# Topological Terms in Lattice Gauge Theories: A Hamiltonian Perspective

**Lena Funcke**

QuantHEP 2026, 14 July 2026

# Why are topological terms interesting?

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| 3+1D: Topological $\theta$-Term | 2+1D: Topological Chern-Simons Term |
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**3+1D: Topological $\theta$-Term**

**Relevance**
The strong CP problem, Grand Unified Theories, …

**2+1D: Topological Chern-Simons Term**

**Relevance**
Quantum Hall effect, fermion/boson dualities, …

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**Parameter**
Continuous angle $\theta \in [0,2\pi)$

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Quantized Chern-Simons coupling $k \in \mathbb{Z}$, called “level”

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Ground state has no $\theta$-dependent degeneracy

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Continuous angle $\theta \in [0,2\pi)$

**Degeneracy**
Ground state has no $\theta$-dependent degeneracy

**Mass generation**
No mass from $\theta$-term, only from QCD and Higgs

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**Mass generation** [3]
Photon mass from Chern-Simons term: $m_\gamma = ke^2/2\pi$

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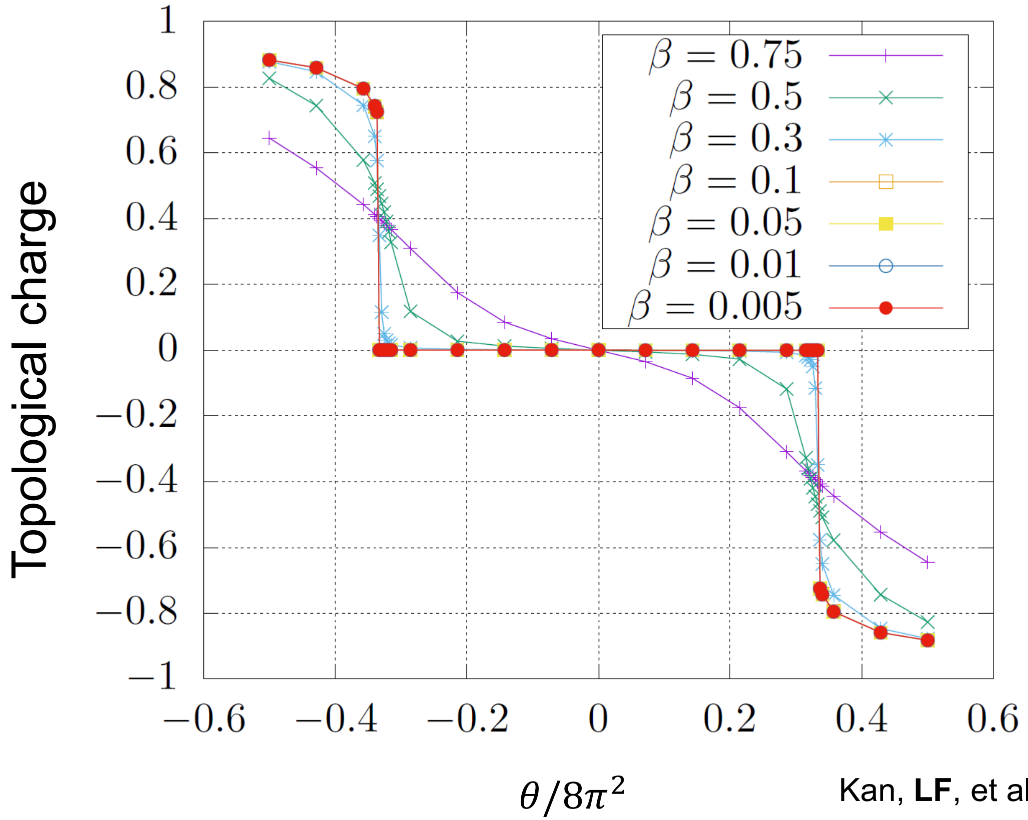
Simulation of $\theta$-term for 3+1D compact U(1)

## Numerical Results for 3+1D $\theta$-Term

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**Why no simulations of topological terms in 2+1D?**
So far, no suitable lattice formulation available!

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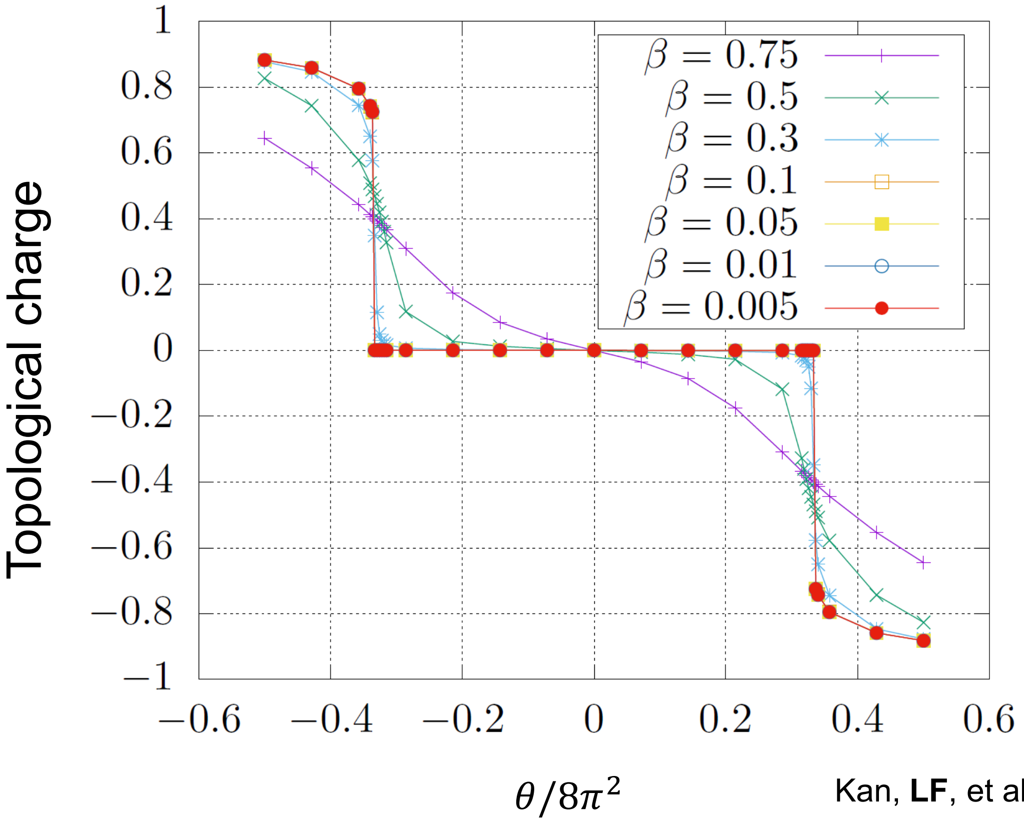


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## Maxwell-Chern-Simons (MCS) Theory

**Recent Hamiltonian lattice formulations**

CS [1] and MCS [2] theory: mainly for analytical studies

MCS theory: targeting numerical simulations [3]

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**Magnetic field term:** similar to QED (without monopoles)
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## 2+1D Maxwell Hamiltonian + CS Extension [3]

**Magnetic field term:** similar to QED (without monopoles)

**Electric field term:** modified by Chern-Simons term

$$\hat{H} = \frac{e^2}{2a^2} \sum_{\text{plaquettes}} \left[ \left( \boxed{\hat{p}_1} \boxed{- \left(\frac{ka^2}{4\pi}\right) \hat{A}_2} \right)^2 + \left( \boxed{\hat{p}_2} \boxed{+ \left(\frac{ka^2}{4\pi}\right) \hat{A}_1} \right)^2 \right] + \frac{1}{2e^2} \sum_{\text{plaquettes}} \left( \boxed{\hat{A}_1 + \hat{A}_2 - \hat{A}_1 - \hat{A}_2} \right)^2$$

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**Photon Mass & Quantized Coupling** | **Ground-State Degeneracy**

**Continuum limit of dispersion relation**

Correctly reproduces photon mass in continuum: [1]

$$\omega^2 \to |\tilde{q}|^2 + \left(\boxed{\frac{ke^2}{2\pi}}\right)^2$$

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### Quantization of Chern-Simons level

Constraint from large gauge transformations:

$$e^{2\pi i \hat{L}_1} e^{2\pi i \hat{L}_2} = \boxed{e^{2\pi i k}} e^{2\pi i \hat{L}_2} e^{2\pi i \hat{L}_1}$$

Correctly reproduces quantization property: [1]

$$\boxed{e^{2\pi i k}} = 1 \qquad \Longrightarrow \quad k \in \mathbb{Z}$$

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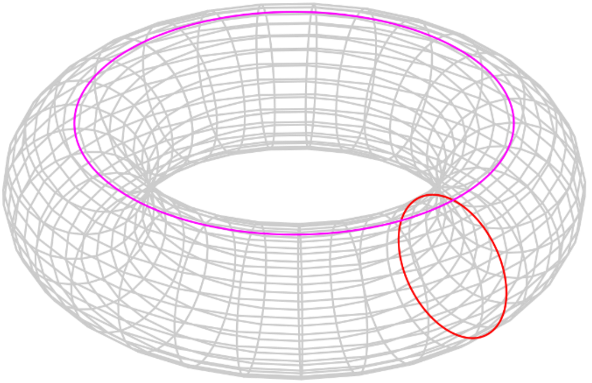
**Ground-State Degeneracy**

**Degeneracy for torus**

Torus: periodic boundary conditions → non-trivial topology



Correctly reproduce $k$-fold degeneracy of ground state [1]

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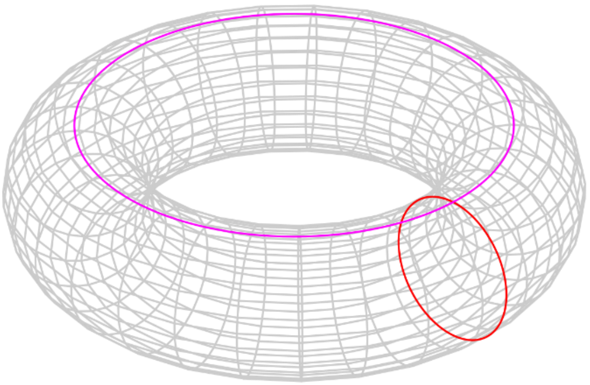

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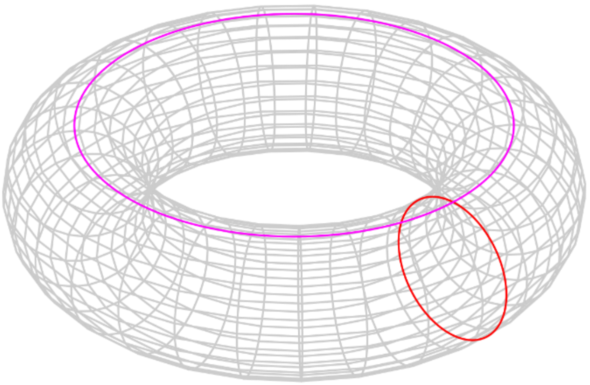

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## Exact Diagonalization

**First step: constant modes** [1]
Study truncation effects in constant-mode sector

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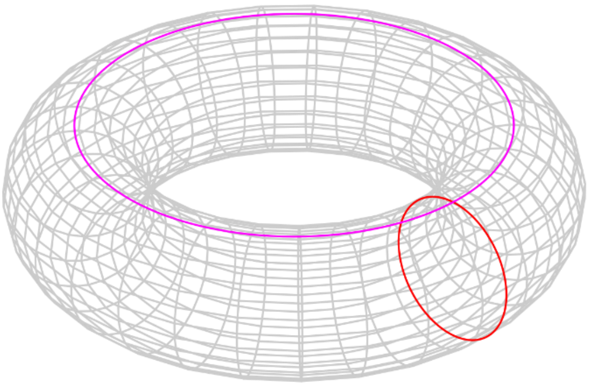

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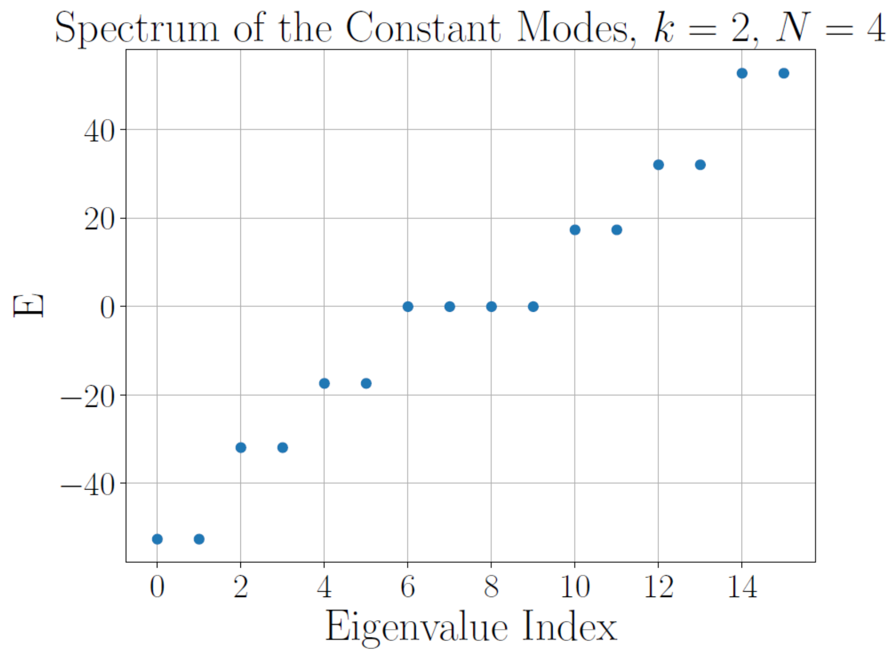


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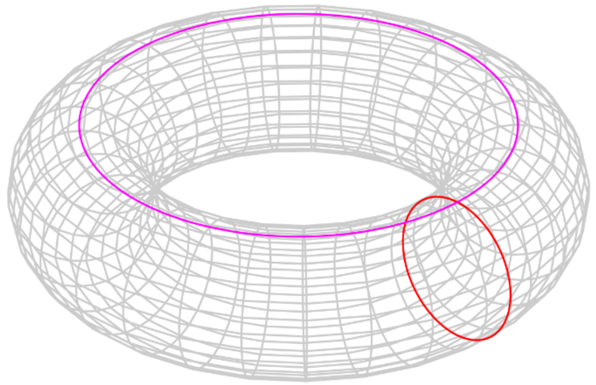

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- Recent claims of $\theta$-independence in QCD [1]
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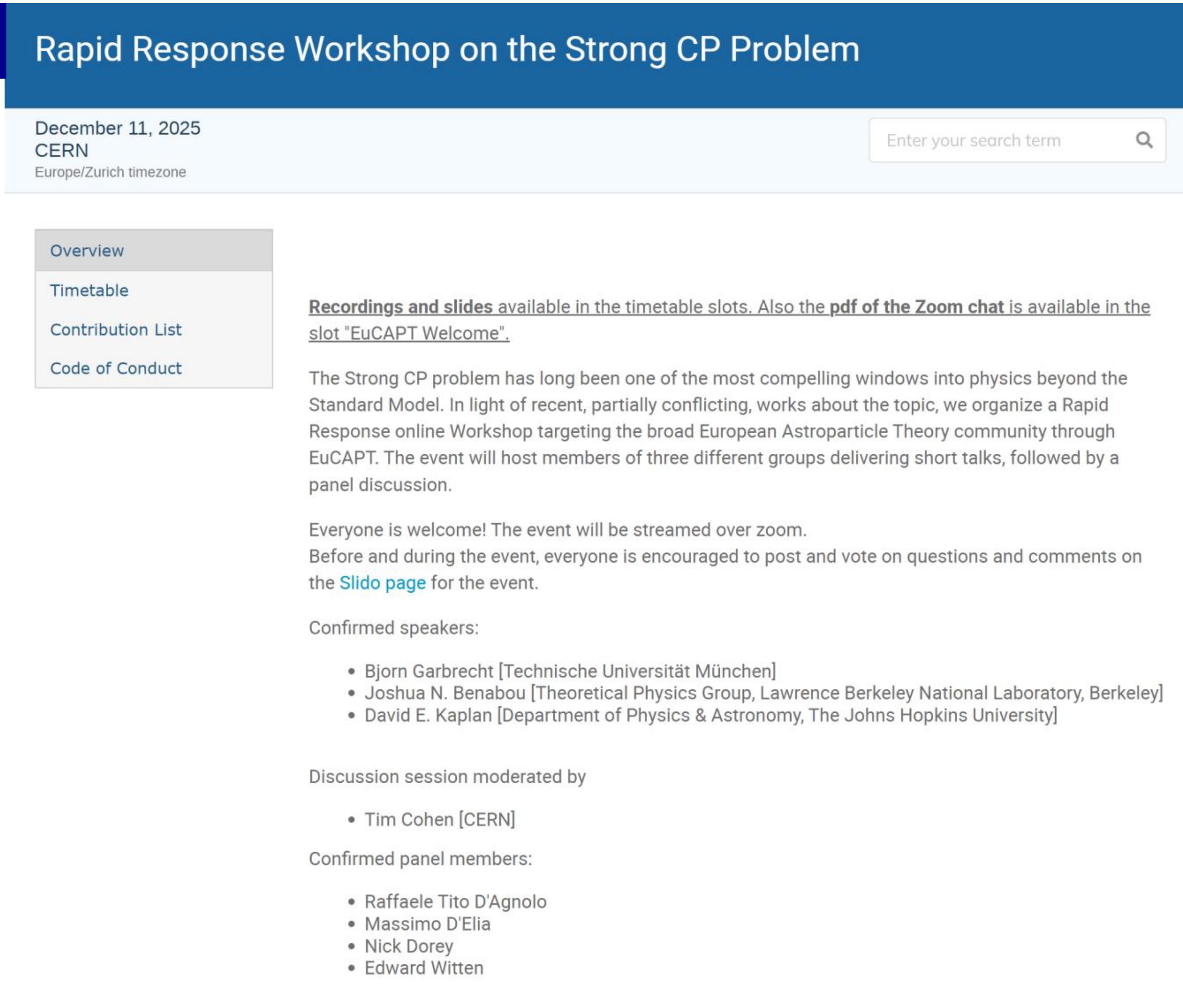

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## Rapid Response Workshop on the Strong CP Problem

December 11, 2025
CERN
Europe/Zurich timezone

Enter your search term

- Overview
- Timetable
- Contribution List
- Code of Conduct

**Recordings and slides** available in the timetable slots. Also the **pdf of the Zoom chat** is available in the slot "EuCAPT Welcome".

The Strong CP problem has long been one of the most compelling windows into physics beyond the Standard Model. In light of recent, partially conflicting, works about the topic, we organize a Rapid Response online Workshop targeting the broad European Astroparticle Theory community through EuCAPT. The event will host members of three different groups delivering short talks, followed by a panel discussion.

Everyone is welcome! The event will be streamed over zoom.
Before and during the event, everyone is encouraged to post and vote on questions and comments on the Slido page for the event.

Confirmed speakers:

- Bjorn Garbrecht [Technische Universität München]
- Joshua N. Benabou [Theoretical Physics Group, Lawrence Berkeley National Laboratory, Berkeley]
- David E. Kaplan [Department of Physics & Astronomy, The Johns Hopkins University]

Discussion session moderated by

- Tim Cohen [CERN]

Confirmed panel members:

- Raffaele Tito D'Agnolo
- Massimo D'Elia
- Nick Dorey
- Edward Witten

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$$H_\theta = -\frac{ig^2\theta}{8\pi^2 a}\sum_{n,i,j,k,b} \varepsilon_{ijk}\mathrm{Tr}\left[E^b_{n,i}\lambda^b\left(U_{n,jk}-U^\dagger_{n,jk}\right)\right]$$

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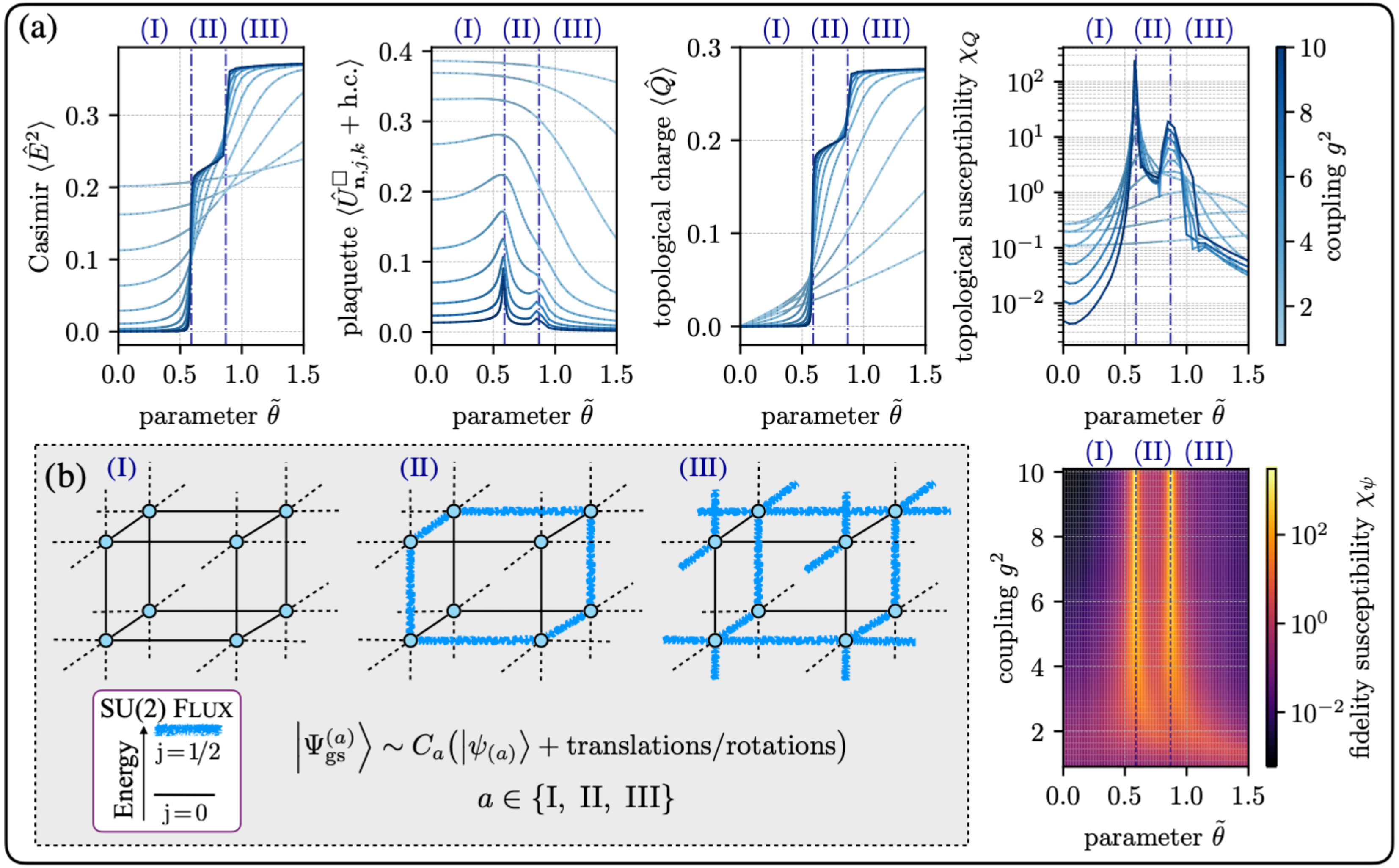


Cataldi, Rosanowski, Silvi, **LF** (in prep)

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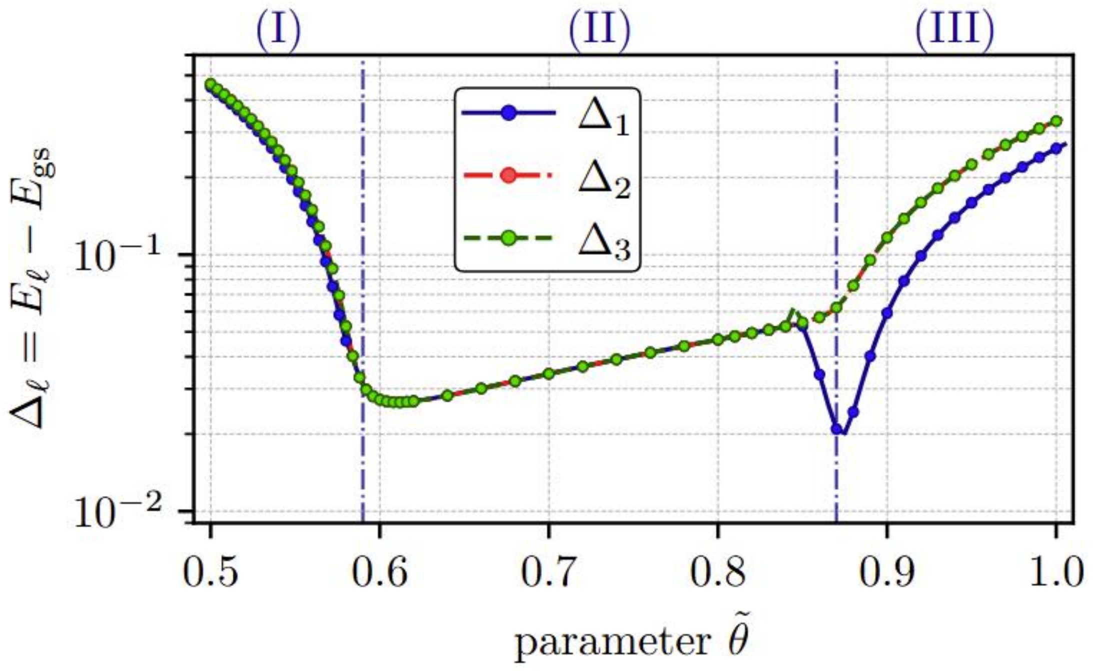


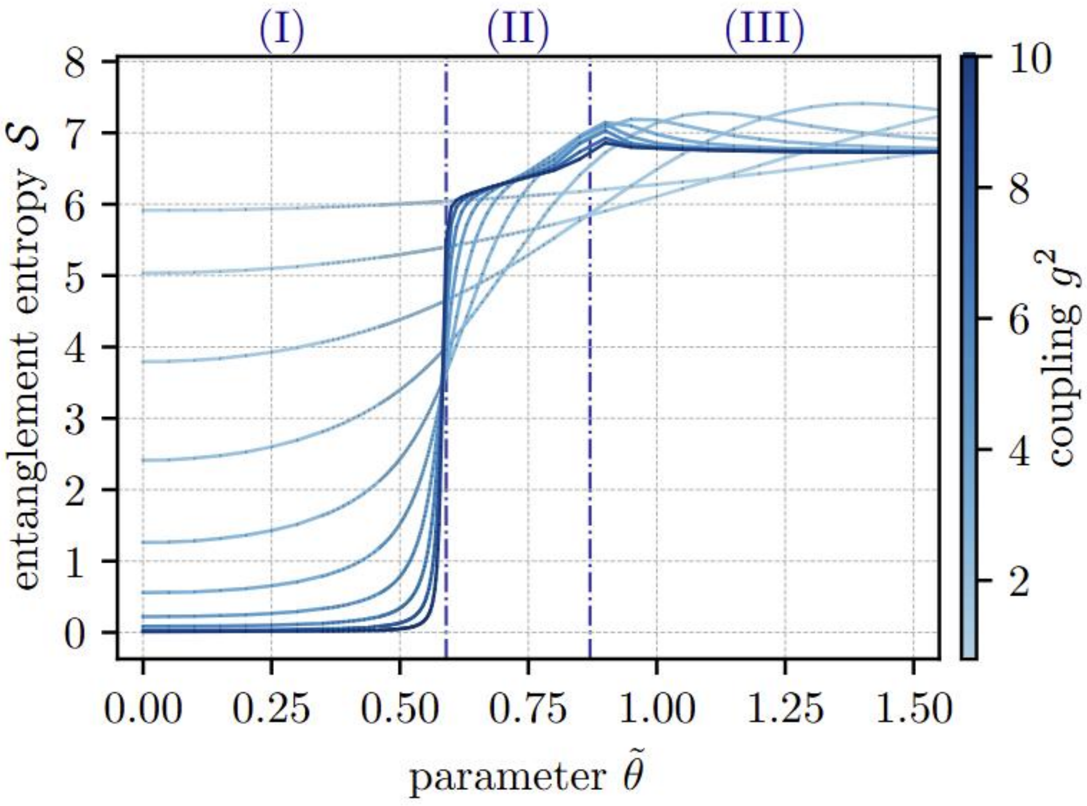


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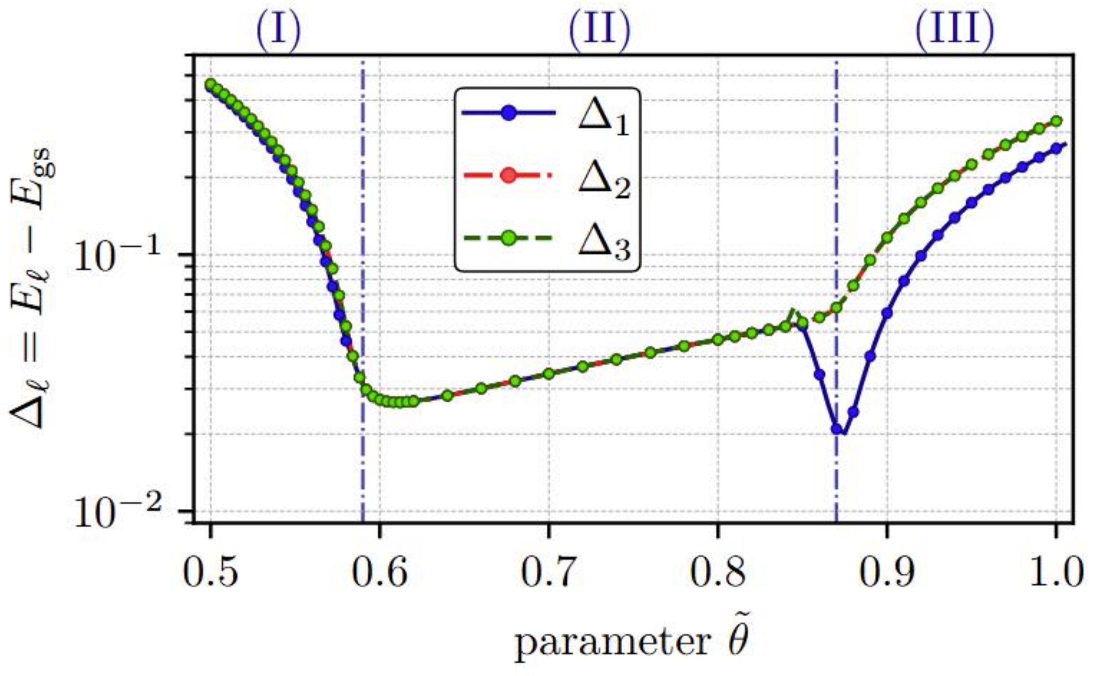


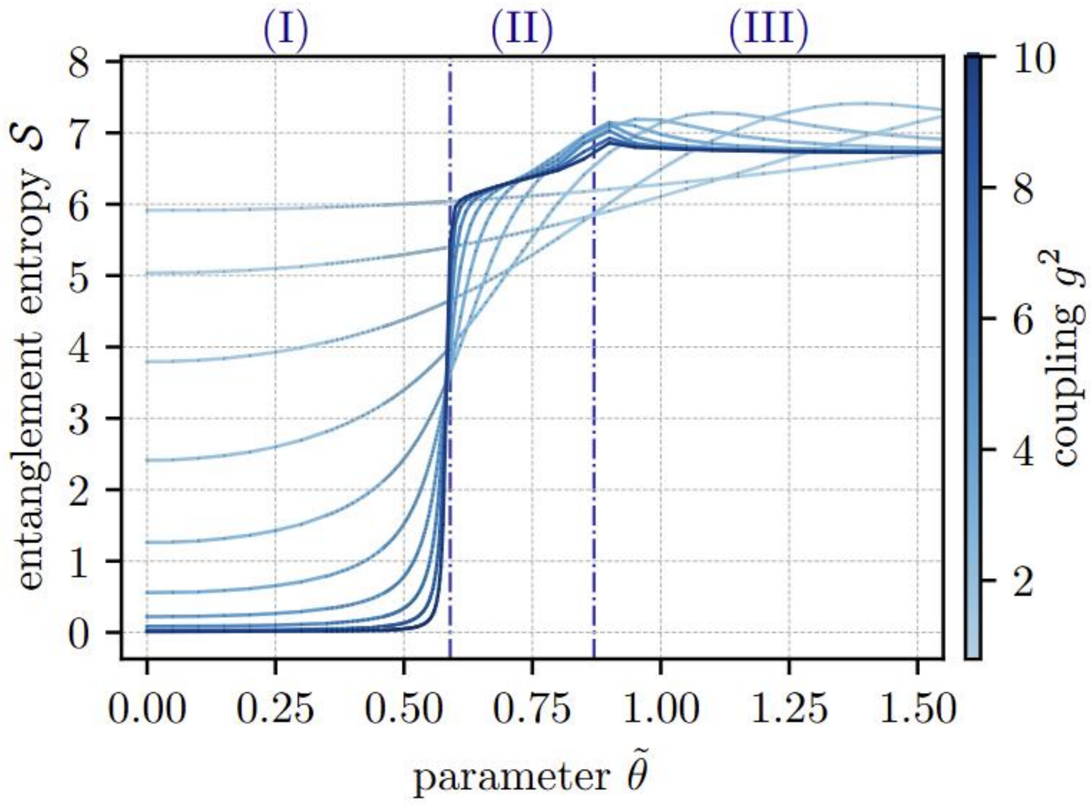


→ Preparatory study for TN and QC simulations

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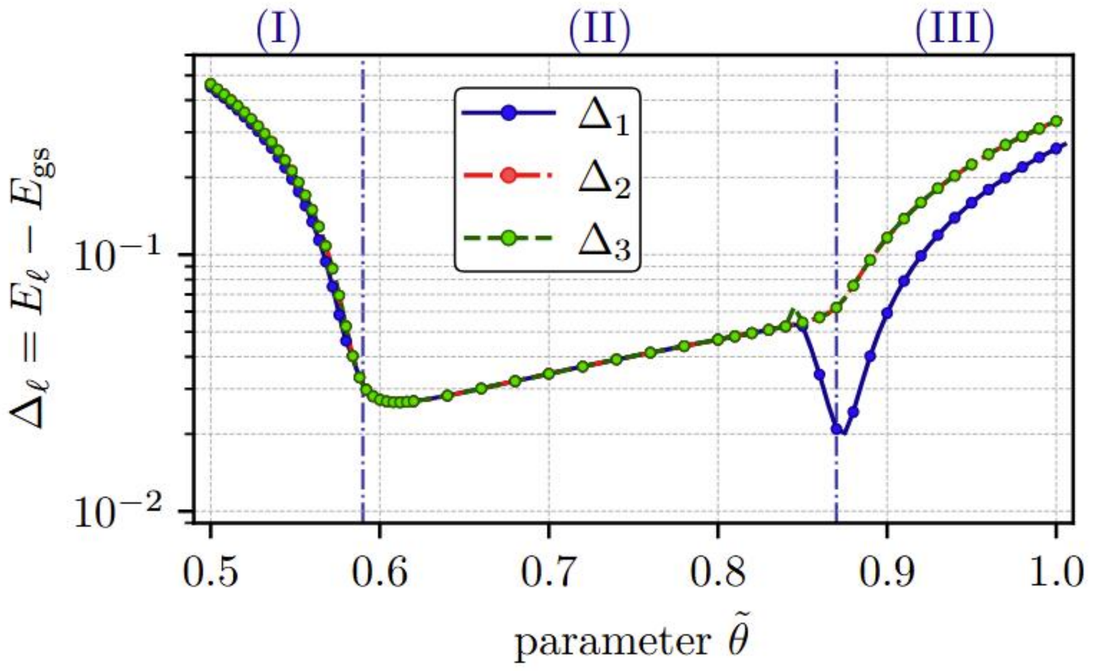


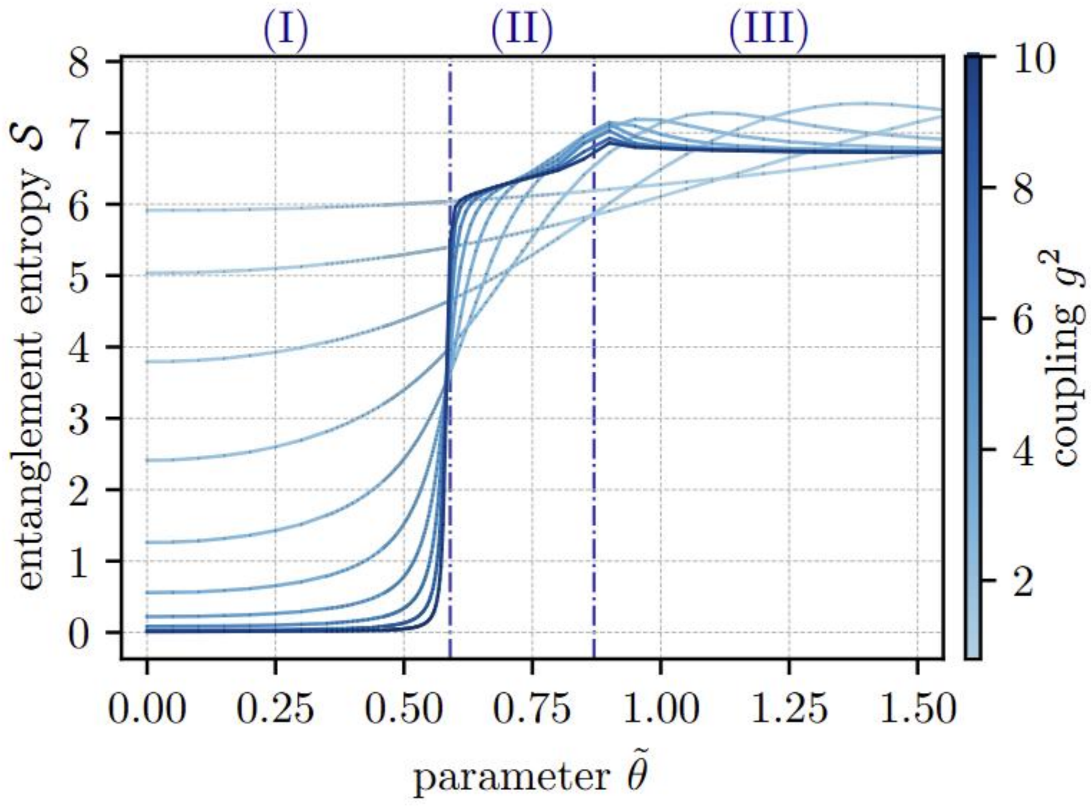


→ Preparatory study for TN and QC simulations

→ Dressed-site Hilbert space dimension for SU(3) would be $\sim 10\times$ larger

Cataldi, Rosanowski, Silvi, **LF** (in prep)

# Summary: Hamiltonian simulations of topological terms

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| 2+1D Chern-Simons Terms | 3+1D $\theta$-Terms |
| --- | --- |

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**Relevance**
- Quantum Hall effect, topological mass generation, …

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- Study truncation effects in full theory
- Tensor network and quantum simulations

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**Relevance**

The strong CP problem, Grand Unified Theories, …

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The strong CP problem, Grand Unified Theories, …

**Recent developments**
- First ED results for U(1) $\theta$-term
- First (preliminary) ED results for SU(2) $\theta$-term

**Future outlook**
- Qudit-based quantum simulations of U(1) $\theta$-term
- Tree tensor network simulations for larger volumes

# Thanks to my group and our collaborators!

**Thanks to you for listening!**

**Questions?**

# Backup: simulating 3+1D SU(2) and SU(3) with TN

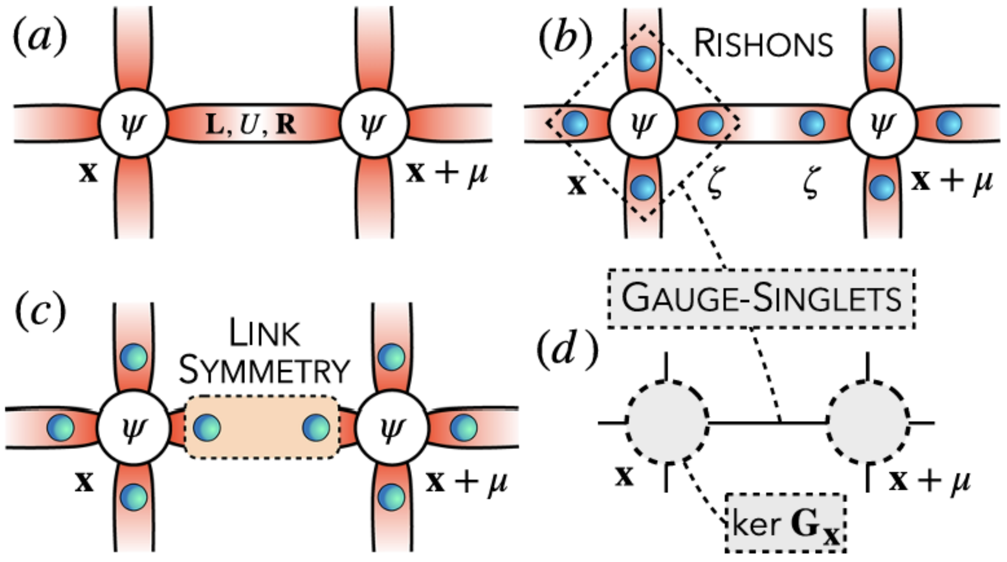



**Fig. 3 | Pictorial representation of the dressed site formalism adopted for TN simulations of LGTs. a** Starting from the original formulation of matter and gauge fields, **b** each truncated gauge link is split into two representations, one per half-link; each is equipped with a proper fermionic rishon mode $\zeta$. **c** All the half-links are absorbed into the attached matter site, forming a gauge-invariant dressed site **d** whose Hilbert space spans all the possible gauge singlets.

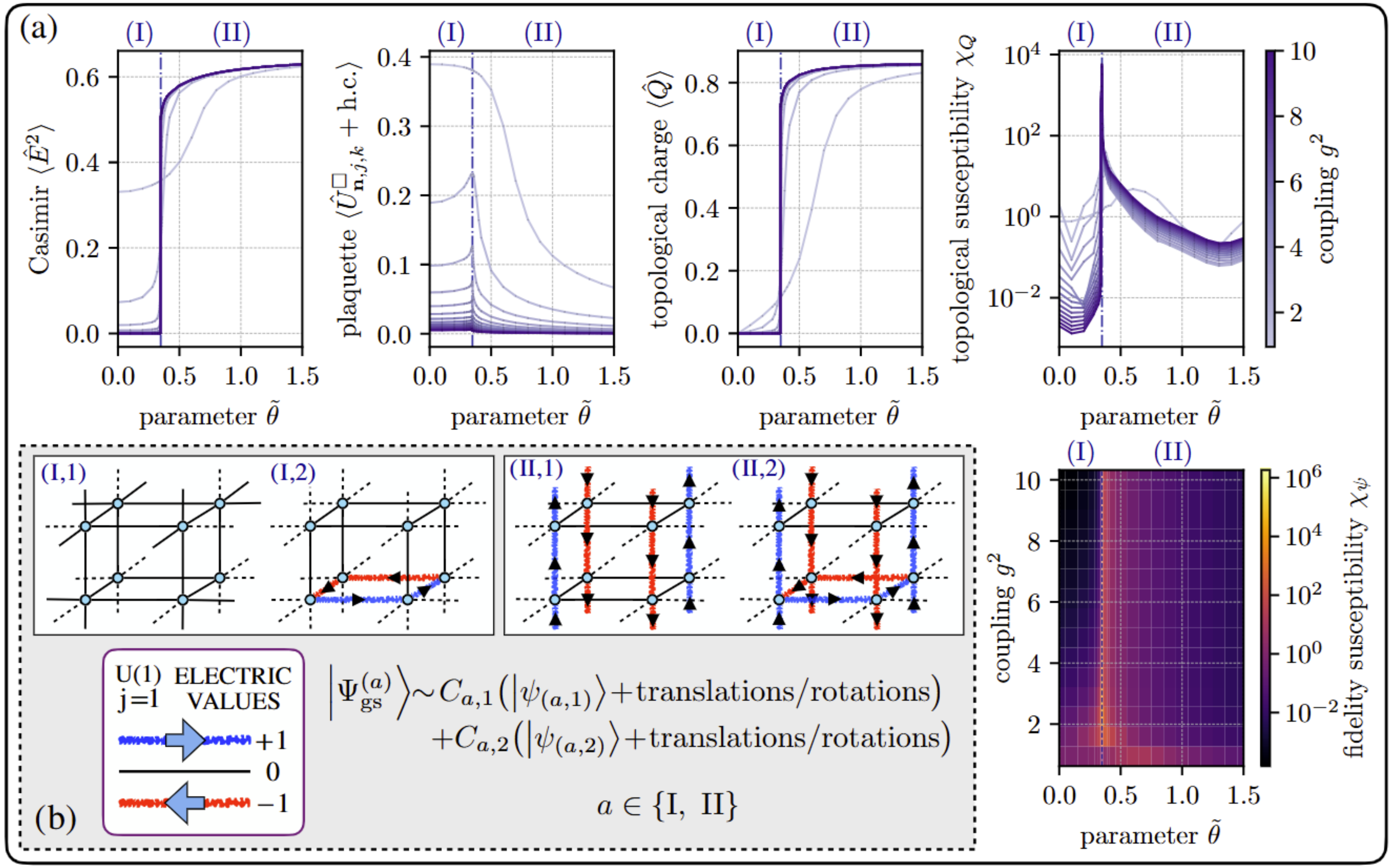
**Table 1 | Dressed site Hilbert space dimensions**

| $\ell$ | *d* | | | | | |
|---|---|---|---|---|---|---|
| | (2 + 1)-dimensions | | | (3 + 1)-dimensions | | |
| | U(1) | SU(2) | SU(3) | U(1) | SU(2) | SU(3) |
| 1 | 35 | 30 | 164 | 267 | 178 | 3096 |
| 2 | 165 | 168 | 752 | 3437 | 3670 | 52,476 |
| 3 | 455 | 600 | 3738 | 18,487 | 35,280 | 813,438 |
| 4 | 969 | 1650 | 19,878 | 64,953 | 214,958 | 17,490,134 |
| 5 | 1771 | 3822 | 43,698 | 177,155 | 967,466 | 69,232,482 |
| 6 | 2925 | 7840 | 82,128 | 408,421 | 3,509,062 | 228,461,186 |
| 7 | 4495 | 14,688 | 212,496 | 835,311 | 10,828,494 | 1,245,755,754 |
| 8 | 6545 | 25,650 | 333,538 | 1,561,841 | 29,473,038 | 2,782,999,996 |

For increasing number $\ell$ of allowed electric energy density levels, the corresponding dimensions *d* in some 2- and 3-dimensional paradigmatic LGTs with dynamical matter and gauge groups U(1), SU(2), and SU(3) are shown.

Magnifico et al., (2025)

# Backup: (preliminary) ED results for 3+1D U(1) $\theta$-term

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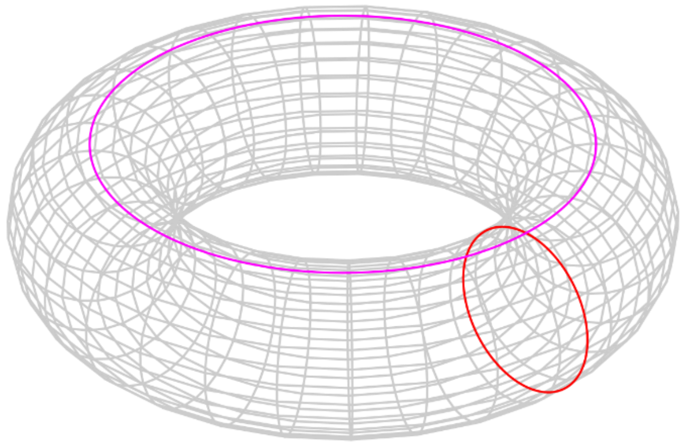
# Backup: Hamiltonian of constant-mode sector

## Landau problem on a torus

**Constant-mode Hamiltonian**

$$H = \frac{Se^2}{2}\left[\left(p_1 - \frac{k}{4\pi}a_2\right)^2 + \left(p_2 + \frac{k}{4\pi}a_1\right)^2\right]$$



Equivalent to particle on a periodic lattice in magnetic background field

**Truncation of constant-mode gauge field**
Discrete $a_i$ → discrete torus:

$$a_1 = \frac{2\pi}{N_x}x = \Delta_x x, \qquad a_2 = \frac{2\pi}{N_y}y = \Delta_y y, \qquad x = 0, \ldots, N_x - 1, \quad y = 0, \ldots, N_y - 1$$

Reduces to (an atypical) Hofstadter problem!

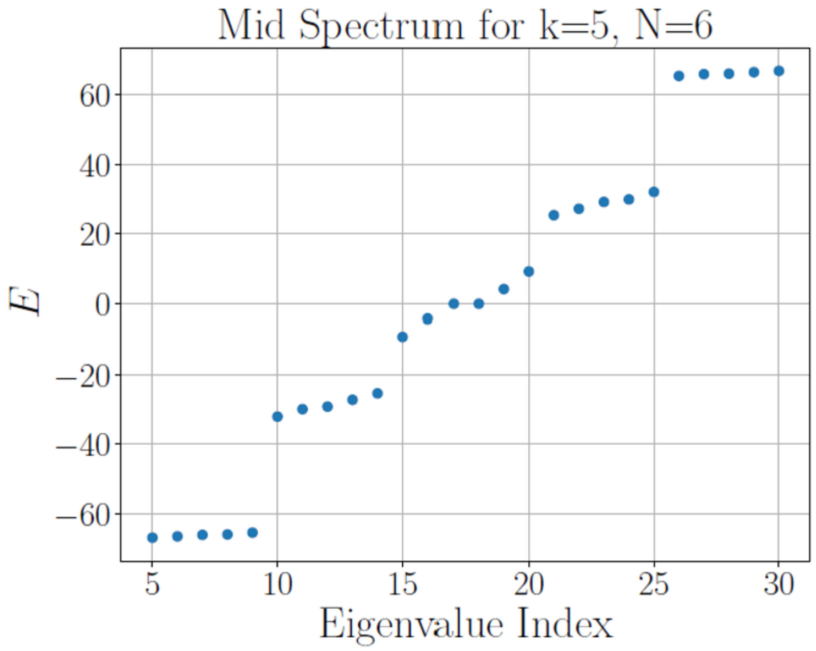
# Backup: Does truncation destroy topological properties?

## Exact Diagonalization

**First step: constant modes** [1]

Study truncation effects in constant-mode sector

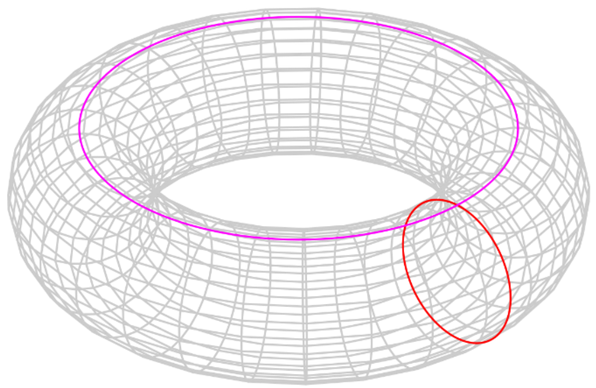
$$A_i(t, \mathbf{x}) = a_i(t)$$




## Ground-State Degeneracy

**Degeneracy for torus**

Torus: periodic boundary conditions → non-trivial topology



Correctly reproduce $k$-fold degeneracy of ground state [2]

→ good cross-check / benchmark for numerical methods!

[1] Bulgarelli, Diamantini, Dichter, **LF**, Hartung, Jansen, Ortega, Singh, Spera (2026)
[2] Peng, Diamantini, **LF**, Hassan, Jansen, Kühn, Luo, Naredi (2024)

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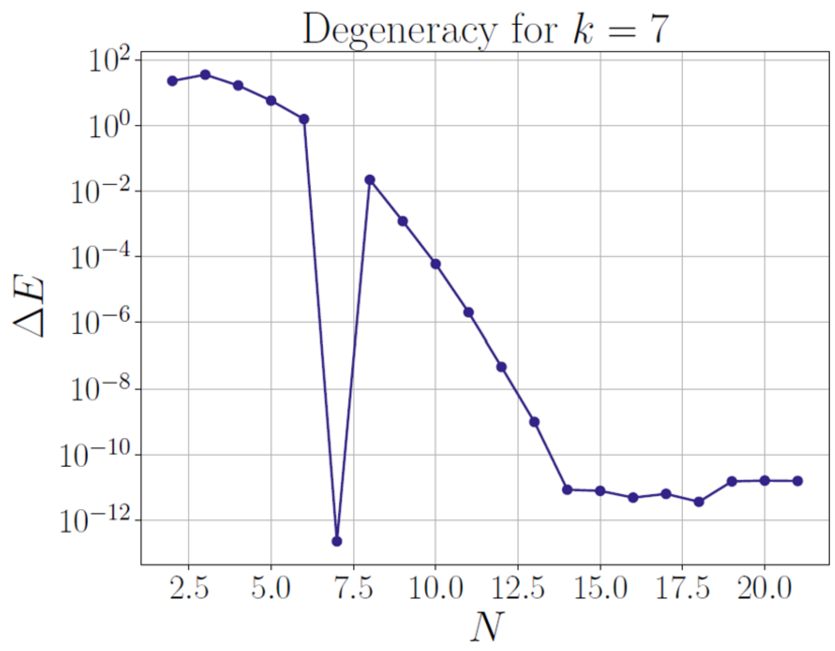


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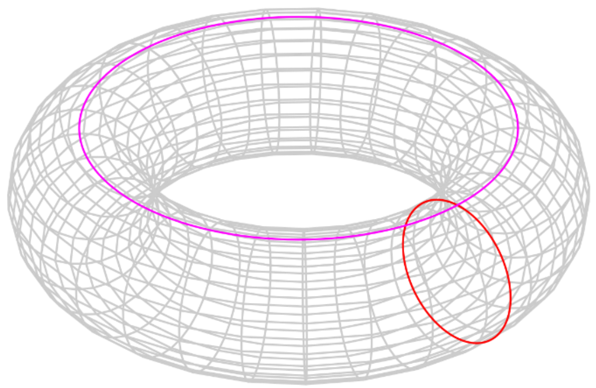

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# Backup: How to formulate MCS theory on the lattice?

| Problem | State-of-the-Art |
|---|---|

**Continuum: 2+1D Chern-Simons term**

$$S_{CS}(A) = -\frac{ik}{4\pi}\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

**Naïve lattice discretization:**

$$S_{CS}(A) = -\frac{ik}{4\pi} a^2 d\tau \sum_{x \in \text{sites}} \epsilon^{\mu\nu\rho} A_{x;\mu} \Delta_\nu A_{x+\hat{\mu};\rho}$$

**Problem: *compact* gauge fields → monopoles**
**→ Chern-Simons term violates gauge invariance!** [1]

[1] Pisarski (1986), Affleck et al. (1989)

**Lattice formulation of *non-compact* MCS Hamiltonian** [2]
However, non-compact gauge field → no simulations

**Lattice formulation of *compact* CS Hamiltonian** [3]
Villain approach, monopoles eliminated → gauge invariant
However, non-commuting geometry → no simulations

**Our work: Derived *compact* MCS lattice Hamiltonian** [4]
Paves the way for simulations on (quantum) computers

[2] Lüscher (1989), …
[3] Jacobson, Sulejmanpasic (2024)
[4] Peng, Diamantini, **LF**, Hassan, Jansen, Kühn, Luo, Naredi (2024)

# Backup: Compact MCS lattice Hamiltonian

## 2+1D Maxwell Lattice Hamiltonian + Extension

**Magnetic field term:** similar to QED (without monopoles)
**Electric field term:** modified by Chern-Simons term

$$\hat{H} = \frac{e^2}{2a^2} \sum_{\text{plaquettes}} \left[ \left( \hat{p}_1 - \left(\frac{ka^2}{4\pi}\right) \hat{A}_2 \right)^2 + \left( \hat{p}_2 + \left(\frac{ka^2}{4\pi}\right) \hat{A}_1 \right)^2 \right]$$

$$+ \frac{1}{2e^2} \sum_{\text{plaquettes}} \left( -\hat{A}_1 \quad -\hat{A}_2 \quad \hat{A}_2 \quad \hat{A}_1 \right)^2$$

**Quadratic Hamiltonian:** can be solved analytically!

## Compact Gauge Fields

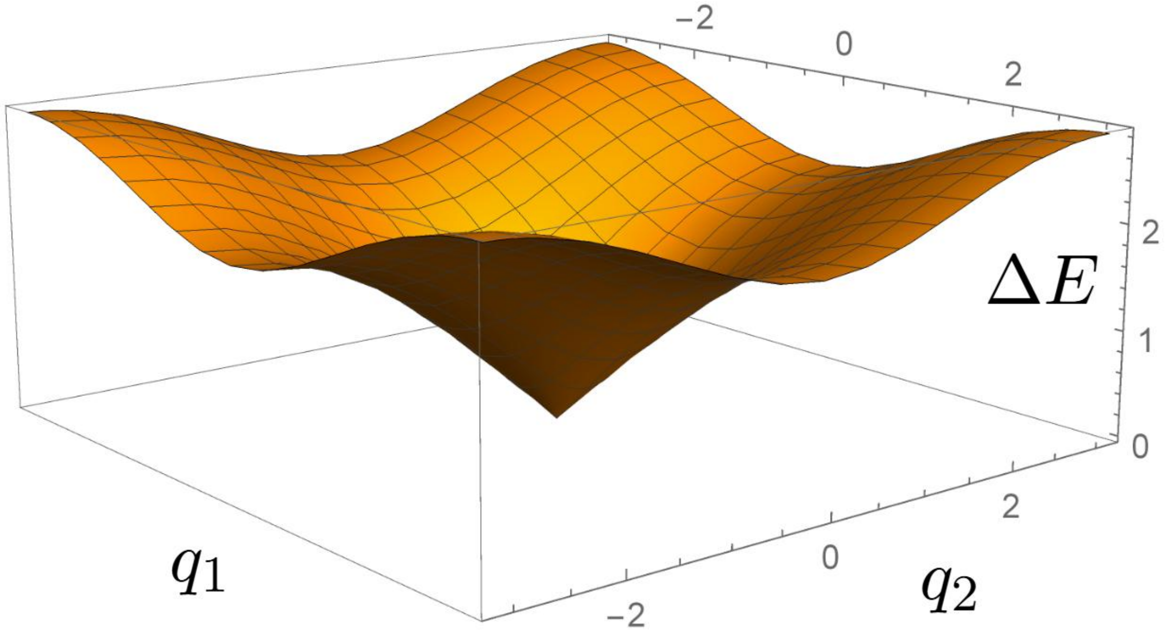
**Gauge configurations:** can take values in $(-\infty, +\infty)$
**Compactness:** ensured by constraints on Hilbert space

**Commutation relations:** $[\hat{A}_x, \hat{p}_y] = i\delta_{x,y}$

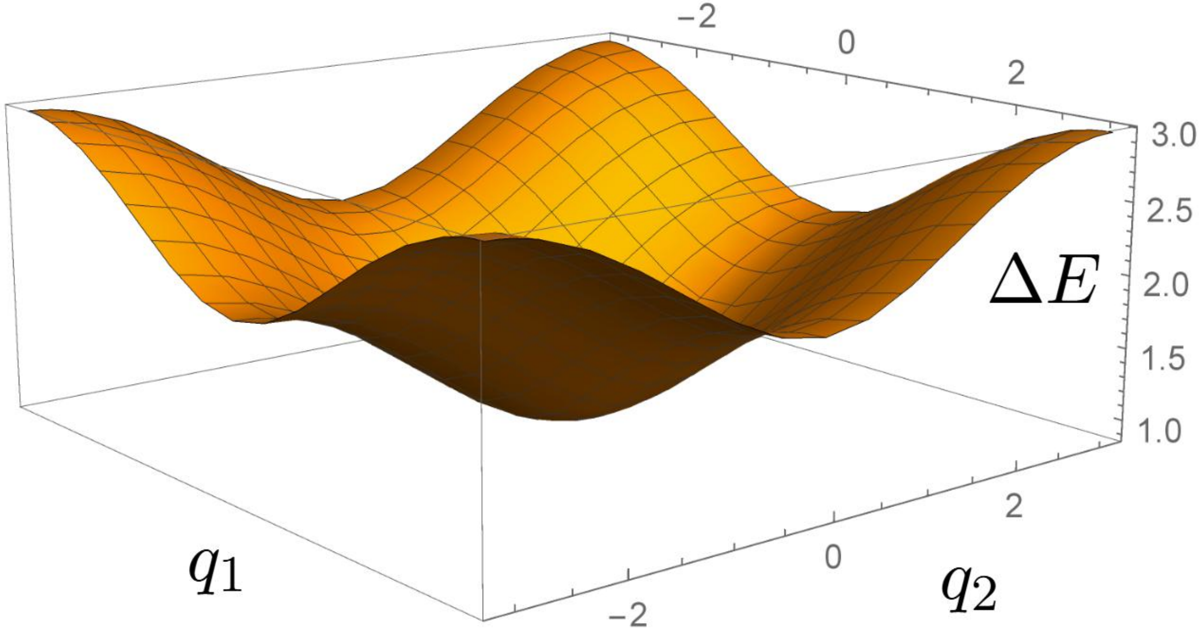
# Backup: Exact solution of lattice Hamiltonian

| Pure Maxwell Theory | Maxwell-Chern-Simons Theory |
|---|---|

**No gap:** Linear dispersion relation of gapless photon




**Gap:** Chern-Simons term gives mass to photon




**Plot of dispersion relation:** $\Delta E = \omega = \sqrt{\frac{1}{a^2}\left[2(1-\cos q_1) + 2(1-\cos q_2)\right] + \left(\frac{ke^2}{4\pi}\right)^2 \left[2 + 2\cos(q_1+q_2)\right]}$

# Backup: Monopole problem of compact MCS theory

**1+1D Lattice Theories** | **Modified Villain Approach**

**Compact Maxwell Theory**
Existence of quantized magnetic fluxes:

$$\int_{\Sigma} dA \in 2\pi\mathbb{Z}$$

If closed surface is contractible, **monopole** inside

**Compact MCS Theory**
Monopole configuration: large gauge transformation → changes Chern-Simons action → boundary terms at spatial infinity → action **violates gauge invariance**

**Conventional Villain Approach**
Add discrete plaquette variables $n$, encode magnetic flux
Interpret $n$ as discrete gauge fields for shift symmetry:

$$A_0 \to A_0 + \frac{2\pi}{d\tau}, \quad A_i \to A_i + \frac{2\pi}{a}, \quad i = 1, 2$$

Gauge the discrete shifts → study compact gauge theory:

$$U(1) = \mathbb{R}/2\pi\mathbb{Z}$$

**Modified Villain Approach**
Eliminate monopoles with Lagrange multiplier

# Backup: 2+1D QED with(out) monopoles

| **Lagrangian Formalism** | **Hamiltonian Formalism** |
| --- | --- |

**Compact variables**
For compact $\theta_i$, action contains terms $\cos(\sum_i c_i \theta_i)$

**Conventional Villain approach**
Replace cosine terms by periodic Gaussian potential:

$$e^{\beta \cos(\theta)} \approx \sum_{n=-\infty}^{\infty} e^{-\frac{\tilde{\beta}}{2}(\theta + 2\pi n)^2}$$

→ discrete gauge fields $n$ can take values in $(-\infty, +\infty)$
→ compactness ensured by constraints on Hilbert space

**Modified Villain approach**
Eliminate monopoles with Lagrange multiplier → flat $n$

**2+1D Compact QED …**

**… including monopoles / instantons**

$$H = \frac{e^2}{2a^2} \sum_{\text{links}} E_i^2 + \frac{1}{e^2 a^2} \sum_{\text{plaquettes}} \left(1 - \cos a^2 B\right)$$

**… with monopoles / instantons removed**

$$H = \frac{e^2}{2a^2} \sum_{\text{links}} E_i^2 + \frac{a^2}{2e^2} \sum_{\text{plaquettes}} B^2$$

# Backup: Constraints on Hilbert space

| Local Gauge Transformations | Large Gauge Transformations |
|---|---|

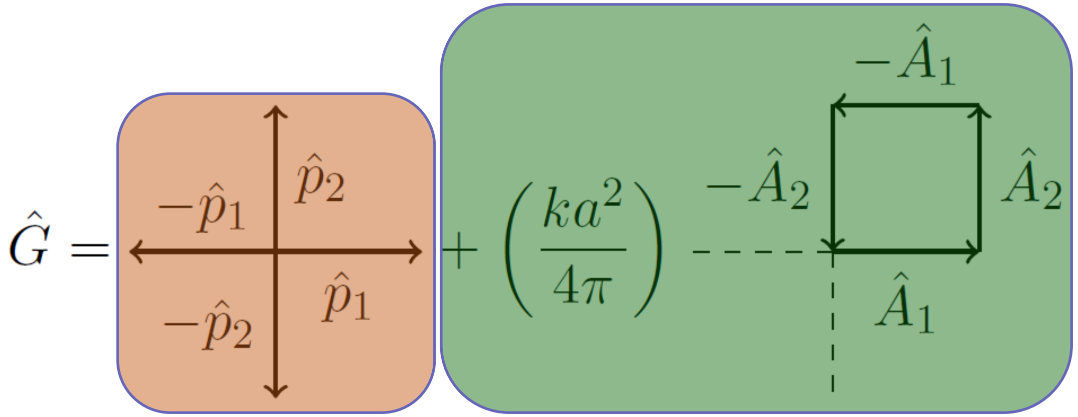
**Gauss' law**

Constraints on physical states in Hilbert space:

$$e^{i\lambda\hat{G}}|\psi\rangle = |\psi\rangle, \quad \forall\lambda\in\mathbb{R}, \qquad [\hat{H},\hat{G}]=0$$

**Generator for local gauge transformations**

Similar as for QED, but modified by Chern-Simons term:




**Two additional constraints**

$$e^{2\pi i\hat{L}_1}|\psi\rangle = e^{i\theta_1}|\psi\rangle, \quad e^{2\pi i\hat{L}_2}|\psi\rangle = e^{i\theta_2}|\psi\rangle$$

→ compactify gauge field configurations

→ similar as for QED (without monopoles)

**Generators for large gauge transformations**

Similar as for QED, but modified by Chern-Simons term

→ enforce quantized Chern-Simons coupling

# Backup: Compactification through constraints

| Large Gauge Transformations | Invariance of Partition Function |
| --- | --- |

**Constraints on Hilbert space …**

$$e^{2\pi i \hat{L}_i}|\psi\rangle = e^{i\theta_i}|\psi\rangle$$

**… allow only certain gauge field configurations**

$$\hat{L}_i|\psi\rangle = \left(\frac{\theta_i}{2\pi} + m_i\right)|\psi\rangle, \; m_i \in \mathbb{Z}, \; i = 1,2$$

**Different topological sectors of theory**

Transformation $|\psi\rangle \to e^{2\pi i \hat{L}_j}|\psi\rangle$

… changes sector $m_i \to m_i + k\,\epsilon_{ij}$

… due to non-zero commutator $[\hat{L}_1, \hat{L}_2] = -\frac{k}{2\pi} i$

**Partition function**

Sum over all sectors / values of $m_i$ with equal weights
→ stays invariant under large gauge transformations

**Similarity to Villain approximation:**

$$e^{\beta\cos(\theta)} \approx \sum_{n=-\infty}^{\infty} e^{-\frac{\tilde{\beta}}{2}(\theta+2\pi n)^2}$$

→ obtain periodic function (i.e., compact gauge field) by summing over multiple non-periodic functions

**Numerical simulation**

Truncation of infinite sum, neglect terms with large $n$

# Backup: Large gauge transformation constraint #1

**2+1D Compact QED …** | **… With Chern-Simons Term**

### Constraint on Hilbert space

$$e^{2\pi i \hat{L}_1} |\psi\rangle = e^{i\theta_1} |\psi\rangle$$

### Generator for large gauge transformation

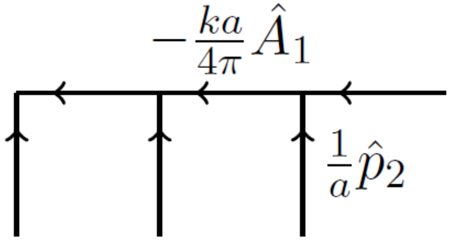
$$\hat{L}_1 = \cdots$$


$$\cdots$$

$$= \frac{1}{a} \sum_{x_1=0}^{N_1-1} \hat{p}_{(x_1,N_2-1);2}$$

### Modified generator for large gauge transformation

$$\hat{L}_1 = \cdots$$




$$\cdots$$

$$= \sum_{x_1 \in \{0,1,\ldots,N_1-1\}} \left( \frac{1}{a} \hat{p}_{(x_1,x_2);2} - \frac{ka}{4\pi} \hat{A}_{(x_1,x_2+1);1} \right)$$

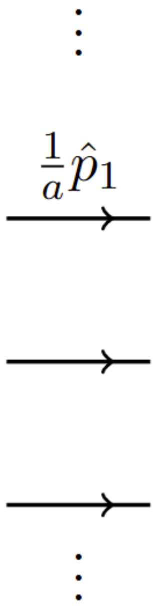
# Backup: Large gauge transformation constraint #2

| **2+1D Compact QED …** | **… With Chern-Simons Term** |
|---|---|

**Constraint on Hilbert space**

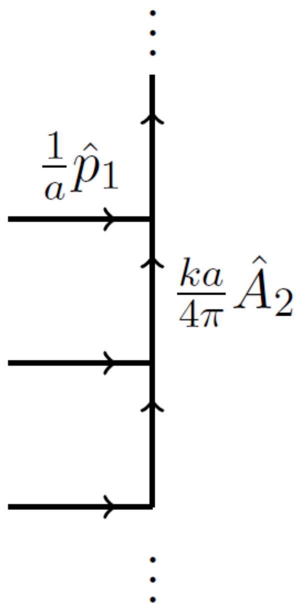
$$e^{2\pi i \hat{L}_2}|\psi\rangle = e^{i\theta_2}|\psi\rangle$$

**Generator for large gauge transformation**

$\hat{L}_2 =$ 

$= \frac{1}{a}\sum_{x_2=0}^{N_2-1} \hat{p}_{(N_1-1,x_2);1}$

**Modified generator for large gauge transformation**

$\hat{L}_2 =$ 


$$= \sum_{x_2 \in \{0,1,\ldots,N_2-1\}} \left( \frac{1}{a}\hat{p}_{(x_1,x_2);1} + \frac{ka}{4\pi}\hat{A}_{(x_1+1,x_2);2} \right)$$

# Backup: Compatibility of constraints

## Commutators

**Constraints need to commute with Hamiltonian**

$$[\hat{H}, \hat{G}] = 0$$

$$[\hat{H}, \hat{L}_i] = 0, \quad i = 1, 2$$

**Constraints need to commute with Gauss' law**

$$[\hat{G}, \hat{L}_i] = 0, \quad i = 1, 2$$

**But: constraints do not commute with each other!**

$$[\hat{L}_1, \hat{L}_2] = -\frac{k}{2\pi} i$$

## Quantization of Chern-Simons Level

**Non-zero commutator yields quantization condition!**

Constraint from large gauge transformations:

$$e^{2\pi i \hat{L}_1} e^{2\pi i \hat{L}_2} = \boxed{e^{2\pi i k}} e^{2\pi i \hat{L}_2} e^{2\pi i \hat{L}_1}$$

Correctly reproduces quantization property: [1]

$$\boxed{e^{2\pi i k}} = 1 \qquad \Longrightarrow \quad k \in \mathbb{Z}$$

[1] Pisarski (1986)

# Backup: Plaquette visualization of lattice Hamiltonian

## Compact MCS Hamiltonian

**Magnetic field term:** similar to QED (without monopoles)
**Electric field term:** modified by Chern-Simons term

$$\hat{H} = \sum_{x\in\text{sites}} \frac{e^2}{2a^2}\left[\left(\hat{p}_{x;1} - \frac{ka^2}{4\pi}\hat{A}_{x-\hat{2};2}\right)^2 + \left(\hat{p}_{x;2} + \frac{ka^2}{4\pi}\hat{A}_{x-\hat{1};1}\right)^2\right] + \frac{1}{2e^2}\left(\Box\hat{A}_{x;1,2}\right)^2$$

$$= \frac{e^2}{2a^2}\sum_{\text{plaquettes}}\left[\left(\hat{p}_1 - \left(\frac{ka^2}{4\pi}\right)\hat{A}_2\right)^2 + \left(\hat{p}_2 + \left(\frac{ka^2}{4\pi}\right)\hat{A}_1\right)^2\right]$$

$$+ \frac{1}{2e^2}\sum_{\text{plaquettes}}\left(\begin{tikzcd} \cdot \arrow[d, "-\hat{A}_2"'] & \cdot \arrow[l, "-\hat{A}_1"'] \\ \cdot \arrow[r, "\hat{A}_1"'] & \cdot \arrow[u, "\hat{A}_2"'] \end{tikzcd}\right)^2$$

**Plaquette operator:**

$$\Box\hat{A}_{x;1,2} \equiv \hat{A}_{x;1} + \hat{A}_{x+\hat{1};2} - \hat{A}_{x+\hat{2};1} - \hat{A}_{x;2}$$