A ribbon ZX-Calculus for gauge theory

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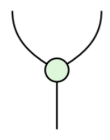
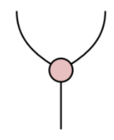
Introduction

ZX calculus is a diagrammatic language for quantum information processing. It has become a useful tool in quantum computing, with applications to circuit optimization , quantum error correction, measurement-based quantum computation and quantum chemistry. On the other hand, its relationship to quantum field theory has only recently begun to be explored. In this work, we further develop this connection by providing a generalization of ZX-calculus to two dimensional Yang–Mills theory with a compact gauge group. We formulate a ribbon ZX calculus in which the re-write rules become manifest via a 2D topology, and a novel interpretation of ribbon diagrams in terms of entangled particles.

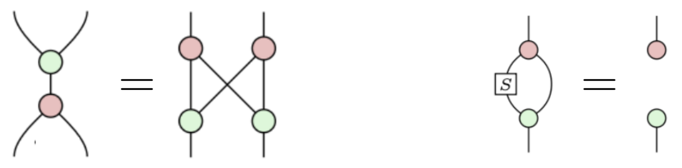
What is ZX

ZX is a graphical calculus built from the two "spiders" describing two product structures on the Hilbert space of a qubit.

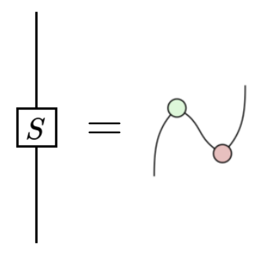
 $= |0\rangle\langle 00| + |1\rangle\langle 11|$  $= |0\rangle\,\langle 00| + \langle 11|) + |0\rangle\,(\langle 01| + \langle 10|)$

They define two strongly complementary Frobenius algebras. Acting on qubit wavefunctions, which are functions in $L^2(\mathbb{Z}_2)$, the X spider (red) is the convolution product while the Z spider (green) is pointwise multiplication. The Z spider generalizes to a copy tensor with n inputs and m outputs. Similarly the X spider copies in the X basis $(|+\rangle\langle ++| + |-\rangle\langle --|)$)and generalizes in the same manner.

The key advantage of ZX-calculus over a generic tensor network is a concise set of rewrite rules for simplifying complex algebraic expressions. Two important rewrites are the bi-algebra rule (strong complementarity) and the Hopf rule:

 (1)

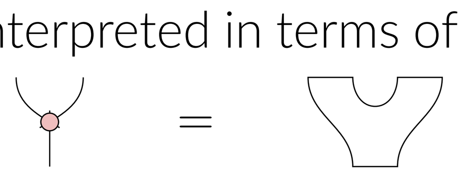
where the anti-pode S can be defined by:

 (2)

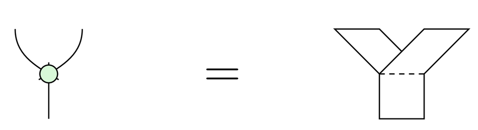
Introduction to gauge theory and 2D Yang Mills

Non-abelian gauge theory is the framework that underlies the interactions of gluons, the mediators of the strong force. It is based on a continuous gauge group G, and describes a Lie- algebra valued gauge field A_μ that subject to the gauge redundancy $A_\mu \to gA_\mu g^{-1} + g^{-1}dg$, where $g(x)$ is a spacetime dependent group element. We will take G to be a compact Lie group.

2D Yang Mills is a gauge theory that depends on the spacetime metric only via the area form: as a result it is an almost topological theory. Given a suitable boundary condition, the Hilbert space on an interval is $L^2(G)$, which is infinite dimensional. Just like the Hilbert space of a qubit, it supports a convolution and pointwise product, which plays the role of the X and Z spider. In this case, these products can be interpreted in terms of path integral processes. The X spider is

 $: |g\rangle \otimes |h\rangle \to |gh\rangle\ .$ (3)

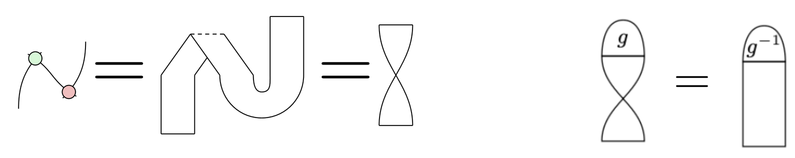
Read from top to bottom, this describes the fusion of two intervals into one; read from bottom to top, this is a co-product describes the splitting of the interval. The Z spider is:

 $: |g\rangle \otimes |h\rangle \to \delta_{g,h}\,|g\rangle\ ,$ (4)

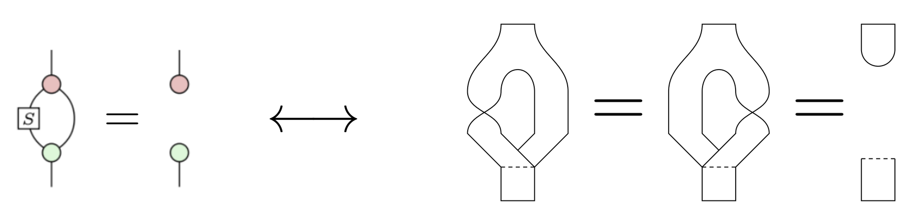
which describes the stacking of two intervals on top of each other. In terms of wavefunctions on the group these correspond to

$$f_1 * f_2(g) = \int dh f_1(gh) f_2(h^{-1}) dh \qquad f_1 \cdot f_2(g) = f_1(g) f_2(g). \tag{5}$$

The ribbon description allows us to describe the antipode topologically

 (6)

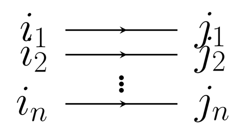
The Hopf rule becomes

 (7)

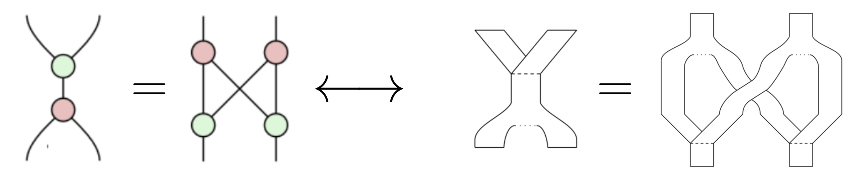
We will derive this diagrammatically in the right panel.

Ribbons as superposition of Worldsheets

When G is a matrix Lie group, $L^2(G)$ has a string basis given by products of matrix elements. Given $U \in G$ in the fundamental $N \times N$ representation, each matrix element U_{ij} represents a single open string with Chan Paton indices $i, j = 1, \cdots N$ labeling the "quarks" at its endpoints. A general basis element is a product of n strings

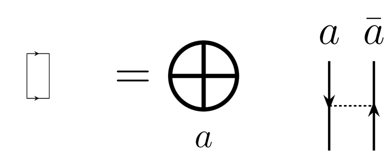
$U_{IJ} = U_{i_1 j_1} \cdots U_{i_n j_n},$  (8)

with n ranging from 0 (the constant function $f(U) = 1$) to ∞. This describes a stack of n open strings. The Z spider can be interpreted as the stacking of worldsheets swept out by these open strings, while the X spider is the string interaction that fuses the open strings endpoints. This worldsheet point of view makes the bi-algebra rule topologically manifest. It corresponds to "peeling apart" the co-product:

 (9)

Ribbons as entangled partic;es

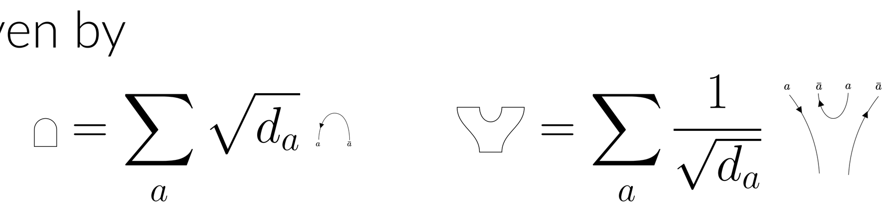
Since, G acts on itself by left and right multiplication, $L^2(G)$ decomposes into irreps of G, denoted V_a below:

$L^2(G) = V_a \otimes V_a^*$  (10)

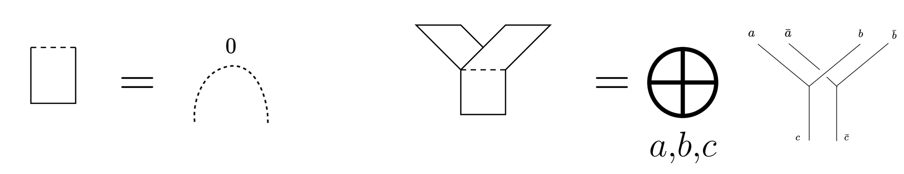
This decomposition means that the Euclidean time evolution of an interval representing $L^2(G)$ is equivalent to the entangled worldlines of particles and anti-particles labelled by a. Concretely, this means there is a basis for $L^2(G)$ spanned by representation matrix elements:

$$a_{ij}(U), \quad i, j = 1, \cdots d_a, \qquad \int dU\ a_{ij}(U) b_{kl}(U^{-1}) = \frac{\delta_{ab}\delta_{il}\delta_{kj}}{d_a}, \tag{11}$$

where d_a is the dimension of the representation. In this representation basis, the path integral processes reduce to worldline processes that follow the diagrammatic rules of the Tannakian category Rep(G). For example, the X spider and its unit are given by

 (12)

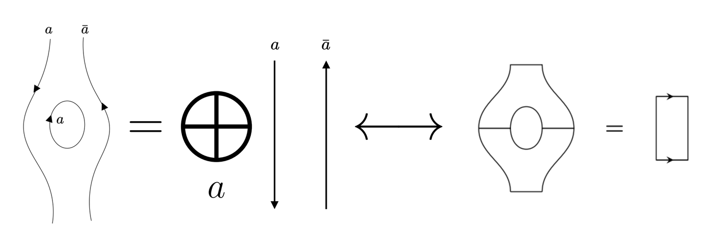
The left diagram describes a sum over pair creation of all particles and antiparticles from the vacuum;attaching these worldlines to the right diagram "caps off" one of the input to reproduce the propagator on $L^2(G)$.Similarly, the Z-unit and Z spider are:

 (13)

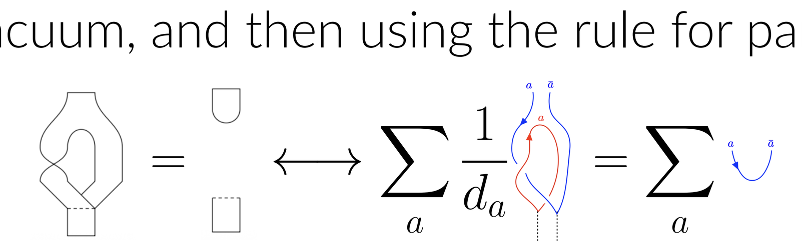
The left diagram describes a vacuum particle corresponding to the function $f(U) = 1$; attaching to a Z spider input trivializes it. The Z spider itself is an entangled sum of particle fusion diagrams given by a trivalent vertex: this describes the Clebsch-Gordan decomposition from tensoring two irreps.

Particle loops and the Hopf rule

Every loop of particle worldline describes a trace that produces a dimension factor d_a. This is an important rule that makes manifest the fact the X spider is an isometry:

 (14)

The Hopf rule can be derived by observing that a particle and anti- particle must annihilate into the vacuum, and then using the rule for particle loops.

 (15)

Outlook: q-Deformation and large N

Our ribbon description of Z and X spiders on $L^2(G)$ generalize to the case when G is a quantum group. When G is compact, Rep$G$ becomes a braided ribbon category, and our formalism describes q-deformed two dimensional yang mills. The large N limit produces topological string theory, where the q-deformed ZX calculus has been employed for computations of anyonic entanglement entropy (arxiv:2607.03526) .