Gauge invariant scalable quantum simulation of QED beyond one spatial dimension

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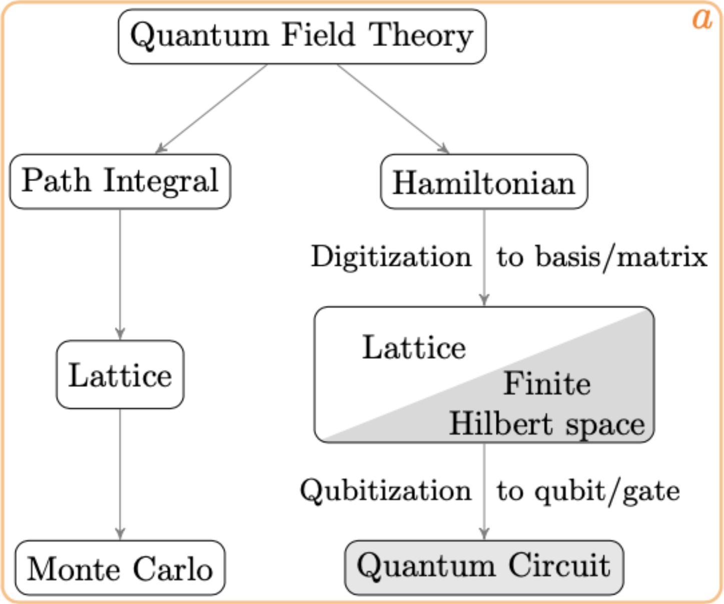
Candidates for quantum simulation of gauge theories



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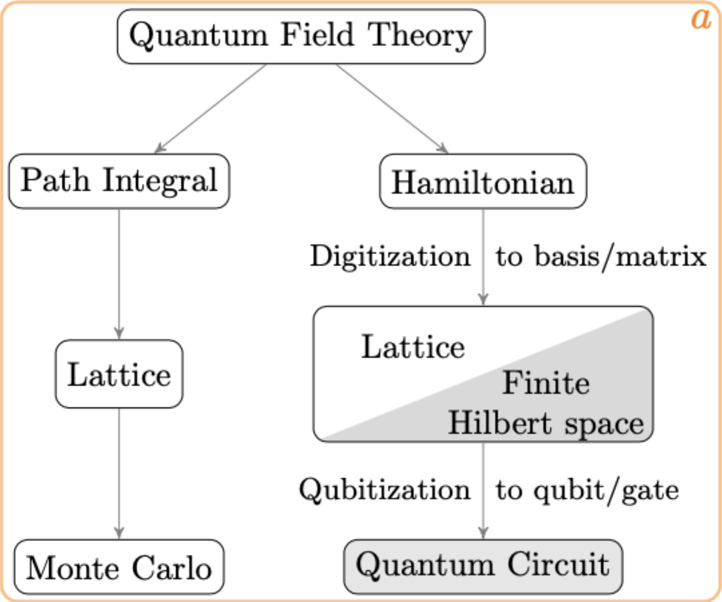

b

$\mathrm{d}x \to 0$	Continuum limit
$L \to \infty$	Infinite volume
$\mathrm{d}t \to 0$	Lie-Trotter-Suzuki limit
$N \to \infty$	Canonical Commutation Relation
	Z_N-gauge to compact QED
	Infinite energy range

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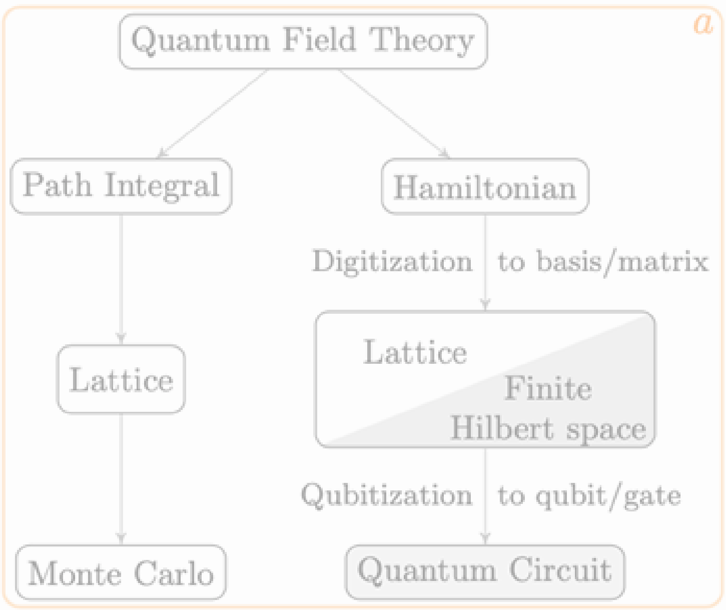

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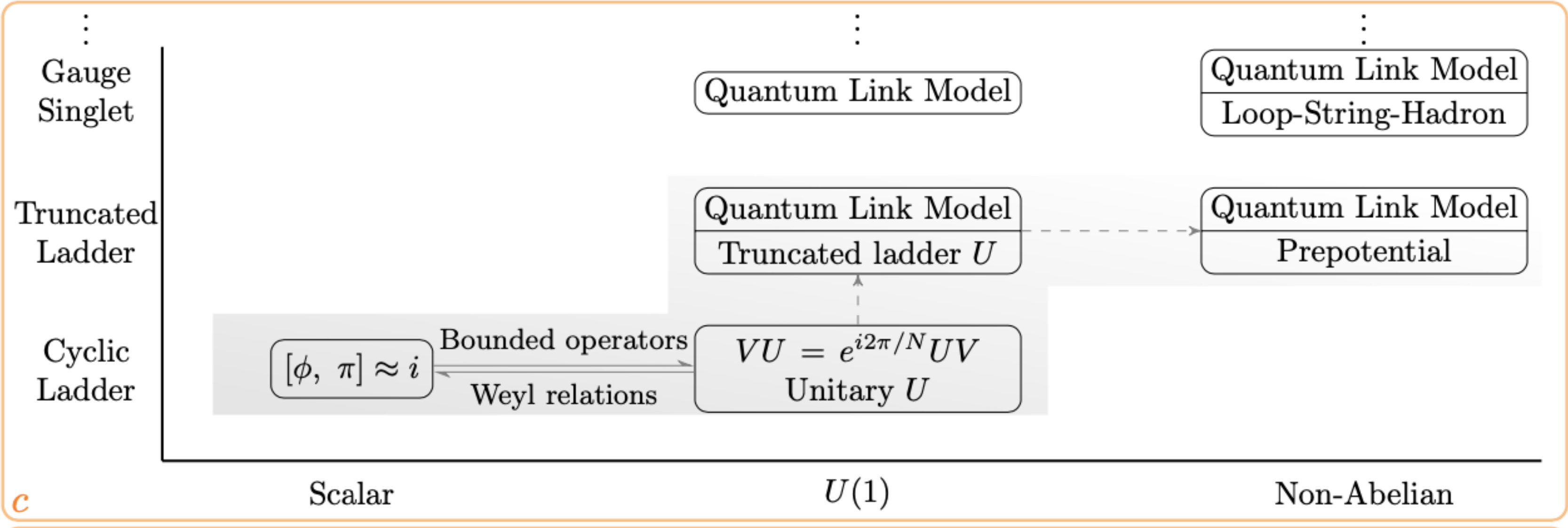
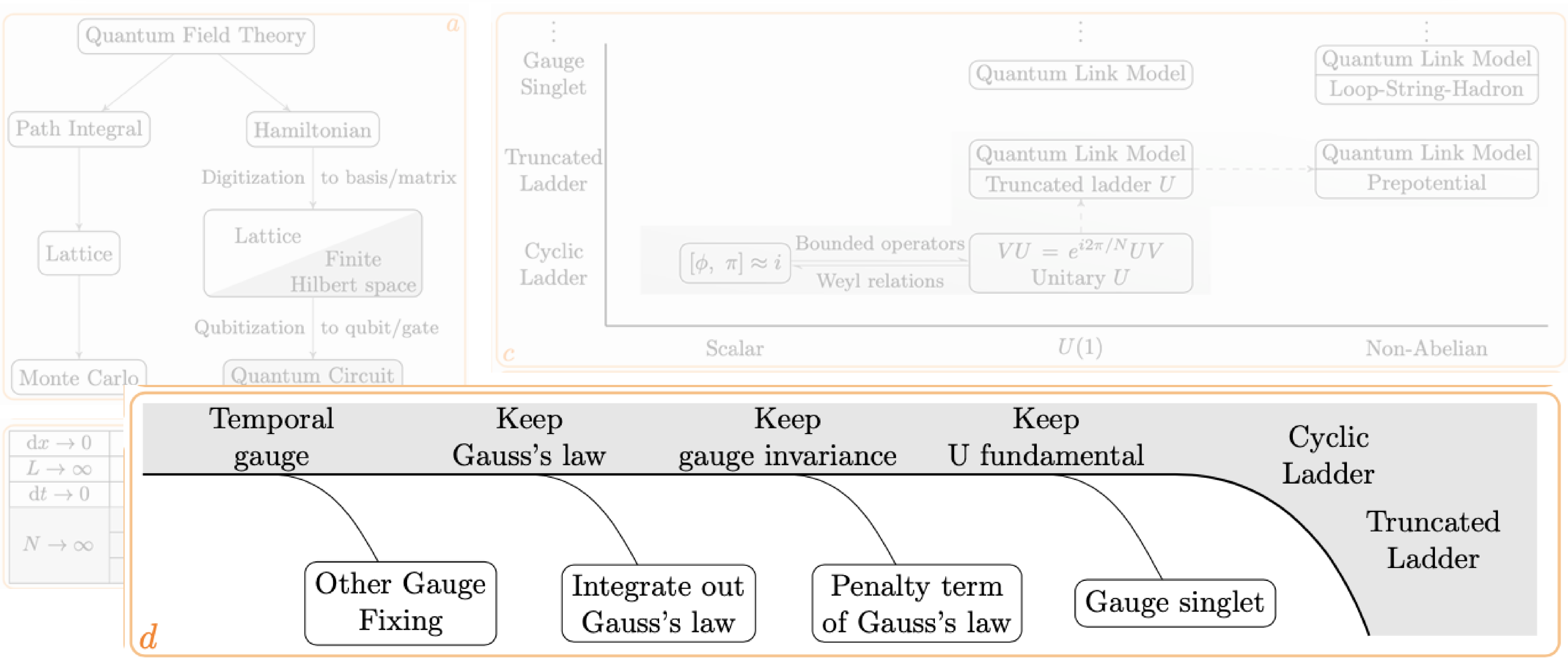


- Quantum link model or qubitisation formulations of gauge theories
- Irreducible representation or electric-field bases
- Prepotential and loop-string-hadron bases
- Group-element bases
- Dual or magnetic basis which diagonalises the magnetic Hamiltonian
- Light-cone quantisation of fields
- Matrix models using dimensional reduction

Candidates for quantum simulation of gauge theories



Our approach for QED

QED Hamiltonian in the continuum

Hamiltonian in Temporal gauge:

$$\boxed{H = \int \mathrm{d}^3x \left[\frac{1}{2}\mathbf{\Pi}^2 + \frac{1}{2}\boldsymbol{B}^2 + \bar{\psi}\gamma^i \left(\partial_i - igA_i\right)\psi + m\bar{\psi}\psi \right]}$$

Where canonical commutation relation:

$$[A_i(x), \Pi^j(y)] = i\delta_i^j \delta^3(x-y), \; \{\psi(x), \psi^\dagger(y)\} = \delta^3(x-y)$$

Electric field: $\boldsymbol{E} = -\mathbf{\Pi}$ Magnetic field: $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

Remaining constraint – Gauss's law: $\boxed{\nabla \cdot \mathbf{\Pi} + g\psi^\dagger\psi = 0}$

Lattice discretization (staggered)

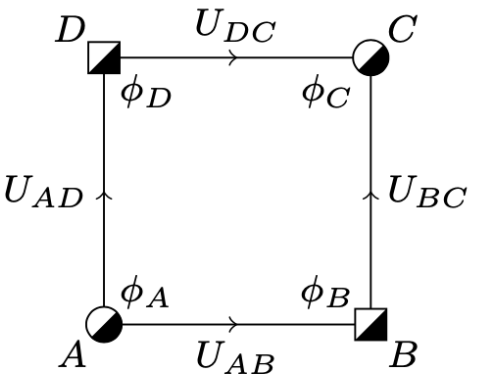
$$H_{KS} = \mathrm{d}^3x \sum_x \left[\frac{1}{2}\Pi^2 + \frac{2 - P - P^\dagger}{4g^2(\mathrm{d}x)^4} + i \sum_j \eta^j(x) \frac{\phi^\dagger(x)U_j(x)\phi(x+j) - \phi^\dagger(x+j)U_j^\dagger(x)\phi(x)}{2\mathrm{d}x_j} + m(-)^n \phi^\dagger(x)\phi(x) \right]$$

[Kogut and Susskind, 1975]

$$P_{ij}(x) = U_i(x)U_j(x+i)U_i^\dagger(x+j)U_j^\dagger(x)$$

$$U_i(x) \sim \exp\left(-igA_i(x)\mathrm{d}x\right)$$

$$\mathrm{d}x \to 0.$$



Non-trivial commutation relations in discrete form:

$$[\Pi_i, U_j] = -\delta_{ij}U_j, \quad [\Pi_i, U_j^\dagger] = \delta_{ij}U_j^\dagger,$$
$$\{\phi_i, \phi_j^\dagger\} = \delta_{ij}$$

Gauss's law on a single site: $G(x) \equiv \sum_i \frac{\Pi_i(x) - \Pi_i(x-i)}{\mathrm{d}x} + g\phi^\dagger(x)(x) = 0$

Discretization of the representation

Infinite dimensional operator algebra:

$$[\Pi,\ U] = -U, \quad [\Pi,\ U^\dagger] = U^\dagger, \quad [U,\ U^\dagger] = 0.$$

$$\Pi = \begin{pmatrix} \ddots & & & \\ & \text{-1} & & \\ & & 0 & \\ & & & 1 \\ & & & & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} \ddots & \ddots & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}$$

Ladder operators in Electric basis

Hard truncation violates Unitarity of the gauge fields:

$$[\Pi,\ U'] = -U', \quad [\Pi,\ U'^\dagger] = U'^\dagger, \quad [U',\ U'^\dagger] \approx 0,$$

$$\Pi = \begin{pmatrix} \text{-2} & & & \\ & \text{-1} & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad U' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Our approach

Restoring the unitarity will violate the commutation relation between the electric field and the gauge-link field $[\Pi,\ U] \approx -U, \quad [\Pi,\ U^\dagger] \approx U^\dagger, \quad [U,\ U^\dagger] = 0,$

Cyclic ladder

e.g.

$$, \ \Pi = \begin{pmatrix} \text{-}2 & & & \\ & \text{-}1 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We rephrase the unbounded operators into bounded ones

$$[\Pi, U] \approx -U \ \Leftrightarrow e^{-i\alpha\Pi} U e^{i\alpha\Pi} = e^{-i\alpha} U$$

U(1) gauge theory is reduced to Z_N theory in the unitary matrix approach and will Be recovered in the large-N limit (*Phys. Rev. D 19 (1979) 3715.*)

Unitary matrix and Fourier Transform

	Canonical		Gauge-Link		Weyl
Infinite	$[A,\Pi]=i$ Hermitian A Hermitian Π	$\Leftrightarrow$	$[\Pi,U]=-U$ unitary U Hermitian Π	$\Leftrightarrow$	$V^\alpha U^\beta = e^{i\alpha\beta}U^\beta V^\alpha$ unitary U unitary V
Finite			$[\Pi,U]=-U$ truncated U		
	$[A,\Pi]\approx i$	$\Leftrightarrow$	$[\Pi,U]\approx -U$	$\Leftrightarrow$	$V^k U^m = e^{i\frac{2km\pi}{N}}U^m V^k$

Note:
The unitary representation of the gauge-link field coincides with both the standard digitisation of the scalar field (*Science 336 (2012) 1130–1133*) and the finite representation of Weyl commutation relation (*Weyl, 1950; Found Phys 6 (1976) 583–587*).

A finite dimensional representation exists via Sylvester's generalization of Pauli matrices with the shift matrix and the clock matrix, we can choose:

$$U = -F^\dagger \exp\left(-i\frac{2\pi}{N}\Pi\right)F, \quad V = \exp\left(-i\frac{2\pi}{N}\Pi\right)$$

$$V^k U^m = U^m V^k e^{i2mk\pi/N}.$$

For example, for N = 4, we get

$$\Pi = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$F = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix}, \quad \omega = e^{i2\pi/4}.$$

The advantages of our formulation:

- Weyl commutation relation ⟶ smooth N → ∞ limit ⟶ recover the canonical commutation relation ⟶ restore the U(1) gauge theory.

- The Fourier transform matrix F allows us to switch the eigen-basis between the electric and gauge potential fields ⟶ crucial when we construct the quantum circuit.

Qubitization

Gauge field: n-qubit , N = 2^n dimensional Hilbert space
Fermionic field: 1-qubit (Jordan-Wigner mapping)

$$H = \mathrm{d}^3x \sum_x \left[\mathcal{H}_\Pi(x) + \mathcal{H}_P(x) + \mathcal{H}_I(x) + \mathcal{H}_Q(x)\right]$$

$$e^{-i\mathrm{d}tH} \approx \prod e^{-i\mathrm{d}t\mathrm{d}^3x\mathcal{H}_\Pi} e^{-i\mathrm{d}t\mathrm{d}^3x\mathcal{H}_P} e^{-i\mathrm{d}t\mathrm{d}^3x\mathcal{H}_I} e^{-i\mathrm{d}t\mathrm{d}^3x\mathcal{H}_Q}$$

$$\begin{aligned} e^{-i\mathrm{d}tH} \approx & \exp\left(-i\frac{\theta_Q}{2}\left[\phi_A^\dagger \phi_A\right]\right)\cdots \\ & \exp\left(-i\frac{\theta_\Pi}{2}\left[\Pi_A^2\right]\right)\cdots \\ & \boxed{\exp\left(-i\frac{\theta_I}{2}\left[i\phi_A^\dagger U_{AB}\phi_B + h.c\right]\right)}\cdots \\ & \boxed{\exp\left(-i\frac{\theta_P}{2}\left[P_{ABCD} + h.c\right]\right)}\cdots, \end{aligned}$$

Each elementary term should be implemented exactly to keep the gauge invariance.

Unitary matrix to unitary gates: decomposition

For a matrix M of a large dimension, the precise qubit implementation of the rotation will rely on the eigen-decomposition,

$$e^{-i\theta f(M,M^\dagger)} = \mathscr{U}^\dagger e^{-i\theta f(\Lambda,\Lambda^\dagger)} \mathscr{U}, \quad M = \mathscr{U}^\dagger \Lambda \mathscr{U}$$

In special cases we can also use singular value decomposition: $M = \mathscr{V} S \mathscr{W}^\dagger$

$$\begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \mathscr{V} & 0 \\ 0 & \mathscr{W} \end{pmatrix} \mathbf{H} \begin{pmatrix} S & \\ & \text{-}S \end{pmatrix} \mathbf{H}^\dagger \begin{pmatrix} \mathscr{V}^\dagger & 0 \\ 0 & \mathscr{W}^\dagger \end{pmatrix}$$

Gauge-fermion interaction

The plaquette term: we split a ladder matrix into two nilpotent matrices where $N^2 = 0$

$$\begin{pmatrix} & N+N^\dagger \\ N+N^\dagger & \end{pmatrix} = \begin{pmatrix} & N \\ N^\dagger & \end{pmatrix} \begin{pmatrix} & N^\dagger \\ N & \end{pmatrix}$$

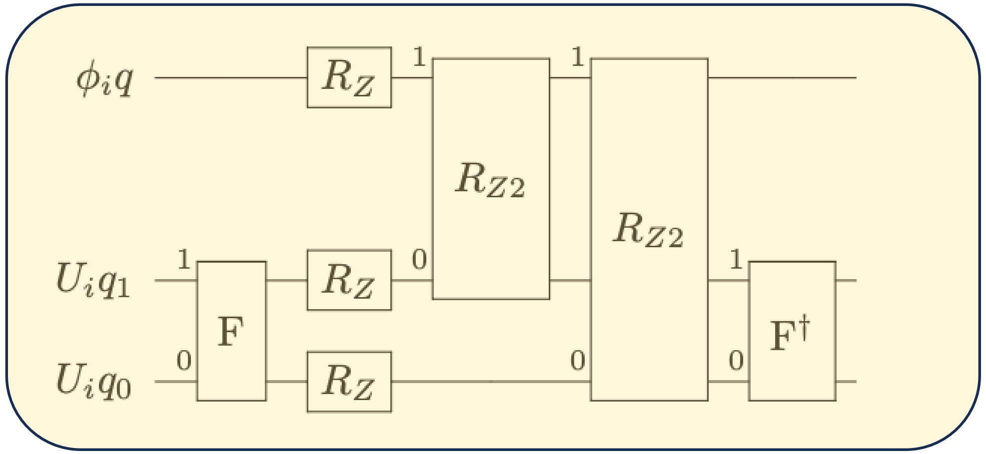
Therefore, it can be shown that we can introduce an ancilla to perform the plaquette interaction.

QED in qubit: gauge-fermion interaction

Qubitization of fermionic fields:

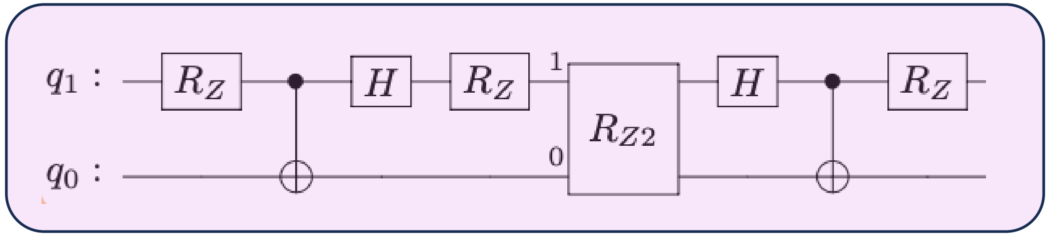
A typical gauge-fermion interaction term can be split into three two-field-operations through

$$\exp\left(-i\frac{\theta}{2}\left[i\phi_P^\dagger U_{PQ}\phi_Q - i\phi_Q^\dagger U_{PQ}^\dagger \phi_P\right]\right)$$
$$= R_P \exp\left(-i\frac{\theta}{2}\left[i\phi_P^\dagger \phi_Q - i\phi_Q^\dagger \phi_P\right]\right) R_P^\dagger,$$



with the unitary transformation

$$R_P = \begin{pmatrix} U_{PQ} & \\ & I \end{pmatrix} = \exp\left(\phi_P^\dagger \phi_P \ln(U_{PQ})\right)$$



QED in qubit: plaquette term

Qubitization of gauge fields:

Via the Fourier transform, we can simply switch between the electric basis and the potential basis

$$\Pi = -\frac{1}{2}\sum_{j=0}^{n-1} 2^j Z_j - \frac{1}{2}, \quad \text{with } Z_j \equiv \underset{\substack{\uparrow \\ (n-1)\text{-th}}}{I} \otimes \cdots \overset{\substack{j\text{-th from right} \\ \downarrow}}{Z} \cdots \otimes \underset{\substack{\uparrow \\ 0\text{-th}}}{I} \qquad \Bigg| \qquad U = F^\dagger \exp\left(-i\frac{2\pi}{N}\left(\Pi + \frac{N}{2}\right)\right) F.$$

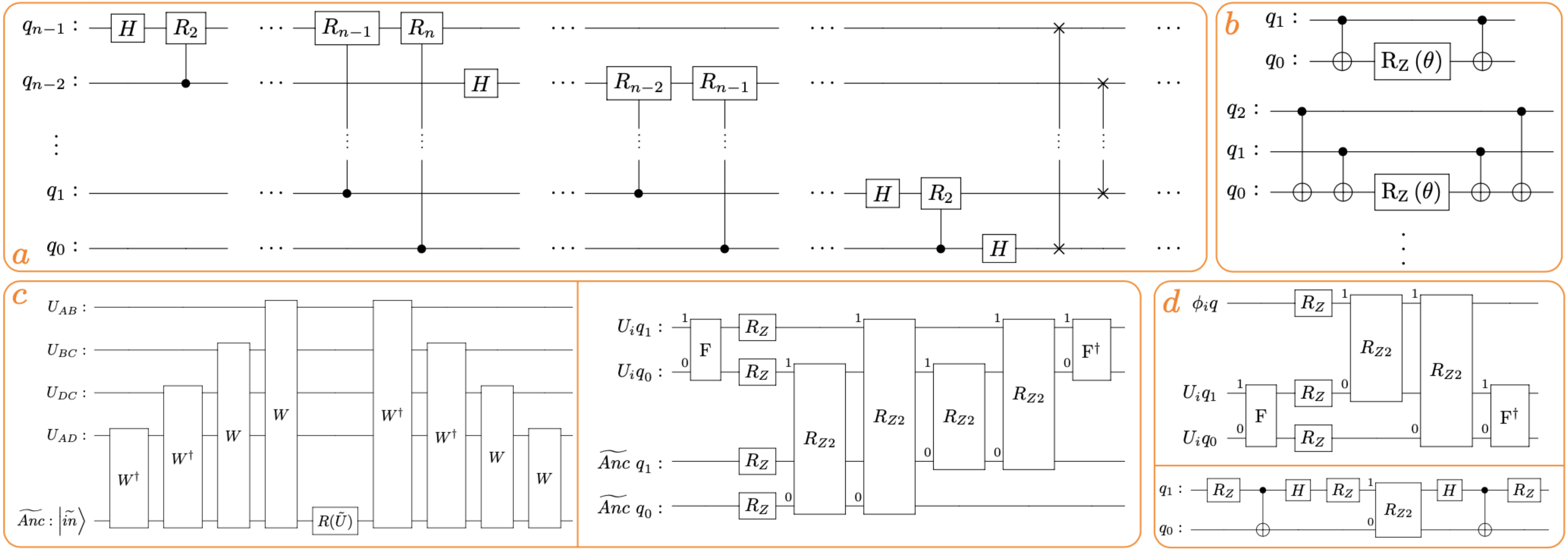
- In the potential basis, the U- relevant terms is in a diagonal shape, and the relevant Trotterization consists of only rotations of Z's, so can be implemented precisely.

Thus, we rely on two basic operations – Quantum Fourier Transform, and multi-Z rotations

"Stator" method can reduce the number of CX gates with the use of ancillary qubits.

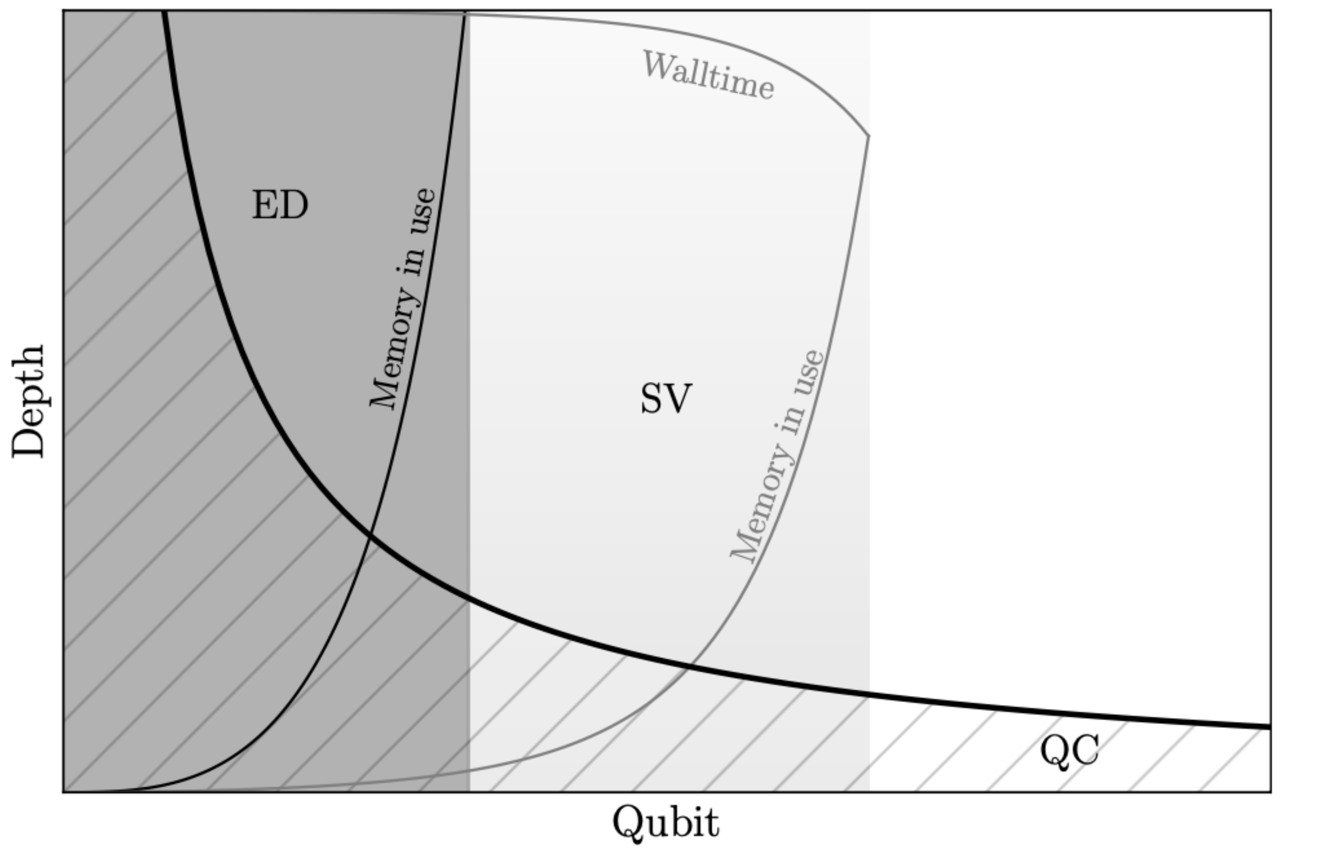
(*Phys. Rev. A 65 no. 3, (2002); Phys. Rev. A 95 no. 2, (2017) ; Phys. Rev. Lett. 118 no. 7, (2017) 070501*)

An illustration of quantum circuits used



(a) Quantum Fourier Transform
(b) Multi-Z rotations.
(c) The plaquette rotation in the "stator" method:
(L) the four links and the ancilla; (R) the W-operation illustrated with 2-qubit for the gauge degree.
(d) The gauge-fermion rotation:
(Upper) the unitary transformation which decouples the gauge degree; (Lower) the two-fermion rotation.

Computational resources

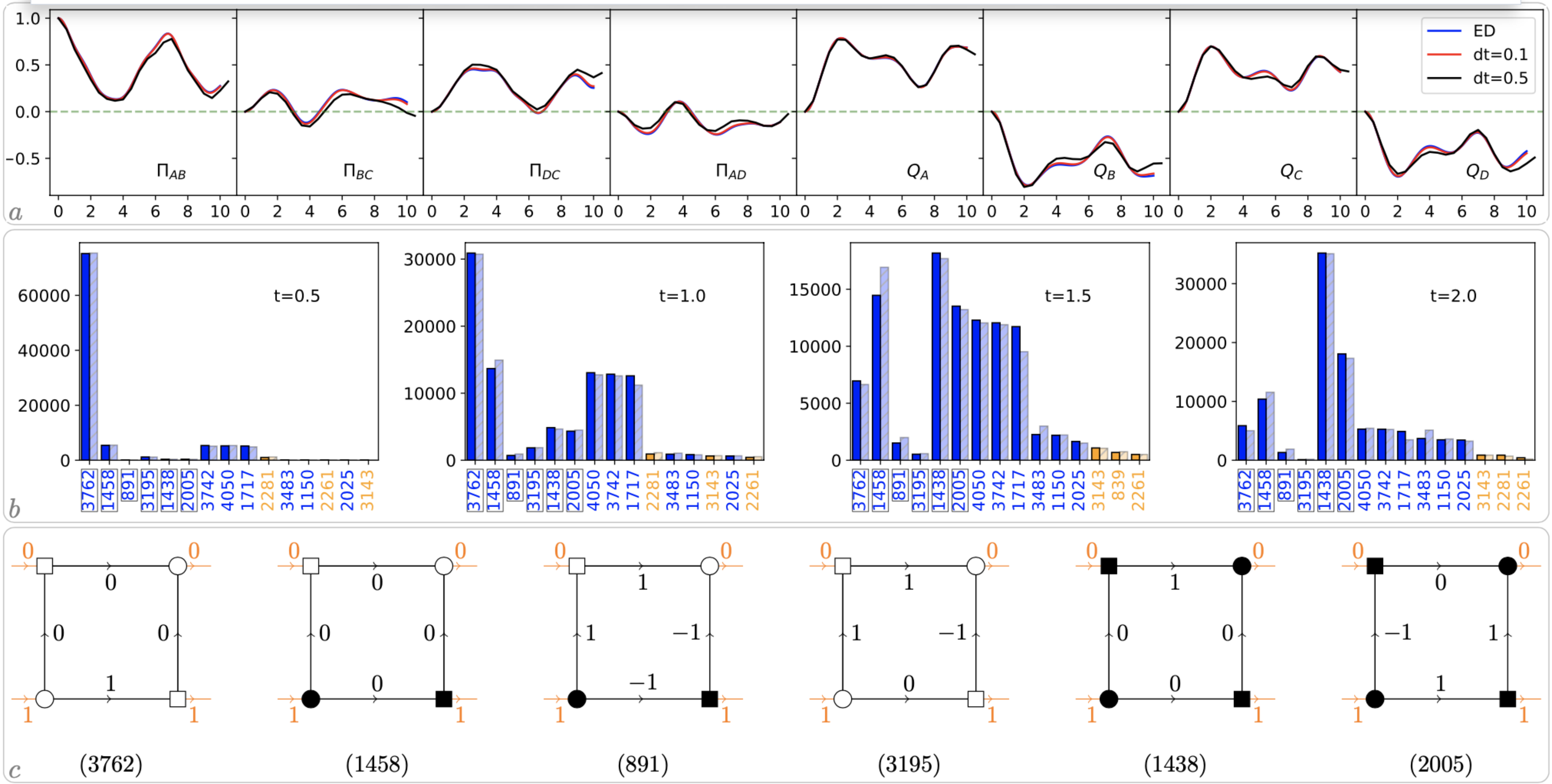


Resources for one Trotter step

	2D			3D		
	n=1	n=2	n=3	n=1	n=2	n=3
Qubit:	9	14	19	22	36	50
CZ:	75	244	734	598	1644	4481
Depth:	172	399	894	740	1776	3454

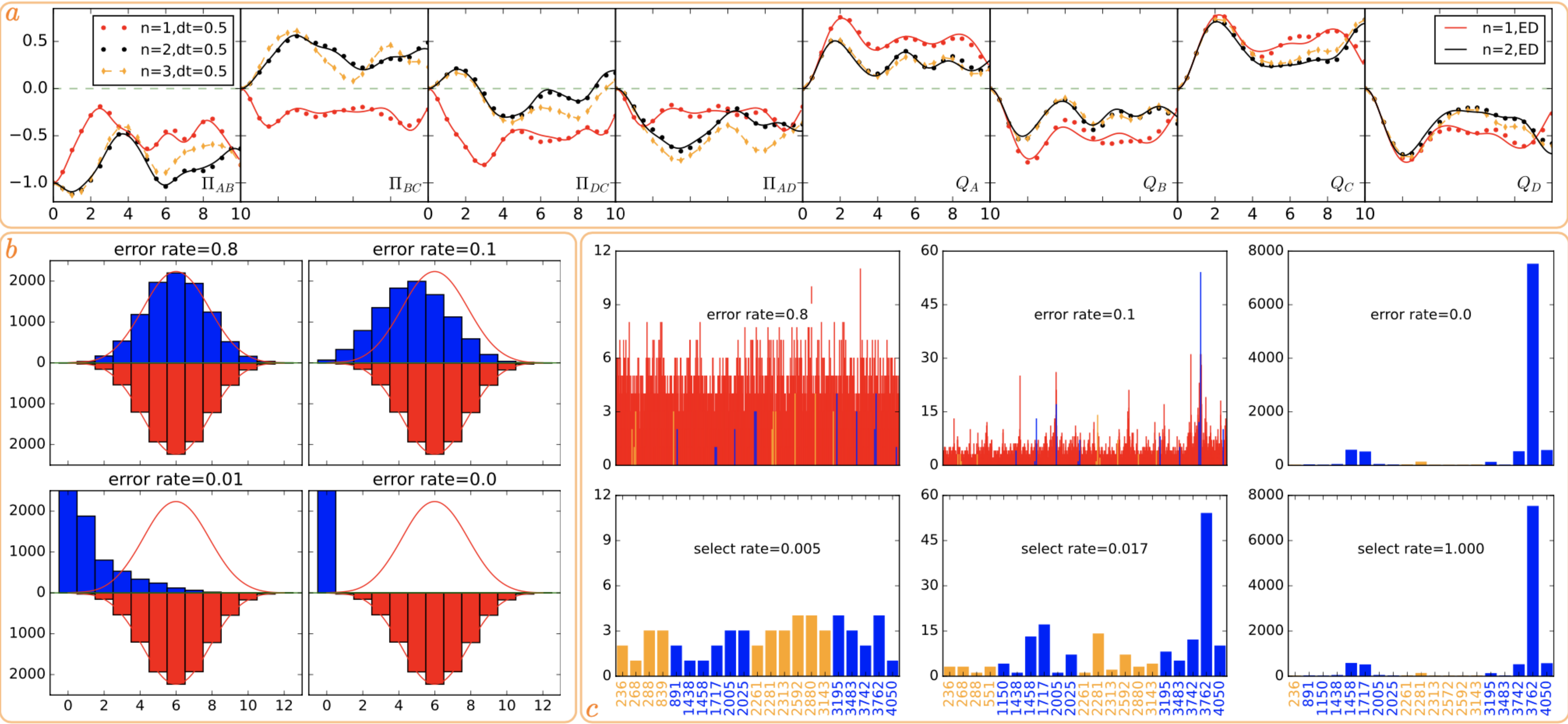
Exact Diagonalization + Qiskit + IBM
Qiskit + IBM
IBM

ED & QISKIT Simulation and results



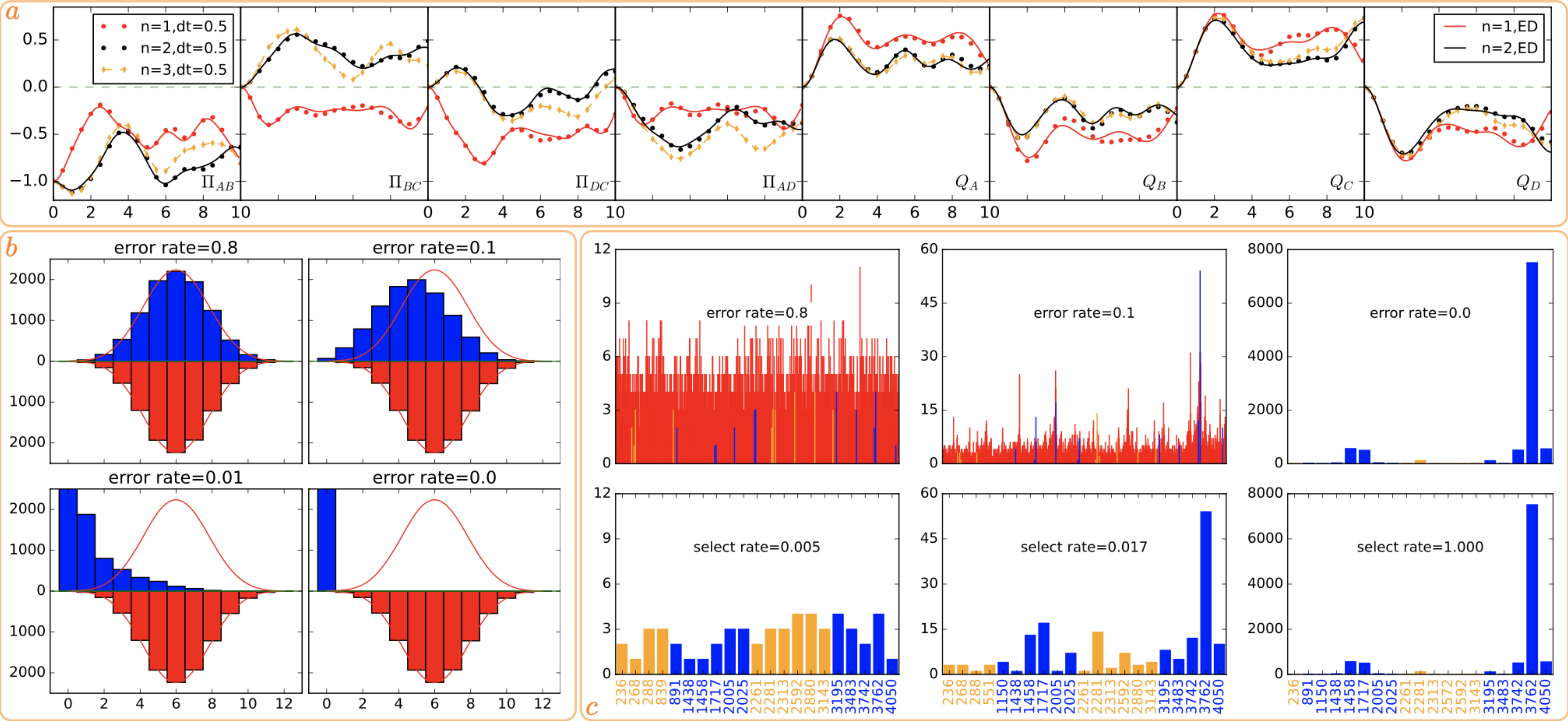
(a) The evolution of the observables over time. Results from ED and Qiskit simulation with dt = 0.1 and 0.5 are plotted in blue, red and black individually, (b) The histogram of the states over time, (c) An illustration of the dominating states.

QISKIT Simulation and results (2D square)



(a) The observables, including electric fields and fermion charges, evolve along time on the 2D square, (b) The distance (or the bit-flip number) plot with varying quantum error rate in the reference/trivial runs, simulated via Qiskit. For a contrast, the worst case, where the measurement is totally random, is plotted in red. (c) The histogram of the final states in the target runs.

QISKIT Simulation and results (2D square)

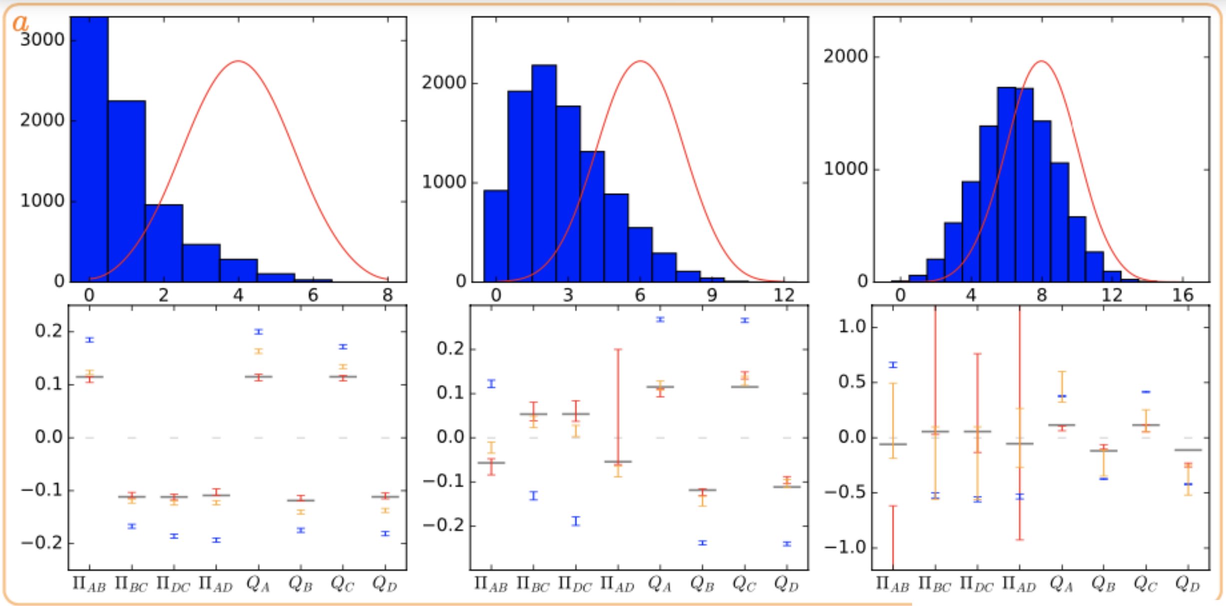


QEM techniques: 1. Post-selection method

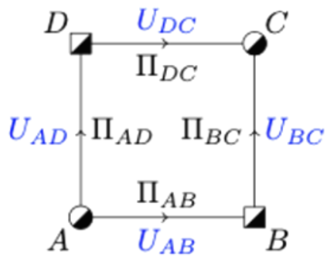
2. Calibration method (Operator Decoherence Renormalization method,

PRX Quantum 5 no. 2, (2024) 020315)

IBM computation and results (2D square)

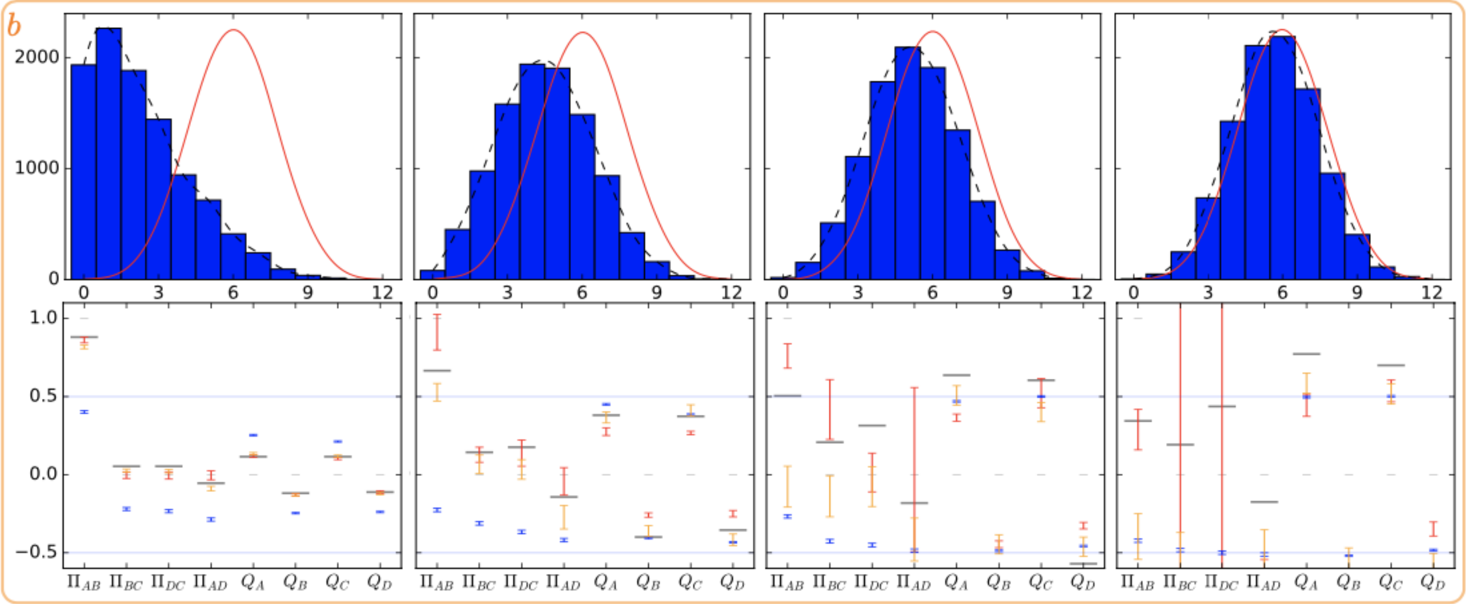


Results for n = 1, 2, 3 (left, middle, right): the distance (upper) and the observables (lower), where the blue, red and orange error bars denote respectively the results from the direct, calibrated and post-selected measurements, and the exact values are drawn as the grey bars.

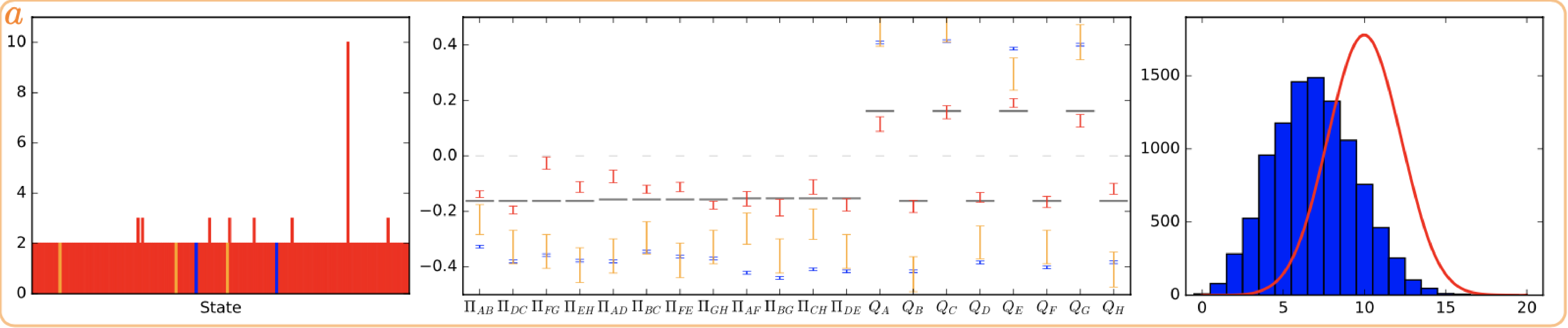


n	1	2	3
Qubit	9	14	19
Depth	172	399	894
CZ	75	244	734

Results for 1, 2, 3, and 4 Trotter steps, of n = 2.



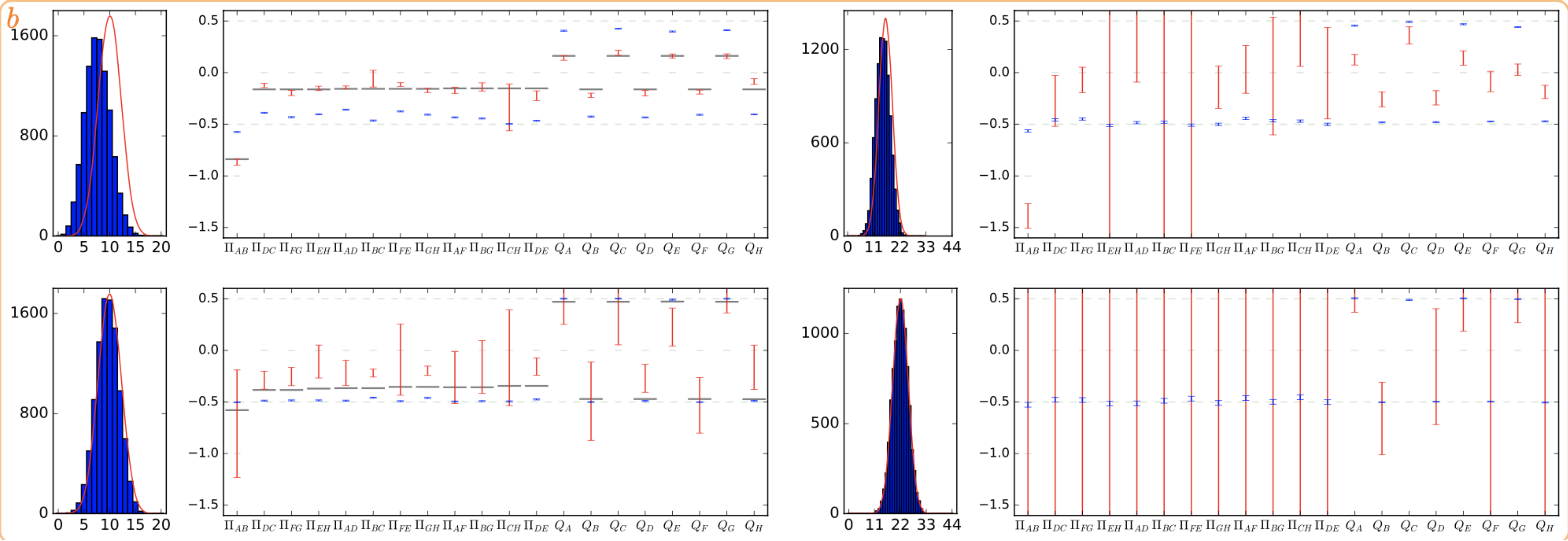
IBM computation and results (3D cube)



n = 1: the histogram of the final states (Left), the distance (Right) and the observables (Middle), where the blue, red and orange error bars denote respectively the results from the direct, calibrated and post-selected measurements, and for comparison, the exact values are drawn as the grey bars.

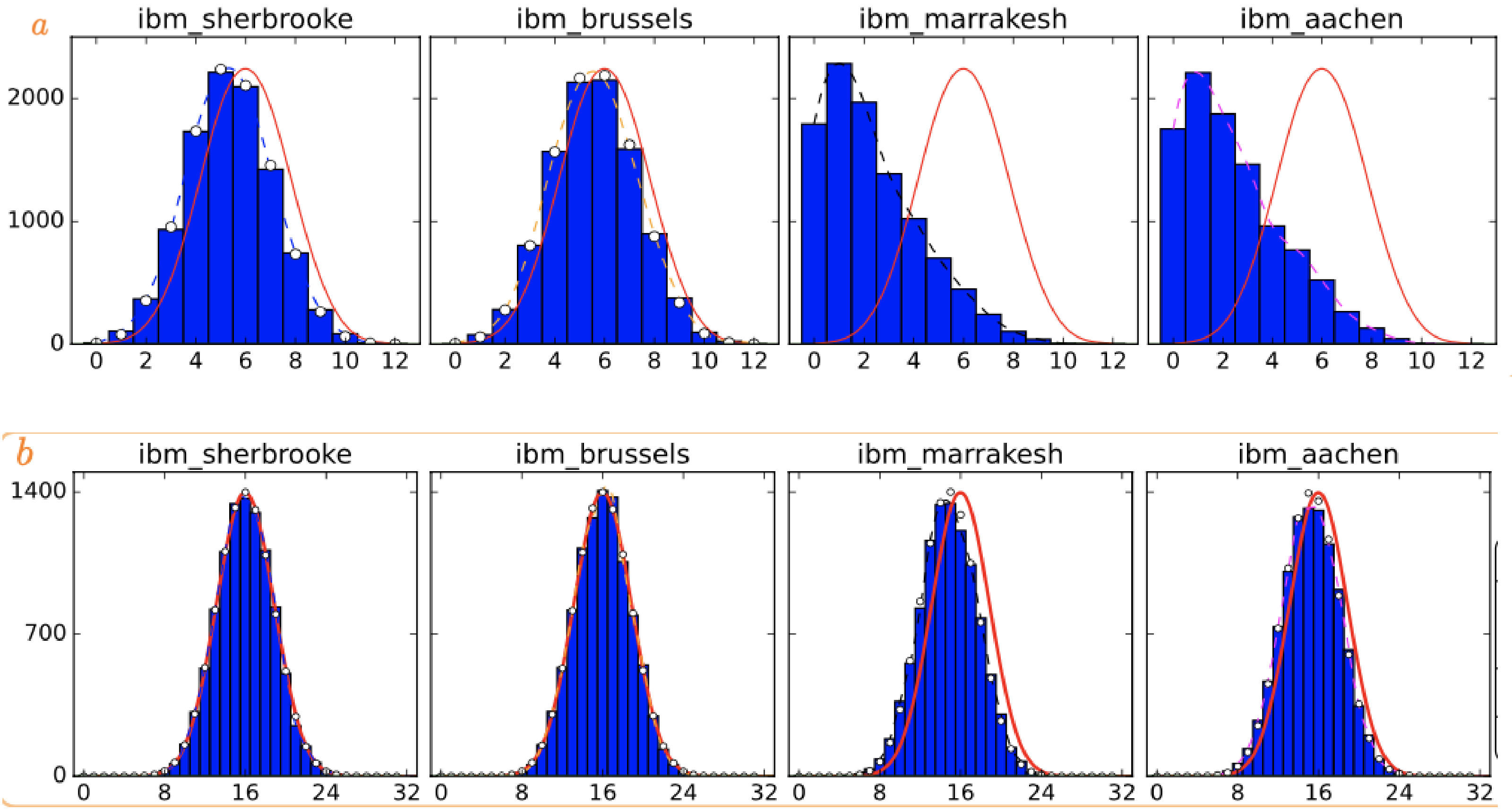
n	Qubit	Depth	CZ
1	22	740	598
2	36	1776	1644
3	50	3454	4481
1	22	1441	1263

IBM computation and results (3D cube)



Computations with initial state of ΠAB = −1: (Left) n = 1 with one Trotter step (upper) and two Trotter steps (lower); (Right) n = 2 (upper) and n = 3 (lower) with one Trotter step.

IBM results across platforms



IBM	sherbrooke	brussels	marrakesh	aachen
Type	Eagle r3 127 150K	Eagle r3 127 220K	Heron r2 156 195K	Heron r2 156 250K
Depth	560	548	389	383
Gates	(ecr,257)	(ecr,257)	(cz,244)	(cz,244)

2D

3D

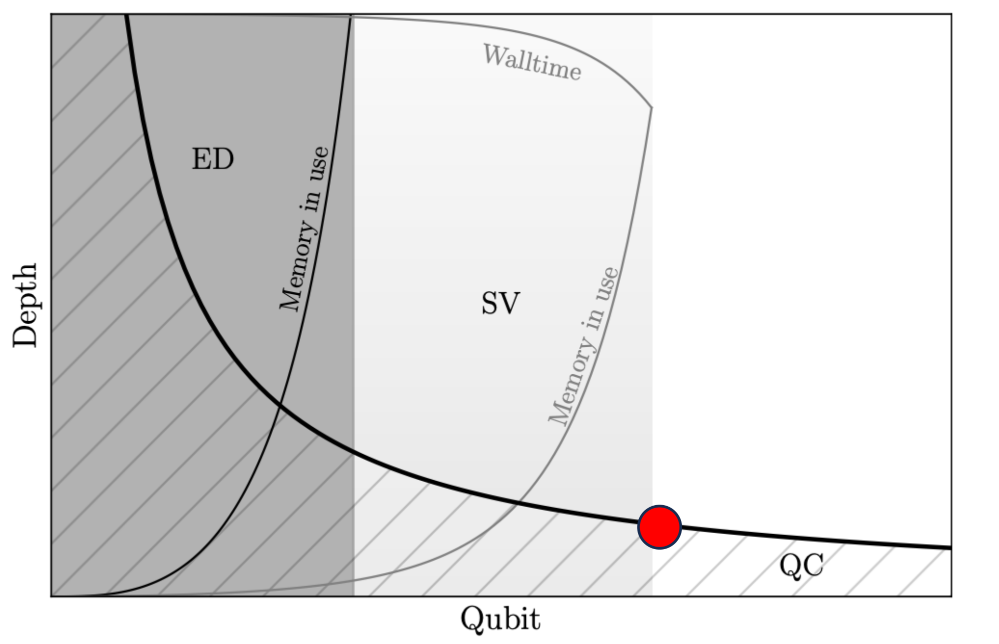
IBM	sherbrooke	brussels	marrakesh	aachen
Type	Eagle r3 127 150K	Eagle r3 127 220K	Heron r2 156 195K	Heron r2 156 250K
Depth	2293	2282	1591	1600
Gates	(ecr,1679)	(ecr,1679)	(cz,1652)	(cz,1652)

n = 2 and one Trotter step: (a) 2D Square (b) 3D Cube

Summary

QED on 2+1D and 3+1D can be implemented into qubit with the quantum algorithm similar to the scalar field theory – “Standard approach” > Scalable and gauge invariant formalism

Signal on IBM Heron for one Trotter step with n = 2 on 3D cube (36×1776)



Example of quantum advantage in comparison to HPC

Extension to non-Abelian gauge theory is possible, however, only with truncated ladder with SVD approach