Error-Mitigated Quantum Simulation of Lattice Gauge Theory

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Motivation

Quantum simulation of LGTs

Hamiltonian quantum simulation provides direct access to real-time dynamics.

Long-term goal

3+1D SU(3) gauge theory with dynamical fermions

SU(2) as an intermediate step

Simpler than SU(3), but already non-Abelian

Common simplifications in 1D demonstrations

- Gauge fields are often integrated out; this simplification does not extend naturally to $d \geq 2$
- Explicit-gauge-field studies often use the lowest nontrivial truncation $j_{\max} = 1/2$

This work: explicit gauge fields and electric-flux sectors beyond $j_{\max} = 1/2$

Loop-String-Hadron (LSH) formulation

Theoretical details:
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Local gauge-invariant quantum numbers

- n_l: bosonic loop / electric-flux quantum number
- $n_i, n_o \in \{0, 1\}$: incoming and outgoing string occupation numbers
- $n_i = n_o = 1$: local hadron occupation

Key features

- The local non-Abelian Gauss-law constraints are solved by construction.
- Only a link-local Abelian flux-matching constraint (AGL) remains:

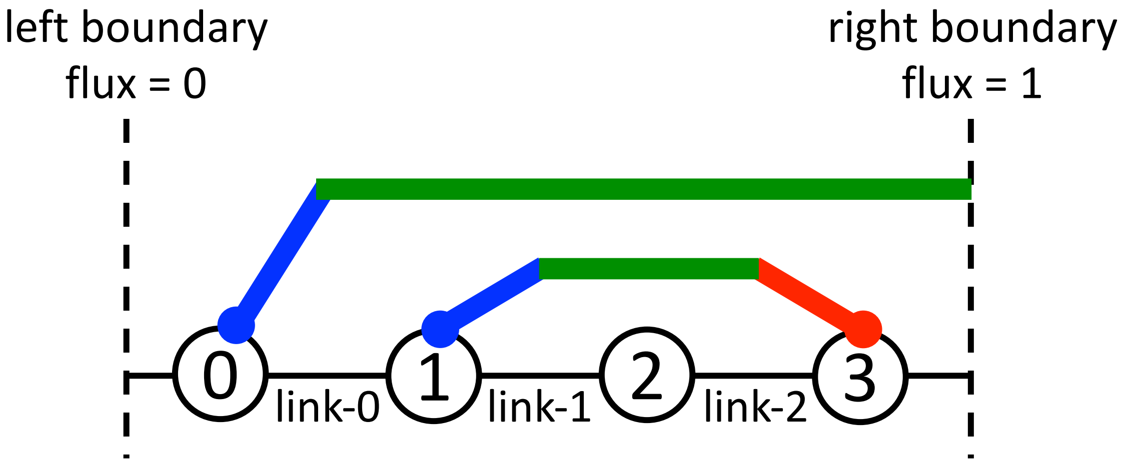
$$N_L(r) = N_R(r)$$

Two local expressions for the flux on link r:
$N_L(r)$: computed from site r, $N_R(r)$: computed from site $r+1$

- Natural truncation in the electric basis
- Gauge-field degrees of freedom remain explicit

Minimal system with an explicit multiqubit gauge-field register

Schematic illustration of the initial LSH basis state



$$n_i = 0 \quad 0 \quad 0 \quad 1$$
$$n_o = 1 \quad 1 \quad 0 \quad 0$$
$$n_l = 0 \quad 1 \quad 2 \quad 1$$

- 4 staggered lattice sites, 3 gauge links
- Open boundaries
- Boundary flux: left=0, right=1
- Local gauge-invariant basis: $|n_i, n_o, n_l\rangle$
- Accessible bosonic levels:
  - $n_l(0) = 0$
  - $n_l(1), n_l(3) \leq 1$
  - $n_l(2) \leq 2$
- Two-qubit register for $n_l(2)$
- Access to the $j = 1$ electric-flux sector

Key ingredients:

dynamical fermions, explicit gauge fields, a genuine multiqubit bosonic register

Hamiltonian $H = H_M + H_E + H_I$

Mass term

$$H_M = \mu \sum_{r=0}^{L-1} (-1)^r \left(n_i(r) + n_o(r)\right)$$

Staggered fermion mass
Alternating sign on even and odd sites
Diagonal in the computational basis

Electric term

$$H_E = \sum_{r=0}^{L-2} \frac{N_L(r)}{2} \left(\frac{N_L(r)}{2} + 1\right)$$

SU(2) electric Casimir energy on each link
$j(r) = N_L(r)/2$
Diagonal in the computational basis and directly measurable

Interaction term

H_I: gauge-matter hopping

Fermions hop between neighboring sites.
The relevant quantum numbers change coherently.
The AGL constraints are preserved.

The Hamiltonian remains local in the LSH variables: the mass and electric terms are local, and the interaction term couples only neighboring sites.

Electric flux and measured observable

Two local descriptions of the same link flux:

$$N_L(r) = n_l(r) + n_o(r)(1 - n_i(r))$$
$$N_R(r) = n_l(r+1) + n_i(r+1)(1 - n_o(r+1))$$

AGL: $N_L(r) = N_R(r)$

Boundary condition

left: $N_R(-1) = 0$
right: $N_L(3) = 1$

Electric energy observable:

$$h_E(r) = j(j+1)$$
$$= \frac{N_L(r)}{2}\left(\frac{N_L(r)}{2} + 1\right)$$

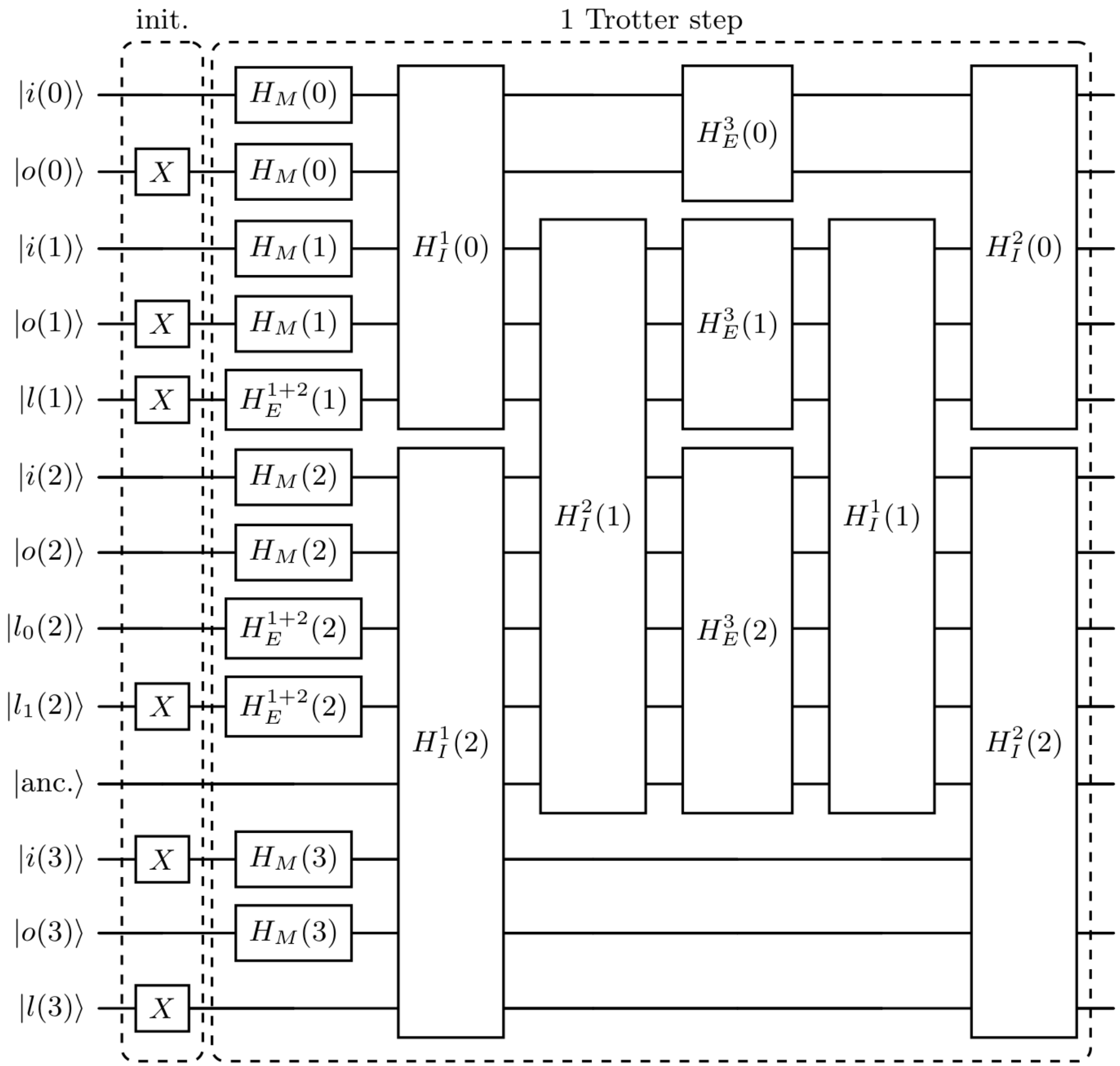
Observables for $r = 0, 1, 2$

$N = 0,1,2$ correspond to $j = 0, \frac{1}{2}, 1$, respectively.

Sensitive to electric-flux redistribution and string-breaking dynamics

Because $h_E(r)$ is diagonal in the computational basis,
it is evaluated directly from the measured (n_i, n_o, n_l) bitstrings.

Block-level circuit diagram for one Trotter step



- 13 qubits in total, including one ancilla
- First-order Trotterization with $\Delta t = 1.25$
- Benchmark: corresponding noiseless Trotterized circuit
- 226 two-qubit (ZZPhase) gates per Trotter step
- Depth reduction through optimized term ordering and ladder-operator cancellation
- Compilation with pytket and execution on Quantinuum H2-2
- Repeated one-step block for multiple Trotter steps

Error mitigation by post-selection

The ideal circuit exactly preserves the AGL constraints.
-> Violations of the AGL constraints can therefore be used for error detection.

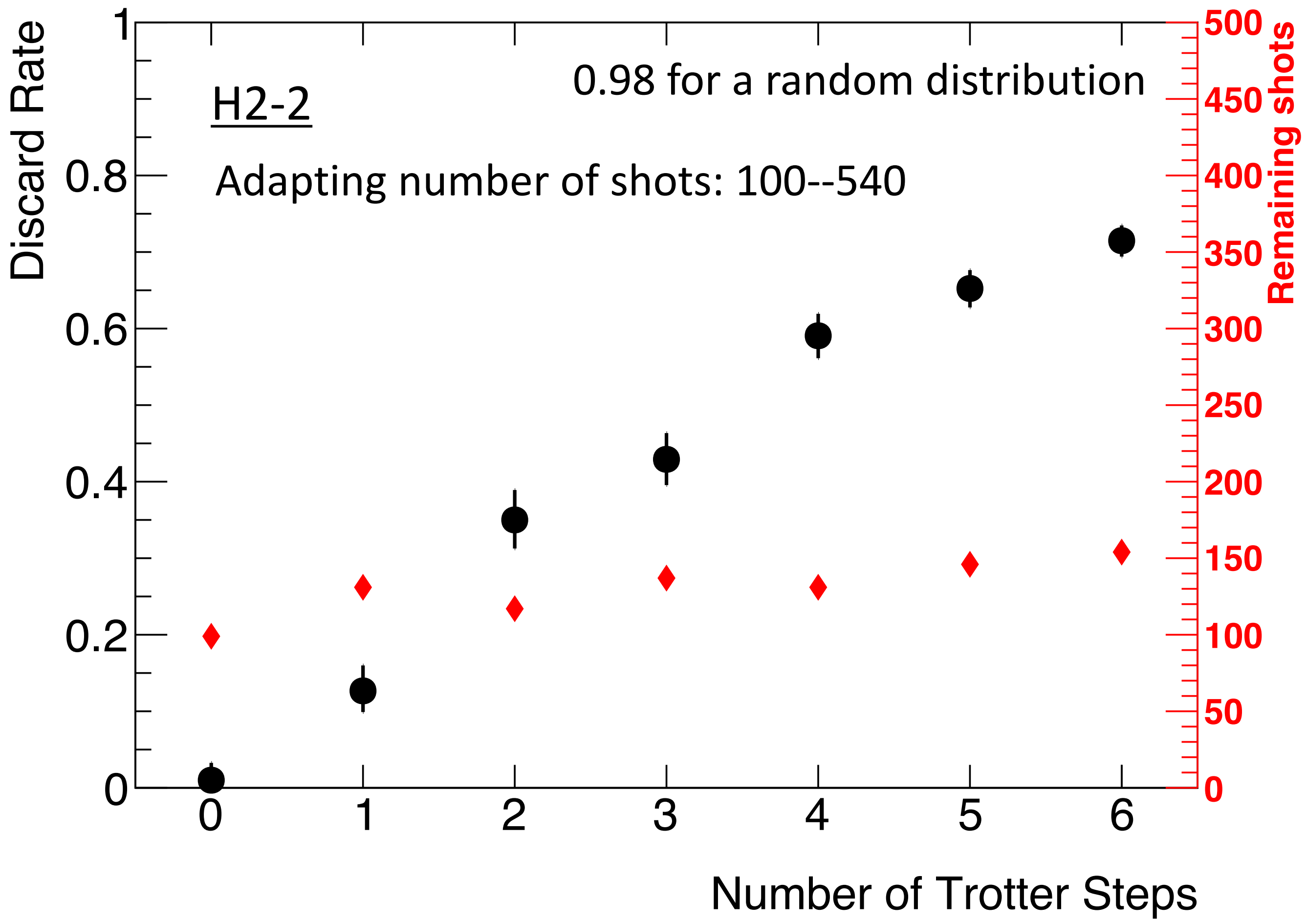
AGL condition

$$N_L(r) = N_R(r)$$
$$N_L(r) = n_l(r) + n_o(r)(1 - n_i(r))$$
$$N_R(r) = n_l(r+1) + n_i(r+1)(1 - n_o(r+1))$$

Post-selection conditions:

- AGL conditions for $0 \leq r \leq 2$
- Left boundary $N_R(-1) = 0$
- $|anc.\rangle = 0$

Discard rate with post-selection



Error mitigation by depolarizing-model rescaling

<u>Depolarizing noise model:</u>

The quantum state is gradually randomized toward the maximally mixed state.

$$\langle O \rangle_{\mathrm{noisy}} = (1-p)\langle O \rangle_{\mathrm{ideal}} + o_{\mathrm{tr}} p$$

$$p = 1.0 - \frac{\langle O \rangle_{\mathrm{noisy}} - o_{\mathrm{tr}}}{\langle O \rangle_{\mathrm{ideal}} - o_{\mathrm{tr}}}$$

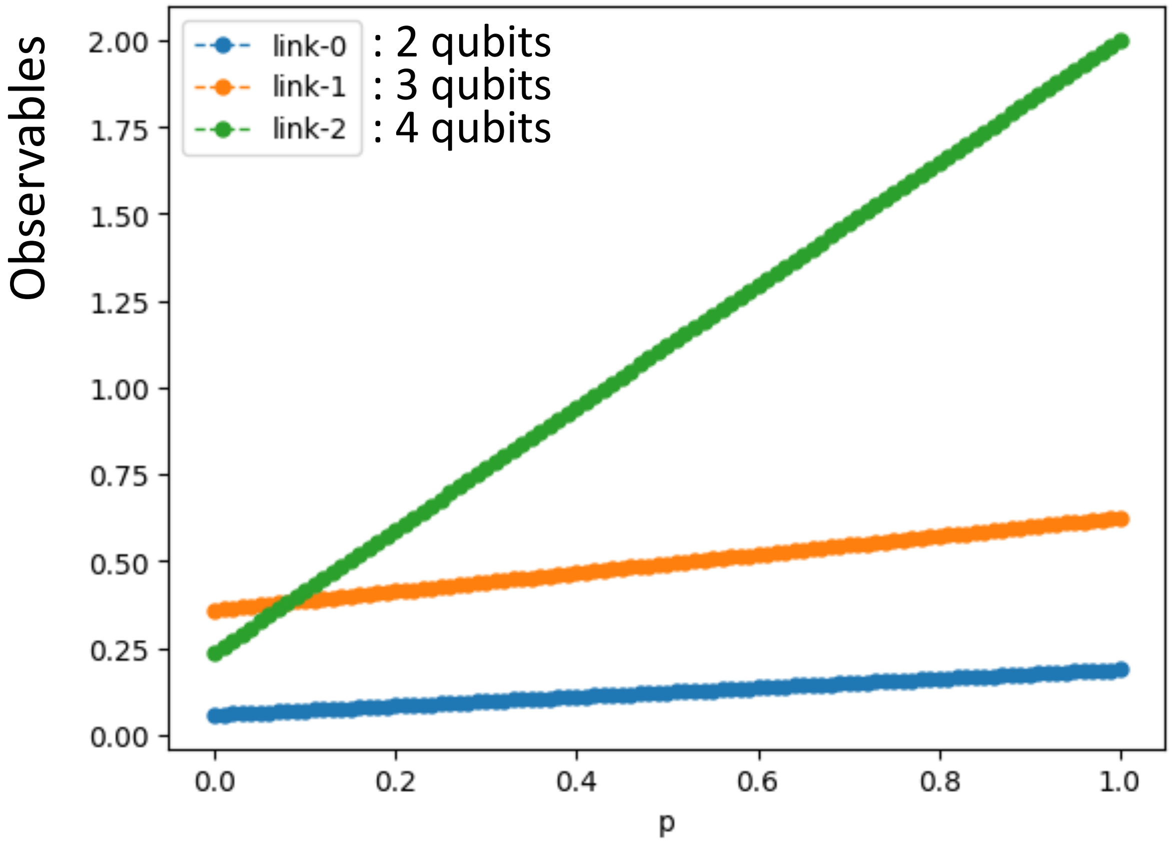
$o_{\mathrm{tr}} \equiv \mathrm{Tr}[O]/2^n$ is determined by the observables

p should be estimated from an independent test circuit

Once p is estimated, the error-mitigated expectation value can be obtained by rescaling:

$$\langle O \rangle_{\mathrm{mit}} = \frac{\langle O \rangle_{\mathrm{raw}} - o_{\mathrm{tr}}}{1-p} + o_{\mathrm{tr}}$$

<u>Expected value of the observable as a function of p</u>



$p = 0$: Ideal state

$p = 1$: Maximally mixed state

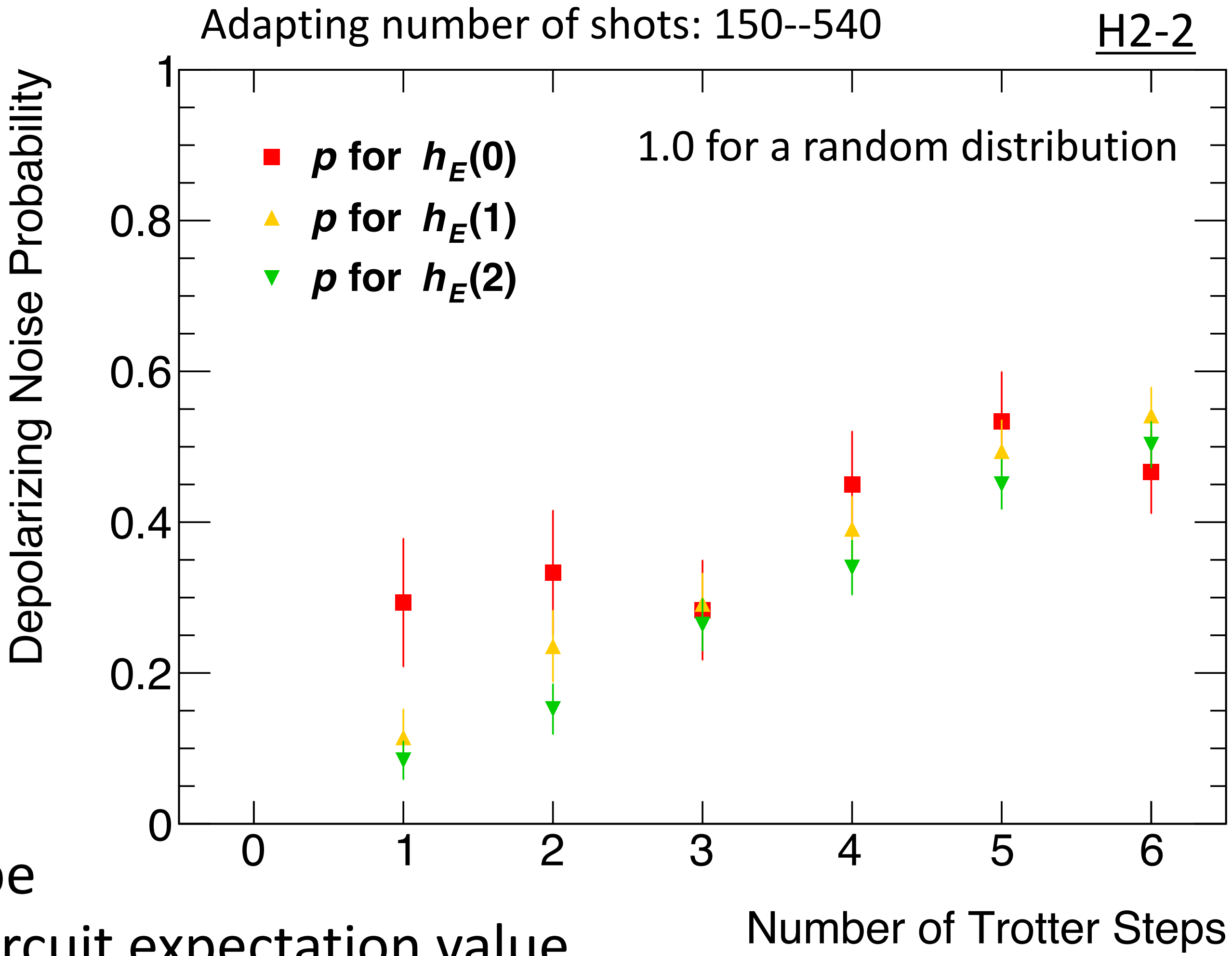
Estimation of the depolarizing noise probability

In this work, we introduce an effective depolarizing parameter p_i for each observable O_i.

We evaluated several test circuits using the emulator and selected the circuit initialized to the zero state as the test circuit.

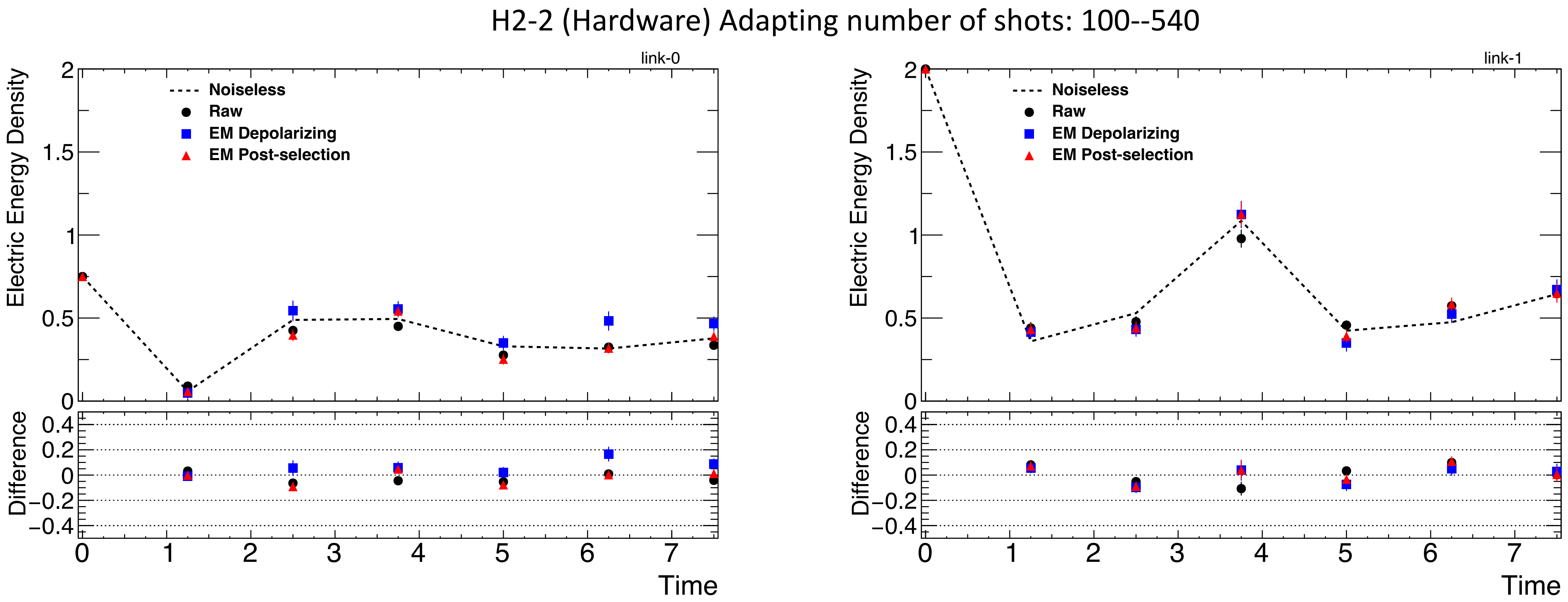
$$\langle O_i \rangle_{\text{ideal}} = 0$$

$$p_i = 1.0 - \frac{\langle O_i \rangle_{\text{noisy}} - o_{\text{tr},i}}{\langle O_i \rangle_{\text{ideal}} - o_{\text{tr},i}} = \frac{\langle O_i \rangle_{\text{noisy}}}{o_{\text{tr},i}}$$



The effective depolarizing parameter p can be estimated directly from the measured test-circuit expectation value.

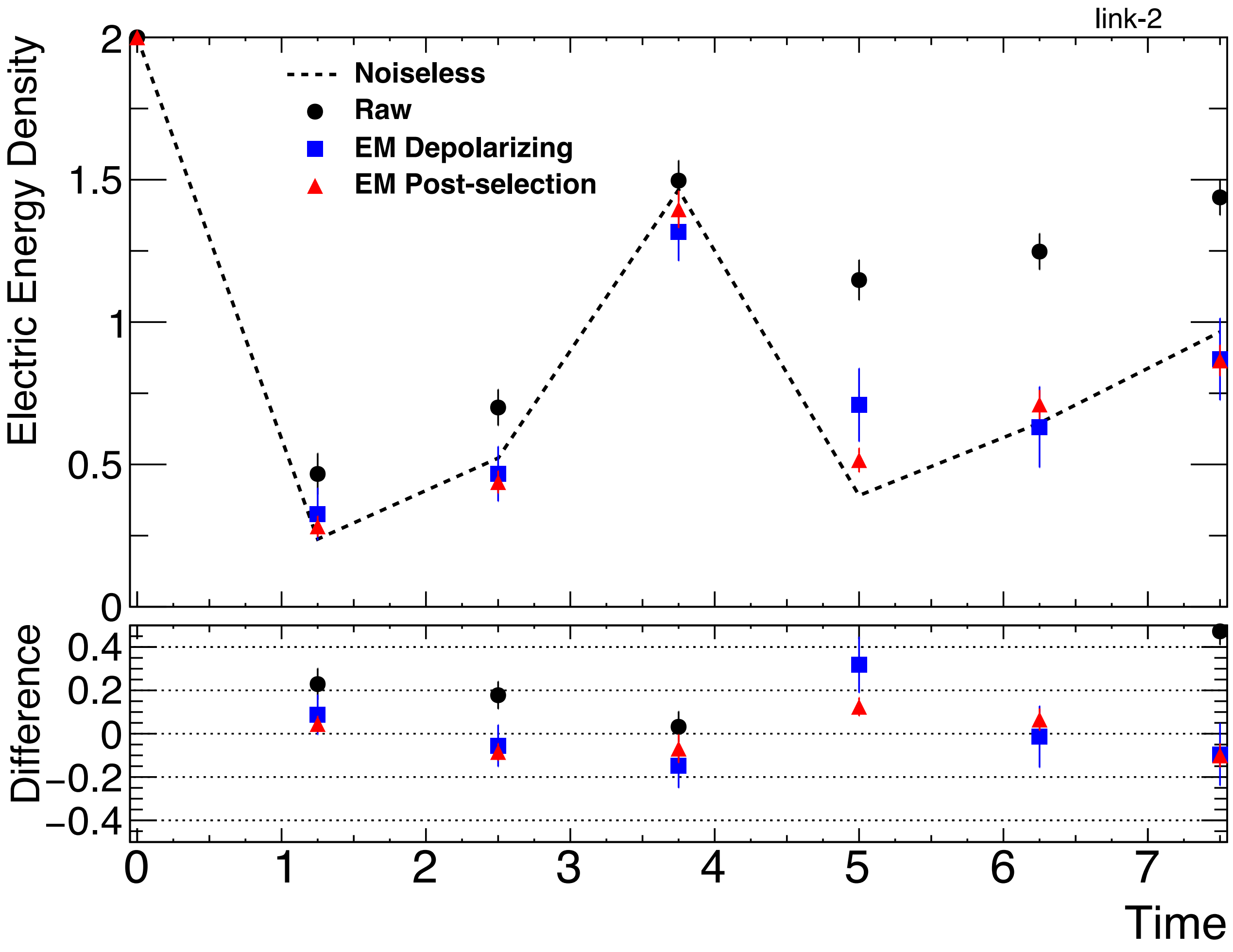
Hardware results: link-0, link-1



- $\Delta t = 1.25$, the number of Trotter steps: 0 to 6
- Even before mitigation, these results are close to the noiseless Trotterized-circuit baseline.

Hardware results: link-2

H2-2 (Hardware) Adapting number of shots: 100--540



- Link-2 is the most challenging observable.
- Raw results for link-2 show visible deviations from the noiseless results, but remain far from the uniform-random limit of 2.0.
- Noise indicators suggest that meaningful results may still be obtained for deeper circuits.
- Both mitigation methods substantially improve agreement

Summary

- We performed a quantum simulation of a 1+1D SU(2) lattice gauge theory based on the LSH formulation on Quantinuum's trapped-ion quantum computer H2-2, retaining explicit gauge-field degrees of freedom and going beyond the conventional $j_{\mathrm{max}} = 1/2$ truncation.
- We executed the first-order Trotterized circuits with fixed $\Delta t = 1.25$ up to 6 Trotter steps; the results and noise indicators suggest that meaningful results may still be obtainable with deeper circuits.
- Both depolarizing-noise rescaling and post-selection substantially improved the results, leading to good agreement with noiseless simulations for the most challenging observable.

Backup

Hamiltonian for the SU(2) LGT

$$\hat{H} = \hat{H}_M + \hat{H}_E + \hat{H}_I$$

It will be trotterized for digitized time evolution.

Interaction $\hat{H}_I = x \sum_{r=0}^{L-2} \hat{\psi}^\dagger(r)\hat{U}(r)\hat{\psi}(r+1) + \text{H.c.}$

Mass $H_M = \mu \sum_{r=0}^{L-1} (-1)^r \left(n_i(r) + n_o(r)\right)$

Electric $H_E = \sum_{r=0}^{L-2} \frac{N_L(r)}{2}\left(\frac{N_L(r)}{2} + 1\right)$

2 sub-terms per link

Each sub-term can be diagonalized by using SVD.

$$H_I^j(r) \rightarrow U^\dagger \underbrace{U H_I^j(r) U^\dagger}_{\text{Diagonal part}} U$$

Label	$H_I^{\mathrm{LSH}(j)}$ translation	
j	1	2
x'	$(r,i) \sim 2r$	$(r+1,o) \sim 2r+3$
x	$(r,o) \sim 2r+1$	$(r+1,i) \sim 2r+2$
y'	$(r+1,i) \sim 2r+2$	$(r,o) \sim 2r+1$
y	$(r+1,o) \sim 2r+3$	$(r,i) \sim 2r$
p	(r,ℓ)	$(r+1,\ell)$
q	$(r+1,\ell)$	(r,ℓ)

Suppose $x = 1$

<u>Diagonal part</u>

phase factor

control factors

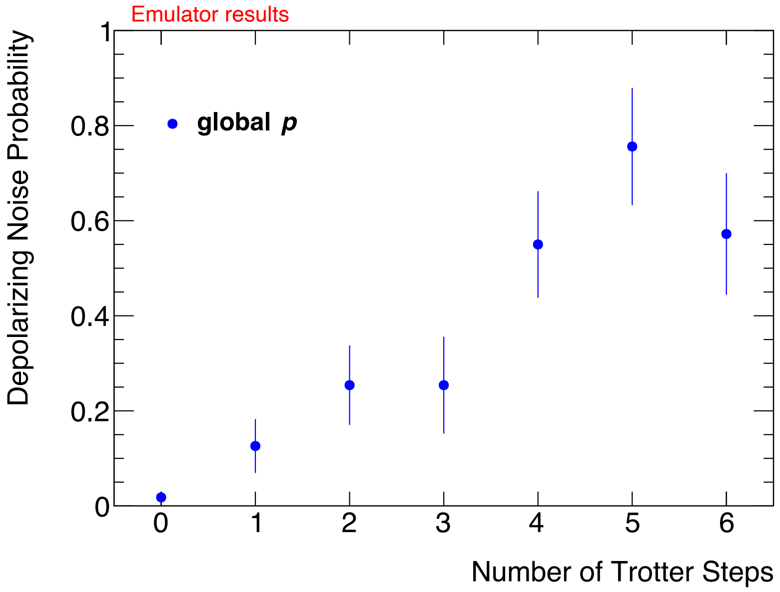
$$\mathscr{U}_{\mathrm{SVD}}^{\mathrm{LSH}} H_I^{\mathrm{LSH}(j)} \mathscr{U}_{\mathrm{SVD}}^{\mathrm{LSH}\dagger} = x \boxed{|1\rangle\langle 1|_{\mathrm{x}'}} \boxed{Z_{\mathrm{y}'} Z_{\mathrm{x}}} \underbrace{\boxed{\left(1 - |1\rangle\langle 1|_{\mathrm{y}} |\Lambda\rangle\langle\Lambda|_{\mathrm{p}}\right)}}_{=\ 0\text{, if } (n_y == 1\ \&\&\ n_p == \Lambda)} \underbrace{\boxed{\left(1 - |0\rangle\langle 0|_{\mathrm{x}} |\Lambda\rangle\langle\Lambda|_{\mathrm{p}}\right)}}_{=\ 0\text{, if } (n_x == 0\ \&\&\ n_p == \Lambda)} \mathcal{D}^{\mathrm{LSH}}(p, n_{\mathrm{x}}, n_{\mathrm{y}})$$

$$\mathscr{U}_{\mathrm{SVD}}^{\mathrm{LSH}} \equiv \mathrm{H}_{\mathrm{y}'}\left(|0\rangle\langle 0|_{\mathrm{y}'} + |1\rangle\langle 1|_{\mathrm{y}'} X_{\mathrm{x}'} (\lambda_{\mathrm{q}}^-)^{n_{\mathrm{y}}} (\lambda_{\mathrm{p}}^-)^{1-n_{\mathrm{x}}}\right).$$

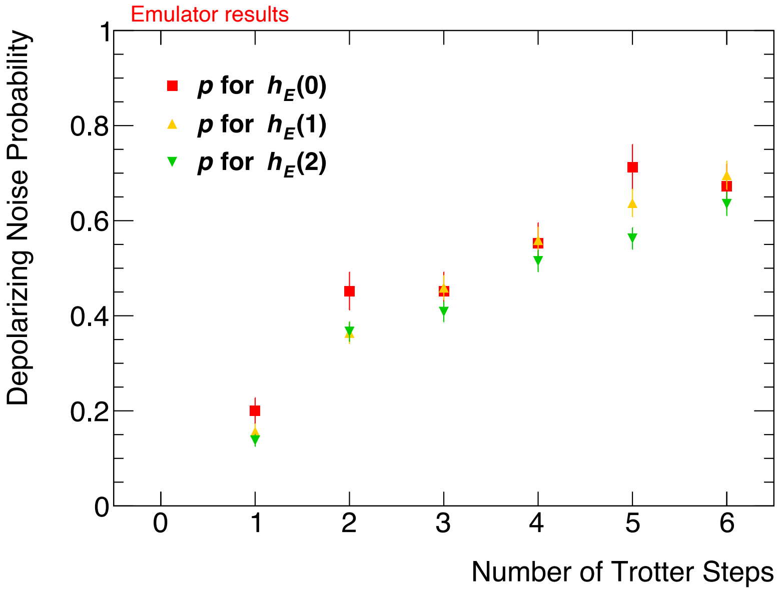
$$\mathcal{D}^{\mathrm{LSH}}(p, n, n') \equiv \sqrt{\frac{p+1+n}{p+1+n'}}$$

Depolarizing Noise Probability Estimation

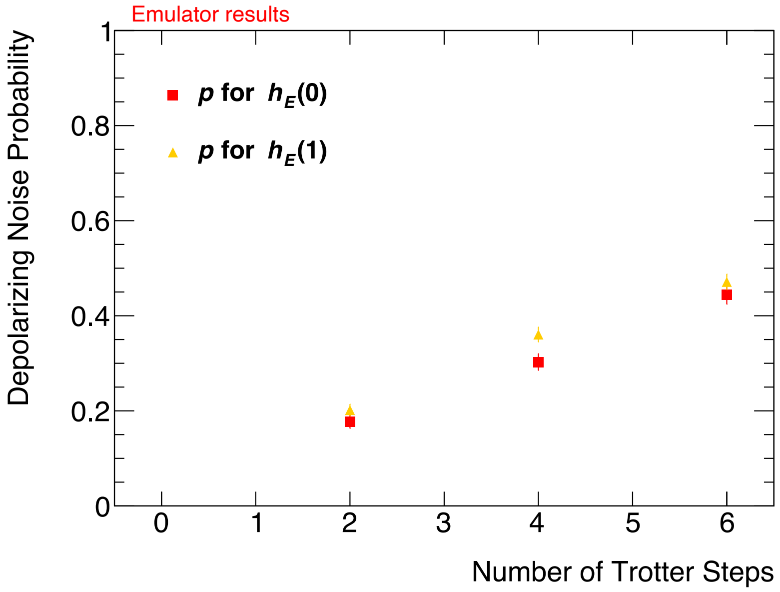
Method 1: Global noise probability



Method 2: Local probability (zero-state init.)

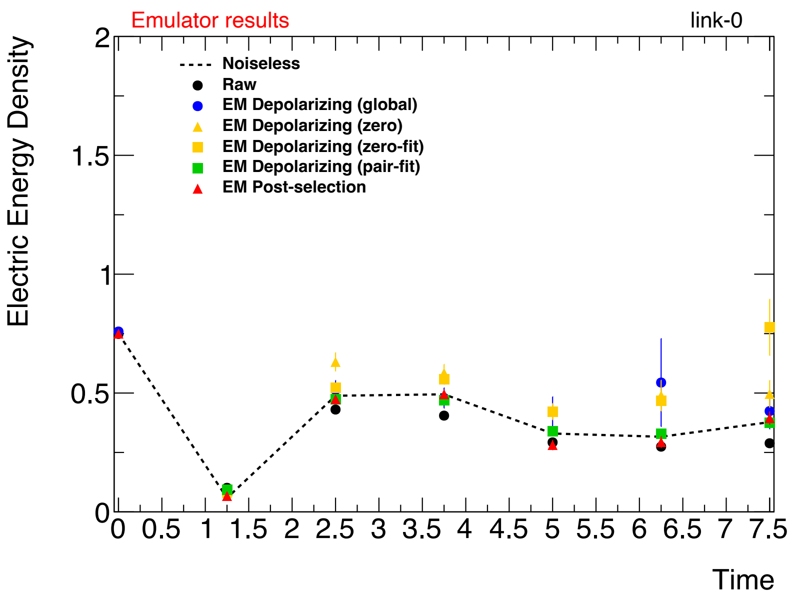


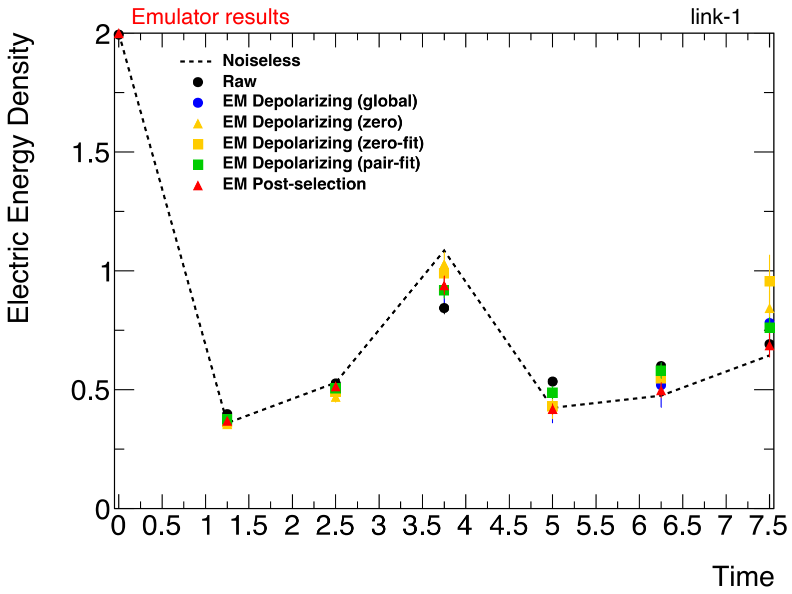
Method 3: Local probability (Inverse pair)

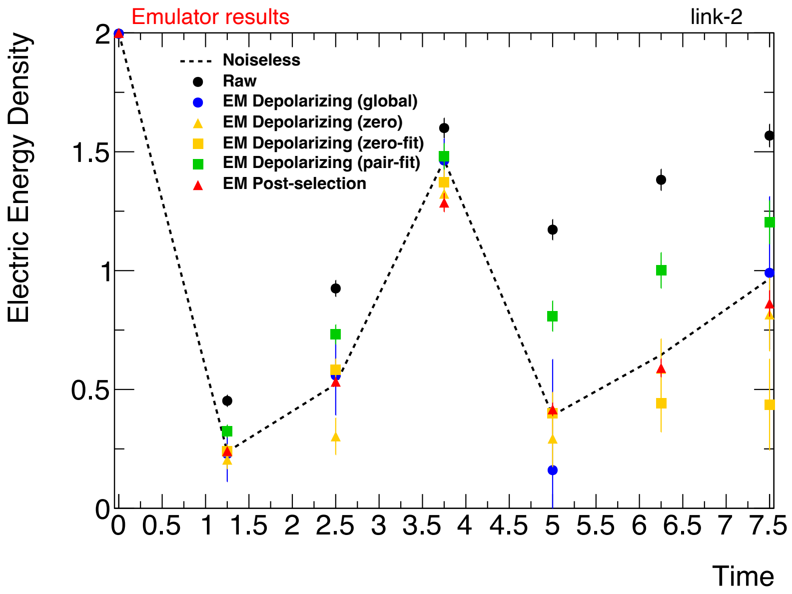


In any approach, the noise probability scales with the number of Trotter steps.
However, the estimated values are different from each other.
Since there is no significant difference among the observables in Method 3,
we use the probability obtained for the link-0 in place of that for the link-2.

Error Mitigated Results

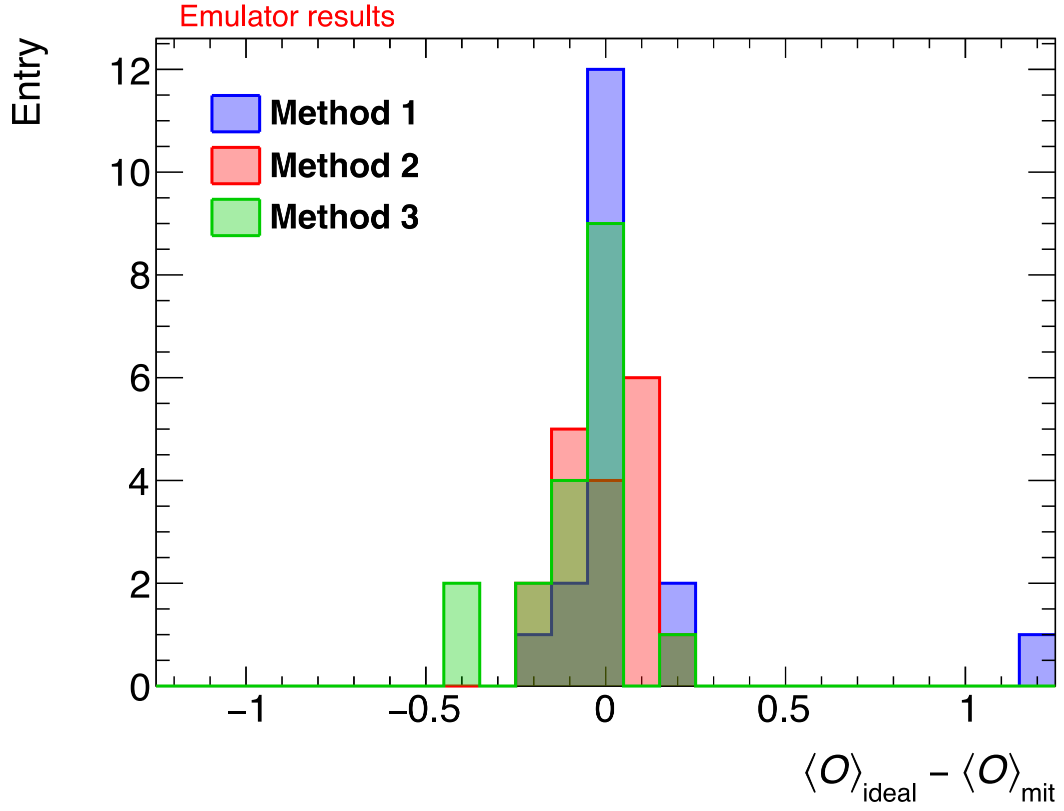


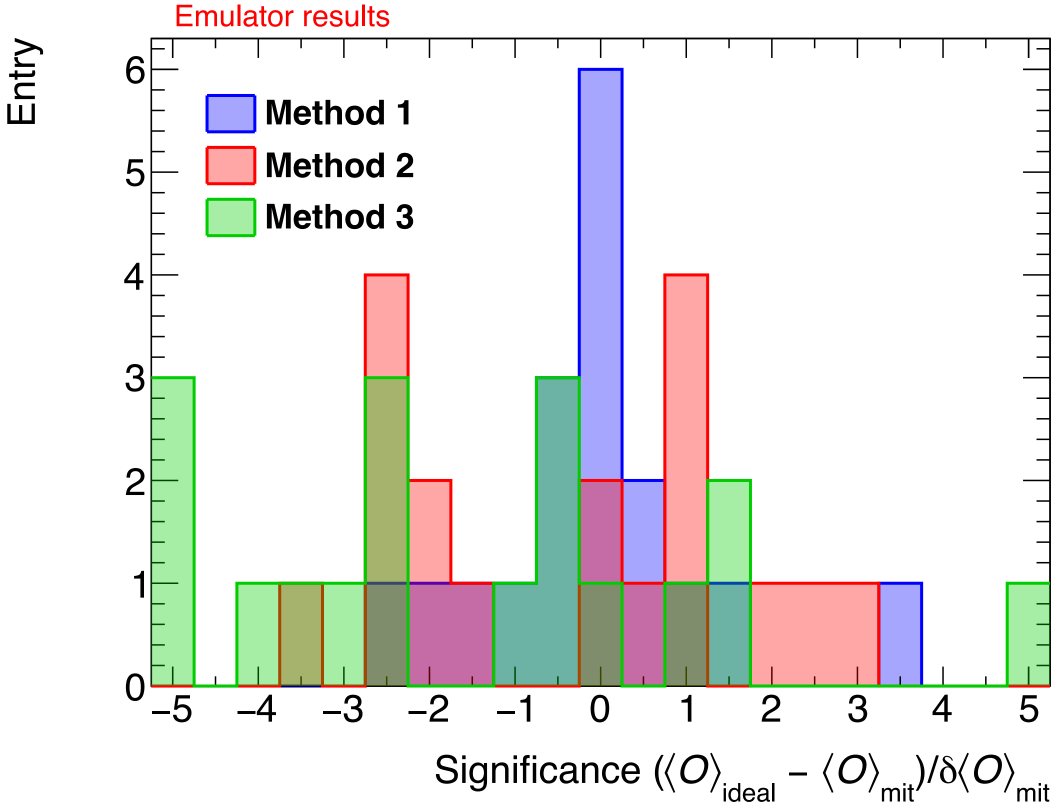




Despite the discrepancy at link-0, I think Method 2 (zero initial state) would be the best.
I will explain the reason on the next page.

Evaluation of Estimation Methods



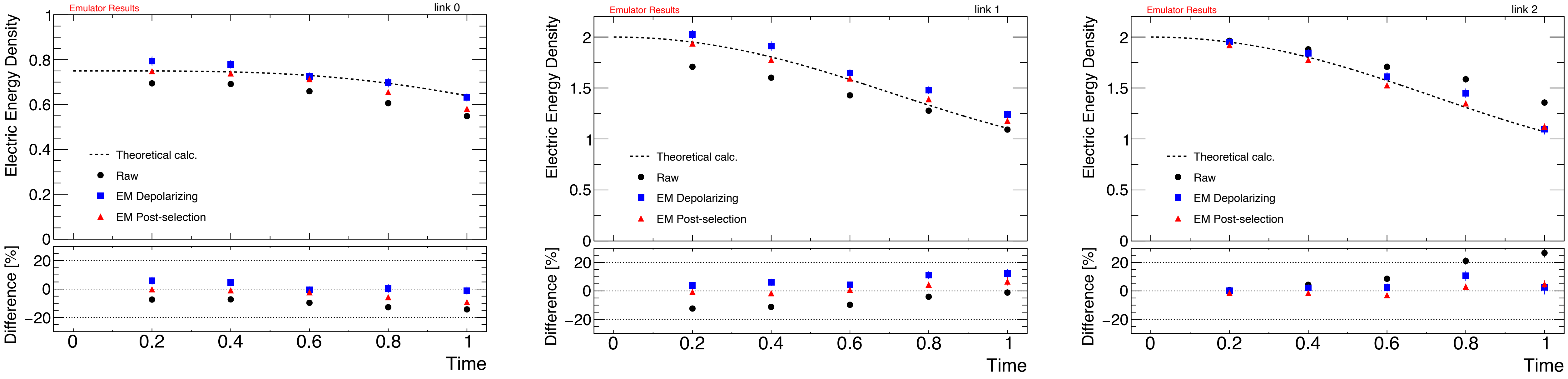


Method 1 generally performs well, but it exhibits the largest discrepancy and also has the largest uncertainty in its estimated probability.

Although the discrepancy in Method 3 is relatively small, its statistical significance is high.
This indicates that the deviation in Method 3 cannot be explained solely by statistical fluctuations.

Therefore, I conclude that Method 2 is the most reliable approach.

2nd Order Trotterization



The Trotter error at t=1.0 is approximately 8–10% with 1-step circuits, and 0–3% with 2-step circuits.
So it is a trade-off between the Trotter error and the circuit noise.
According to emulator results, 1-step circuits give better results.
Therefore, all data points shown here are calculated by using the 1-step circuit.