Onset of thermalization of q-deformed SU(2) Yang-Mills theory on a trapped-ion quantum computer

Tomoya Hayata

Keio University, School of Medicine

Collaborators: Yoshimasa Hidaka (YITP), Yuta Kikuchi (Quantinuum)

References:

Tomoya Hayata, Yoshimasa Hidaka , Yuta Kikuchi, arXiv: 2601.13530 [hep-lat]

QuantHEP 2026

Queen Mary, University of London

I want to solve nonequilibrium dynamics of nonabelian lattice gauge theories

Approximate gauge group keeping its <u>nonabelian feature</u>

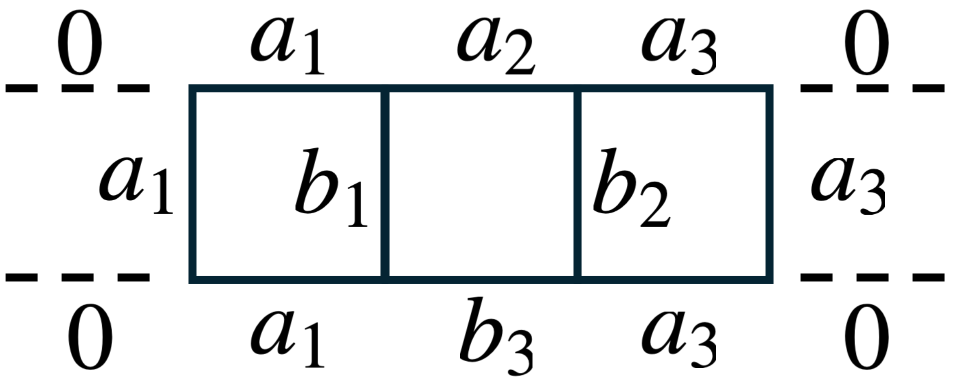
➡ nonabelian anyons

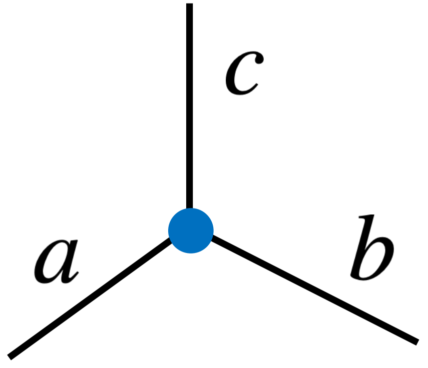
2+1 dimensions or 3+1 dimensions

(with fermions)

SU(2)$_k$ Yang-Mills model on a two-leg ladder

$N \times 1$ square lattice/ chain of plaquette





Gauge field on a link = (k+1)-qudit

$$a_i, b_i = 0, \frac{1}{2}, \ldots, \frac{k}{2}$$

constraints $\boxed{N_{ab}^{c} = 1}$

Fusion rules (Star operators)

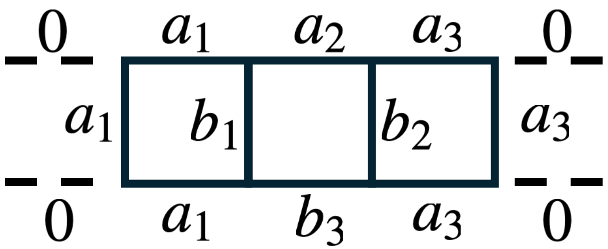
$$N_{ab}^{c} := \begin{cases} 1 & a+b \geq c, b+c \geq a, c+a \geq b, \\ & a+b+c \in \mathbb{Z}, \text{ and } a+b+c \leq k \; . \\ 0 & \text{else} \end{cases}$$

$$|\Psi\rangle = \sum_{\{a\},\{b\}} \psi(a_1, a_2, a_3, b_1, b_2, b_3)|a_1, a_2, a_3, b_1, b_2, b_3\rangle$$

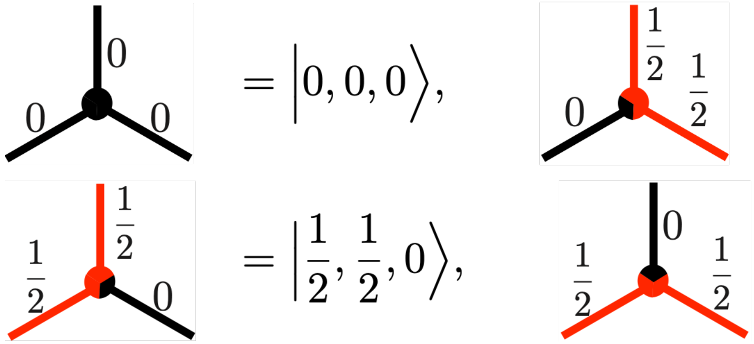
SU(2)$_1$ Yang-Mills model on a two-leg ladder

Gauge field on a link = qubit

$$a_i, b_i = 0, \frac{1}{2}$$



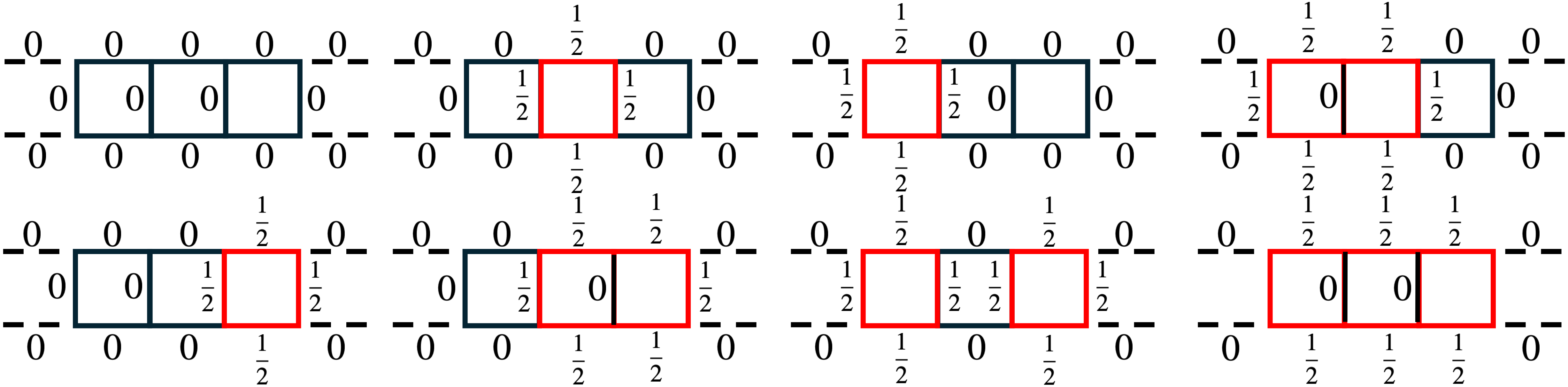
Fusion rules $\quad a \times b = (a+b)\,\mathrm{mod}2$



$= |0,0,0\rangle,\quad = |0,\frac{1}{2},\frac{1}{2}\rangle,$

$= |\frac{1}{2},\frac{1}{2},0\rangle,\quad = |\frac{1}{2},0,\frac{1}{2}\rangle$

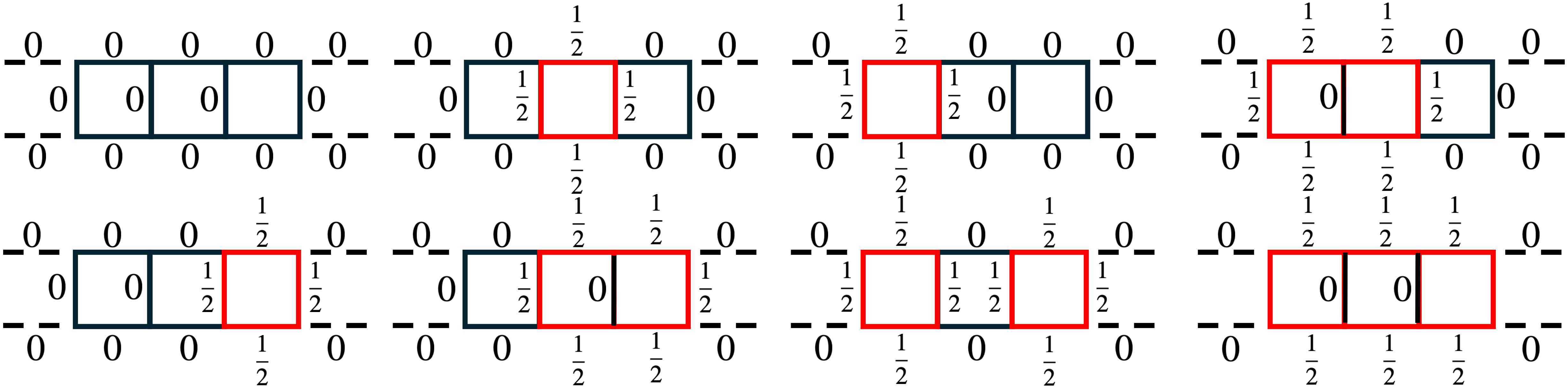
"Logical" states



SU(2)$_1$ Yang-Mills model on a two-leg ladder

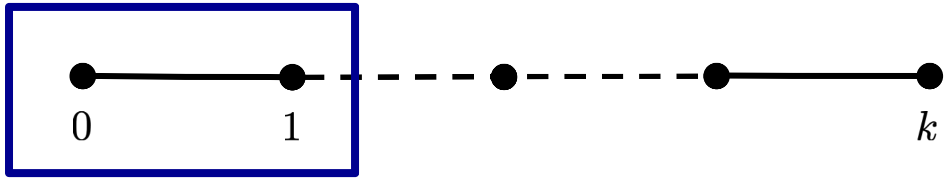
- Fusion rules = $\mathbf{Z_2}$ group/Abelian anyons
- Basis is constructed from closed loops
- Basis is constructed by product of Wilson loops $\quad \mathrm{tr}U(1)^{a_1}\cdots\mathrm{tr}U(N)^{a_N}|0\rangle$
- Spin chain on a dual graph $\quad a_i = 0, 1$

"Logical" states



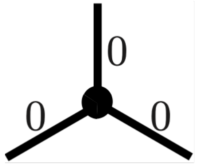
SU(2)$_1$ Yang-Mills model on a two-leg ladder

Gauge field on a link = qubit

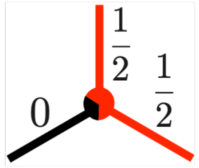


=qubit

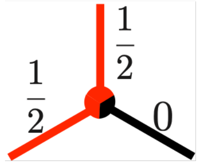
Fusion rules



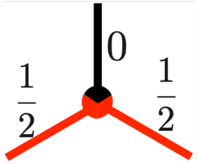
$= \left|0, 0, 0\right\rangle,$



$= \left|0, \frac{1}{2}, \frac{1}{2}\right\rangle,$

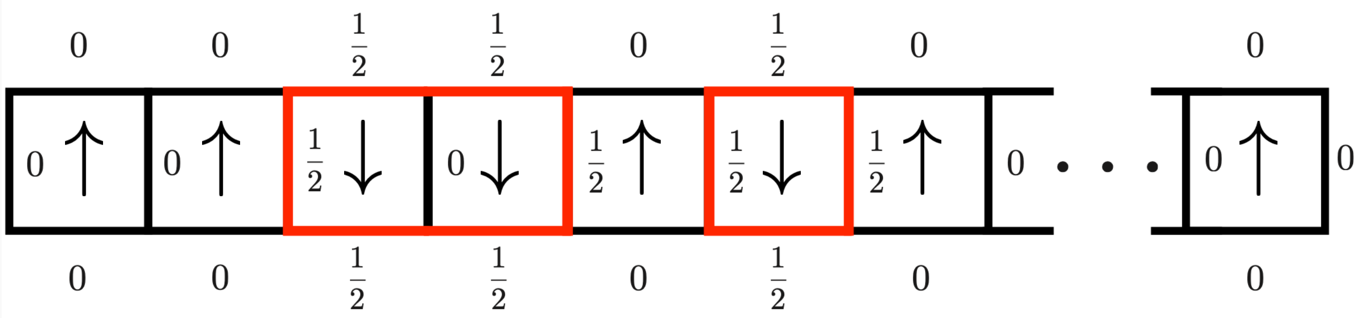


$= \left|\frac{1}{2}, \frac{1}{2}, 0\right\rangle,$



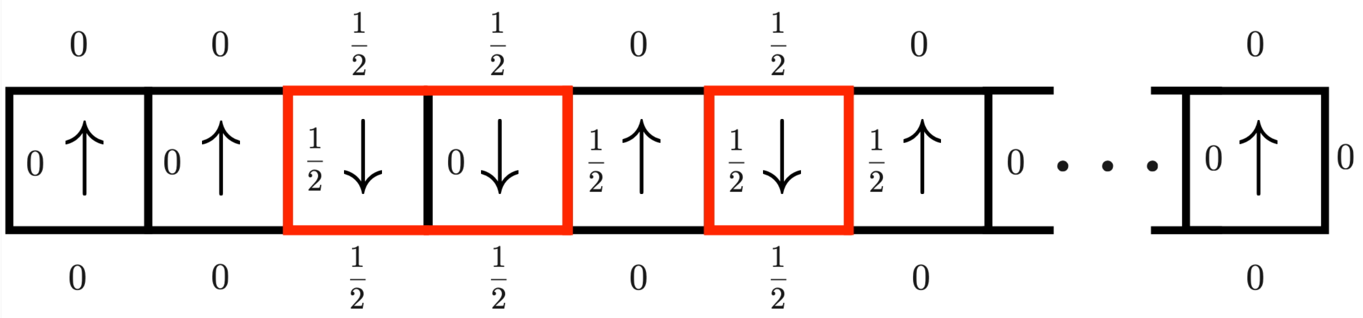
$= \left|\frac{1}{2}, 0, \frac{1}{2}\right\rangle$

Plaquette chain = Qubit-chain



Apply Wilson loop = Flip spin on a dual graph

$SU(2)_1$ Yang-Mills model on a two-leg ladder



$Z_0 = Z_{N+1} = 1$

$$H = \frac{c}{2}\sum_{n=1}^{N}(1 - Z_n) + \frac{c}{4}\sum_{n=1}^{N-1}(1 - Z_n Z_{n+1}) - K\sum_{n=1}^{N} Z_{n-1}X_n Z_{n+1}$$

This model is unitary equivalent to the transverse-field Ising (TFI) model

$$U = \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}\sum_n (-1)^n Z_n Z_{n+1}}$$

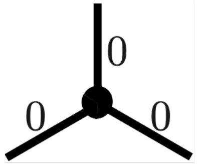
$$H = \frac{c}{2}\sum_{n=1}^{N}(1 - Z_n) + \frac{c}{4}\sum_{n=1}^{N-1}(1 - Z_n Z_{n+1}) - K\sum_{n=1}^{N} X_n$$

$$|\psi_{\mathrm{str}}(t)\rangle = \mathrm{e}^{-\mathrm{i}H_{\mathrm{YM}}t}|\psi_{\mathrm{str}}(0)\rangle = U\mathrm{e}^{-\mathrm{i}H_{\mathrm{TFI}}t}U^\dagger|\psi_{\mathrm{str}}(0)\rangle$$

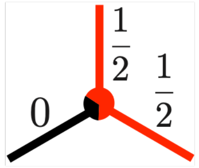
$SU(2)_1$ Yang-Mills model on a honeycomb lattice

Levin, Wen, PRB 71 045110 (2005)

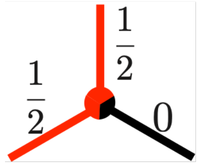
Trivalent graph = Honeycomb lattice



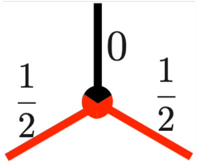
$= |0,0,0\rangle,$



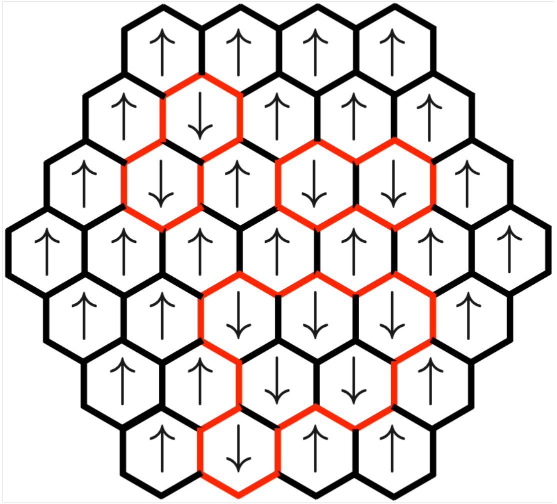
$= |0,\frac{1}{2},\frac{1}{2}\rangle,$

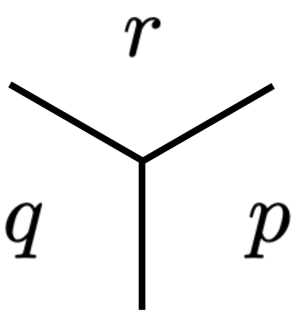


$= |\frac{1}{2},\frac{1}{2},0\rangle,$



$= |\frac{1}{2},0,\frac{1}{2}\rangle$





$$H = \frac{c}{4}\sum_{(n,m)}(1 - Z_n Z_m) - K\sum_p X_p \prod_{(q,r)} \mathrm{i}^{\frac{1-Z_q Z_r}{2}}$$

This model is unitary equivalent to the TFI model

$$U = \prod_{<pqr>} \mathrm{e}^{-\mathrm{i}\frac{\pi}{24}(3Z_p Z_q Z_r - Z_p - Z_q - Z_r)}$$

$$H = \frac{c}{4}\sum_{(n,m)}(1 - Z_n Z_m) - K\sum_p X_p$$

Levin, Gu PRB 86 115109 (2012)

$$|\psi_{\mathrm{str}}(t)\rangle = \mathrm{e}^{-\mathrm{i}H_{\mathrm{YM}}t}|\psi_{\mathrm{str}}(0)\rangle = U\mathrm{e}^{-\mathrm{i}H_{\mathrm{TFI}}t}U^{\dagger}|\psi_{\mathrm{str}}(0)\rangle$$

$SU(N)_1$ Yang-Mills model on a two-leg ladder

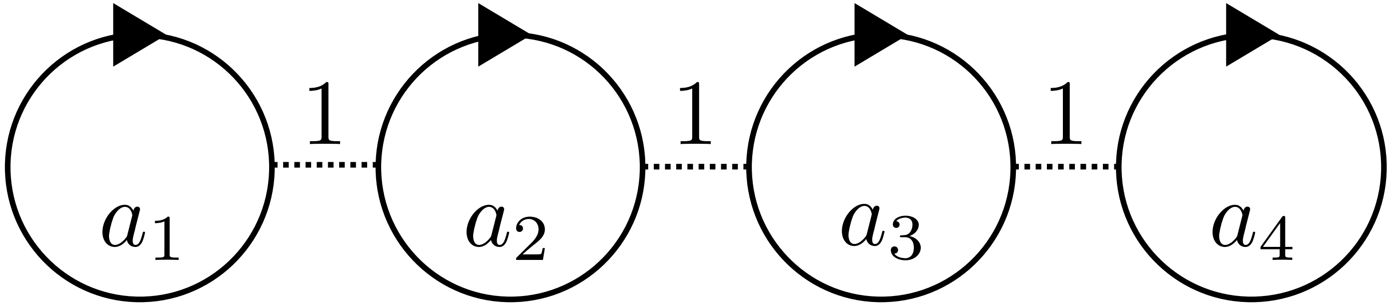
TH, Hidaka, Fujikura, in preparation

Fusion rules

$\mathbf{Z}_N$ group/Abelian anyons $\qquad a \times b = a + b \ \text{ modulo } \ N$

Hilbert space

Dual basis = Plaquette basis



$$|\psi\rangle = \sum_{a_1,a_2,a_3,a_4 \in \{1,\ldots,N\}} \psi(a_1,a_2,a_3,a_4)|a_1,a_2,a_3,a_4\rangle$$

$\Longleftrightarrow$ perturbed $\mathbf{Z}_N$ toric code

$\boxed{N\text{-qudit chain and N-level quantum clock model}}$

Let us consider $SU(2)_k$

$$k = 2, 3$$

$SU(2)_2$ Yang-Mills model on a two-leg ladder

Fusion rules

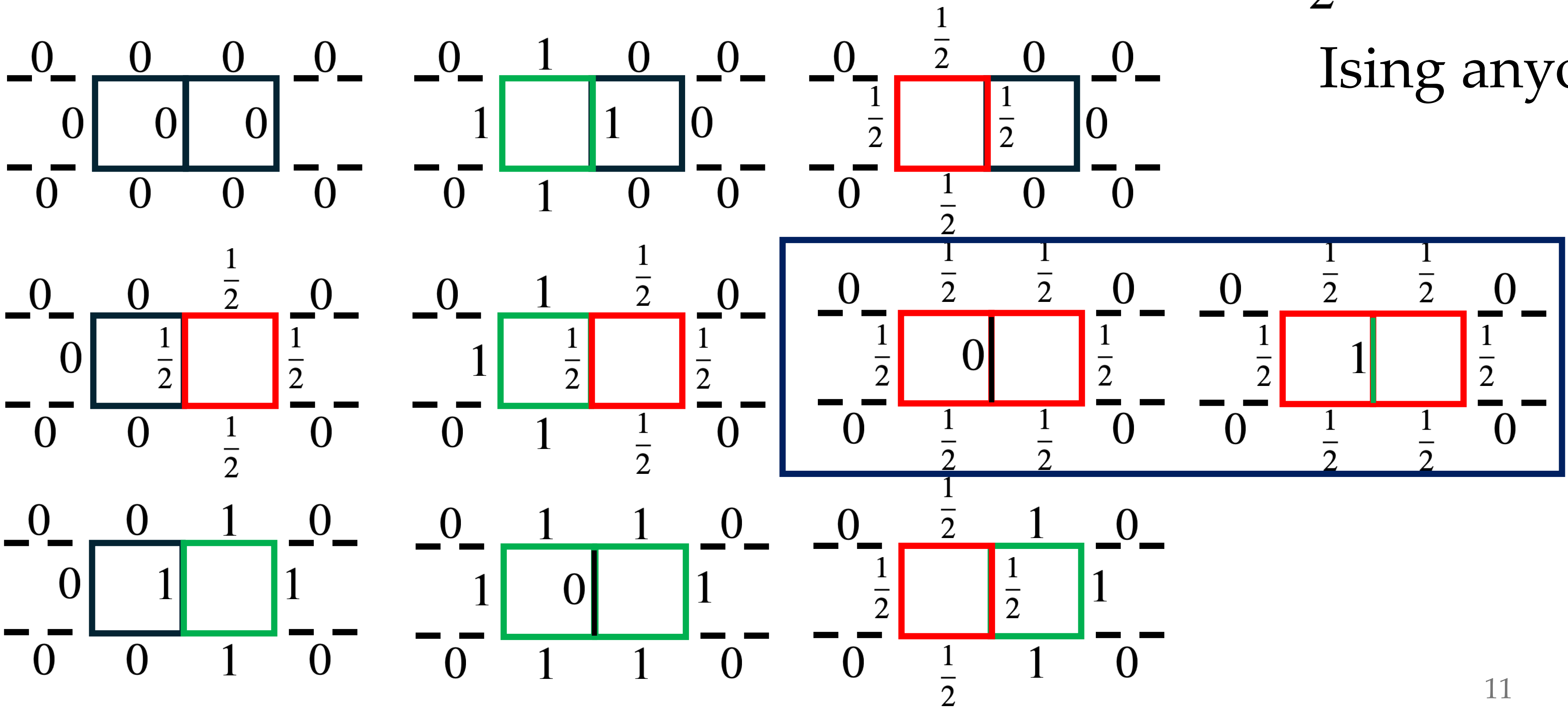
$d_0 = d_1 = 1 \quad \boxed{d_{\frac{1}{2}} = \sqrt{2}}$

$$0 \times a = a \times 0 = a \quad \frac{1}{2} \times 1 = \frac{1}{2} \quad \boxed{\frac{1}{2} \times \frac{1}{2} = 0 + 1} \quad 1 \times 1 = 0$$

"Logical" states $\neq$ Qutrit chain on a dual graph

$0 \equiv 1,\ \frac{1}{2} \equiv \sigma,\ 1 \equiv \psi$

Ising anyon



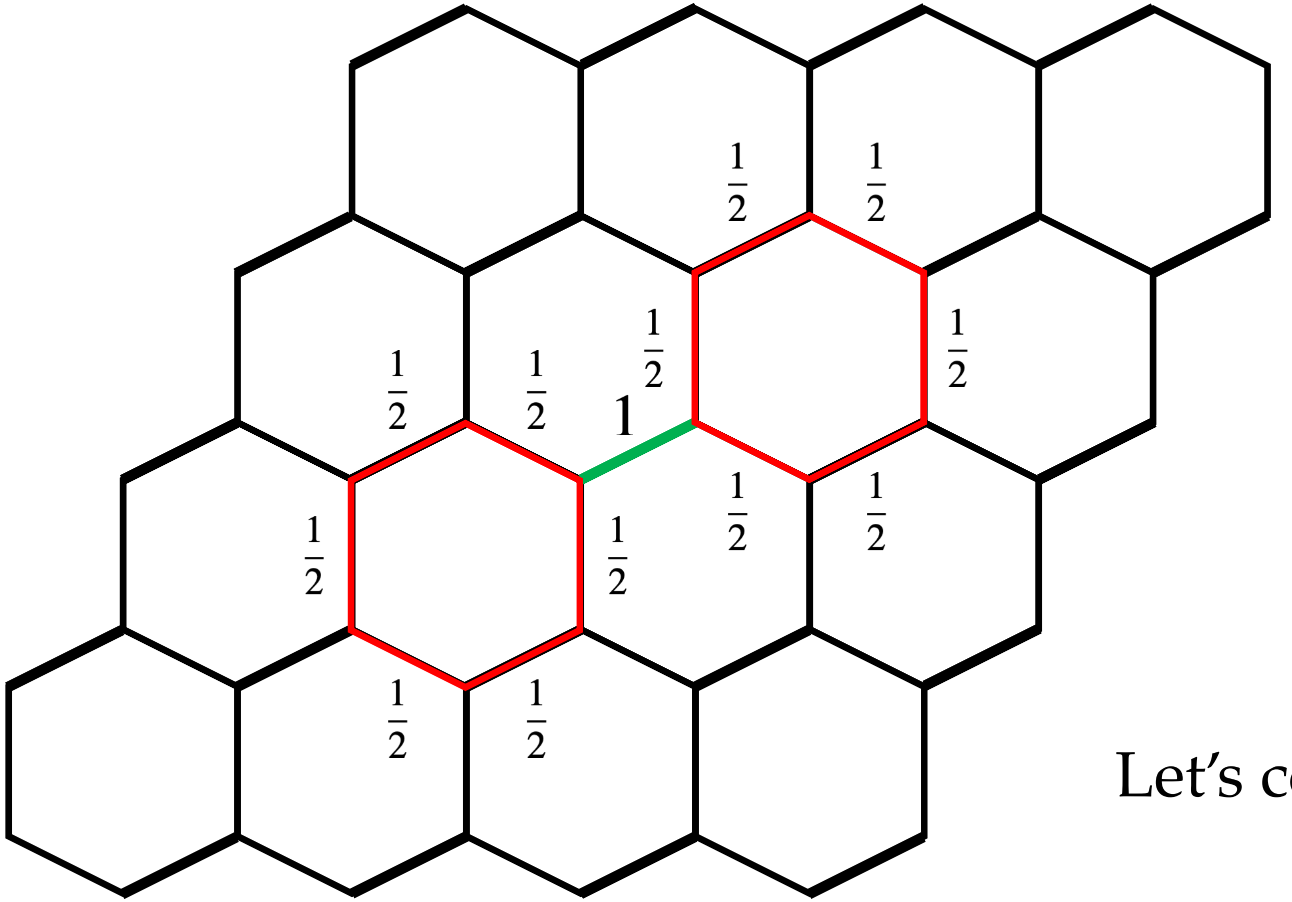
$SU(2)_2$ Yang-Mills model on a two-leg ladder

Fusion rules

$$d_0 = d_1 = 1 \quad \boxed{d_{\frac{1}{2}} = \sqrt{2}}$$

$$0 \times a = a \times 0 = a \quad \frac{1}{2} \times 1 = \frac{1}{2} \quad \frac{1}{2} \times \frac{1}{2} = 0 + 1 \quad 1 \times 1 = 0$$

"Logical" states $\neq$ product states of Wilson loop



Nonabelian feature

- Nonabelian anyon fusion rules
- Glasses-like states

Let's construct a minimal model keeping it!

<u>Integer sector</u> of $SU(2)_3$ Yang-Mills model on a two-leg ladder

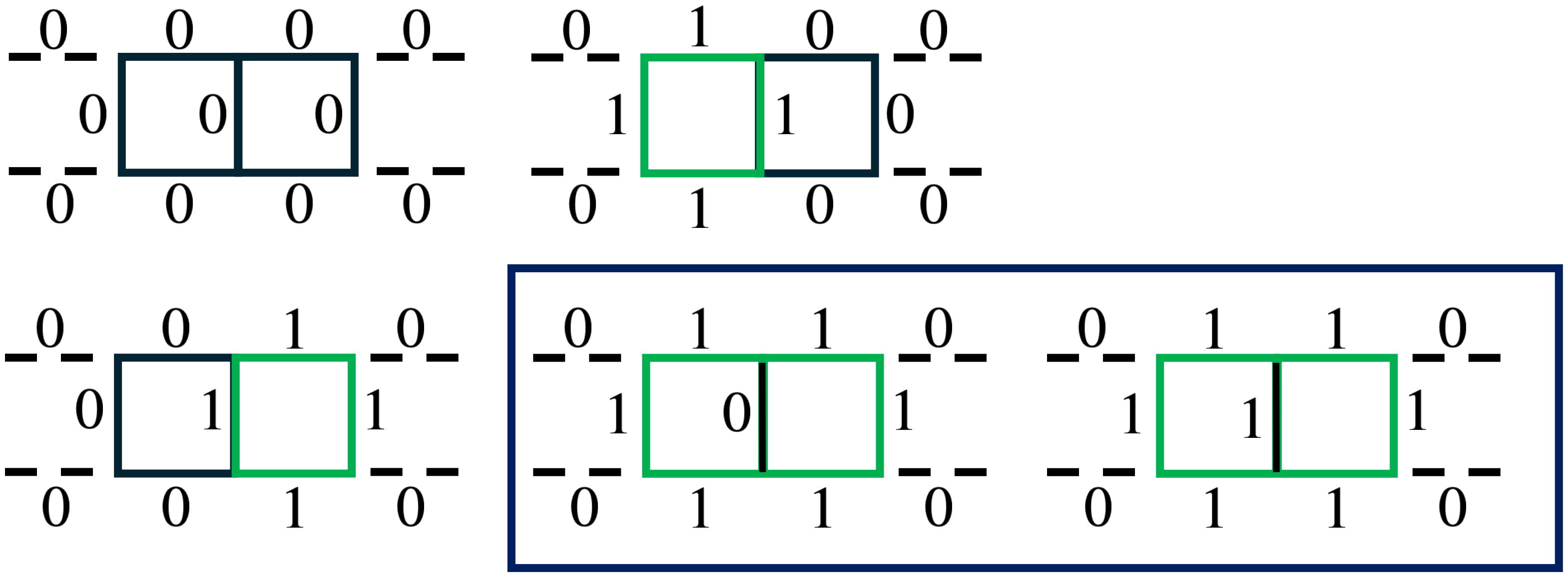
Fusion rules $0, \cancel{\frac{1}{2}}, 1, \cancel{\frac{3}{2}}$ = Fibonacci anyons

$$0 \times a = a \times 0 = a \qquad 1 \times 1 = 0 + 1$$

$$d_0 = 1 \qquad d_1 = \frac{1+\sqrt{5}}{2}$$

$$0 \equiv 1,\ 1 \equiv \tau$$

"Logical" states



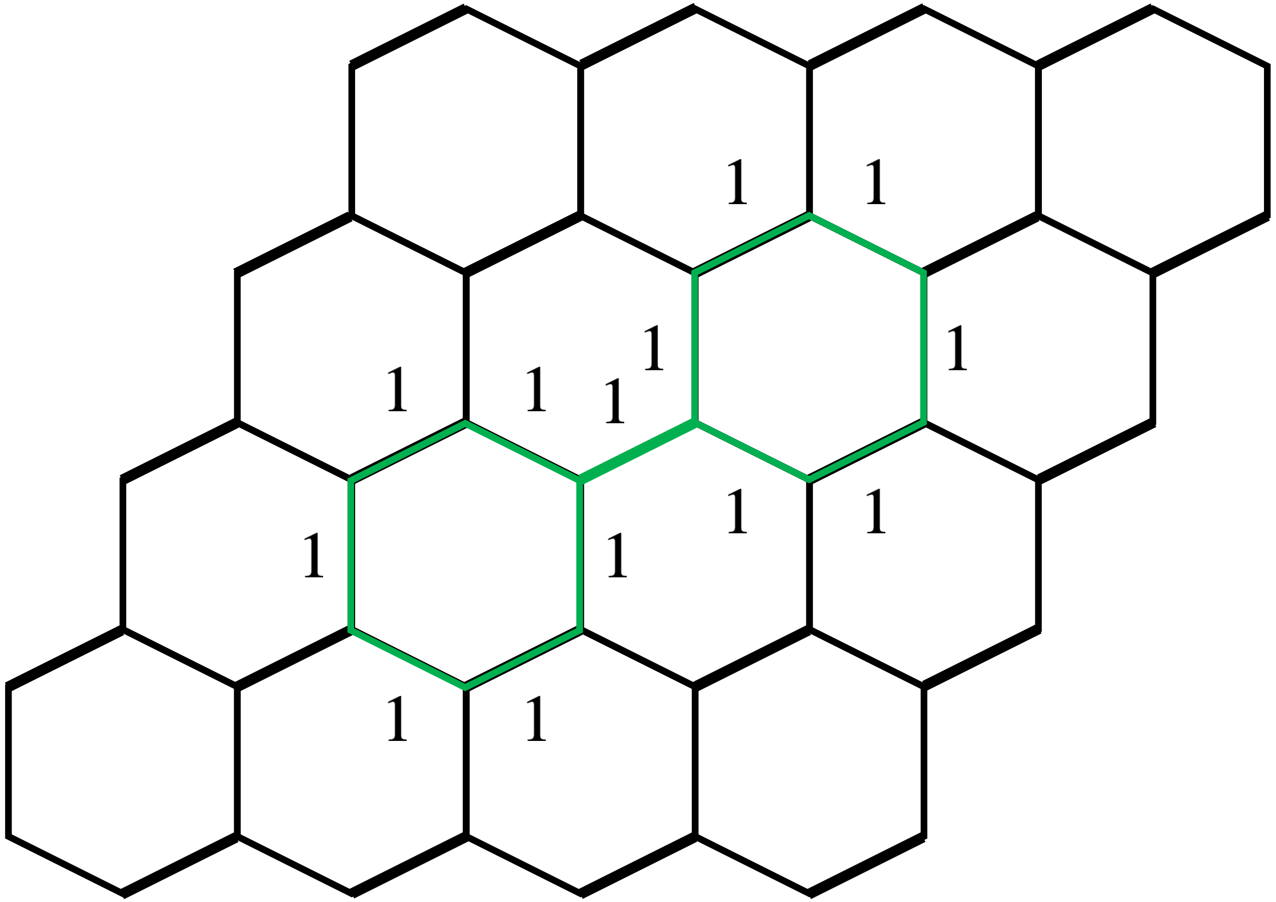
Integer sector of $SU(2)_3$ Yang-Mills model on a two-leg ladder

Fusion rules $0, \cancel{\frac{1}{2}}, 1, \cancel{\frac{3}{2}}$

$0 \times a = a \times 0 = a \qquad 1 \times 1 = 0 + 1$

$d_0 = 1 \qquad d_1 = \dfrac{1 + \sqrt{5}}{2}$

"Logical" states $\neq$ product state of Wilson loop



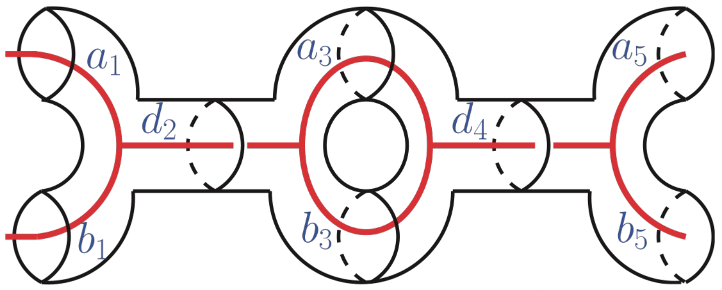
Let us introduce
a “minimal” nonabelian (anyon) model
motivated by Yang-Mills theory

(2+1)-dimensional Interacting Fibonacci anyon model

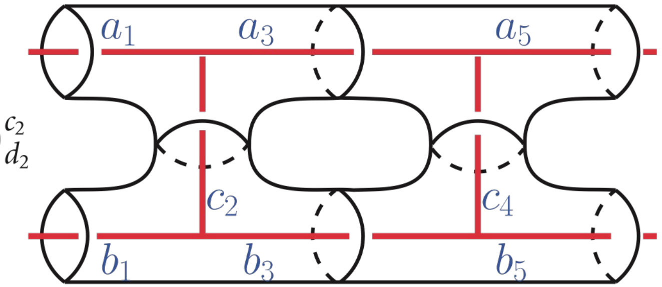
○ Interacting Fibonacci anyon model on a two-leg ladder

Gils, Trebst, Kitaev, Ludwig, Troyer, Wang, Nature Physics 5, 834 (2009).

$$H = -J_r \sum_{\text{rungs } r} \delta_{\ell(r),1} - J_p \sum_{\text{plaq } p} \delta_{\phi(p),1}$$

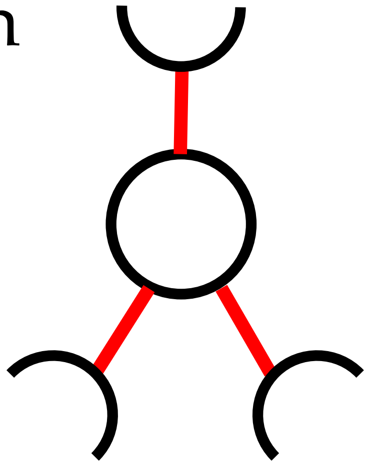


$$= \sum_{c2,c4} (F^{b_3}_{b_1 a_1 a_3})^{c_2}_{d_2}$$



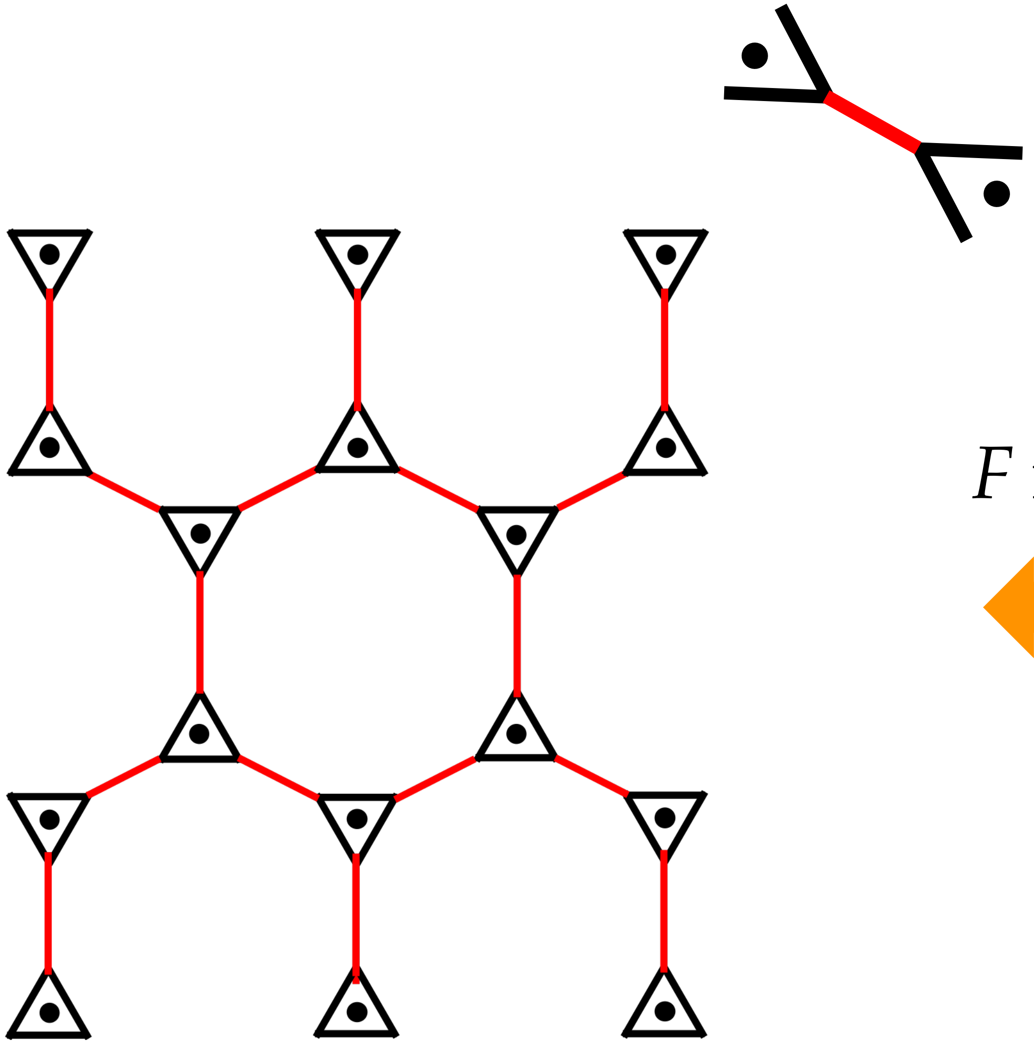
$$(F^{b_5}_{b_3 a_3 a_5})^{c_4}_{d_4}$$

○ A (2+1)-dimensional extension



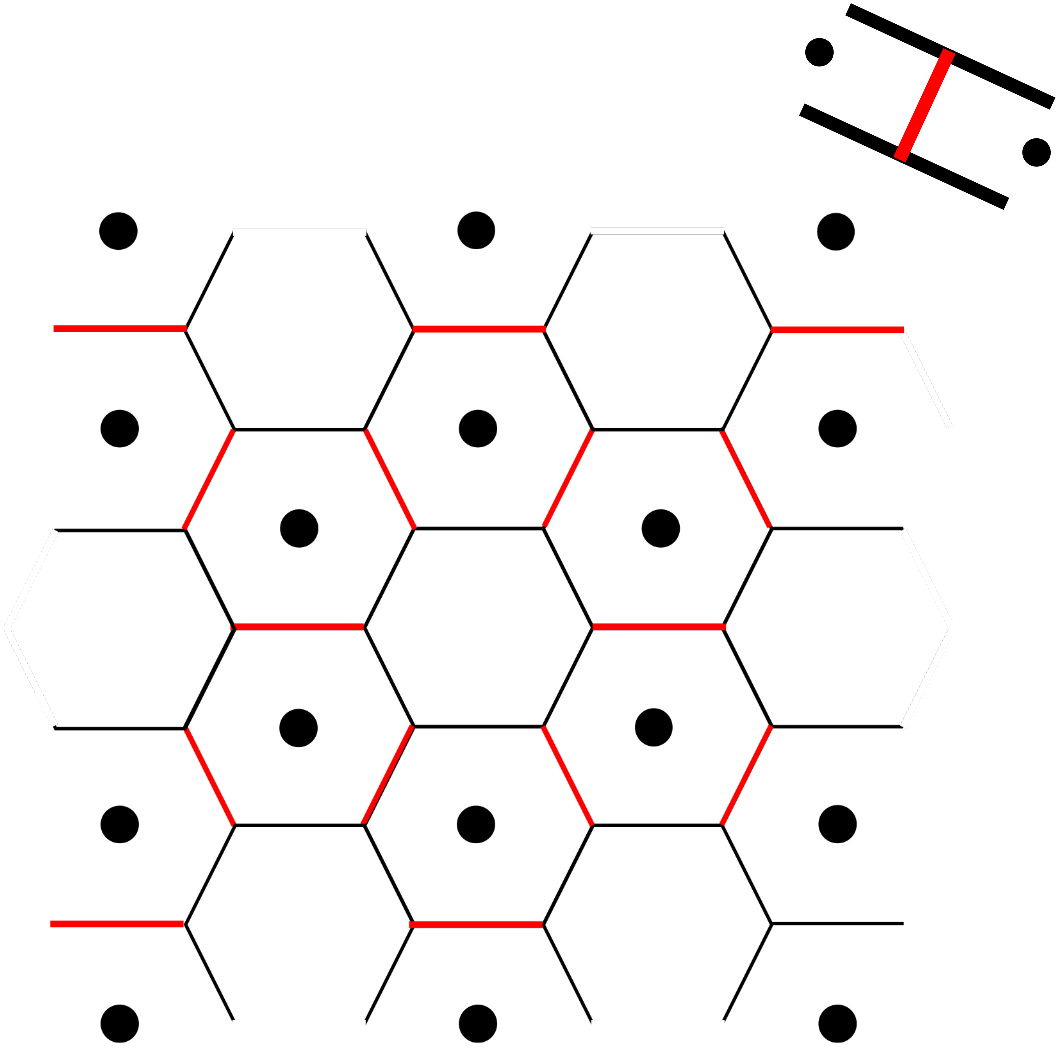
Fibonacci–Yang–Mills model in 2+1 dimensions

○Decorated honeycomb lattice



F moves

○Triangular lattice



Fibonacci–Yang–Mills model in 2+1 dimensions

○Hamiltonian

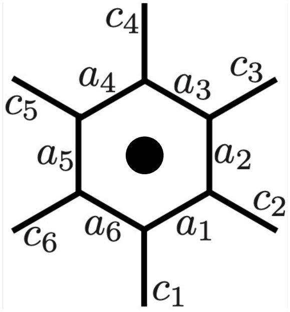
$$H = \frac{1}{2}\sum_{e\in\mathcal{E}} E^2(e) - \frac{K}{2}\sum_{f\in\mathcal{F}} \mathrm{tr}U(f) + \mathrm{tr}U^\dagger(f)$$

○ Electric field acts on red edges

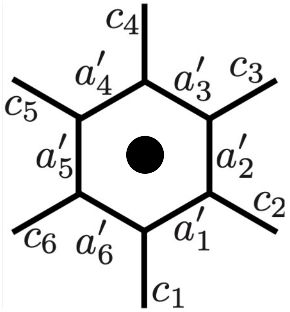
$$E^2 \uparrow_a \;=\; a(a+1) \uparrow_a$$

○ Magnetic field acts on faces with a hole

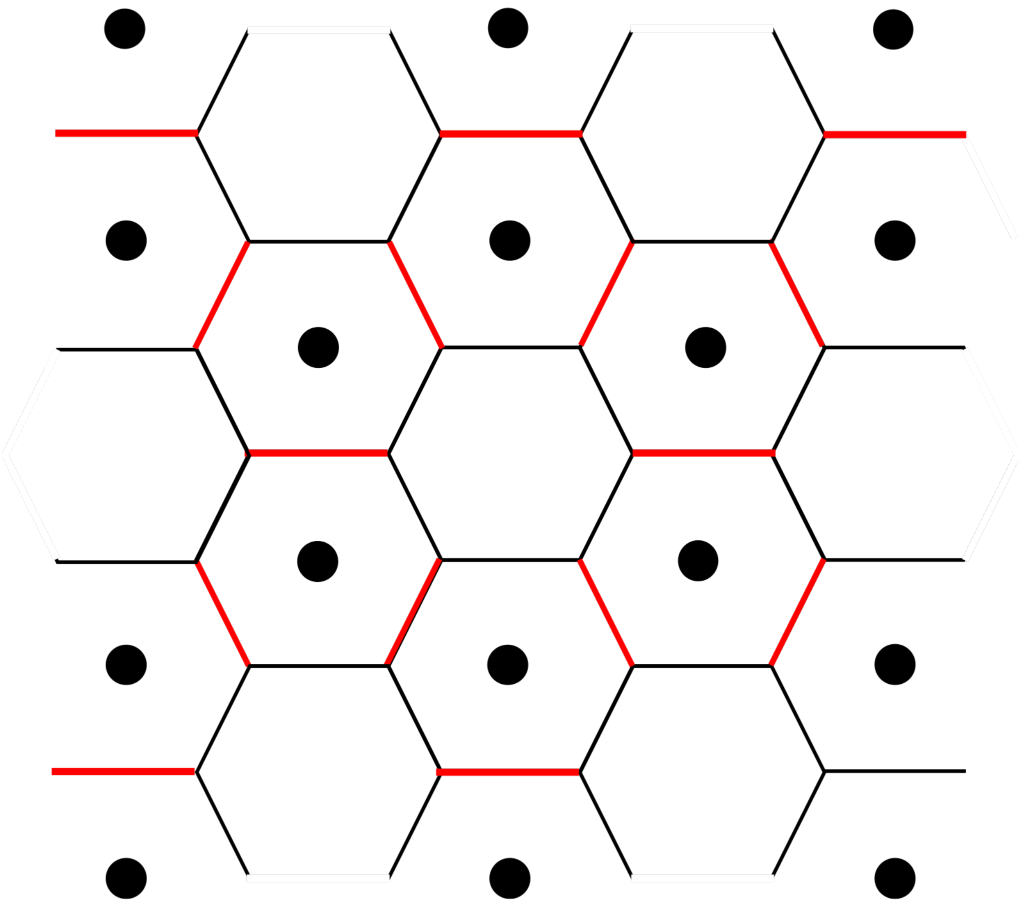
$\mathrm{tr}U_{\frac{1}{2}}$



$$= \prod_{i=1}^{6} \sum_{a'_i} [F^{c_i a_{i-1} \frac{1}{2}}_{a'_i}]_{a_i a'_{i-1}}$$



○Triangular lattice

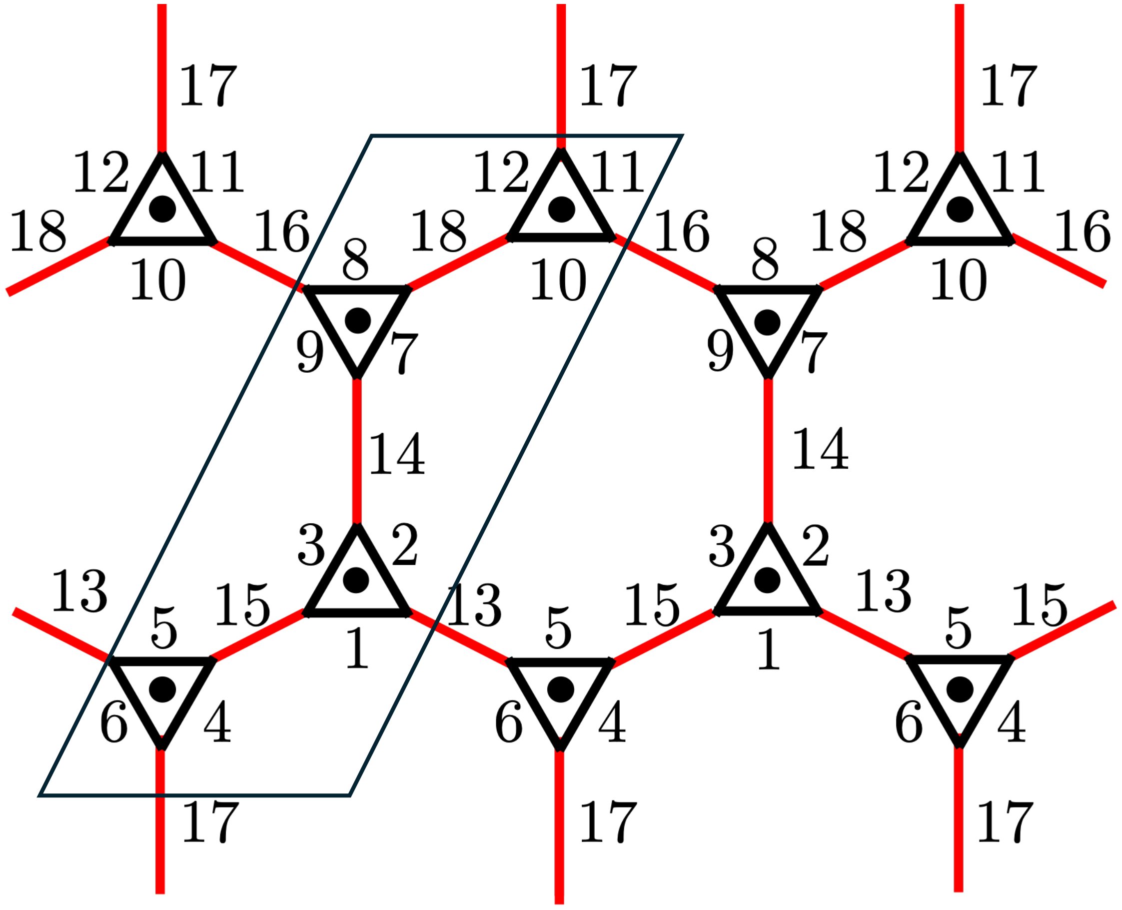


Experimental setup

Fibonacci–Yang–Mills model in 2+1 dimensions

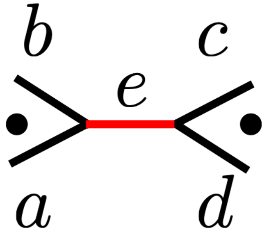
○ We simulate 18-qubit system

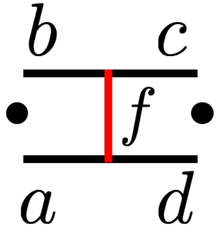
$2 \times 2 = 4$ plaquette system with PBCs

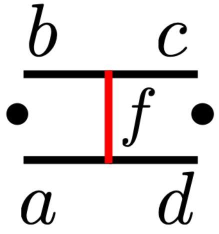


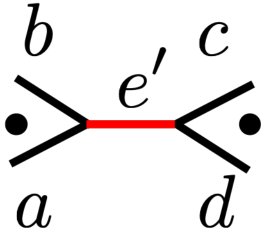
Quantum circuit of Trotter evolution

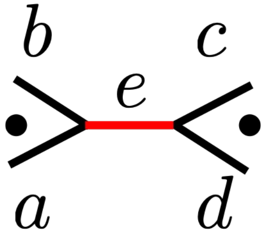
○ Electric terms

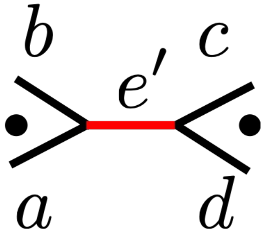
E^2 

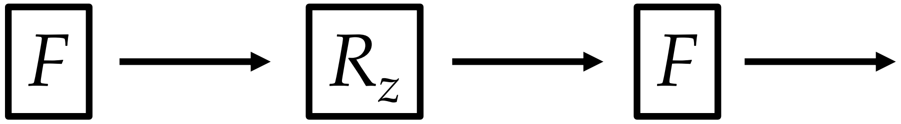
$= \sum_f [F^{abc}_d]_{ef} E^2$ 

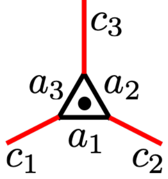
$= \sum_f [F^{abc}_d]_{ef} \delta_{f\tau}$ 

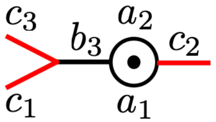
$= \sum_{f,e'} [F^{abc}_d]_{ef} [F^{abc}_d]_{e'f} \delta_{f\tau}$ 

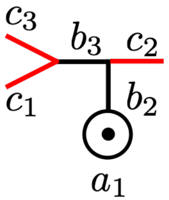
e^{-idtE^2} 

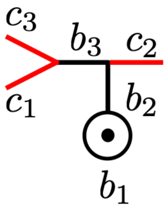
$= \sum_{f,e'} [F^{abc}_d]_{ef} [F^{abc}_d]_{e'f} e^{-idt\delta_{f\tau}}$ 



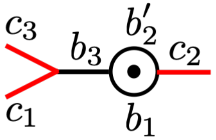
$\mathrm{tr}U_\tau$ 

$= \sum_{b_3} [F^{a_1 c_1 c_3}_{a_2}]_{a_3 b_3} \mathrm{tr}U_\tau$ 

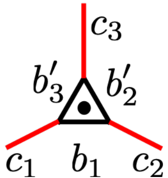
$= \sum_{b_2,b_3} [F^{a_1 c_1 c_3}_{a_2}]_{a_3 b_3} [F^{a_1 b_3 c_2}_{a_1}]_{a_2 b_2} \mathrm{tr}U_\tau$ 

$= \sum_{b_1,b_2,b_3} [F^{a_1 c_1 c_3}_{a_2}]_{a_3 b_3} [F^{a_1 b_3 c_2}_{a_1}]_{a_2 b_2} [S^{b_2}]_{a_1 b_1}$ 

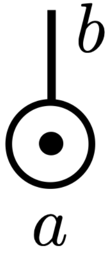
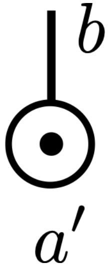
$= \sum_{b_1,b_2,b_3,b_2'} [F^{a_1 c_1 c_3}_{a_2}]_{a_3 b_3} [F^{a_1 b_3 c_2}_{a_1}]_{a_2 b_2} [S^{b_2}]_{a_1 b_1}$

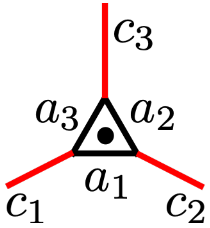
$\times [F^{b_1 b_3 c_2}_{b_1}]_{b_2' b_2}$ 

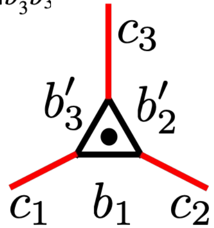
$= \sum_{b_1,b_2,b_3,b_2',b_3'} [F^{a_1 c_1 c_3}_{a_2}]_{a_3 b_3} [F^{a_1 b_3 c_2}_{a_1}]_{a_2 b_2} [S^{b_2}]_{a_1 b_1}$

$\times [F^{b_1 b_3 c_2}_{b_1}]_{b_2' b_2} [F^{b_1 c_1 c_3}_{b_2'}]_{b_3' b_3}$ 

○ Magnetic terms

$\mathrm{tr}U_\tau$  $= \sum_{a'} [S^b]_{aa'}$ 

$\mathrm{e}^{\mathrm{i}\frac{\sqrt{5}Kdt}{2}\mathrm{tr}U_{\mathrm{P}}}$  $=$

$\sum_{b_1,b_2,b_3,b_2',b_3'} [F^{a_1 c_1 c_3}_{a_2}]_{a_3 b_3} [F^{a_1 b_3 c_2}_{a_1}]_{a_2 b_2} [\mathrm{e}^{\mathrm{i}\frac{\sqrt{5}Kdt}{2}\mathrm{tr}S^{b_2}_{\mathrm{P}}}]_{a_1 b_1} [F^{b_1 b_3 c_2}_{b_1}]_{b_2' b_2} [F^{b_1 c_1 c_3}_{b_2'}]_{b_3' b_3}$ 

$\boxed{F} \longrightarrow \boxed{F} \longrightarrow \boxed{S_{\mathrm{P}}}$

$\longrightarrow \boxed{F} \longrightarrow \boxed{F} \longrightarrow$

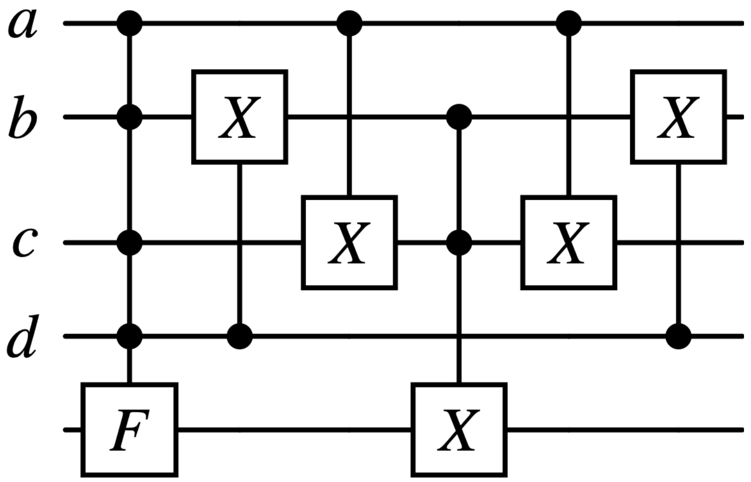
We need circuits of F and S_{P}

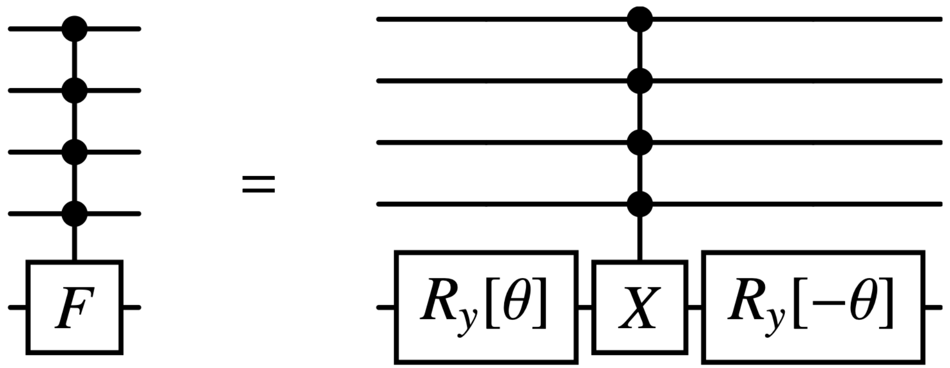
Quantum circuit of Trotter evolution

Bonesteel-DiVincenzo, PRB 86, 165113 (2012)

○ *F*-move (Fibonacci anyons)

$[F^{abc}_{d}] =$

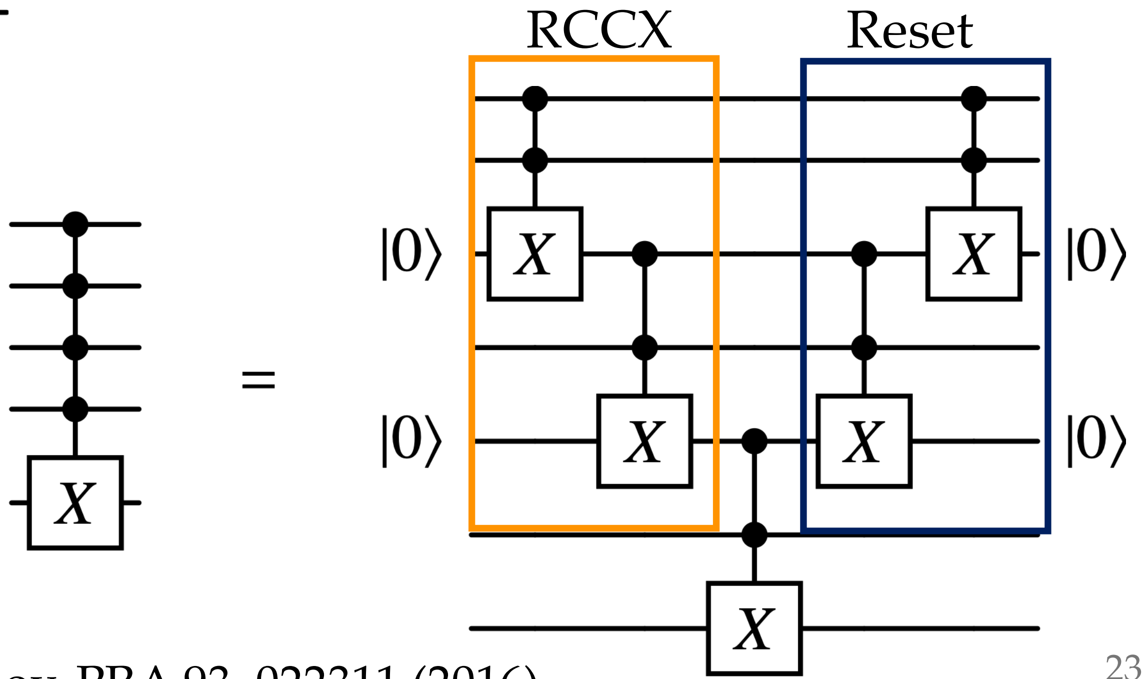




six Toffoli gates

40 CX gates

➡ 24 CX gates



Maslov, PRA 93, 022311 (2016)

Quantum circuit of Trotter evolution

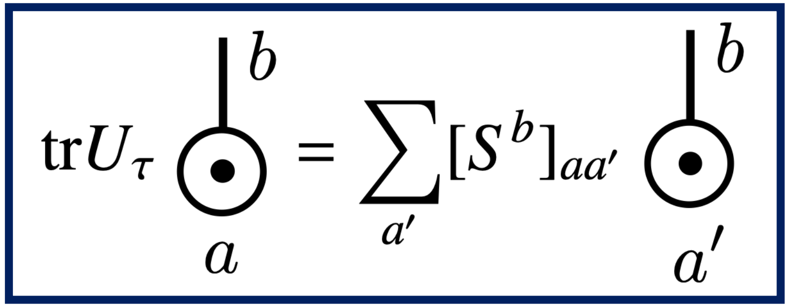
○ Phase gate of Wilson loop

$$[S^1_{\mathrm{P}}] = \begin{pmatrix} \frac{1-\varphi^2}{1+\varphi^2} & \frac{2\varphi}{1+\varphi^2} \\ \frac{2\varphi}{1+\varphi^2} & \frac{1+2\varphi-\varphi^2}{1+\varphi^2} \end{pmatrix}$$

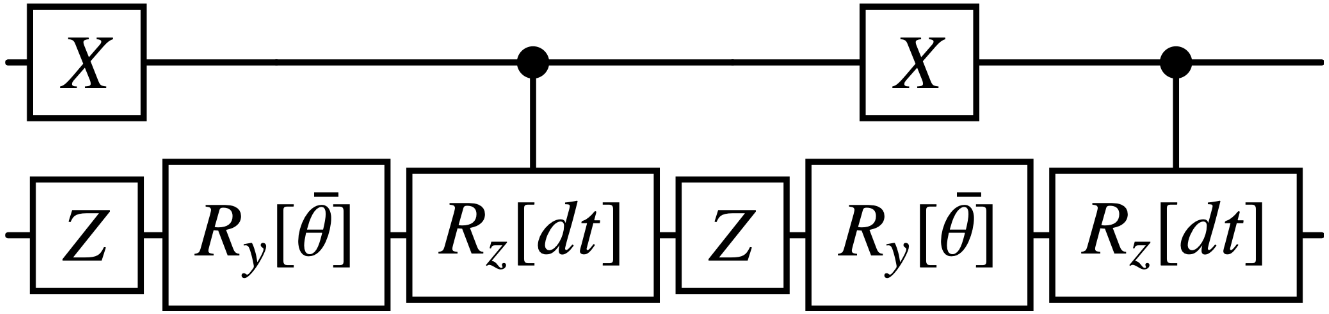
$$= \frac{1}{\sqrt{1+\varphi^2}}\begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{1+\varphi^2}}\begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$$

$$= R_y(\bar{\theta})ZZR_y(\bar{\theta})Z$$

$$[S^{\tau}_{\mathrm{P}}] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



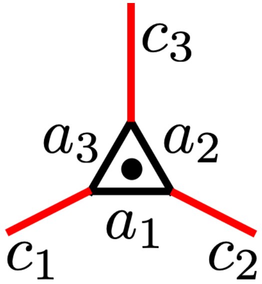
$$e^{-\mathrm{i}\frac{dt}{2}S_{\mathrm{P}}} =$$



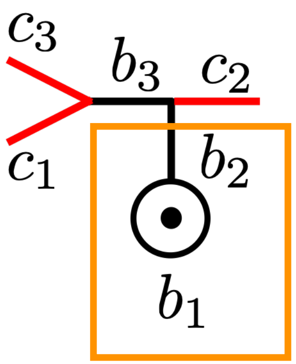
Circuit for measurement of Wilson loop

○ Measure Z in computational basis of Wilson loop

$\mathrm{tr}U_\mathrm{P}$



$$= \sum_{b_1,b_2,b_3} [F^{a_1 c_1 c_3}_{a_2}]_{a_3 b_3} [F^{a_1 b_3 c_2}_{a_1}]_{a_2 b_2} [S^{\,b_2}_\mathrm{P}]_{a_1 b_1}$$

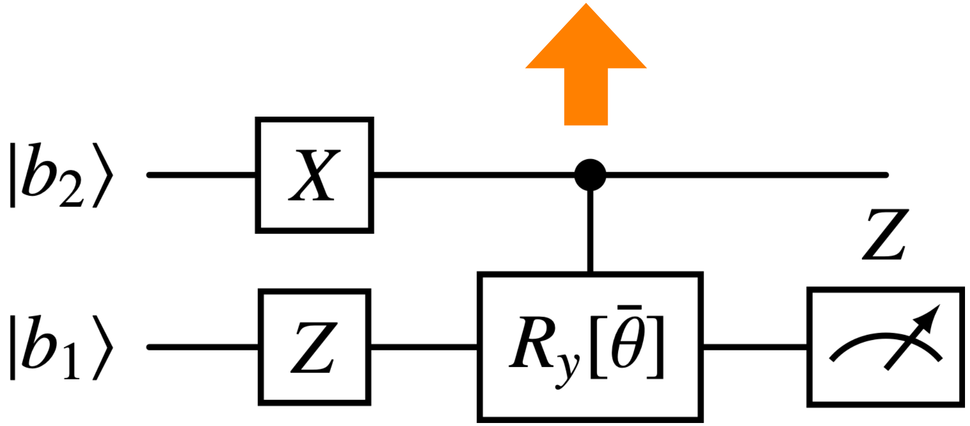


Measure

$\boxed{F} \longrightarrow \boxed{F} \longrightarrow \boxed{S_\mathrm{P}}$

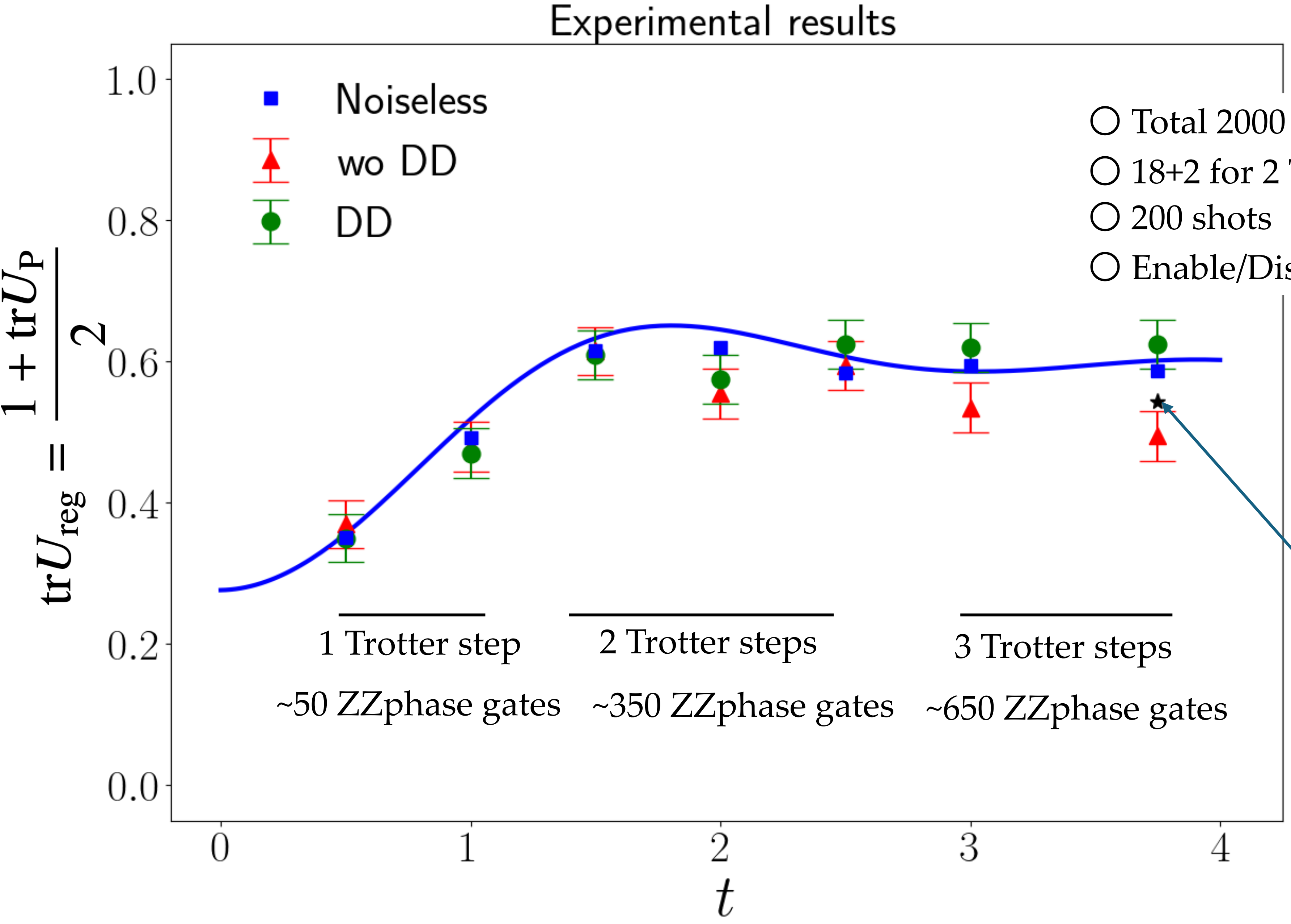
$$S^{\,1}_\mathrm{P} = R_y(\bar{\theta}) Z Z R_y(\bar{\theta}) Z$$

$$S^{\,\tau}_\mathrm{P} = Z$$



$$\mathrm{tr}U_\mathrm{reg} = \frac{1 + \mathrm{tr}U_\mathrm{P}}{2} \sim \frac{1 + Z}{2}$$

Thermalization dynamics on H2-1 device



- Total 2000 HQCs
- 18+2 for 2 Trotter steps, 18+8 for 3 Trotter step
- 200 shots
- Enable/Disable dynamical decoupling

$0.99895^{650} \sim 0.5$

Summary

- A simple nonabelian (anyon) model is simulated using H2-1

- After gate reduction, memory error is dominant source of error in a few Trotter step

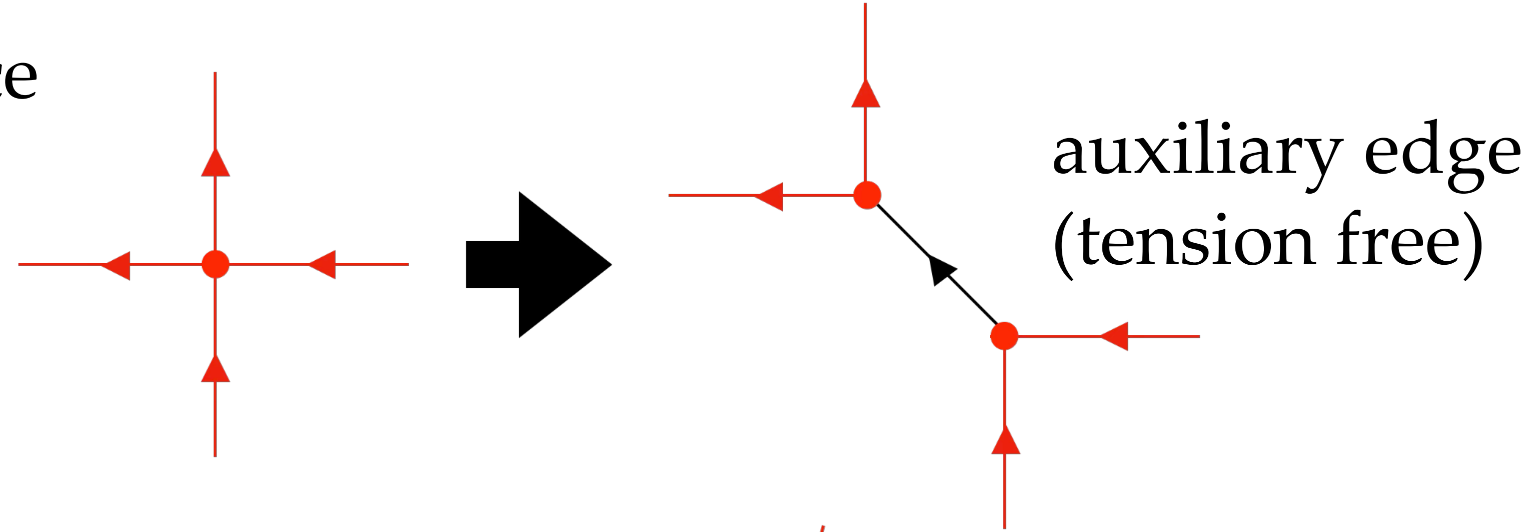
- Longer time evolution with more qubits and error mitigation techniques

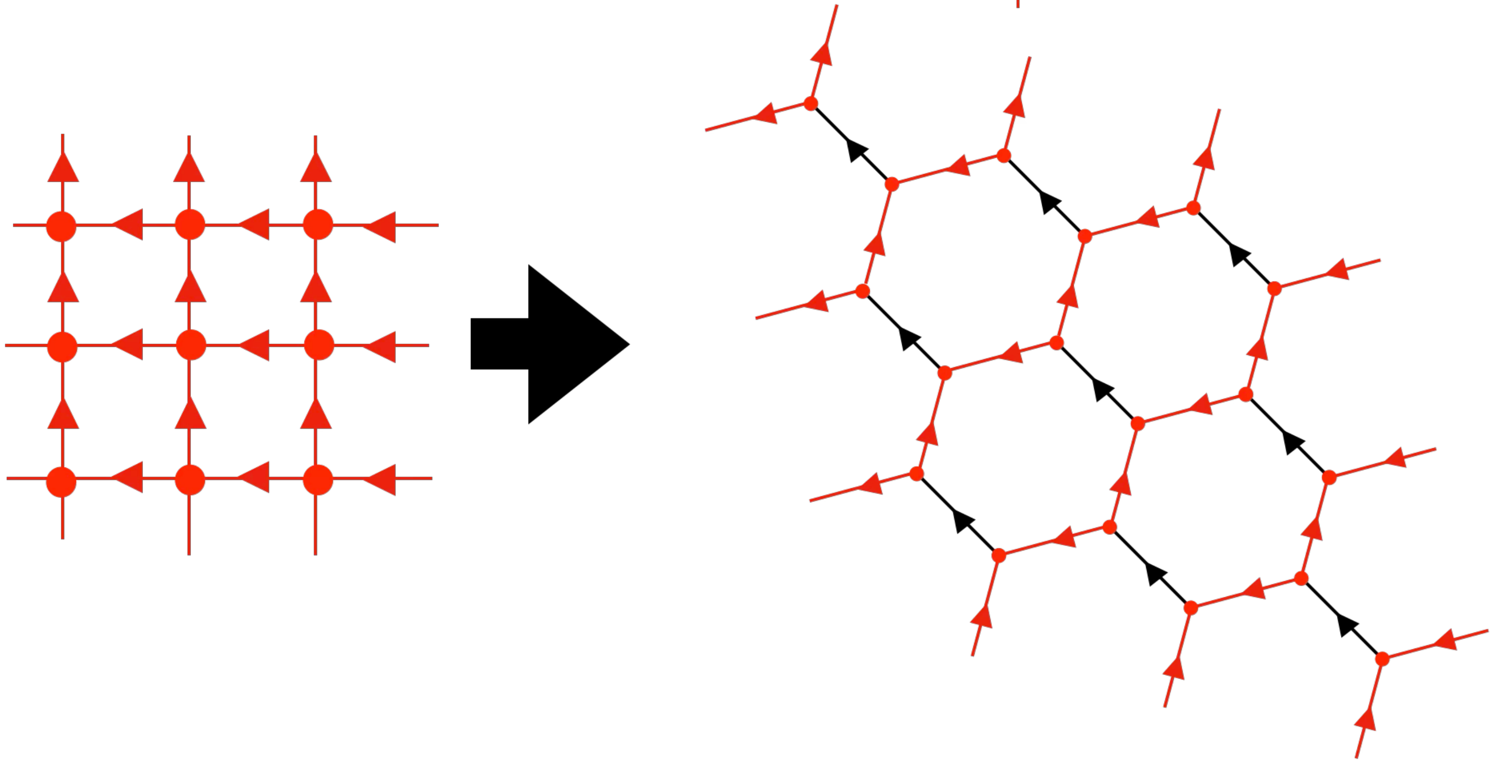
- Other group, fermions, 3+1 dimensions (Walker-Wang model)

Thank you!

Fibonacci–Yang–Mills model in 2+1 dimensions

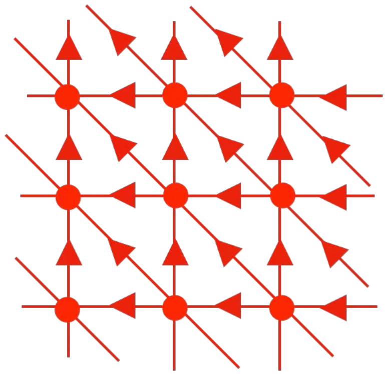
〇Square lattice



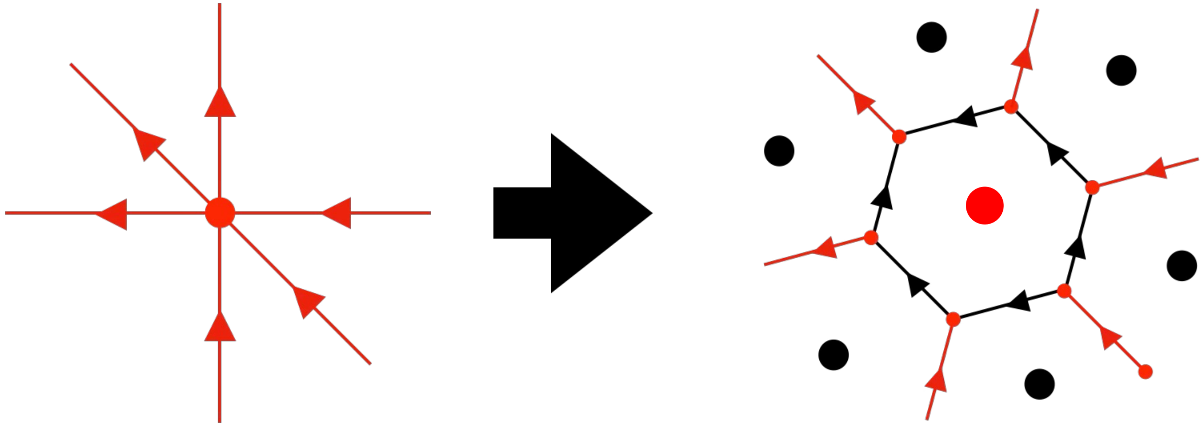


Fibonacci–Yang–Mills model in 2+1 dimensions

○Triangular lattice



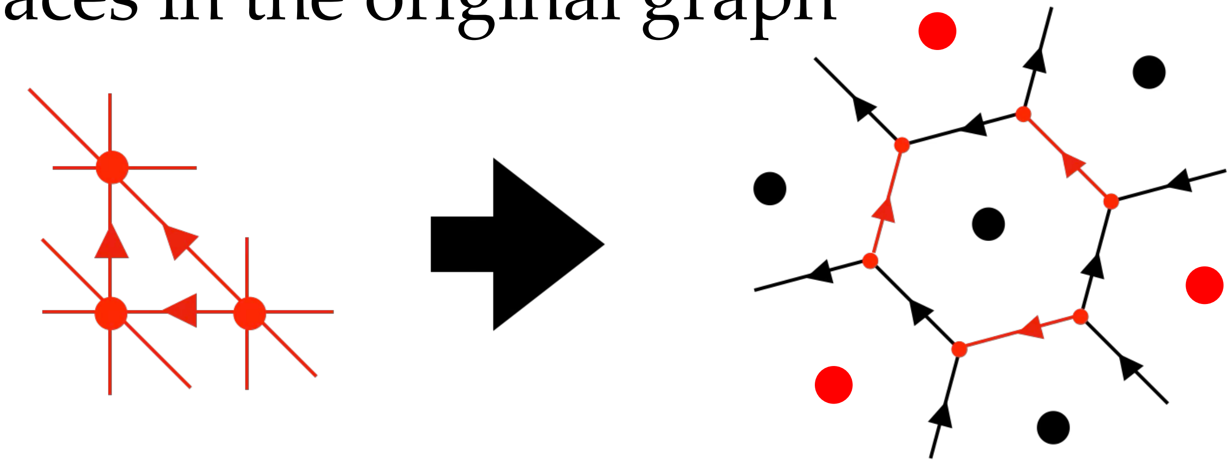
Vertices in the original graph



auxiliary edge
(tension free)

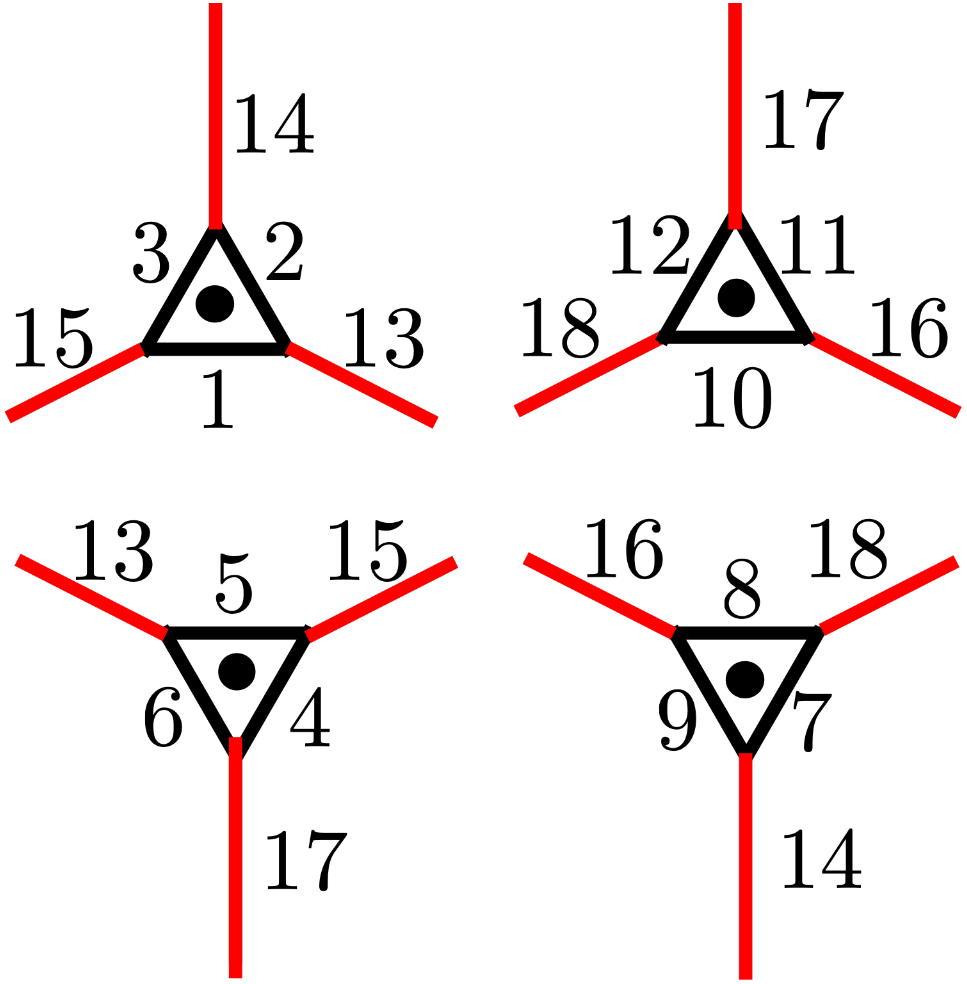
no plaquette term

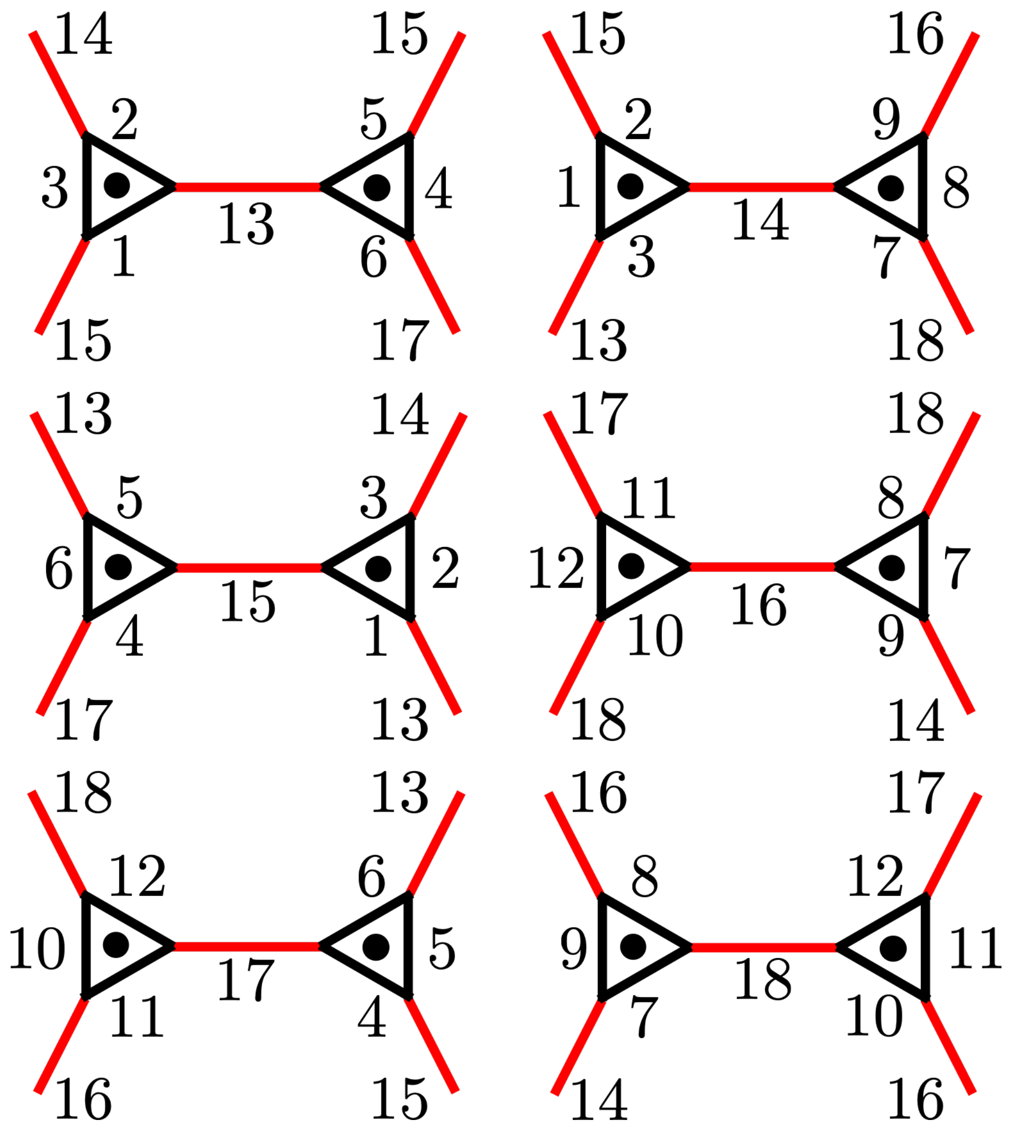
Faces in the original graph



Fibonacci–Yang–Mills model in 2+1 dimensions

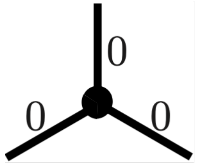
- We simulate 18-qubit system



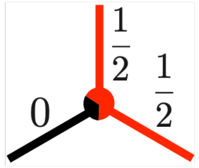


SU(2)$_1$ Yang-Mills model on a two-leg ladder

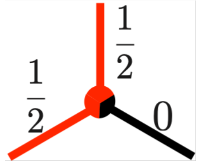
Fusion rules



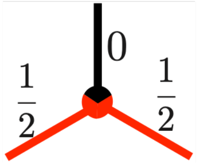
$= \left|0, 0, 0\right\rangle,$



$= \left|0, \frac{1}{2}, \frac{1}{2}\right\rangle,$

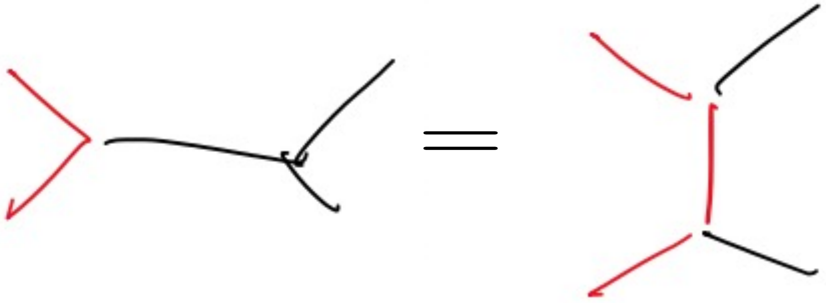


$= \left|\frac{1}{2}, \frac{1}{2}, 0\right\rangle,$

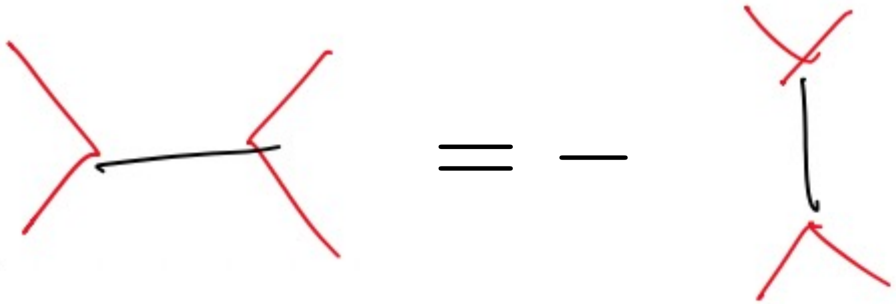


$= \left|\frac{1}{2}, 0, \frac{1}{2}\right\rangle$

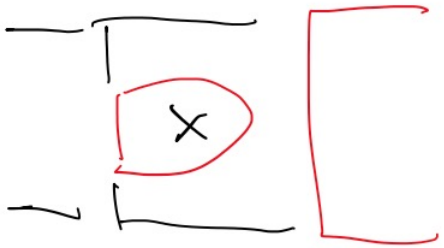
F moves

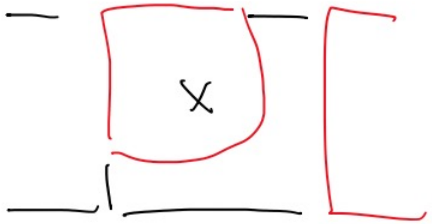


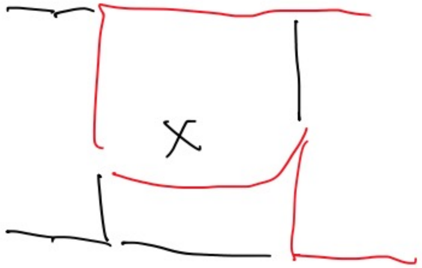
Only closed strings due to BCs

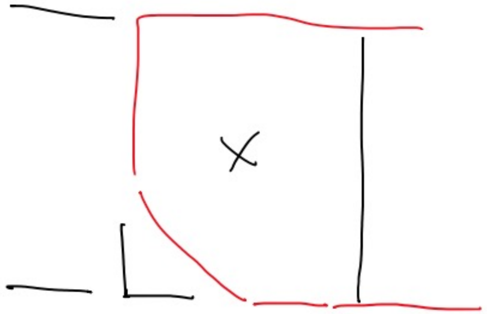


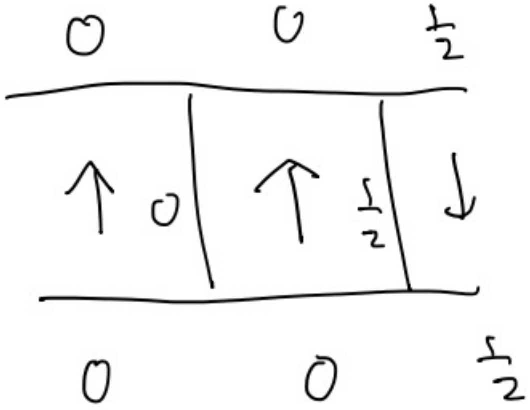
Recombination of strings gives -1

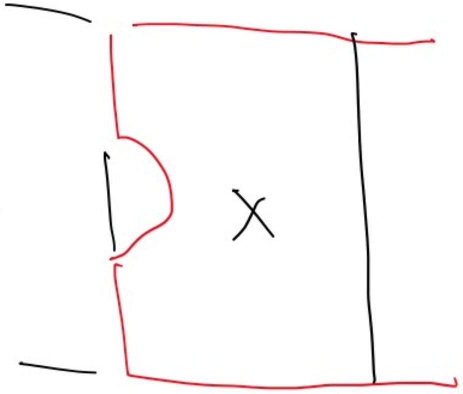
$$\mathrm{tr}U_{\frac{1}{2}} | \uparrow\uparrow\downarrow\rangle = \overset{\mathrm{I}}{\frac{1}{\sqrt{d_0}}}$$ 

$$= \overset{\mathrm{II}}{\frac{1}{\sqrt{d_0}}}$$ 

$$= \overset{\mathrm{III}}{-\frac{1}{\sqrt{d_0}}}$$ 

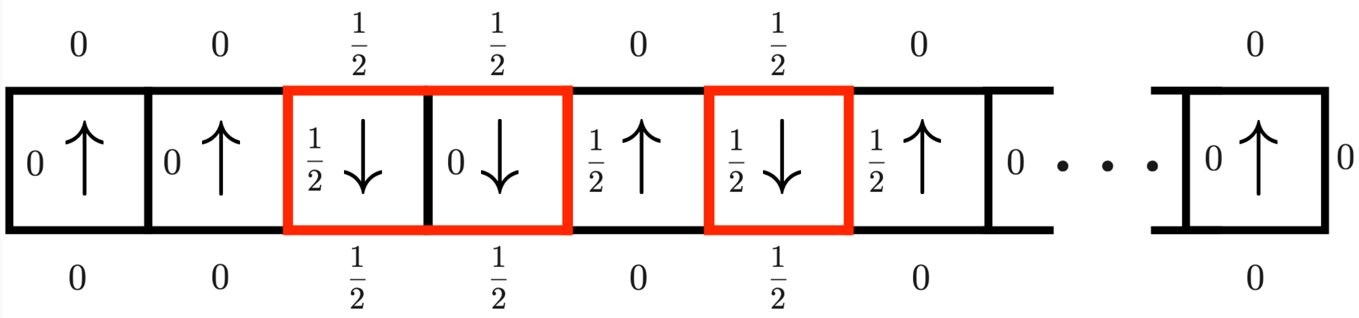
$$= \overset{\mathrm{IV}}{-\frac{1}{\sqrt{d_0}}}$$ 



$$= \overset{\mathrm{V}}{-\frac{1}{\sqrt{d_0}}}$$ 

$$\overset{\mathrm{VI}}{=} -| \uparrow\downarrow\downarrow\rangle$$

$SU(2)_1$ Yang-Mills model on a two-leg ladder



$$\mathrm{tr}U_\Box(i)|\uparrow\uparrow\uparrow\rangle = |\uparrow\downarrow\uparrow\rangle$$

$$\mathrm{tr}U_\Box(i)|\uparrow\downarrow\uparrow\rangle = |\uparrow\uparrow\uparrow\rangle$$

$$\mathrm{tr}U_\Box(i)|\uparrow\uparrow\downarrow\rangle = -|\uparrow\downarrow\downarrow\rangle$$

$$\mathrm{tr}U_\Box(i)|\uparrow\downarrow\downarrow\rangle = -|\uparrow\uparrow\downarrow\rangle$$

$$\mathrm{tr}U_\Box(i)|\downarrow\downarrow\uparrow\rangle = -|\downarrow\uparrow\uparrow\rangle$$

$$\mathrm{tr}U_\Box(i)|\downarrow\uparrow\uparrow\rangle = -|\downarrow\downarrow\uparrow\rangle$$

$$\mathrm{tr}U_\Box(i)|\downarrow\uparrow\downarrow\rangle = |\downarrow\downarrow\downarrow\rangle$$

$$\mathrm{tr}U_\Box(i)|\downarrow\downarrow\downarrow\rangle = |\downarrow\uparrow\downarrow\rangle$$

$$\mathrm{tr}U_\Box(i) = Z_{i-1}X_iZ_{i+1}$$

SU(3)$_1$ Yang-Mills model on a two-leg ladder

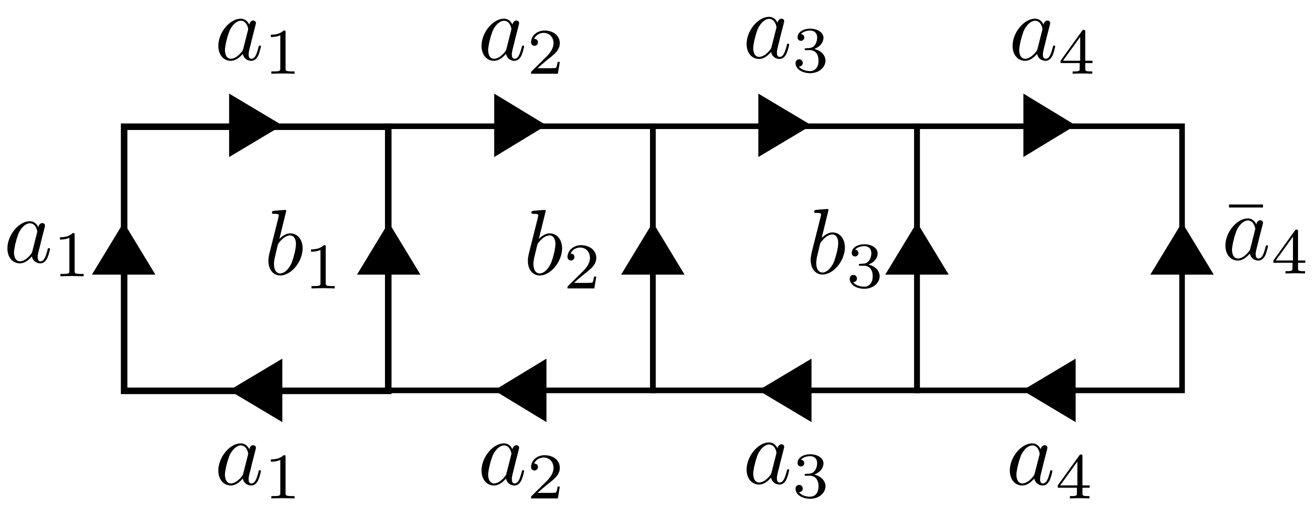
Fusion rules = $\mathbf{Z}_3$ group

$$1 \times 1 = 1, \;\; 1 \times 3 = 3 \times 1 = 3, \;\; 1 \times \bar{3} = \bar{3} \times 1 = \bar{3},$$
$$3 \times 3 = \bar{3}, \;\; 3 \times \bar{3} = 1, \;\; \bar{3} \times \bar{3} = 3$$

$1 \equiv 0, \; 3 \equiv 1, \; \bar{3} \equiv 2$

$a \times b = a + b \;\; \text{modulo} \;\; 3$

Hilbert space



$$|\psi\rangle = \sum_{a_1, a_2, a_3, a_4 \in \{1, 3, \bar{3}\}} \psi(a_1, a_2, a_3, a_4) |a_1, a_2, a_3, a_4\rangle$$

Qutrit chain

SU(3)$_1$ Yang-Mills model on a two-leg ladder

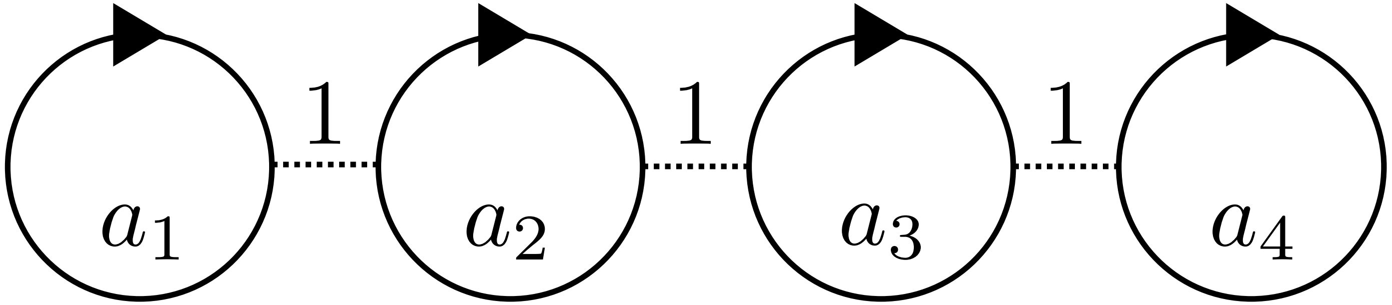
Fusion rules = $\mathbf{Z}_3$ group

$$1 \times 1 = 1, \quad 1 \times 3 = 3 \times 1 = 3, \quad 1 \times \bar{3} = \bar{3} \times 1 = \bar{3},$$
$$3 \times 3 = \bar{3}, \quad 3 \times \bar{3} = 1, \quad \bar{3} \times \bar{3} = 3$$

$$1 \equiv 0, \; 3 \equiv 1, \; \bar{3} \equiv 2$$
$$a \times b = a + b \;\; \text{modulo} \;\; 3$$

Hilbert space

Dual basis = Plaquette basis

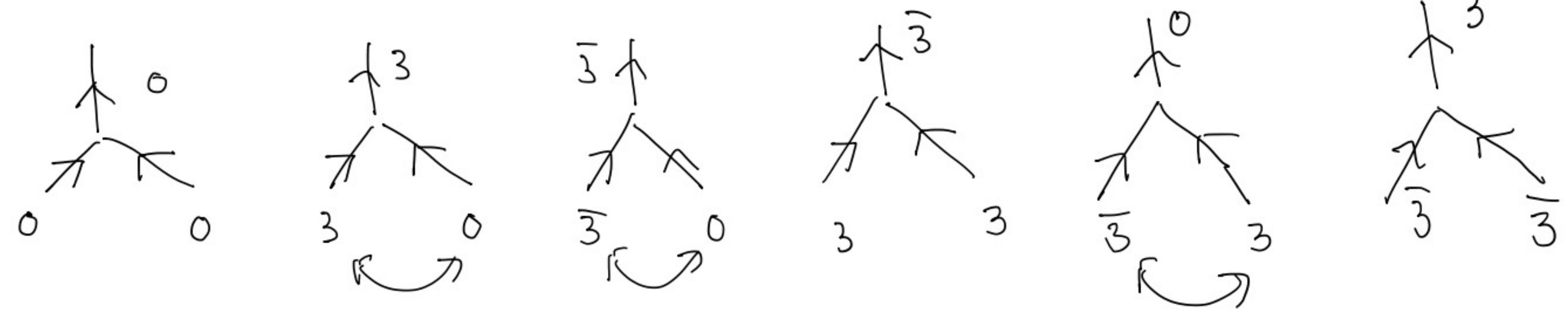


$$|\psi\rangle = \sum_{a_1, a_2, a_3, a_4 \in \{1, 3, \bar{3}\}} \psi(a_1, a_2, a_3, a_4) |a_1, a_2, a_3, a_4\rangle$$

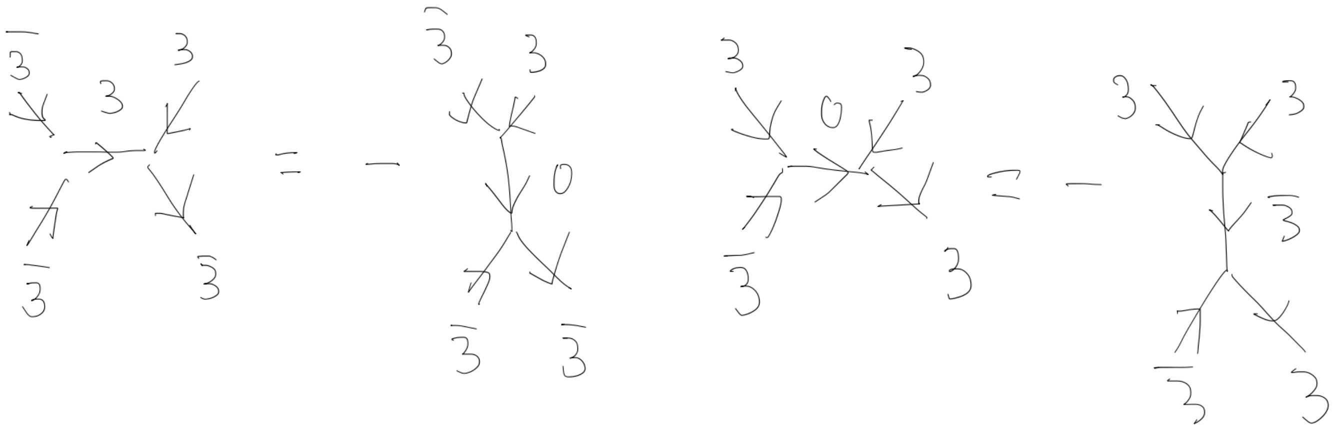
Qutrit chain and spin-1 TFI model

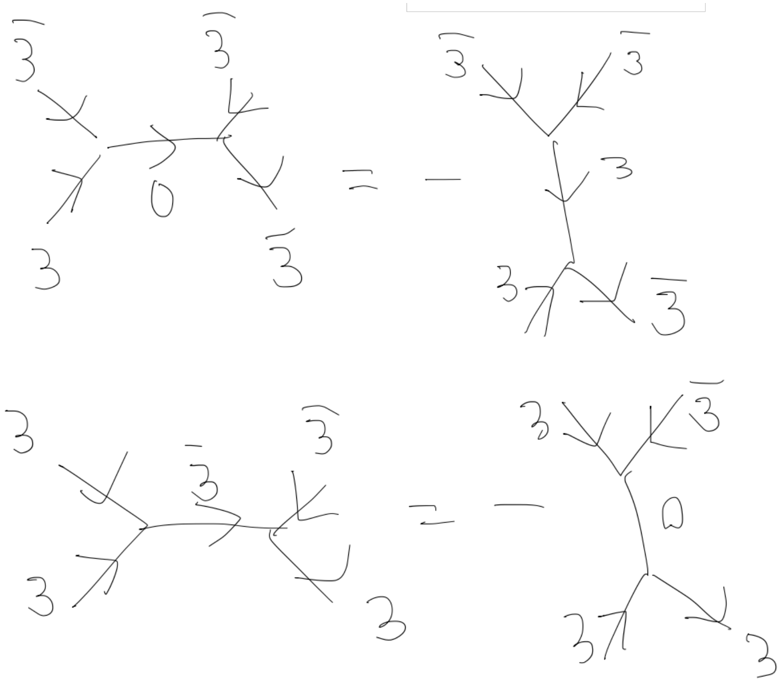
SU(3)$_1$ Yang-Mills model on a two-leg ladder

Fusion rules



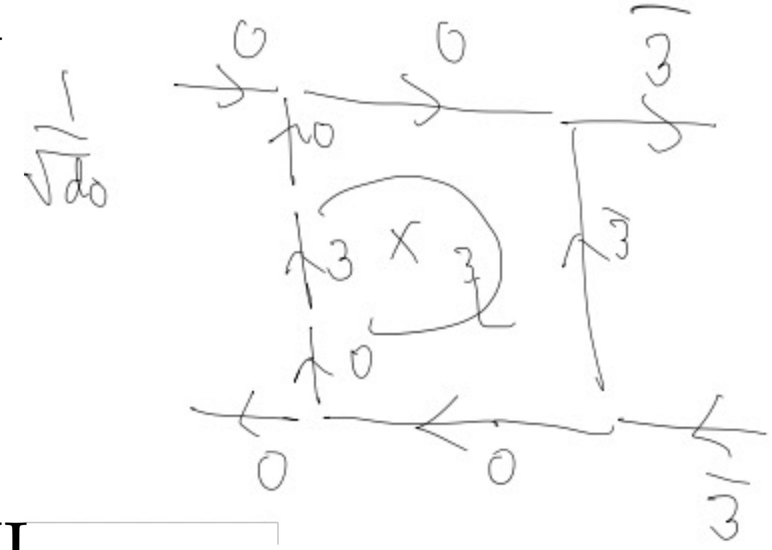
F moves





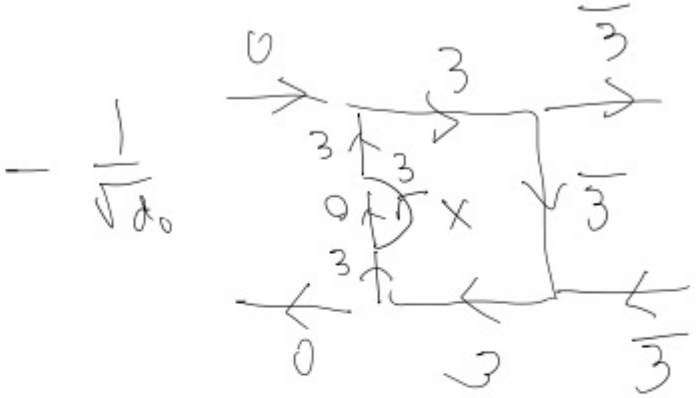
$$\mathrm{tr}U_3|00\bar{3}\rangle =$$

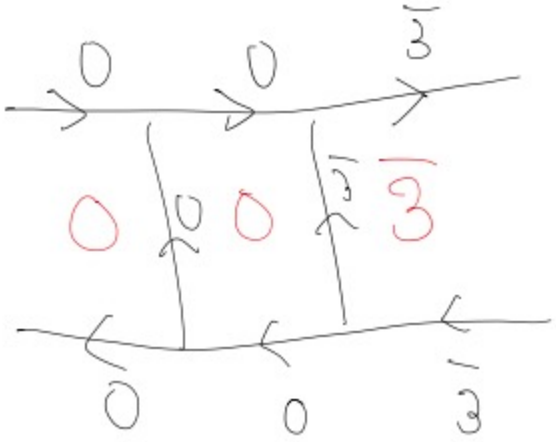
I



$=$

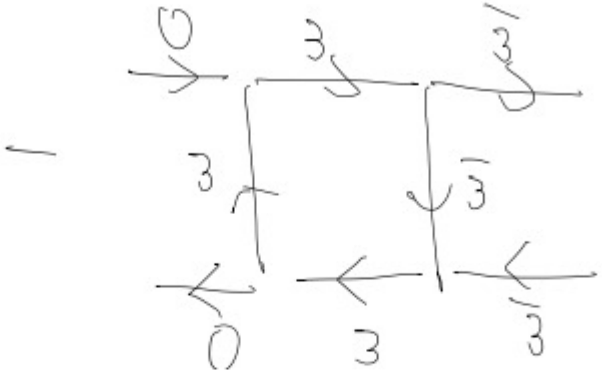
II





$=$

III



$=$

IV

$$-|03\bar{3}\rangle$$

$3 \times 3 \times 3$ patterns ➡ unitary transformation ➡ spin model

SU(3)$_1$ Yang-Mills model on a two-leg ladder

○ Hamiltonian of a qutrit chain

$$H_E = \frac{c}{2}\sum_{x=1}^{N} M(x) + \frac{c}{2}\sum_{x=1}^{N-1} M^r(x, x+1)$$

$$M = 2\sum_{a=3,\bar{3}} |a\rangle\langle a| \qquad M^r = \sum_{a=3,\bar{3}} \left(|a,1\rangle\langle a,1| + |1,a\rangle\langle 1,a| + |a,\bar{a}\rangle\langle a,\bar{a}|\right)$$

$$H_B = -\frac{K}{2}\sum_{x=1}^{N}\left[W(x) + W^\dagger(x)\right] \qquad W = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$