SU(2) Quantum Link Model on the (2+1)-d Square and Honeycomb Lattices

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Motivation and Aim

Wilson's formulation of lattice gauge theories use an infinite dimensional Hilbert space, while quantum link models realise exact gauge invariance using generalised link operators [1].

We investigate the SU(2) gauge theory quantum link model in (2+1)-dimensions on both the square and honeycomb lattices.

Quantum Link Models

For SU(2) pure gauge theory, we can parametrise the links as [2]

$$U_{x,\mu} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}_{x,\mu} = U^0_{x,\mu}\mathbb{1} + i\vec{U}_{x,\mu}\vec{\sigma}$$

$$U_{x,\mu} = \begin{pmatrix} U^0 + iU^3 & iU^1 + U^2 \\ iU^1 - U^2 & U^0 - iU^3 \end{pmatrix}_{x,\mu}$$

and we have the Hamiltonian and Gauss law

$$H_\square = \frac{J}{8}\left[\mathrm{Tr}(U_1U_2U_3^\dagger U_4^\dagger) + \mathrm{Tr}(U_4U_3U_2^\dagger U_1^\dagger)\right]$$

$$\vec{G}_x|\Psi\rangle = 0 \quad \Longleftrightarrow \quad \begin{cases} G^3_x|\Psi\rangle = 0 \\ G^+_x|\Psi\rangle = 0 \end{cases}$$

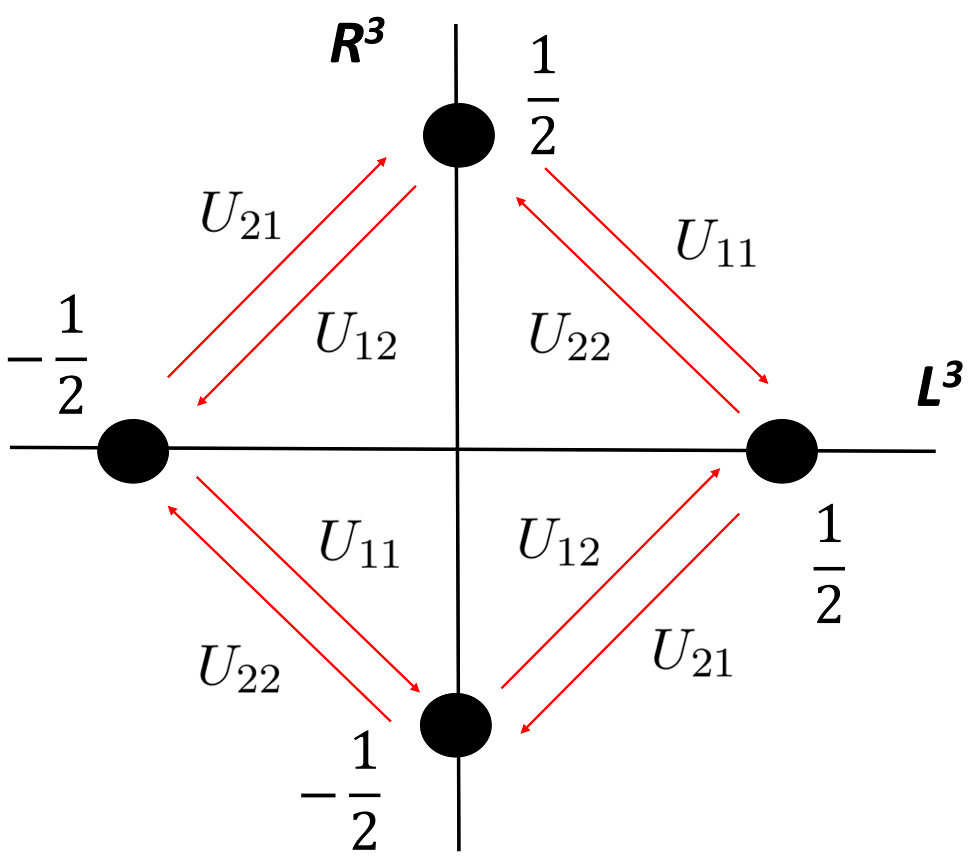
Since the elements are now operators, there are non-trivial commutation relations. By first considering the left and right contributions of the SU(2) gauge transformation generator

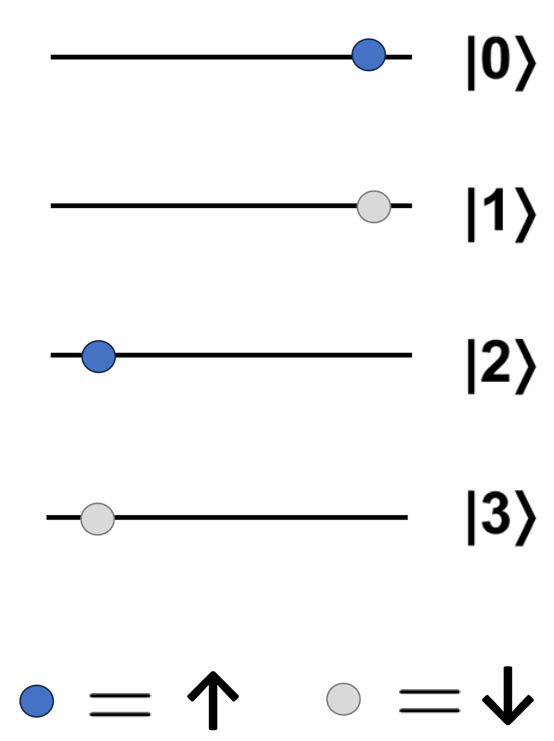
$$\vec{G}_x = \sum_\mu (\vec{R}_{x-\hat{\mu},\mu} + \vec{L}_{x,\mu})$$

the new commutation relations are

$$\begin{aligned} [U^0_{x,\mu}, U^0_{y,\nu}] &= 0, \\ [U^0_{x,\mu}, U^i_{y,\nu}] &= 2i\delta_{x,y}\delta_{\mu\nu}(R^i_{x,\mu} - L^i_{x,\mu}), \\ [U^i_{x,\mu}, U^j_{y,\nu}] &= 2i\delta_{x,y}\delta_{\mu\nu}\epsilon_{ijk}(R^k_{x,\mu} + L^k_{x,\mu}) \end{aligned}$$

The full commutation relations form an SO(5) algebra, and for this work we use the spinorial representation (for work with the vector representation see [3]). The third component of the left and right flux operators can take values in {-1/2,+1/2}, so each link has four degrees of freedom, called "rishons". The corresponding weight diagram is [4]



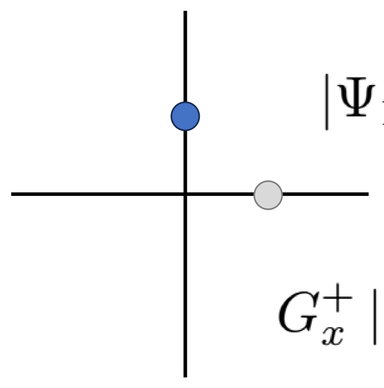


Local Gauge Invariant Basis

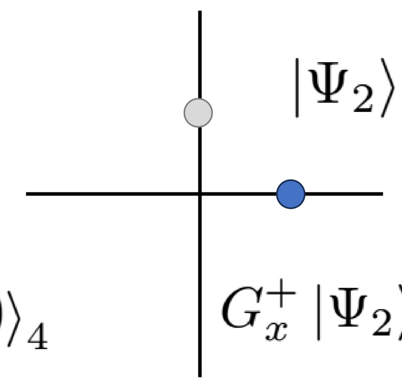
Instead of worrying about all possible microscopic rishon states, we can take advantage of Gauss's law and only consider the possible gauge singlets we can form at a site.

Square Lattice

Consider the following two site configurations



$$|\Psi_1\rangle = |2\rangle_1|3\rangle_2|\emptyset\rangle_3|\emptyset\rangle_4$$
$$G^3_x|\Psi_1\rangle = 0$$
$$G^+_x|\Psi_1\rangle = 2|2\rangle_1|2\rangle_2|\emptyset\rangle_3|\emptyset\rangle_4$$



$$|\Psi_2\rangle = |3\rangle_1|2\rangle_2|\emptyset\rangle_3|\emptyset\rangle_4$$
$$G^3_x|\Psi_2\rangle = 0$$
$$G^+_x|\Psi_2\rangle = 2|2\rangle_1|2\rangle_2|\emptyset\rangle_3|\emptyset\rangle_4$$

Neither obey Gauss law but we can take a linear combination of the two states which does. Thus for where *i* and *j* are the links with rishons and *k*'s are the links without rishons, we have six gauge singlets of the form

$$|B_{ij}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle)\prod_{k\neq i,j}|\emptyset_k\rangle$$

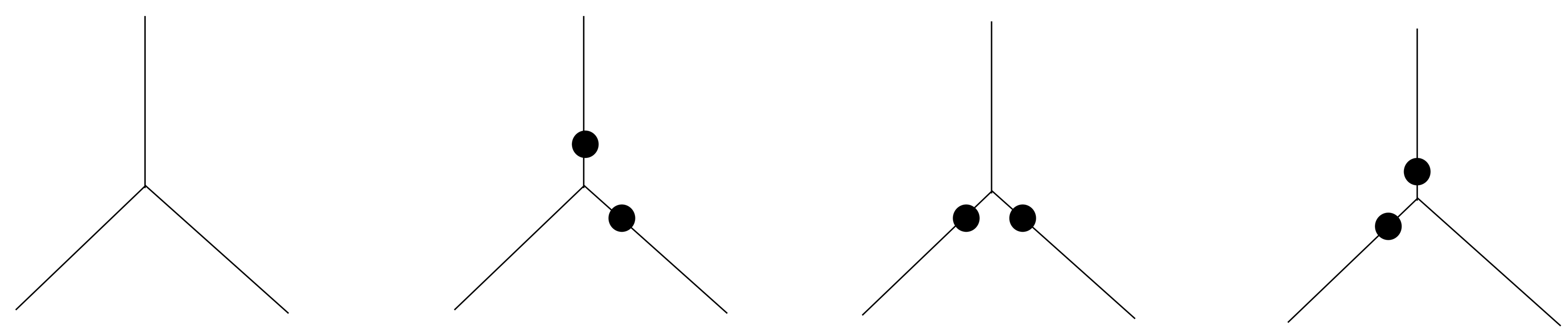
We also have a trivial gauge singlet with no rishons at a site, and two singlets with four beads at a site, constructed by coupling two pairs of spin-1/2 singlets and triplets respectively to total spin 0.

$$|S_1\rangle = \frac{1}{2}(|\uparrow\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle)$$

$$|T_1\rangle = -\frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle) + \frac{1}{2\sqrt{3}}(|\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle)$$

Honeycomb Lattice

The honeycomb lattice can take advantage of the gauge invariant basis even more so, since with only three legs per site, we can only form a singlet with no beads at a site, or two beads. This results in only four total states, where we use black beads to represent the appropriate linear combination.



Advantages

This formulation has advantage since using the black bead notation, constructing a full lattice amounts to assigning a state to each site under the constraint of exactly one bead per link, whilst the Hamiltonian acts to move each bead of a plaquette to the other end of the link. More notably, consider these examples for the number of full lattice states with and without using the gauge invariant basis.

Lattice Size	Non-Gauge Invariant Basis	Gauge Invariant Basis
2 x 2	*65536*	*50*
4 x 2	*4.3x10^{9}*	*1170*
4 x 4	*1.84x10^{19}*	*635450*

References: *[1] Nucl.Phys. B492 (1997) 455-474, [2] https://wiese.itp.unibe.ch/theses/hager_master.pdf, [3] arXiv:2602.23213Phys, [4] Rev. B 97, 205108 (2018)*