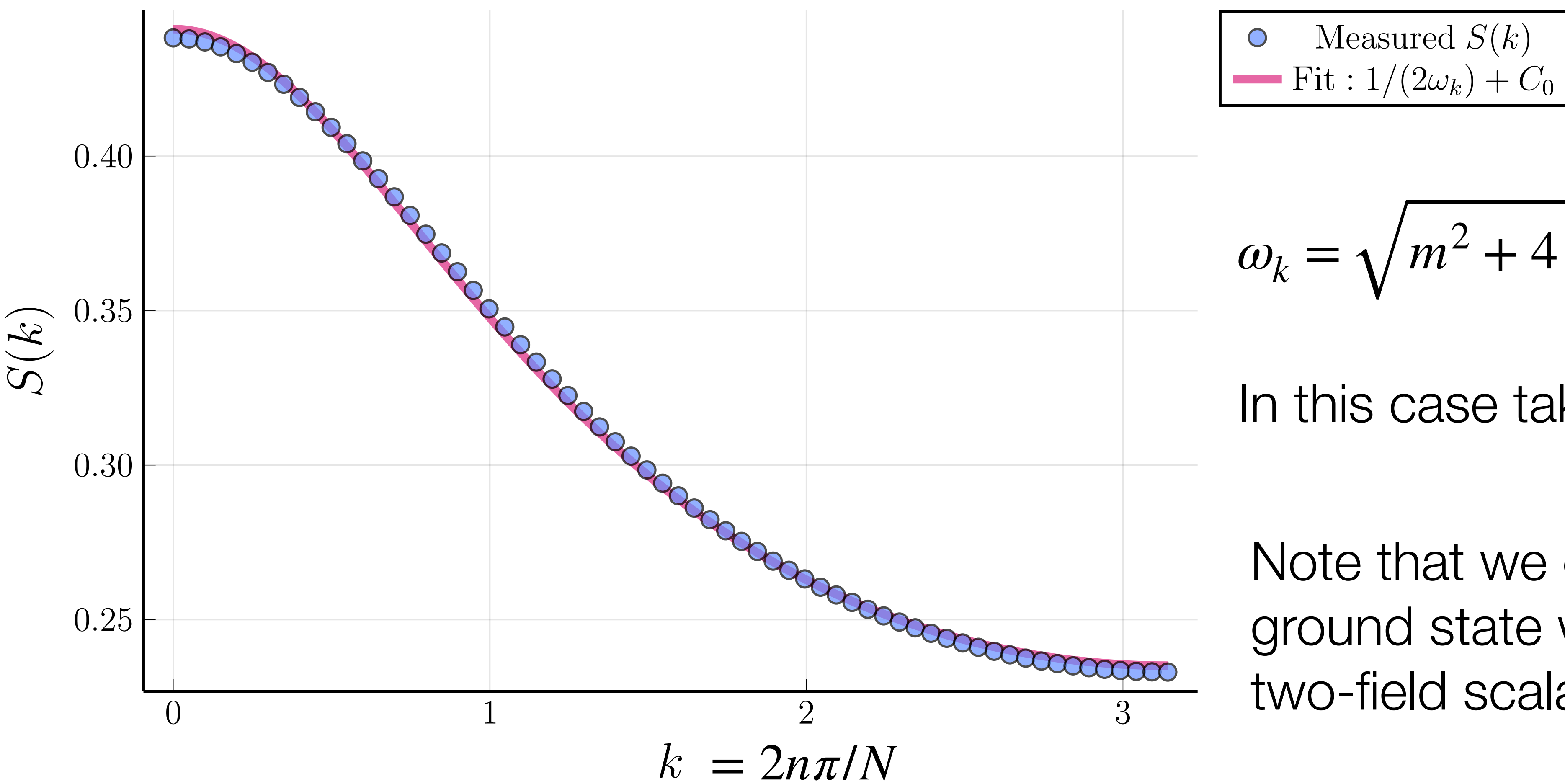
Example: $U_{\text{hop},2}$ for ϕ_A. It couples two leaf nodes on different branches …



Recipe:

1. find Lowest Common Ancestor of the two leaves (N_1 in this case)
2. find path (in red) to it from the leaves
3. construct tensor along this path (the ***path-product***)
4. use SVD to commute the leaf indices to the LCA
5. act with gate locally
6. redistribute leaf indices by SVD down arms to correct positions in path-product
7. sew the new path-product tensors back into the TTN

The state converges on the configuration with lowest H (i.e. the groundstate). We can check this by measuring the dispersion relation (c.f. fourier transform of $S_{nm} = \langle \hat{q}_n \hat{q}_m \rangle$).



$$\omega_k = \sqrt{m^2 + 4\sin^2(n\pi/N)}$$

In this case take $N = 128$

Note that we can learn the ground state with arbitrary two-field scalar potential.

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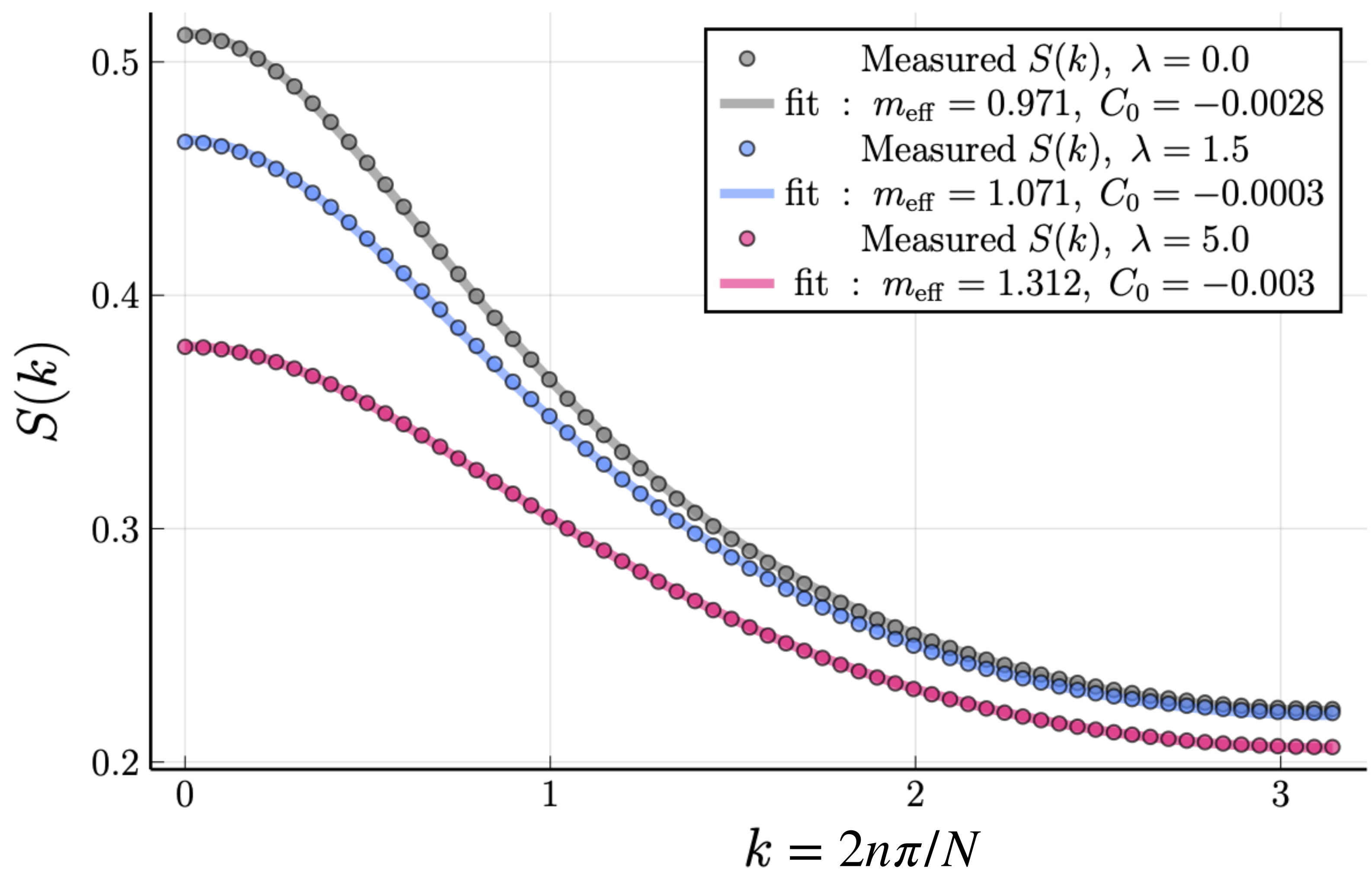
e.g. $V = \frac{\lambda}{4}\phi_A^2\phi_B^2$



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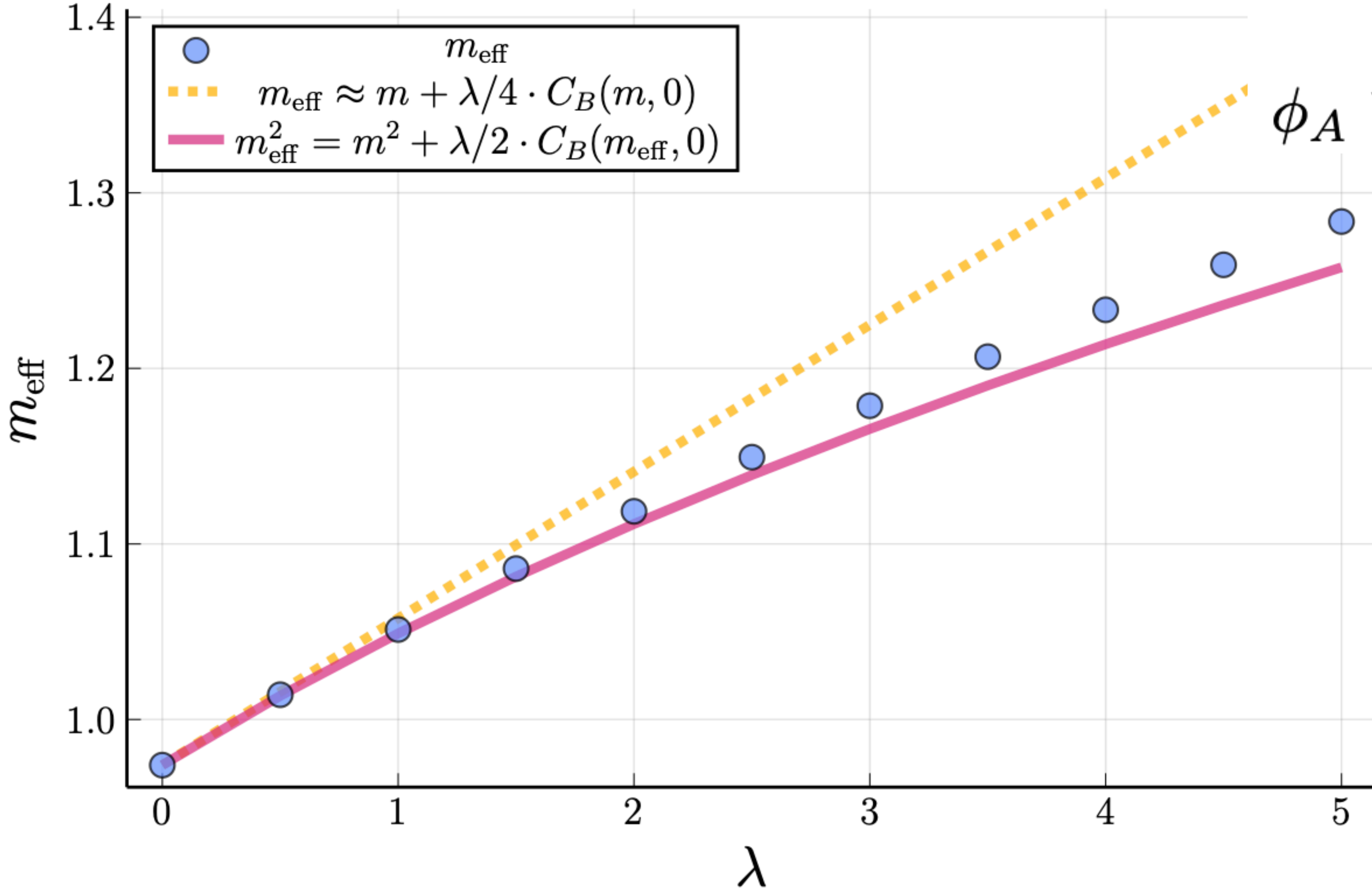
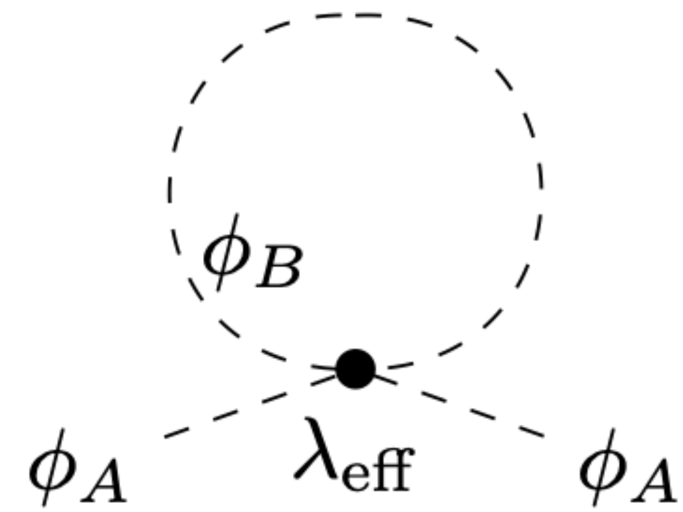
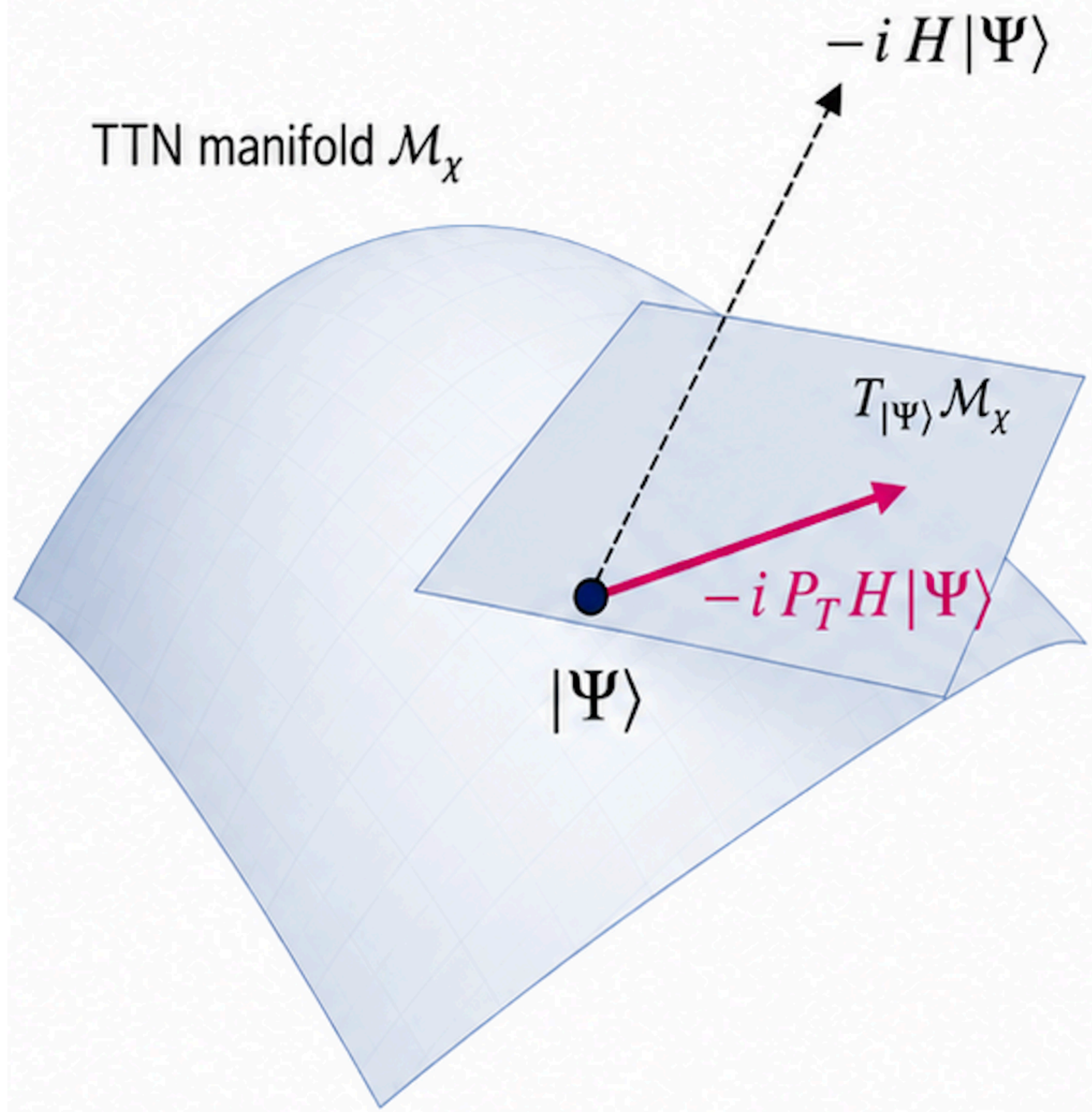




Evolution by Time-dependent variational principle (TDVP)

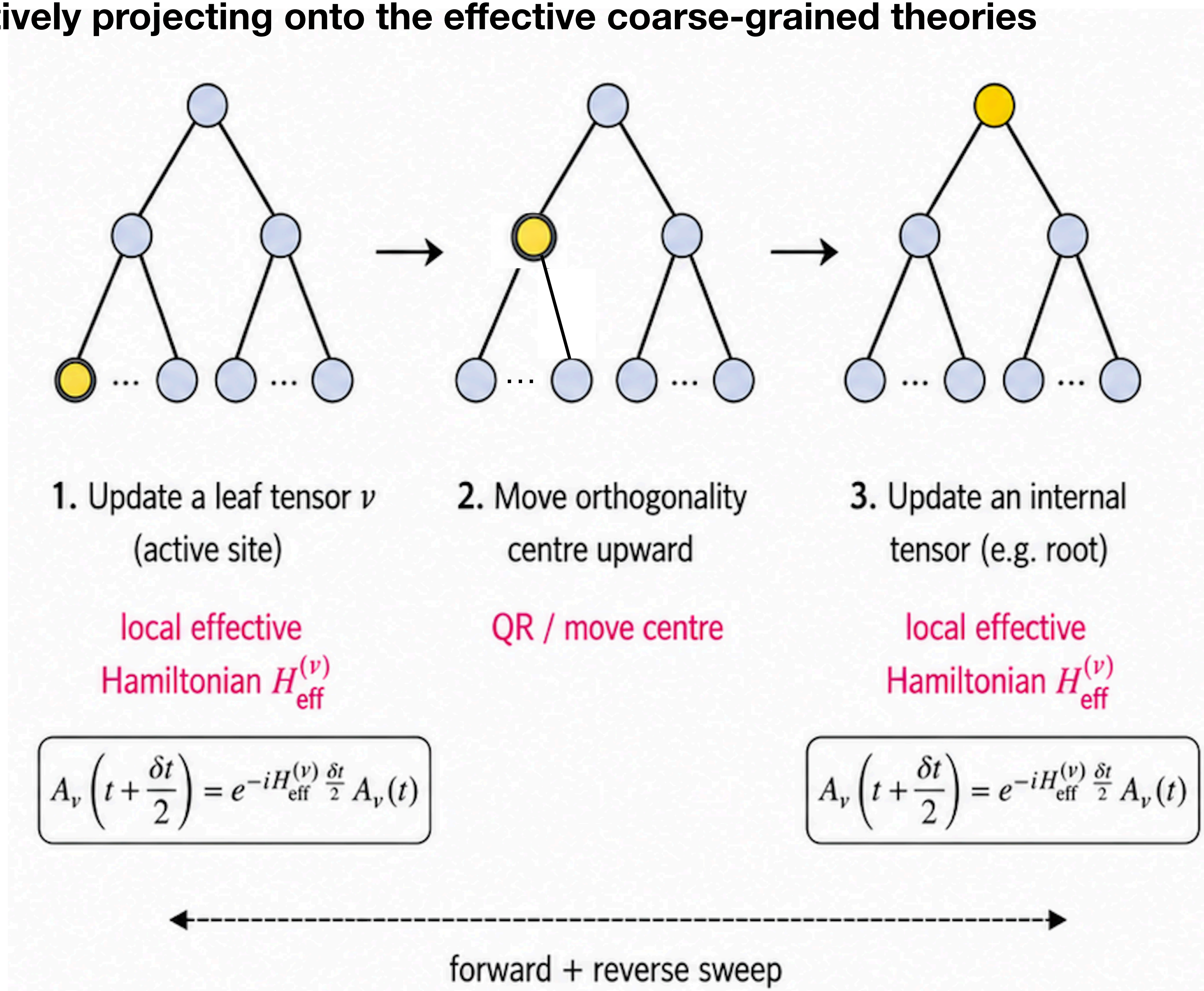
a) Geometric picture — P_T = projector to TTN



$$i\,\frac{d}{dt}\,|\Psi\rangle = P_T H\,|\Psi\rangle$$

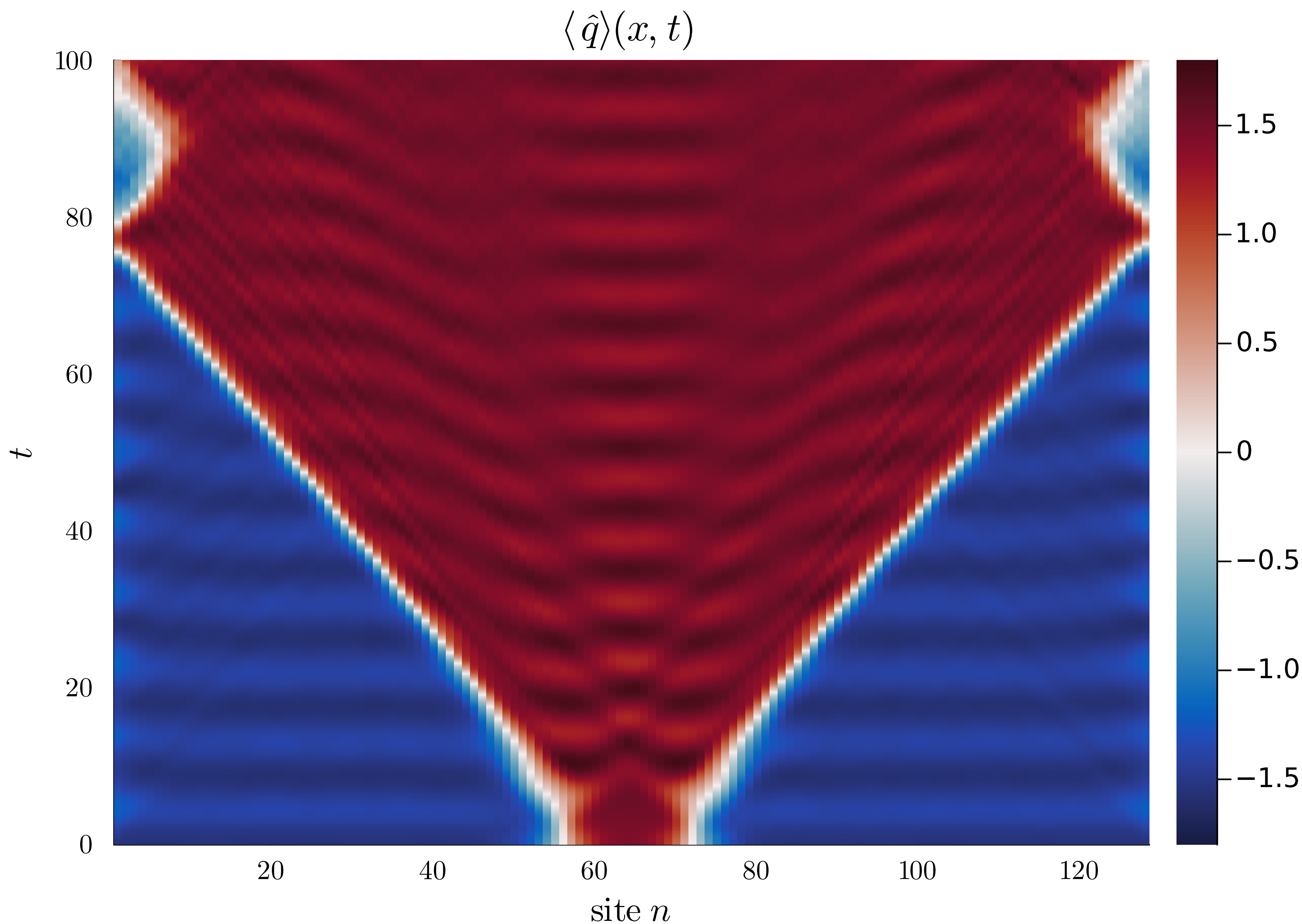
Evolution by Time-dependent variational principle (TDVP)

b) One-site TDVP sweeps of the tree —
effectively projecting onto the effective coarse-grained theories

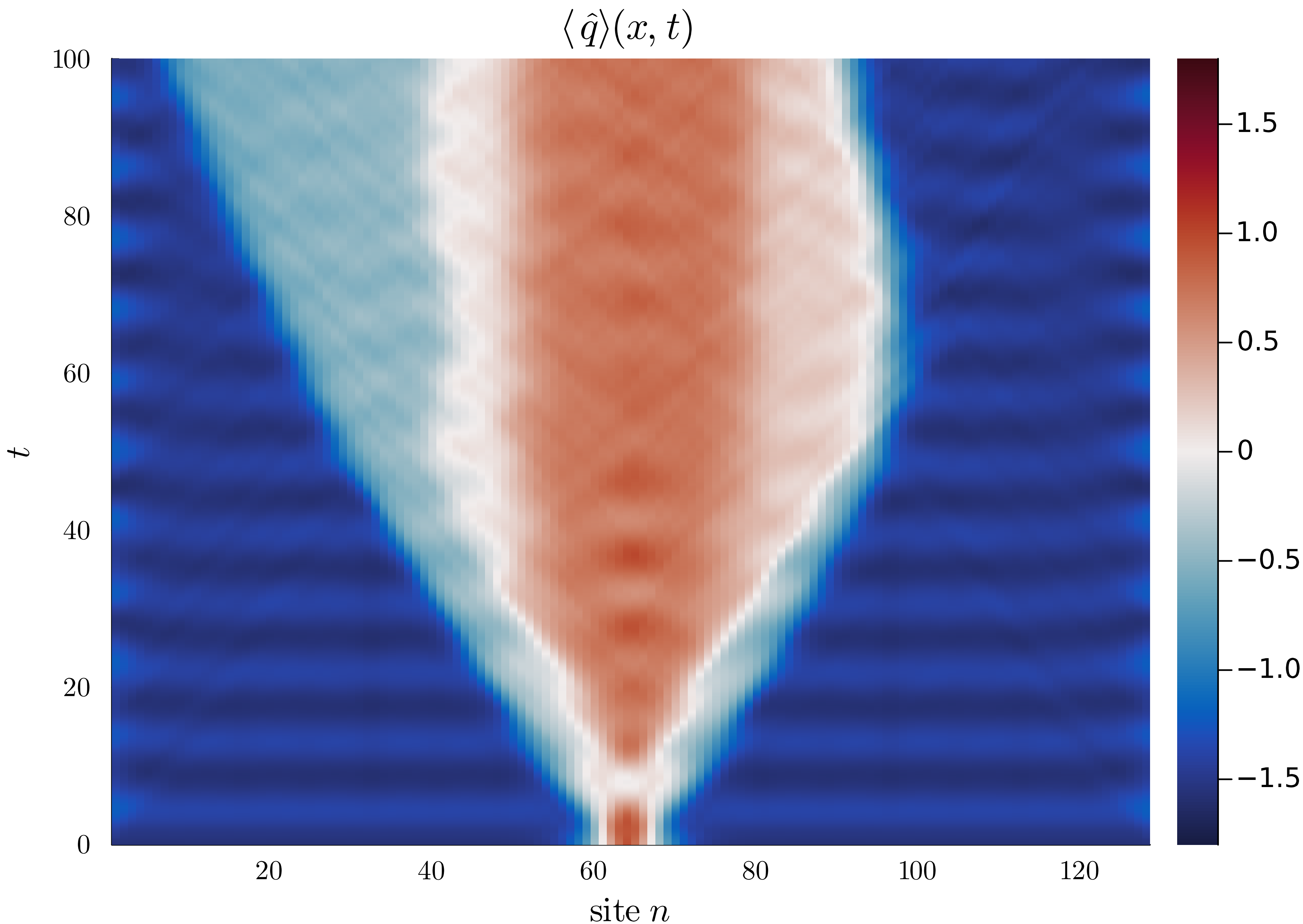


Results:

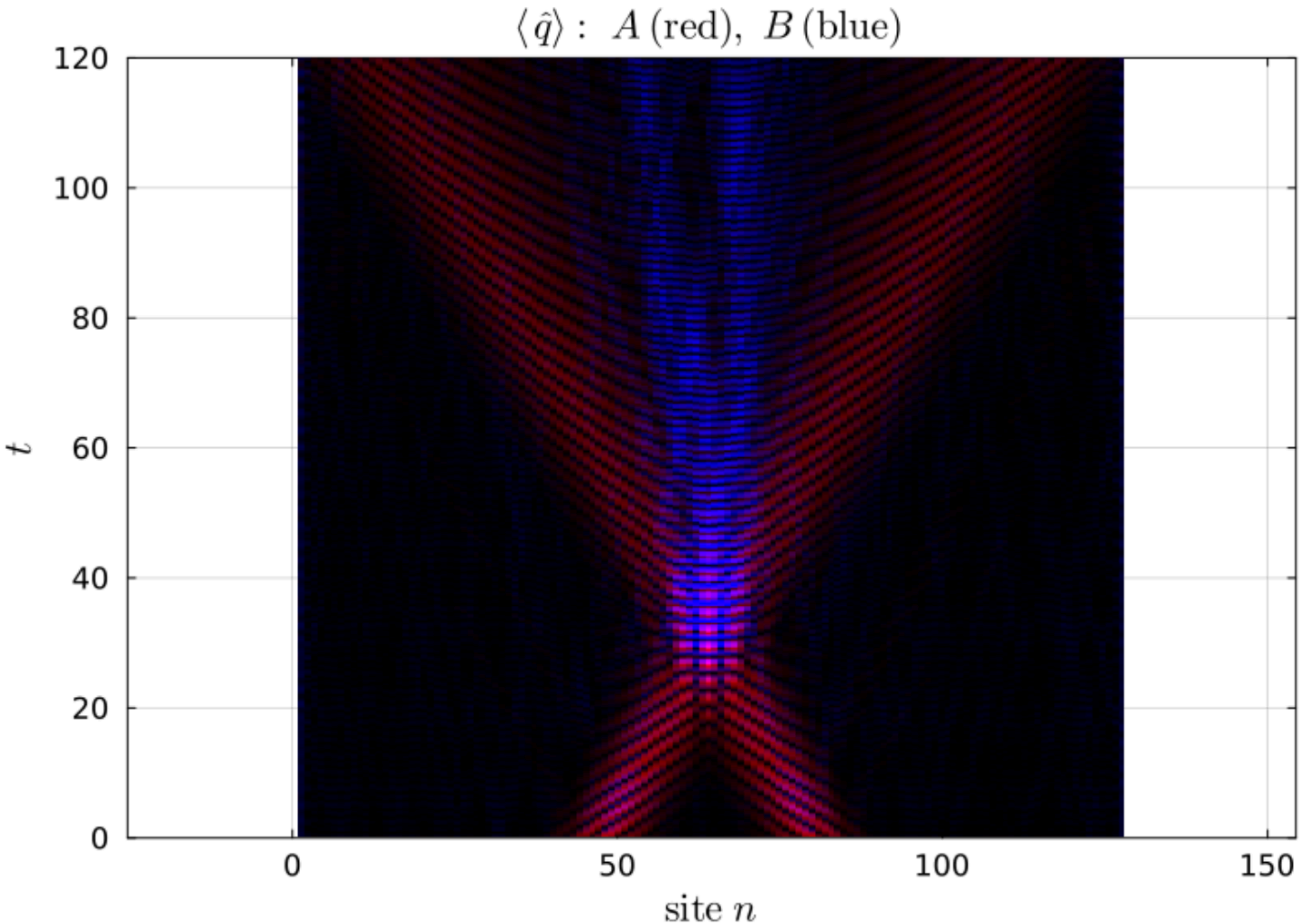
Metastable vacuum decay and bubble growth



Two-wall bound state in 1+1D (c.f. QCD string)



Scattering in $\lambda\phi_A^2\phi_B$ theory with s-channel ϕ_B particle production



Possible applications ...

- Decay of ϕ_A and friction due to ϕ_B coupling. (Terminal wall velocity)
- Important for baryogenesis. Also CP violating couplings to complex ϕ_B
- 4D Coleman instanton using spherically symmetric ansatz (1D problem)
- Tunnelling in gravitational backgrounds — e.g. black hole catalysed phase transitions
- Time dependent (cosmological) backgrounds

Summary

- TTNs are a promising platform for doing all sorts of non-perturbative physics
- Allow a geometric interpretation of renormalisation, and efficient at modelling entanglement
- Here we presented a general method for simulating coupled scalar QFT
- Many applications (suggestions welcome!)
- Code will be made available on GitHub