Reviewing Time Evolution Methods for Quantum Systems

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Suzuki-Trotter decompositions, aka splitting methods or (Lie) product formulae

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Error estimation and efficiency [Omelyan + *CPC* **146** (2002), *CPC* **151** (2003)]

$$\mathrm{e}^{(A+B)h+\mathcal{O}_1\,h+\mathcal{O}_3\,h^3+\mathcal{O}_5\,h^5+\cdots} = \mathrm{e}^{Aa_1h}\,\mathrm{e}^{Bb_1h}\cdots\mathrm{e}^{Bb_qh}\,\mathrm{e}^{Aa_{q+1}h}$$

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\mathrm{Eff}_4 &= \frac{1}{q^4\sqrt{\sum_{j=1}^{6}|\gamma_j|^2}}
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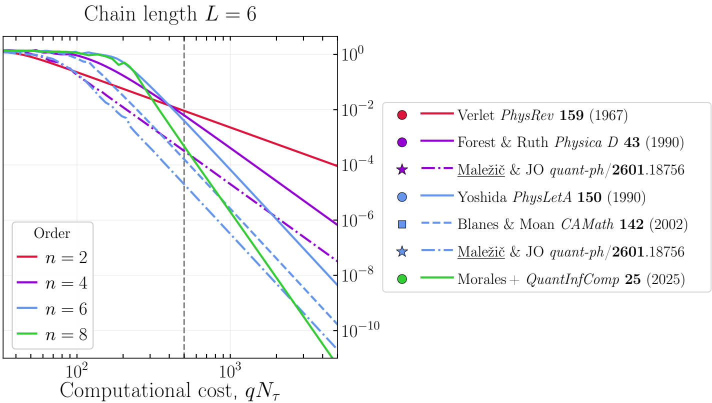
Scheme construction and efficiency verification

[Maležič & JO *quant-ph*/**2601**.18756; Maležič *Qiskit* (2026)]

[Maležič & JO *PoS* **LATTICE** (2025)]



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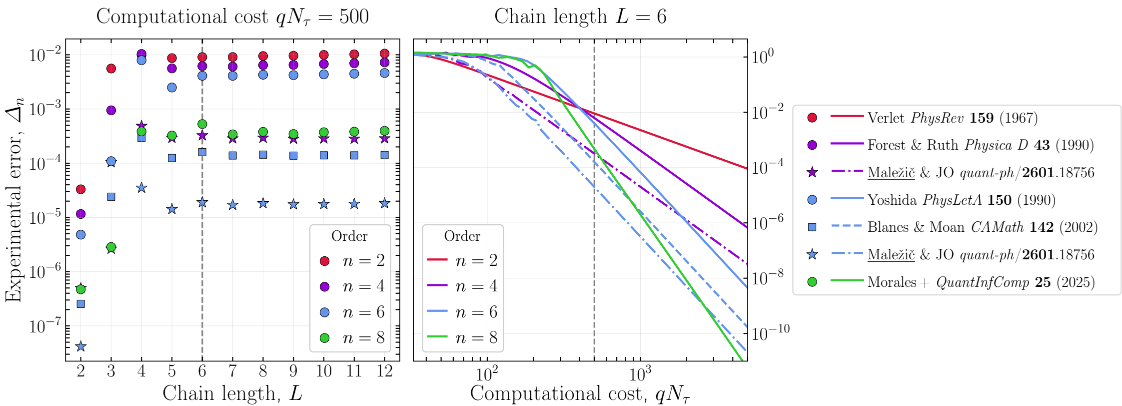

General error bounds [Childs+ *PRX* **11** (2021); Heyl, Hauke+ *SciAdv* **5** (2019)]

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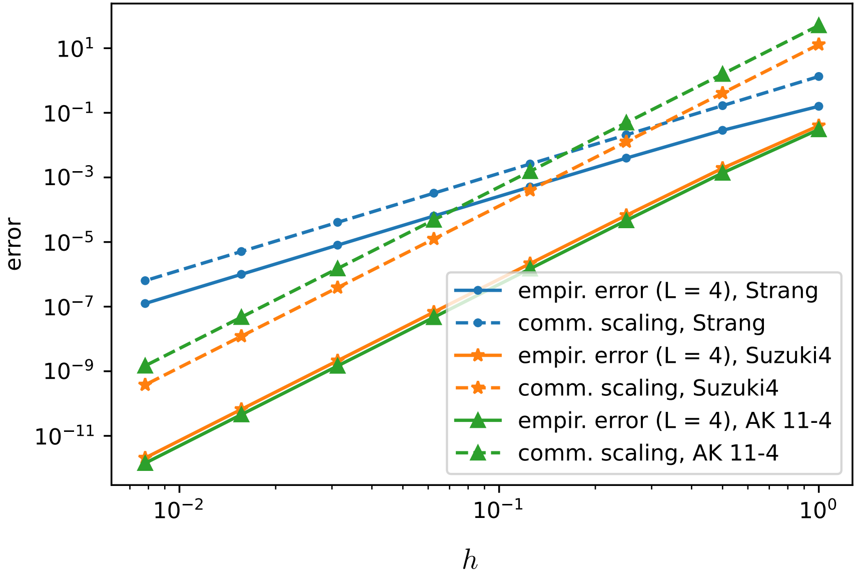
$$\left. + \sum_{j=s+1}^{K} \sum_{\substack{p_{s+1} + \cdots + p_j = n \\ p_j \neq 0}} \binom{n}{p_{s+1}, \ldots, p_j} \left\| \mathrm{ad}_{A_{s+1}}^{p_{s+1}} \cdots \mathrm{ad}_{A_j}^{p_j} B_j \right\|_2 \right),$$

$$\mathrm{ad}_C^p D := \underbrace{[C, \cdots [C}_{p\text{-times}}, D] \cdots] \quad \textit{and} \quad B_j := \sum_{\ell=1}^{j-1} A_\ell, \quad j = 2, \ldots, K\,.$$

Bound application [Schubert & Mendl *GitHub* (2023), *PRB* **108** (2023)]



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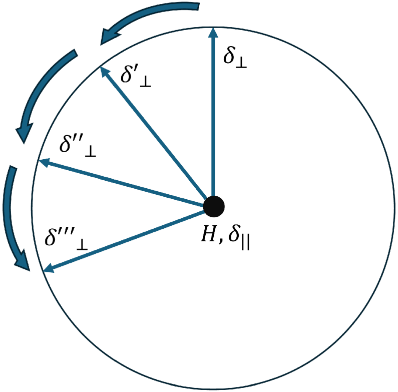

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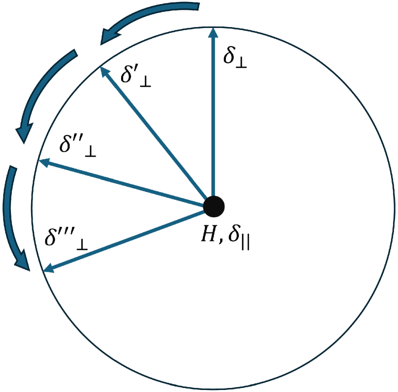

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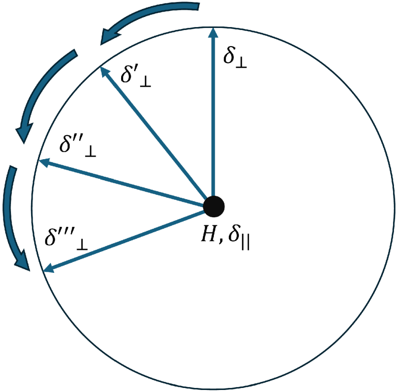

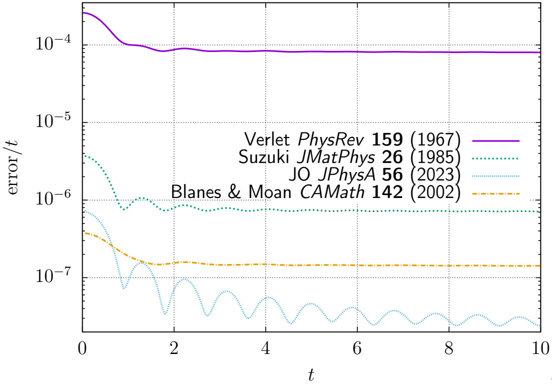

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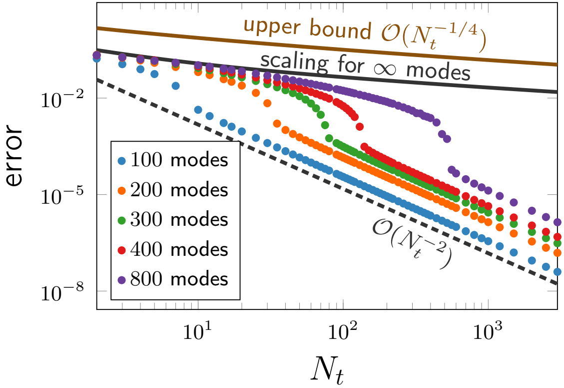
- Finite modes (discrete space) act as regulator.



Lower error bounds [Hahn+ *PRA* **111** (2025)]

$$\left\| \left(\mathrm{e}^{-\mathrm{i}\,\frac{t}{n}A} \mathrm{e}^{-\mathrm{i}\,\frac{t}{n}B} \right)^n - \mathrm{e}^{-\mathrm{i}\,t(A+B)} \right\| \geq \frac{t^2}{2n} \left(1 - \frac{t}{2n} \|A\| \right)$$

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$$\times\left(\|[A,B]\|-\frac{t}{2}\|[A+B,[A,B]]\|\right)-\frac{t^3}{24n^2}\|[A,[A,B]]\|-\frac{t^3}{12n^2}\|[B,[B,A]]\|$$

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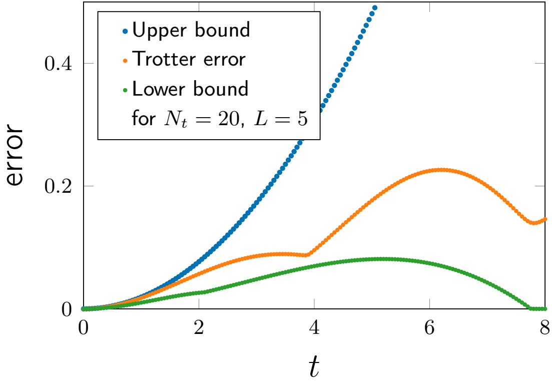

Decompositions into 2 operators [JO *JPhysA* **56** (2023); McLachlan *JSciComp* **16** (1995)]

$$\mathrm{e}^{(A+B)h+\mathcal{O}\left(h^{n+1}\right)} = \mathrm{e}^{Aa_1 h}\,\mathrm{e}^{Bb_1 h}\,\mathrm{e}^{Aa_2 h}\cdots\mathrm{e}^{Bb_q h}\,\mathrm{e}^{Aa_{q+1} h}$$

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\end{aligned}
$$

$$
\Rightarrow \mathrm{e}^{h\sum\limits_{k=1}^{\mathit{\Lambda}} A_k+\mathcal{O}\left(h^{n+1}\right)}
$$

$$
= \left(\prod_{k=1}^{\mathit{\Lambda}} \mathrm{e}^{A_kc_1h}\right)\left(\prod_{k=\mathit{\Lambda}}^{1} \mathrm{e}^{A_kd_1h}\right)\cdots\left(\prod_{k=1}^{\mathit{\Lambda}} \mathrm{e}^{A_kc_qh}\right)\left(\prod_{k=\mathit{\Lambda}}^{1} \mathrm{e}^{A_kd_qh}\right)
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Time-dependent Hamiltonian [Ikeda + *Quantum* **7** (2023); Suzuki *ProcJapAcB* **69** (1993)]

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U(t_1,t_0) &\equiv \mathcal{T}\left\{\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t_1}\mathrm{d}t H(t)}\right\} \\
&= \lim_{N_t\to\infty}\left(\mathrm{e}^{-\mathrm{i}\,H(t_1-h)h}\cdots\mathrm{e}^{-\mathrm{i}\,H(t_0+h)h}\mathrm{e}^{-\mathrm{i}\,H(t_0)h}\right)
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&= \lim_{N_t\to\infty}\left(\mathrm{e}^{-\mathrm{i}\,H(t_1-h)h}\cdots\mathrm{e}^{-\mathrm{i}\,H(t_0+h)h}\mathrm{e}^{-\mathrm{i}\,H(t_0)h}\right)
\end{aligned}
$$

- Introduce symbolic time-shift operator:

$$
\mathrm{e}^{-\mathrm{i}\,A_k(t)}\mathrm{e}^{-\mathrm{i}\,\mathcal{T}h} := \mathrm{e}^{-\mathrm{i}\,\mathcal{T}h}\mathrm{e}^{-\mathrm{i}\,A_k(t+h)}
$$

Time-dependent Hamiltonian [Ikeda+ *Quantum* **7** (2023); Suzuki *ProcJapAcB* **69** (1993)]

$$
\begin{aligned}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle &= H(t)|\psi(t)\rangle \\
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$$
\Rightarrow \mathcal{T}\left\{\mathrm{e}^{-\mathrm{i}\int_t^{t+h}\mathrm{d}s H(s)}\right\} = \mathrm{e}^{-\mathrm{i}(H(t)+\mathcal{T})h}
$$

Generalised Trotter formula for $H(t) = \sum_{k=1}^{\Lambda} A_k(t)$

[Suzuki *ProcJapAcB* **69** (1993); Hatano & Suzuki *math-ph*/**0506007**]

$$\mathcal{T}\left\{\mathrm{e}^{-\mathrm{i}\int_t^{t+h}\mathrm{d}s H(s)}\right\} = \mathrm{e}^{-\mathrm{i}(H(t)+\mathcal{T})h}$$
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$$\Rightarrow S_n(t,h) = \prod_{i=1}^{q}\left[\left(\prod_{k=1}^{\Lambda}\mathrm{e}^{c_i A_k(t+e_i h)h}\right)\left(\prod_{k=\Lambda}^{1}\mathrm{e}^{d_i A_k(t+e_i h)h}\right)\right],$$

$$e_i := \sum_{j=i}^{q} d_j + \sum_{j=i+1}^{q} c_q$$

Asymmetric Trotterization

$$H = H_0 + \varepsilon H_1 \,, \quad \varepsilon \ll 1$$

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$$\alpha = \frac{1-6a}{24} \quad \Rightarrow \text{ set } a = \frac{1}{6}$$

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- Interaction picture [Low & Wiebe *quant-ph*/**1805**.00675; Rajput, Roggero + *Quantum* **6** (2022); Ciavarella & Bauer *PRL* **133** (2024); Bosse + *NatCommun* **16** (2025)]:

$$H_1 = \sum_{\gamma=1}^{\Gamma} H_1^{\gamma}\,,$$

$$\mathrm{e}^{-ihH} = \mathrm{e}^{-ihH_0}\prod_{\gamma}\left(\mathrm{e}^{ihH_0}\mathrm{e}^{-ih(H_0+\varepsilon H_1^{\gamma})}\right) + \mathcal{O}\left(\varepsilon^2 h^2\right)$$

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Processed methods

[Butcher *ConfNumDiffEq* (1969); Blanes + *AplNumMat* **204** (2024); Morales + *QuantInfComp* **25** (2025)]

$$S(h) = P(h)\,\hat{S}(h)\,P(h)^{-1}$$
$$\Rightarrow S(h)^N = P(h)\,\hat{S}(h)^N\,P(h)^{-1}$$

- Kernel $\hat{S}$ cheap, processor P rarely needed

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- Can help with efficient long-time evolution
- Useless if intermediate measurements are required

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Multi-product formulae

[Blanes + *CelMechDynAst* **75** (1999); Chin *CelMechDynAst* **106** (2008); Low + *quant-ph*/**1907**.11679]

- Linear combinations of order n_0 product formula S_{n_0}

$$S^{(s)}_{n,n_0}(h) = \sum_{j=1}^{m} a_j \, S_{n_0}\left(\frac{h}{k_j}\right)^{k_j}$$

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Quantum computing post-processing:

[Carrera Vazquez + *Quantum* **7** (2023); Childs & Wiebe *QantInfComp* **12** (2012); Park + *quant-ph*/**2602**.01713]

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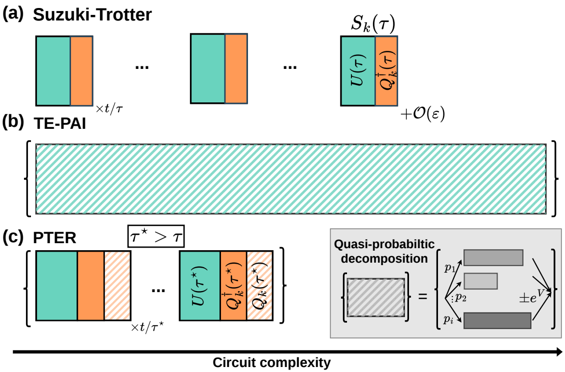
Randomised methods

[Campbell *PRL* **123** (2019); Granet & Dreyer *QuantInf* **10** (2024); Kiss, Roggero + *Quantum* **7** (2023); Kiumi & Koczor *QuantSciTech* **10** (2025); Hayata & Kikuchi *quant-ph*/**2604**.02854; Wang, Sun + *quant-ph*/**2607**.11856] [Murota, Rinaldi + *quant-ph*/**2606**.29741]



- Sample few gates randomly, repeat many times

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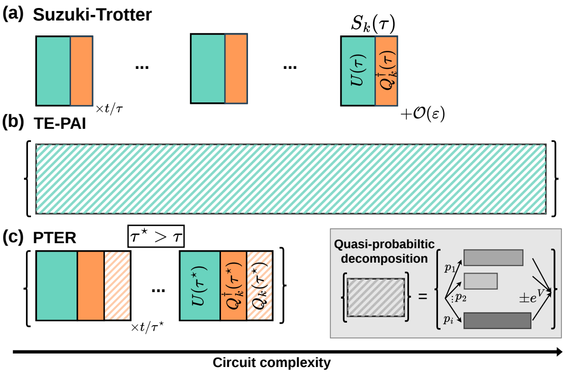

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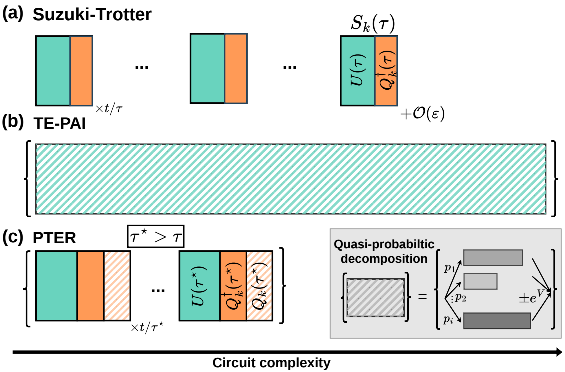

- Sample few gates randomly, repeat many times
- Reduces circuit depth
- Can eliminate Trotter error (continuous formulation)

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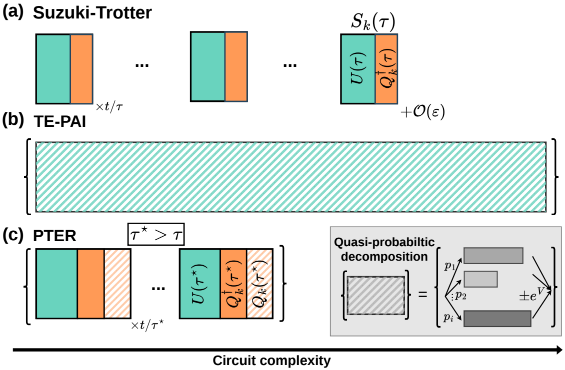

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- Introduces stochastic error

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Symplectic integrators [Omelyan + *CPC* **151** (2003)]

- Numerically solve equations of motion

$$
\begin{aligned}
\dot{x} &= p \\
\dot{p} &= -V'(x)
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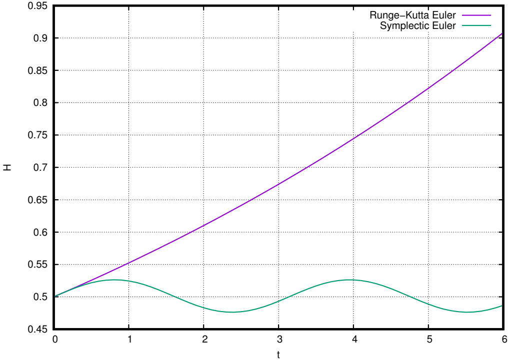

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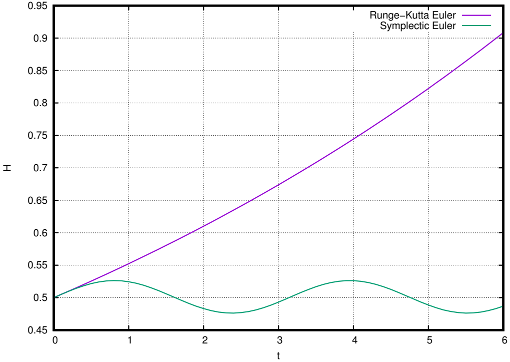

$$x_1 = x_0 + a_1 h p_0$$
$$p_1 = p_0 - b_1 h V'(x_1)$$
$$x_2 = x_1 + a_2 h p_1$$
$$\vdots$$

Comparison to Runge-Kutta

$$\begin{aligned} H &= T(p) + V(x)\,, \\ \dot{x} &= T'(p) \\ \dot{p} &= -V'(x) \end{aligned}$$

Explicit Runge-Kutta	Symplectic	Implicit Runge-Kutta
$x_{n+1} = x_n + hT'(p_n)$	$x_{n+1} = x_n + hT'(p_n)$	$x_{n+1} = x_n + hT'(p_{n+1})$
$p_{n+1} = p_n - hV'(x_n)$	$p_{n+1} = p_n - hV'(x_{n+1})$	$p_{n+1} = p_n - hV'(x_{n+1})$

Special case of Trotterization [Omelyan+ *CPC* **151** (2003)]

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \mathrm{e}^{t\left(p\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial p}\right)} \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}$$

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$$\Rightarrow \mathcal{O}_5 = \gamma_1[A,[A,[A,[A,B]]]] + \gamma_2[A,[A,[B,[A,B]]]] + \gamma_3[B,[A,[A,[A,B]]]] + \cancel{\gamma_4[B,[B,[B,[A,B]]]]} + \gamma_5[B,[B,[A,[A,B]]]] + \cancel{\gamma_6[A,[B,[B,[A,B]]]]}$$

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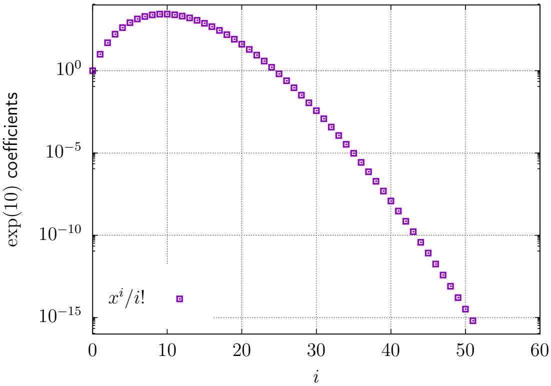
Polynomial expansions

$$\mathrm{e}^{Hh} = \sum_{i=0}^{k} \frac{(Hh)^i}{i!} + \mathcal{O}\left(h^{k+1}\right)$$



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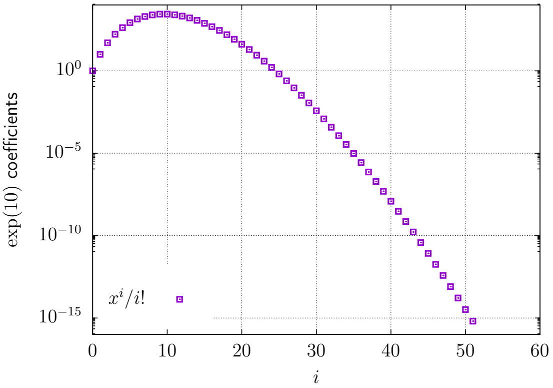

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$$\left.\left. + \frac{h}{n-1}H\left(|\psi\rangle + \frac{h}{n}H|\psi\rangle\right)\cdots\right)\right)$$

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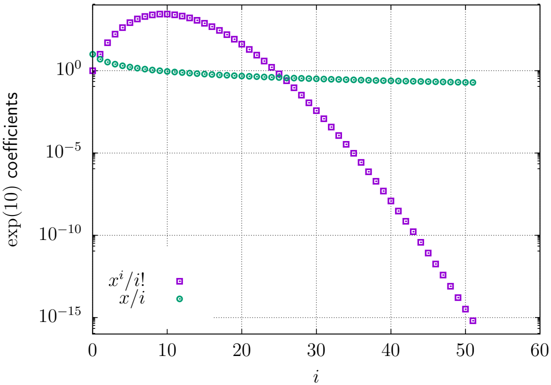

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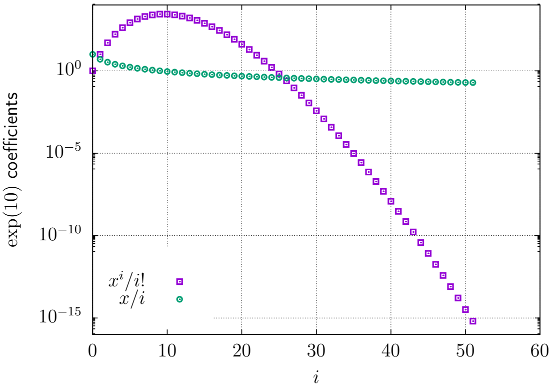

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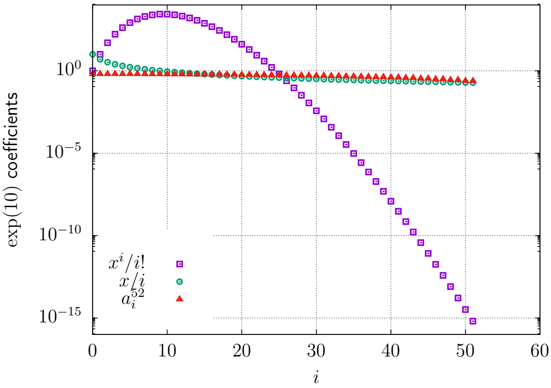

Taylor vs. Chebyshev expansion [JO *PoS* **LATTICE** (2023)]

$$
\mathrm{e}^{Hh} = \lim_{k\to\infty} \sum_{i=0}^{k} \mu_i T_i \left(\frac{H}{\Gamma}\right) ,
$$

$$
\mu_0 = I_0(\Gamma h)\,,\; \mu_i = 2I_i(\Gamma h)\,,
$$

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T_0(H) &= \mathbb{1} \, , \\
T_1(H) &= H \, , \\
T_{i+1}(H) &= 2H \cdot T_i(H) - T_{i-1}(H)
\end{aligned}
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Taylor vs. Chebyshev expansion [JO *PoS* **LATTICE** (2023)]

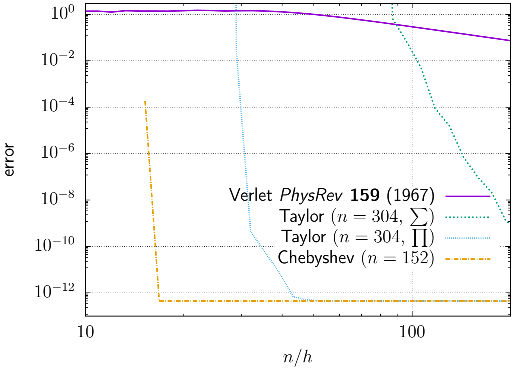
$$\mathrm{e}^{Hh} = \lim_{k\to\infty} \sum_{i=0}^{k} \mu_i T_i \left(\frac{H}{\Gamma}\right) \,,$$

$$\mu_0 = I_0(\Gamma h) \,, \; \mu_i = 2I_i(\Gamma h) \,,$$

$$T_0(H) = \mathbb{1} \,,$$

$$T_1(H) = H \,,$$

$$T_{i+1}(H) = 2H \cdot T_i(H) - T_{i-1}(H)$$



Choose your own adventure!

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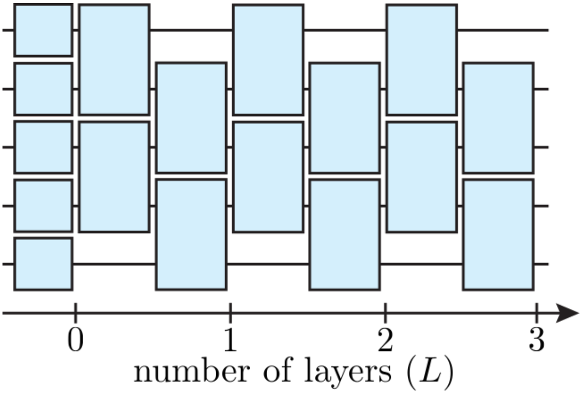
8. Integration on manifolds

Quantum circuit optimisation

[Le + *Quantum* **9** (2025); Mansuroglu + *QuantSciTech* **8** (2023); Tepaske + *SciPostPhys* **14** (2023)]

[Mc Keever & Lubasch *PRR* **5** (2023)]

Ansatz: brickwall circuit



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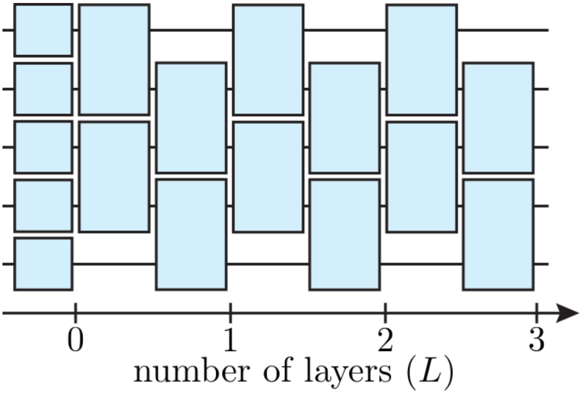

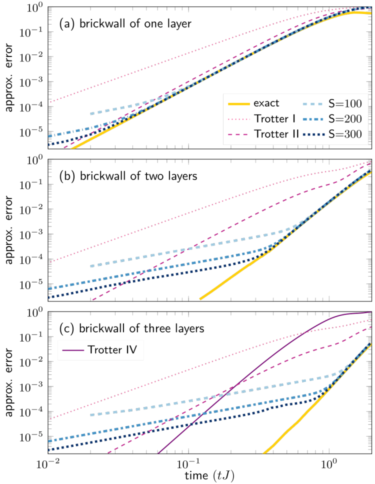

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Time-dependent variational principle (TDVP)

[Haegeman + *PRL* **107** (2011), *PRB* **94** (2016); Paeckel + *AnnalsPhys* **411** (2019)]

- Tensor network state $\psi(X)$ parametrised by X of limited dimension D

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[Haegeman + *PRL* **107** (2011), *PRB* **94** (2016); Paeckel + *AnnalsPhys* **411** (2019)]

- Tensor network state $\psi(X)$ parametrised by X of limited dimension D
- Trotterization "time-evolving block decimation (TEBD)" increases D with every step

$$-\,\mathrm{i}\,H\,|\psi(X)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(X)\rangle$$

Time-dependent variational principle (TDVP)

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- TDVP directly evolves X via projection

$$
\begin{aligned}
-\mathrm{i}\, H \,|\psi(X)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(X)\rangle \\
&= |\nabla_X \psi(X)\rangle \cdot \dot{X}
\end{aligned}
$$

Time-dependent variational principle (TDVP)

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$$\begin{aligned} -\mathrm{i}\,H\,|\psi(X)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(X)\rangle \\ &= |\nabla_X \psi(X)\rangle \cdot \dot{X} \end{aligned}$$

Time-dependent variational principle (TDVP)

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$$
\begin{aligned}
-\mathrm{i}\, H \, |\psi(X)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t} \, |\psi(X)\rangle \\
&= |\nabla_X \psi(X)\rangle \cdot \dot{X}
\end{aligned}
$$

$$
\Rightarrow \left\langle \nabla_{\bar{X}} \psi(\bar{X}) \,\middle|\, \nabla_X \psi(X) \right\rangle \cdot \dot{X} = -\mathrm{i} \left\langle \nabla_{\bar{X}} \psi(\bar{X}) \right| H \left| \psi(X) \right\rangle
$$

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Numerical integration on manifolds

[Crouch & Grossman *JNonLinSci* **3** (1993); Munthe-Kaas *AplNumMat* **29** (1999)]

	Runge-Kutta	Crouch-Grossman
ODE	$\dot{z} = f(t,z)$	$\dot{\psi} = H(t,\psi) \cdot \psi$

Numerical integration on manifolds

[Crouch & Grossman *JNonLinSci* **3** (1993); Munthe-Kaas *AplNumMat* **29** (1999)]

	Runge-Kutta	Crouch-Grossman
ODE	$\dot{z} = f(t,z)$	$\dot{\psi} = H(t,\psi) \cdot \psi$
step	$z(t+h) = z(t) + h \sum_{i=1}^{s} b_i k_i$	$\psi(t+h) = \mathrm{e}^{h \sum_{i=1}^{s} \tilde{b}_i k_i} \cdot \psi(t)$

Numerical integration on manifolds

[Crouch & Grossman *JNonLinSci* **3** (1993); Munthe-Kaas *AplNumMat* **29** (1999)]

	Runge-Kutta	Crouch-Grossman
ODE	$\dot{z} = f(t,z)$	$\dot{\psi} = H(t,\psi) \cdot \psi$
step	$z(t+h) = z(t) + h \sum_{i=1}^{s} b_i k_i$	$\psi(t+h) = \mathrm{e}^{h \sum_{i=1}^{s} \tilde{b}_i k_i} \cdot \psi(t)$
	$u_i = h \sum_{j=1}^{s} a_{ij} k_j$	$u_i = h \sum_{j=1}^{s} \tilde{a}_{ij} k_j$

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	$u_i = h\sum_{j=1}^{s} a_{ij} k_j$	$u_i = h\sum_{j=1}^{s} \tilde{a}_{ij} k_j$
	$k_i = f\left(t + c_i h, z(t) + u_i\right)$	$k_i = H\left(t + \tilde{c}_i h, \mathrm{e}^{u_i} \cdot \psi(t)\right)$

Numerical integration on manifolds

[Crouch & Grossman *JNonLinSci* **3** (1993); Munthe-Kaas *AplNumMat* **29** (1999)]

	Runge-Kutta	Munthe-Kaas
ODE	$\dot{z} = f(t,z)$	$\dot{\psi} = H(t,\psi) \cdot \psi$
step	$z(t+h) = z(t) + h\sum_{i=1}^{s} b_i k_i$	$\psi(t+h) = \mathrm{e}^{h\sum_{i=1}^{s} b_i \tilde{k}_i} \cdot \psi(t)$
	$u_i = h\sum_{j=1}^{s} a_{ij} k_j$	$u_i = h\sum_{j=1}^{s} a_{ij} k_j$
	$k_i = f\left(t + c_i h, z(t) + u_i\right)$	$k_i = H\left(t + c_i h, \mathrm{e}^{u_i} \cdot \psi(t)\right)$
		$\tilde{k}_i := \sum_{l=1}^{n-1} \frac{\mathcal{B}_l}{l!} \mathrm{ad}_{u_i}^{l} k_i$

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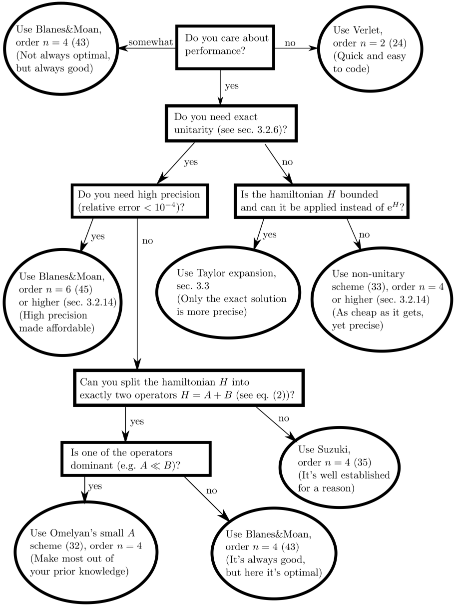
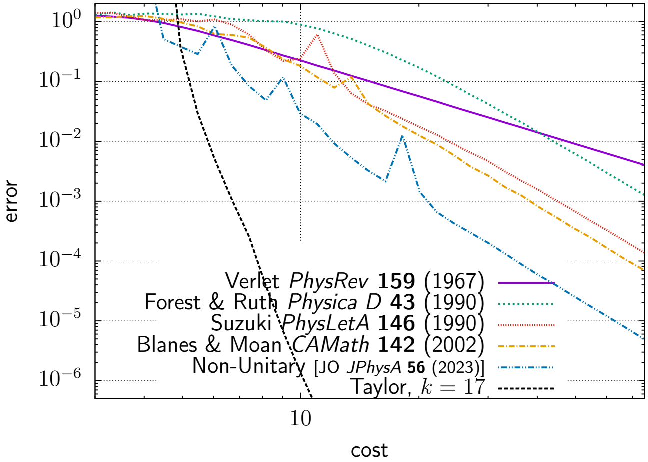
How to choose... [JO *JPhysA* **56** (2023)]



Three simple tricks for better Trotterization



0. Don't use Forest & Ruth.
1. Use 2-operator methods for Λ operators.
2. Use complex coefficients.
3. Consider alternatives.

Heisenberg model and error estimation

[Childs + *PRX* **11** (2021); Heyl, Hauke + *SciAdv* **5** (2019); Schubert & Mendl *PRB* **108** (2023)]

$$H = \sum_{i=1}^{L} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z \right) ,$$

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\,Ht} &\equiv U(t) = U(h)^{t/h} \\ &= S(h)^{t/h} + \mathcal{O}\left(h^n\right) , \end{aligned}$$

$$S(h) = \mathrm{e}^{\mathrm{i}\,H_1^x c_1 h} \mathrm{e}^{\mathrm{i}\,H_1^y c_1 h} \mathrm{e}^{\mathrm{i}\,H_1^z c_1 h} \mathrm{e}^{\mathrm{i}\,H_2^x c_1 h} \mathrm{e}^{\mathrm{i}\,H_2^y c_1 h} \mathrm{e}^{\mathrm{i}\,H_2^z c_1 h} \cdots \mathrm{e}^{\mathrm{i}\,H_1^z d_q h} \mathrm{e}^{\mathrm{i}\,H_1^y d_q h} \mathrm{e}^{\mathrm{i}\,H_1^x d_q h} ,$$

$$\begin{aligned} \text{error} &= \frac{1}{\sqrt{N}} \left\| U(t) - S(h)^{t/h} \right\|_{\mathsf{F}} \\ &= \frac{1}{\sqrt{N}} \sqrt{\sum_v \left| U(t) \cdot v - S(h)^{t/h} \cdot v \right|^2} \end{aligned}$$

Proof of validity of Λ-operator decomposition I

For every $i \in \{1, \ldots, q\}$ (setting $d_0 = 0$)

$$\begin{aligned} c_i + d_{i-1} &= a_i - d_{i-1} + d_{i-1} \\ &= a_i \,, \\ d_i + c_i &= b_i - c_i + c_i \\ &= b_i \,, \end{aligned}$$

i.e. decomposition works trivially for $\Lambda = 2$.

Proof of validity of Λ-operator decomposition II

Use the Baker–Campbell–Hausdorff formula:

$$\left(\prod_{k=1}^{\Lambda} \mathrm{e}^{A_k c_i h}\right) = \mathrm{e}^{C_1 c_i h + C_2 (c_i h)^2 + C_3 (c_i h)^3 + \cdots}$$

$$\left(\prod_{k=\Lambda}^{1} \mathrm{e}^{A_k d_i h}\right) = \mathrm{e}^{D_1 d_i h + D_2 (d_i h)^2 + D_3 (d_i h)^3 + \cdots}$$

where $C_1 = D_1 = \sum_{k=1}^{\Lambda} A_k$ and the remaining operators C_l, D_l are some linear combinations of lth order commutators $[A_{k_1}, [A_{k_2}, \cdots [A_{k_{l-1}}, A_{k_l}] \cdots]]$.

Proof of validity of Λ-operator decomposition III

C_l, D_l are independent of i and for every $\Lambda \geq 2$ all the C_l (D_l) are mutually non-commuting and linearly independent in general. Thus, the right hand side is completely independent of Λ.
For $\Lambda = 2$:

$$\mathrm{e}^{h \sum\limits_{k=1}^{\Lambda} A_k + \mathcal{O}\left(h^{n+1}\right)} = \mathrm{e}^{C_1 c_1 h + C_2 (c_1 h)^2 + \cdots} \mathrm{e}^{D_1 d_1 h + D_2 (d_1 h)^2 + \cdots} \\ \dots \mathrm{e}^{C_1 c_q h + C_2 (c_q h)^2 + \cdots} \mathrm{e}^{D_1 d_q h + D_2 (d_q h)^2 + \cdots}$$

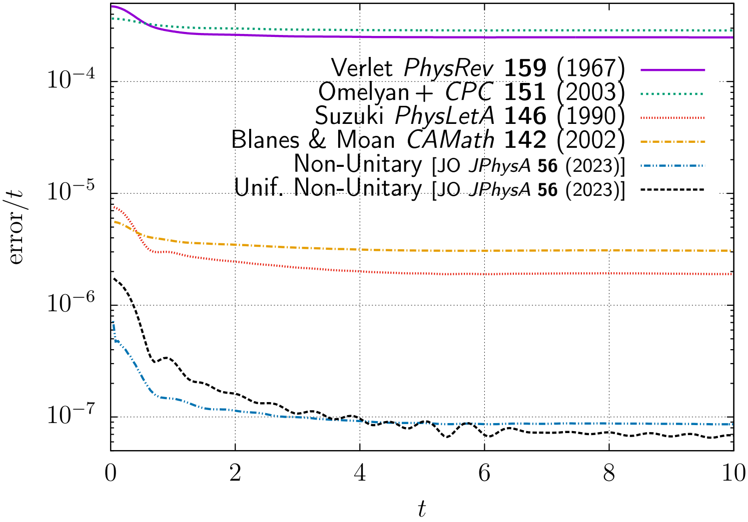
Since the c_i, d_i fulfil this condition for every C_i, D_i (they solve the corresponding system of polynomial equations), they automatically fulfil it for every Λ.

Derivation time-dependent Hamiltonian evolution

Effective Hamiltonian $H_{\mathsf{eff}}(t) \equiv \sum_k A_k(t) + \mathcal{T}$, Trotterize:

$$
\begin{aligned}
S_n(t,h) = {} & \mathrm{e}^{c_1 \mathcal{T} h} \left(\prod_{k=1}^{\Lambda} \mathrm{e}^{c_1 A_k(t) h} \right) \left(\prod_{k=\Lambda}^{1} \mathrm{e}^{d_1 A_k(t) h} \right) \mathrm{e}^{d_1 \mathcal{T} h} \cdots \\
& \left(\prod_{k=\Lambda}^{1} \mathrm{e}^{d_{q-1} A_k(t) h} \right) \mathrm{e}^{d_{q-1} \mathcal{T} h} \, \mathrm{e}^{c_q \mathcal{T} h} \left(\prod_{k=1}^{\Lambda} \mathrm{e}^{c_q A_k(t) h} \right) \left(\prod_{k=\Lambda}^{1} \mathrm{e}^{d_q A_k(t) h} \right) \mathrm{e}^{d_q \mathcal{T} h} \\
= {} & \mathrm{e}^{\mathcal{T} h} \left(\prod_{k=1}^{\Lambda} \mathrm{e}^{c_1 A_k(t+(1-c_1)h) h} \right) \left(\prod_{k=\Lambda}^{1} \mathrm{e}^{d_1 A_k(t+(1-c_1)h) h} \right) \cdots \\
& \left(\prod_{k=\Lambda}^{1} \mathrm{e}^{d_{q-1} A_k(t+(d_q+c_q+d_{q-1})h) h} \right) \left(\prod_{k=1}^{\Lambda} \mathrm{e}^{c_q A_k(t+d_q h) h} \right) \left(\prod_{k=\Lambda}^{1} \mathrm{e}^{d_q A_k(t+d_q h) h} \right)
\end{aligned}
$$

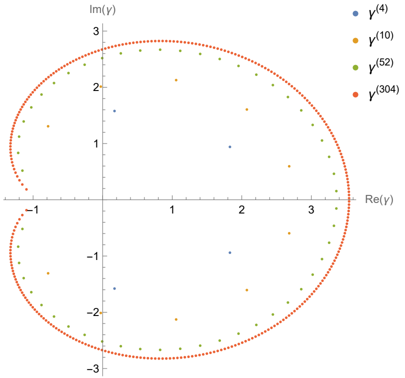
Error accumulation II

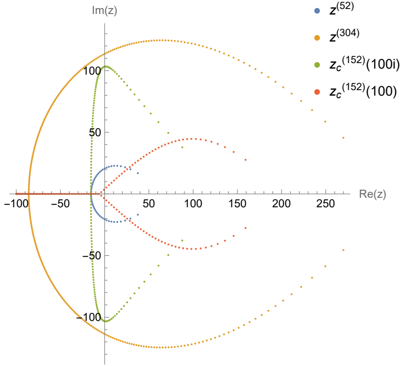


$$c_1 = 0.1 + 0.025\,\mathrm{i}$$
$$d_1 = 0.1 + 0.025\,\mathrm{i}$$
$$c_2 = 0.1 - 0.066\,\mathrm{i}$$
$$d_2 = 0.1 - 0.066\,\mathrm{i}$$
$$c_3 = 0.1 + 0.082\,\mathrm{i}$$
$$\text{order} = 4\,,$$
$$\text{cycles} = 5\,,$$
$$\mathrm{Eff}_4 = 6.38$$

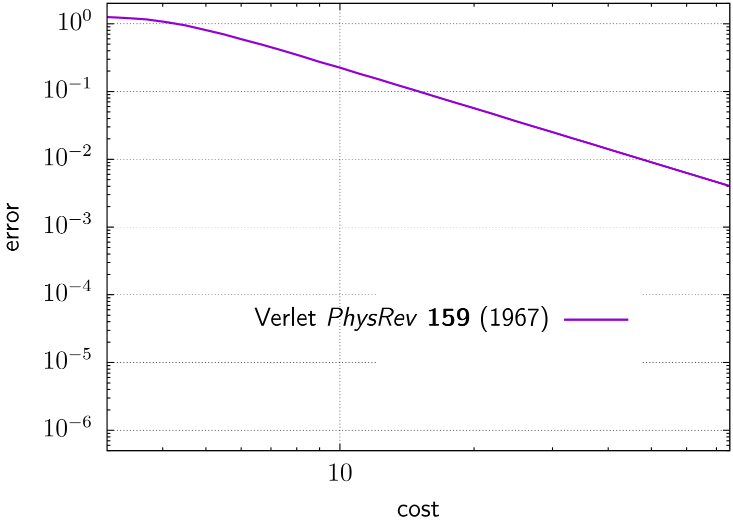
Convergence regions [JO *PoS* **LATTICE** (2023)]

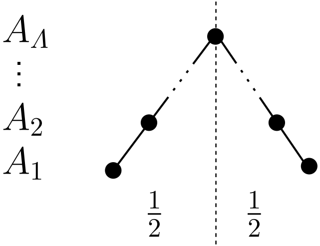
$$\mathrm{e}^{Hh} = \prod_{i=1}^{k} \left(1 + \frac{\gamma_i^{(k)}}{k} Hh \right) + \mathcal{O}\left(h^{k+1}\right) \qquad \propto \prod_{i=1}^{k} \left(Hh - z_i^{(k)} \right) + \mathcal{O}\left(h^{k+1}\right)$$





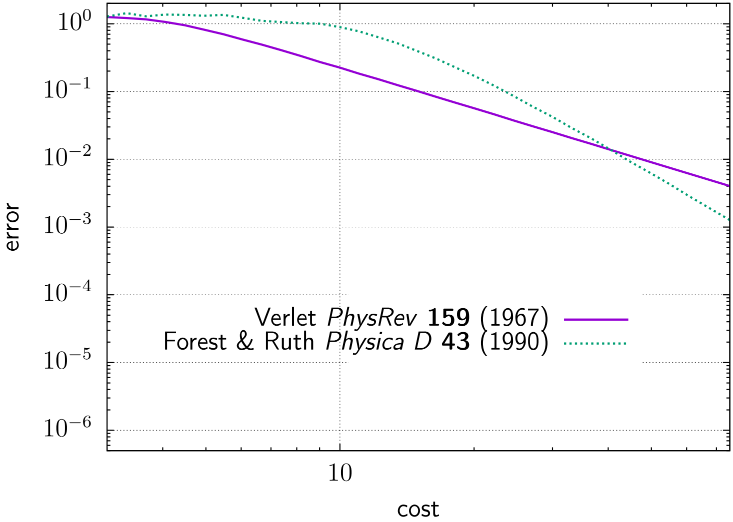
Benchmarking the Heisenberg model

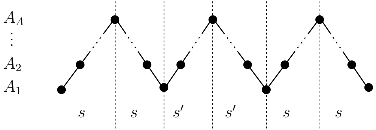




$$\text{order} = 2\,,$$
$$\text{cycles} = 1$$

Benchmarking the Heisenberg model





$$s = \frac{1}{2(2-\sqrt[3]{2})}$$

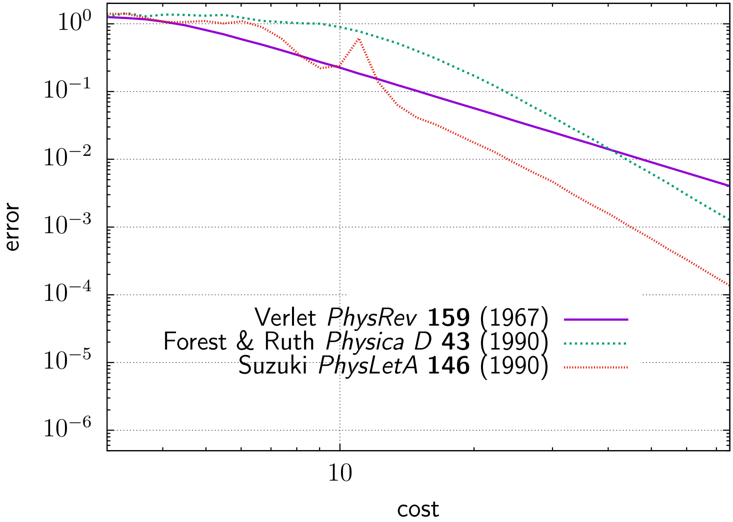
$$s' = \frac{1}{2} - 2s$$

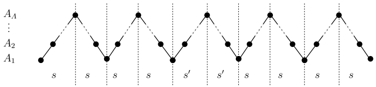
$$\text{order} = 4\,,$$

$$\text{cycles} = 3\,,$$

$$\text{Eff}_4 = 0.315$$

Benchmarking the Heisenberg model





$$s = \frac{1}{2(4-\sqrt[3]{4})}$$

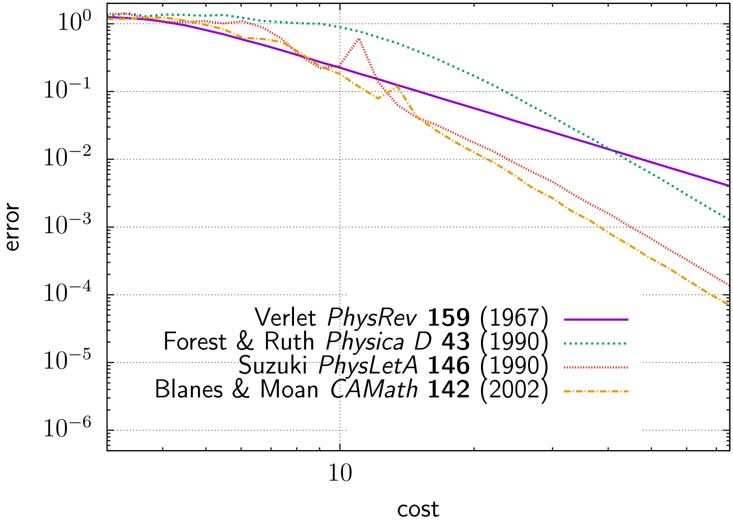
$$s' = \frac{1}{2} - 4s$$

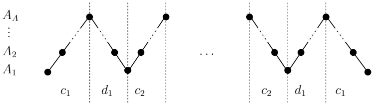
$$\text{order} = 4\,,$$

$$\text{cycles} = 5\,,$$

$$\mathrm{Eff}_4 = 1.10$$

Benchmarking the Heisenberg model





$c_1 \approx 0.08 \quad d_1 \approx 0.13$

$c_2 \approx 0.22 \quad d_2 \approx -0.36$

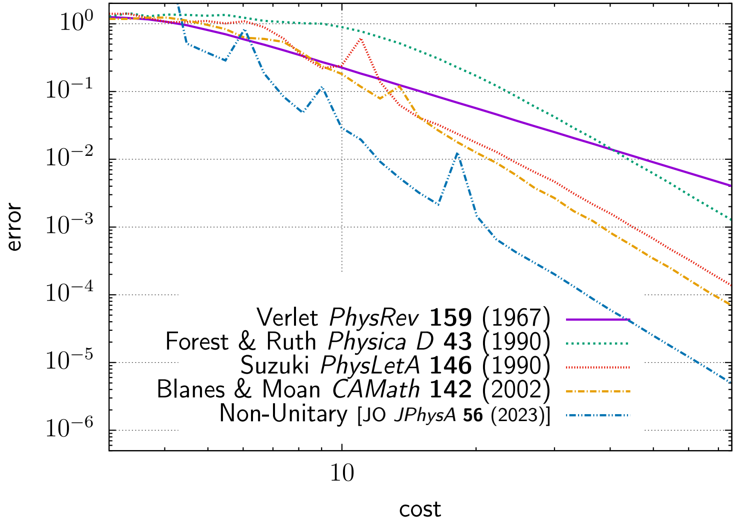
$c_3 \approx 0.32 \quad d_3 \approx 0.11$

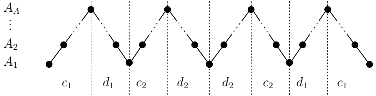
$$\text{order} = 4\,,$$

$$\text{cycles} = 6\,,$$

$$\mathrm{Eff}_4 = 10.2$$

Benchmarking the Heisenberg model





$$c_1 \approx 0.10 + 0.02\,\mathrm{i}$$

$$d_1 \approx 0.15 + 0.07\,\mathrm{i}$$

$$c_2 \approx 0.14 + 0.07\,\mathrm{i}$$

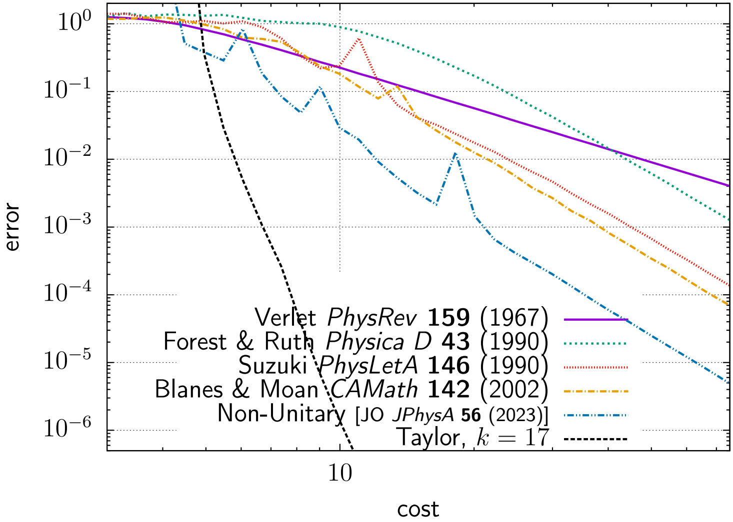
$$d_2 \approx 0.11 - 0.16\,\mathrm{i}$$

$$\text{order} = 4\,,$$

$$\text{cycles} = 4\,,$$

$$\mathrm{Eff}_4 = 29.9$$

Benchmarking the Heisenberg model



$$e^{Hh} = \lim_{k\to\infty} \sum_{i=0}^{k} \frac{(Hh)^i}{i!}$$

$$\left| \frac{(\lambda_{\mathsf{max}}(H)h)^k}{(k+1)!} \right| < \varepsilon$$

Adaptive step size [Blanes + *AplNumMat* **146** (2019); Zhao + *PRXQ* **4** (2023)]