Gauge and Homological Structures in Quantum Error Correction

Junichi Haruna
Kyoto University

QuantHEP 2026, Queen Mary University of London

16 July 2026

Based on arXiv:2511.15224, 2602.14499

Introduction

Why should a gauge theorist care about QEC?

You want: quantum simulation of lattice gauge theories

↓ but

noisy qubits: errors corrupt the computation

↓ so

correct errors on the fly = quantum error correction (QEC)

↓ and in fact

QEC itself = $\mathbb{Z}_2$ lattice gauge theory + homology (this talk)

Gauge – QEC – Homology, a shared backbone

QEC
- Quantum codes
- Logical gates

Logical gates
by gauge fields
→ Main Part I

Gauge theory
- Topological QFT
- Gauge invariance

Homology and
transversality
→ Main Part 2

Wilson loop
= homology class

Homology
- Chain complex
- Homology group

Preliminary

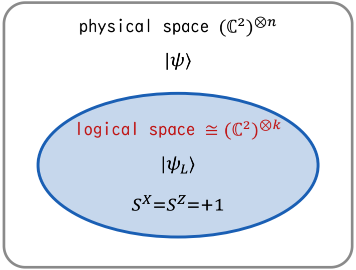
What is Quantum Error Correction?

- Logical info $\hookrightarrow$ larger Hilbert space
- Quantum (CSS) code: X/Z-type commutative "stabilizers"

$$S^X = \prod_{j=1}^{n} (X_j)^{\gamma_j} \quad \text{or} \quad S^Z = \prod_{j=1}^{n} (Z_j)^{\tilde{\gamma}_j}$$

- Logical state $|\psi_L\rangle$: fixed by every stabilizer

$$\forall S \; |\psi_L\rangle = +1 \, |\psi_L\rangle$$

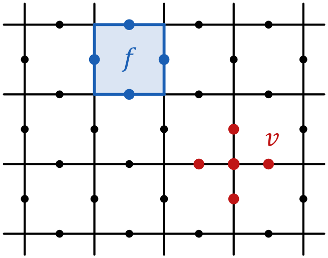


$\mathbb{Z}_2$ lattice gauge theory

- Claim: you already know a quantum code
- Kogut-Susskind Hamiltonian on 2D torus

$$\boxed{H = \cancel{- g^2 \sum_e X_e} - \sum_p B_p - \sum_v A_v}$$

  - $B_p = \prod_{e \in \partial p} Z_e$ (Flux)
  - $A_v = \prod_{e \ni v} X_e$ (Gauss law)
  - $Z_e =$ link variable (gauge field)
- kinetic term dropped $\Rightarrow$ topological



Dictionary: $\mathbb{Z}_2$ lattice gauge theory $\leftrightarrow$ QEC

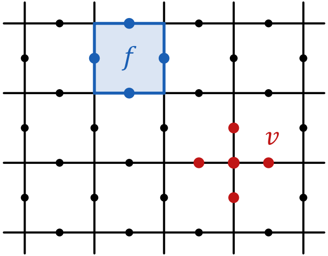
- Topological $\mathbb{Z}_2$ LGT = quantum code

Gauss law A_v	$\leftrightarrow$	X stabilizer
Flux B_p	$\leftrightarrow$	Z stabilizer
link variable Z_e	$\leftrightarrow$	qubit

- Ground states = logical state

$$A_v|\psi\rangle = B_p|\psi\rangle = +1|\psi\rangle$$

- Choice of stabilizers (manifold) = choice of quantum codes



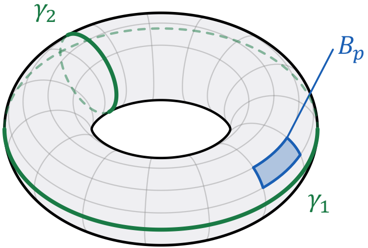
Logical Pauli operator as Wilson loop

- Logical Pauli
  - Commute with all stabilizers
  - Not a product of stabilizers

- X/Z on logical states

$$\bar{Z}|0\rangle_L = |0\rangle_L, \qquad \bar{Z}|1\rangle_L = -|1\rangle_L$$
$$\bar{X}|0\rangle_L = |1\rangle_L, \qquad \bar{X}|1\rangle_L = |0\rangle_L$$

- Logical Pauli = non-contractible loops
  $\Rightarrow$ classified by homology/topology



Logical gates beyond Pauli

- Z: phase flip, X: bit flip $\Rightarrow$ not enough for universal quantum computation
- Solovay-Kitaev theorem: Full Clifford + 1 non-Clifford is enough

generating gate set

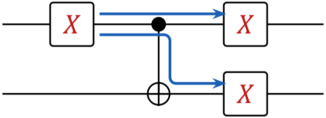
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ S = \begin{pmatrix} 1 & \\ & i \end{pmatrix},\ CZ = \mathrm{diag}(1,1,1,-1),\ T = \begin{pmatrix} 1 & \\ & e^{i\pi/4} \end{pmatrix}$$

Problem I

How to construct logical gates
beyond Pauli in general?

Need for transversal gates

- Multi-qubit gates amplify error

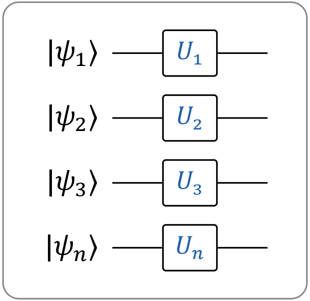


CNOT: one X fault spreads to both qubits

- Destroys error-correction capability
- What kind of implementation is better? → "Transversal gates"

Transversal gate

- An independent single-qubit unitary on every qubit
- Do not amplify error ⇒ fault-tolerant
- Limited class of unitaries
  → Q. Universal? No



When and which transversal gates?

- **Eastin-Knill theorem** [Phys. Rev. Lett. **102**, 110502 (2009)]
  - No universal gate set only with transversal gates
  - ⇒ Limits what is possible, not which gates for a given code

- Many examples, but no unifying rule

<u>Problem 2</u>

Which code can implement which gate transversally?

Main Part 1:
Logical gates by gauge fields

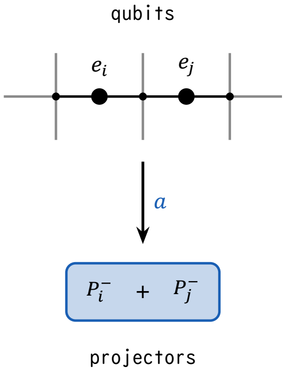
Junichi Haruna
arXiv:2511.15224

Gauge field = qubits → projectors

- Gauge field = linear map from qubit to operator
- Each qubit e_j → projectors onto $|1\rangle_j$:

$$a(e_j) := P_j^- = |1\rangle\langle 1|_j = \frac{I - Z_j}{2}$$

$$a(e_i + e_j) := a(e_i) + a(e_j)$$



Recovering gates from gauge fields

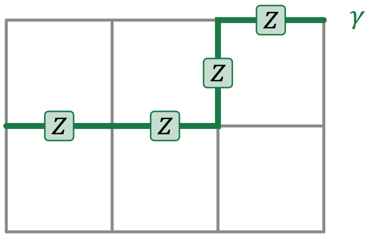
- Single-qubit gates come straight back:

$$Z_j = \exp\left(i\pi a(e_j)\right)$$

- Transversal Z gates = Wilson lines

$$\prod_{i=1}^{n} (Z_i)^{\gamma_i} = \exp\left(i\pi a(\gamma)\right)$$

- How are *other* logical gates related to gauge fields? (→ Problem 1)



Logical operators = gauge invariant

- Logical operator preserves the code space:

$$S^\dagger \, U \, S \simeq U \;\text{ on }\; \mathcal{H}_c$$

- X-stabilizer conjugation = gauge transformation

$$(S_v^X)^\dagger \, a \, S_v^X = a + dv \quad (\mathrm{mod}\ 2)$$

- Logical = gauge-invariant:

$$(S_v^X)^\dagger \, U(a) \, S_v^X = U(a+dv) \simeq U(a)$$

physical

$$\begin{tikzcd} a \arrow[r, "S_v^X"] \arrow[d, "U"'] & a+dv \arrow[d, "U"] \\ U(a) \arrow[r, "S_v^X \simeq I"] & U(a+dv) \end{tikzcd}$$

logical

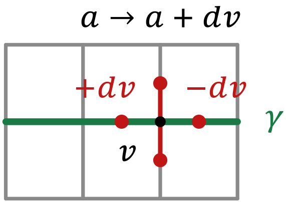
Construction principle

- Any function $U(a(\gamma))$ is gauge invariant:

$$a(\gamma) = \oint_\gamma a \to \oint_\gamma (a + dv) = \oint_\gamma a$$

- Which gate? → Replacement principle
- Replace physical X, Z with logical $\overline{X}, \overline{Z}$:

$$\mathcal{O}(X, Z) \to \overline{\mathcal{O}}(\overline{X}, \overline{Z})$$



Logical gates by gauge field construction

Logical gate	Gauge field expression
$\overline{S}(\gamma)$	$\exp\left(i\pi\, a(\gamma)^2/2\right)$
$\overline{H}(\gamma)$	$\overline{S}(\gamma)\,\exp\left(i\pi\, b(\tilde{\gamma})^2/2\right)\,\overline{S}(\gamma)$
$\overline{CZ}(\gamma_1,\gamma_2)$	$\exp\left(i\pi\, a(\gamma_1)a(\gamma_2)\right)$
$\overline{T}(\gamma)$	$\exp\left(i\pi(2a^3-3a^2+2a)/4\right)$

- 😊 Applicable for general CSS codes
- 😊 Includes universal gate set
- 😢 Not fault-tolerant: many entangling gates

$\Rightarrow$ Answer to Problem I

Main Part 2:
Homological origin of transversal gates

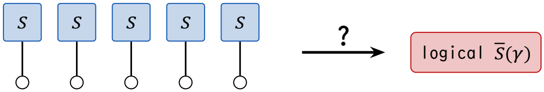
Junichi Haruna
arXiv:2602.14499

Transversal Z rotations

- Transversal S gate (as the simplest example):

$$U(\theta/2) := \prod_{j=1}^{n} \exp\left(i\pi \frac{\theta_j}{2} \frac{I - Z_j}{2} \right) = \prod_{j=1}^{n} (S_j)^{\theta_j}, \quad \theta_j \in \mathbb{Z}_4 = \{0, 1, 2, 3\}$$

- $S = \sqrt{Z}$ is the $\pi/2$ rotation, and $S^2 = Z$
- Goal: find θ such that $U(\theta/2)$ is logical and realizes $\sqrt{\overline{Z}(\gamma)}$
  - Existence not guaranteed (Problem 2 for S)

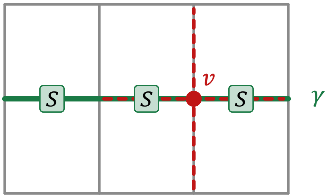


Transversal phase gates in LGT

- Transversal S = square root of Wilson loop:

$$\sqrt{\overline{Z}(\gamma)} = \exp\left(\frac{i\pi}{2}\, a(\gamma)\right)$$

- Gauge invariant? → No: sees $a(\gamma)$ mod 4
- Fix → lift to $\mathbb{Z}_4$ field $\tilde{a}$: $\tilde{a} = a \bmod 2$
- Transversal S exists $\Longleftrightarrow$ liftability



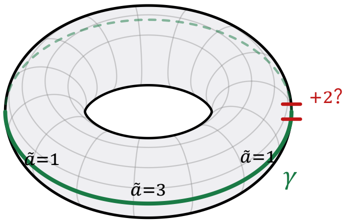
$a \to a + dv$: flips 2 links on γ

$W(\gamma)$: $\times (-1)^2 = +1$

$\sqrt{W(\gamma)}$: $\times\, i^2 = -1$

Liftability is a topological question

- Locally? → always: $\tilde{a}(e) \in \{a(e), a(e)+2\}$
- Globally? → can fail around nontrivial $[\gamma]$
- Decided by? → two topological data:
  - — spacetime: $[\gamma] \in H_1$
  - — gauge group: $\mathbb{Z}_2 \to \mathbb{Z}_4$
- Liftability → determined by homology



each link: lift exists

around $[\gamma]$: +2 mismatch possible

Main theorem

Theorem. Transversal implementability of logical S gate
$\Longleftrightarrow$ vanishing of two obstruction maps for $\overline{Z}(\gamma)$:

$$\boxed{\beta_1([\gamma]) = \beta_2([\gamma]) = 0}$$

- $\beta_1([\gamma]) := [(H_X \odot K)\gamma]$: additional condition mod 2
- $\beta_2([\gamma]) := [H_X(\gamma + H_Z^T \alpha)/2]$: liftability from $\mathbb{Z}_2$ to $\mathbb{Z}_4$

H_X = stack of X-checks, H_Z = stack of Z-checks, K = stack of $\{X\text{-checks}, \gamma\}$, $\odot$: row-wise product, $[\cdot]$ = class modulo trivial part

$\beta_2 =$ Bockstein obstruction

- $Z \to S$: halving the angle $\pi \to \pi/2$ = lifting the coefficients $\mathbb{Z}_2 \to \mathbb{Z}_4$
- β_2 measures its obstruction
- Transversal implementability = Existence of lift

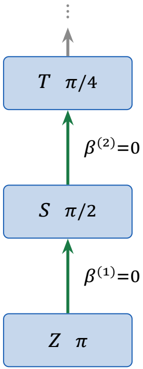
$$
\begin{tikzcd}
 & {[H\gamma/2]} & {[\theta]} & {[\gamma]} & \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\times 2"] & \mathbb{Z}_4 \arrow[r, "\mathrm{mod}\,2"] & \mathbb{Z}_2 \arrow[r] \arrow[ll, bend left] & 0
\end{tikzcd}
$$

β_2: Bockstein connecting map

keeping $[\gamma] = [\theta \bmod 2]$

Beyond Z and S

- Same argument for climbing the Clifford hierarchy: $Z \to S \to T \to \cdots$
- Transversal $\Longleftrightarrow$ a ladder of vanishing obstructions $\beta_1^{(\nu)} = \beta_2^{(\nu)} = 0$
- Gauge-invariant root $\exp(i\pi/2^{m-1} \oint \tilde{a})$ = liftability to $\mathbb{Z}_{2m}$



$\Rightarrow$ Answer to Problem 2

Summary and
future directions

Summary

Gauge theory · quantum error correction · homology theory
share common mathematical structures.

- Logical gates: built systematically from lattice gauge fields
- Transversality: homological obstruction to lift gauge fields to $\mathbb{Z}_{2^m}$
- Their interplay → new, unifying insight

Future directions

- Fault-tolerance written in the language of lattice gauge theory / homology.
- Non-Abelian extension: non-Abelian codes & quantum double model.
- Higher cohomological invariants and non-Clifford gates.

A shared language, still mostly unexplored.

Thank you!

Backup

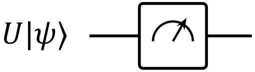
How to detect and correct?

- Measurement of stabilizers ⇒ Error detection

$$|\psi_L\rangle \xrightarrow{\text{error}} U|\psi_L\rangle \xrightarrow{\text{measure}} S = \pm 1$$

- Flipped result (= “syndrome”)
  ⇒ Error occurred!

- Error-correcting capability
  = Min. weight of errors that
  do not flip syndromes



projective measurement
→ ±1 (syndrome)

Two problems, one dictionary

Problem 1

How to construct logical gates beyond Pauli in general?

⇒ Main Part 1: a gauge-field engine arXiv:2511.15224

Problem 2

Which code implements which gate transversally?

⇒ Main Part 2: a homological obstruction arXiv:2602.14499

Work in the exponent

- Gates form a group: closed under the product $U_1 U_2$ (nonlinear)
- Idea: push everything into the exponent

$$U = \exp(i\,G)$$

- Then generators just *add*: $G_1 + G_2$
  $\Rightarrow$ a linear space, much simpler
- Gauge field $=$ the generator G: an operator-valued cochain
  linear map: qubits $\to$ operators

group of gates

$U_1 \cdot U_2$

closed under product

$\downarrow$ exp

Lie algebra

$G_1 + G_2$

closed under sum $=$ linear

Why projectors?

- Every diagonal gate = one master formula:

$$\boxed{U_{\text{diag}} = \exp\left(i\, f\left(P_1^-, \dots, P_n^-\right)\right)}$$

- choosing the gate = choosing f:

$$Z = \exp\left(i\pi\, P^-\right)$$
$$S = \exp\left(\frac{i\pi}{2}\, P^-\right)$$
$$T = \exp\left(\frac{i\pi}{4}\, P^-\right)$$

same P^-,
only the angle changes

$$CZ_{ij} = \exp\left(i\pi\, P_i^- P_j^-\right)$$

product → entangling

Logicality from the commutation relation

- Logicality $\Leftrightarrow$ commutativity on $\mathcal{H}_c$
- Commutation relation with X-stabilizer:

$$S^X(h)^\dagger\, U^\dagger\left(\frac{\theta}{2}\right) S^X(h)\, U\left(\frac{\theta}{2}\right) = \underbrace{e^{i\pi\langle\theta,h\rangle/2}}_{\text{const. phase}}\ \underbrace{U(\theta\circ h)}_{Z\text{-string}}$$

$\theta\circ h$: entrywise product

- Z-string must coincide with a stabilizer (and phase trivial)

$$\begin{tikzcd}
|\psi_L\rangle \arrow[r, "S_v^X \simeq I"] \arrow[d, "U"'] & |\psi_L\rangle \arrow[d, "U"] \\
U|\psi_L\rangle \arrow[r, "S_v^X \simeq I"'] & \begin{array}{c} US_v^X|\psi_L\rangle \\ \simeq S_v^X U|\psi_L\rangle \end{array}
\end{tikzcd}$$

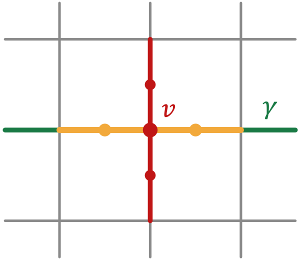
$S^\dagger U^\dagger S U \simeq I$ on $\mathcal{H}_c$

Two logicality conditions on θ

- Two conditions on θ, different moduli:

  1. Phase factor: $H\theta = 0 \pmod 4$
  2. Z-string = stabilizer $(\perp h, \gamma)$: $(H \odot K)\theta = 0 \pmod 2$

- Which logical Z? → further $(S^2 = Z)$ needed



H = stack of $\{h\}$, K = stack of $\{h, \gamma\}$; $\odot$: row-wise product

Simplifying $\overline{S}(\gamma)$ modulo 4

- $\overline{S}(\gamma)$ as a polynomial of gauge fields:

$$\overline{S}(\gamma) = \exp\left(\frac{i\pi}{2}\left(\frac{10}{3}a(\gamma) - 3a(\gamma)^2 + \frac{2}{3}a(\gamma)^3\right)\right)$$

- Simplified modulo 4 (since $S^4 = I$):

$$\boxed{\overline{S}(\gamma) = \exp\left(\frac{i\pi}{2}a(\gamma)^2\right)}$$

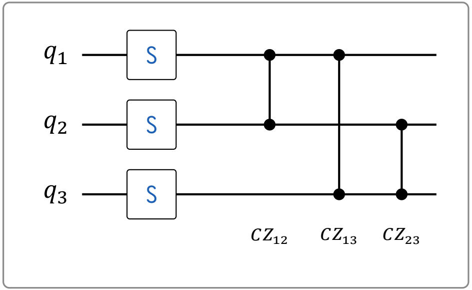
Physical gate decomposition $(n = 3)$

- Expansion of gauge fields polynomial

$$\bar{S}(\gamma) = \exp\left(\frac{i\pi}{2}(z_1 + z_2 + z_3)^2\right)$$
$$= \exp\left(\frac{i\pi}{2}(z_1 + z_2 + z_3)\right)$$
$$\times \exp\left(i\pi(z_1 z_2 + z_1 z_3 + z_2 z_3)\right)$$

- First factor = S on each qubit, second factor = CZ on every pair

$$\boxed{\bar{S}(\gamma) = S_1 S_2 S_3 \cdot CZ_{12} CZ_{13} CZ_{23}}$$



Summary of Main Part I

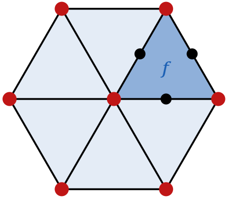
- Dictionary: $\mathbb{Z}_2$ lattice gauge theory $\rightleftarrows$ quantum codes
- Systematic construction:
  - Logical gate = exponential of a polynomial of gauge fields
  - Replacement principle: $Z \to \overline{Z}$, $X \to \overline{X}$
  - Applicable to general quantum (CSS) codes

These gates are neither transversal, nor fault-tolerant in general.
When is a logical gate transversal? → Main Part 2

CSS codes = chain complexes over $\mathbb{Z}_2$

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

- Latticize a space into cells
- Chain complex: ∂ = boundary map
- Qubits on C_1; $H_X = \partial_1$, $H_Z^T = \partial_2$
- CSS condition = $\partial^2 = 0$



face $= C_2$, edge/qubit $= C_1$, vertex $= C_0$

Chain complexes across QEC

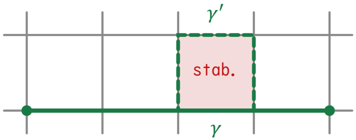
- Measurement error ⇒ longer chain complexes
  - Fault-complex [Hillmann et al. 2025]
  - Single-shot QEC [Campbell 2018; Jacob-McLauchlan-Browne 2508.08191]

- Algebraic operation ⇒ new family of QEC codes
  - Hypergraph product codes → tensor product [Tillich, Zémor 2014]
  - Fiber-bundle codes → fiber bundle [Hastings, Haah, O'Donnell 2021]
  - Lifted product codes → lifted product [Panteleev, Kalachev 2021]

- Code surgery → mapping cone [Ide et al. 2025]

β_1, β_2 as homology maps

- So far: β from *one fixed* γ
- But: same logical $\overline{Z}$ for topologically equivalent loops
- Modding out by stabilizers
  $\Rightarrow$ β_1, β_2 depend only on topological equivalence class $[\gamma]$



$$\gamma' = \gamma + \partial(\text{stab.})$$

$$[\gamma'] = [\gamma] \;\Rightarrow\; \beta(\gamma') = \beta(\gamma)$$

Steane $[[7,1,3]]$: transversal S yes, T no

- Self-dual CSS ($H_X = H_Z = H$, the $[7,4,3]$ Hamming code): a single logical class

gate	lift	obstruction	transversal?
$S\,(\pi/2)$	$1\to 2$	$\beta_1^{(1)} = \beta_2^{(1)} = 0$	yes
$T\,(\pi/4)$	$2\to 3$	$\beta_2^{(2)} \neq 0$	no

- S: lift $\hat{\theta} = 3 \cdot 1_7$ gives $\langle \hat{\theta}, 1_7 \rangle = 1 \pmod 4$, so $\prod_i S_i^\dagger$ is logical S
- T: every logical $\theta^{(3)} = 2c_0 1_7 + 4\sum_i c_i h_i$ is even, so $\langle \theta^{(3)}, 1_7 \rangle \neq 1 \pmod 8$: no transversal T, even for non-uniform $\theta \in \mathbb{Z}_8^7$
- $[[15,1,3]]$ Reed-Muller: transversal T (backup).

The obstruction detects the absence of transversal T: Eastin-Knill, from obstruction theory.

Relation to high-energy physics?

- Transversal Pauli Z rotation gate at level m = "2^{m-1}-root of Wilson loop"
- Transversal implementability = ∃ gauge-invariant 2^{m-1}-root of $W(\gamma)$
- Example: $m = 2$ = transversal S (our running case)

gauge field side QEC operator side

$$W(\gamma) = \exp\left(\oint_\gamma a\right) \quad \leftrightarrow \quad Z(\gamma) = \prod_i (Z_i)^{\gamma_i}$$

$$W^{(2)}(\gamma^{(2)}) = \exp\left(\frac{1}{2}\oint_{\gamma^{(2)}} \tilde{a}\right) \quad \leftrightarrow \quad S(\gamma^{(2)}) = \prod_{i=1}^{n} (S_i)^{\gamma_i^{(2)}}$$

$$\textit{with} \quad W(\gamma) = \left(W^{(2)}(\gamma^{(2)})\right)^2$$

Our transversal S is the case $m = 2$: the square root of the Wilson loop.
$m = 3$ (4th root) = transversal T on $[[15,1,3]]$ → next.

$[[15,1,3]]$ Reed-Muller: setup

- Setup: the $[[15,1,3]]$ punctured Reed-Muller code. Physical $T^\dagger$ on every qubit $\to$ logical T.
- In this framework: uniform angle $\theta_i \equiv 1$, level $m = 3$, $\mathbb{Z}_8$ coefficients.
- Tri-orthogonality (every triple overlap of H^Z rows is even) $\Rightarrow$ $\beta_1^{(\nu)}, \beta_2^{(\nu)}$ vanish for $\nu = 1,2,3$.

$[[15,1,3]]$ Reed-Muller: transversal T

- $\Rightarrow$ A $\pi/4$ rotation of logical Z is transversal = transversal T.
- HEP reading: $m = 3$, so logical $\overline{T}$ is the $2^{m-1} = 4$-th root of the Wilson loop:

$$\boxed{\overline{T}(\gamma) \quad = \quad W(\gamma)^{1/4}}$$

Textbook "tri-orthogonal $\Rightarrow$ transversal T" = both obstructions vanish at uniform angle, level 3.

Recovering known transversality criteria

Known criterion	In this framework
Divisibility: a 2^m-divisible CSS code admits transversal $\exp(i\pi Z/2^m)$	$\beta_2^{(\nu)} = 0$ for uniform θ_ν (the homology class lifts)
Tri-orthogonality $\Rightarrow$ transversal T	level $m = 3$ with $\beta_1^{(\nu)} = \beta_2^{(\nu)} = 0$ (uniform angle)
Quasi- / multi-level generalizations	non-uniform θ_ν: the full classification

Divisibility and tri-orthogonality are the uniform-angle corner of a full homological classification (non-uniform angles give strictly more transversal gates).

Relation to Webster-Quintavalle-Bartlett

- Closest prior work [arXiv:2303.15615]: transversal diagonal gates from a level structure on the code
- At each level, the two classifications coincide
- Difference: closed-form / homological (here) vs algorithmic (there)

Same classification, complementary content; β_2 is the Bockstein-type refinement.

Two circuits for one $\overline{S}$

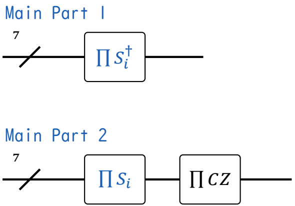
- Main Part 1 says

$$\overline{S} = \prod_{i=1}^{7} S_i^{\dagger}$$

- Main Part 2 says

$$\overline{S} = \prod_{i=1}^{7} S_i \cdot \prod_{1 \le i < j \le 7} CZ_{ij}$$

- Contradiction? What happens?



Two circuits for the same $\overline{S}$

Resolved by a logical identity

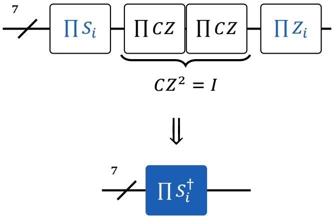
- CZ part + logical Z = logical identity

$$\prod_{1\le i<j\le 7} CZ_{ij} \cdot \overline{Z} \simeq I$$

- Deform by a logical identity:

$$\prod_i S_i \cdot \prod_{i<j} CZ_{ij} \simeq \prod_i S_i \cdot \prod_i Z_i = \prod_i S_i^\dagger$$

- "Nice" logical gate ⇔ "nice" logical identity



Deform $\overline{S}$ by a logical identity
⇒ transversal $S^\dagger$

Relation to gauge-covariant codes

- Active line in this community: encode a gauge theory as a code
  - Gauss law at each vertex → a stabilizer [Roggero et al. 2024; Yao 2025]
  - Goal: fault-tolerant simulation of the gauge theory itself
- This work: the opposite arrow
  - Use gauge / homological structure to classify transversal gates of any CSS code

Same dictionary, opposite direction:
gauge theory → code (simulation) vs code → gauge structure (fault-tolerant logic).

Non-Abelian code and quantum double model

- Non-Abelian gauge groups → non-Abelian anyons = key to topological QC
- Quantum double model (Kitaev 1997):
  - A lattice model with the quantum double of a finite group G
  - $G = \mathbb{Z}_2$: reduces to the toric code
  - G is non-Abelian: realizes non-Abelian anyons (e.g. $G = S_3, D_4$)