# Pion properties and quark condensate in isospin-asymmetric nuclear matter

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# Introduction

- Dynamical chiral symmetry breaking is supposed to be the origin of the masses of hadrons in quantum chromodynamics (QCD).

- $\langle\Omega|[Q_A^a, P^a]|\Omega\rangle = \left\langle \bar{u}u + \bar{d}d \right\rangle = \langle \bar{q}q \rangle$ : a sufficient condition for the spontaneous breaking of the chiral symmetry in the chiral limit.

- One expects the broken chiral symmetry to be partially restored in some conditions, such as at a finite density or temperature.

- Once the partial restoration of chiral symmetry in a nuclear medium is confirmed, we can guess that the properties of the pions are influenced by in-medium effects.

# Previous researches

- In Ref. [1], the partial restoration of the chiral symmetry in isospin-symmetric nuclear matter was investigated from the in-medium quark condensate.

- The in-medium pion properties in isospin-symmetric nuclear matter were calculated in Ref. [2]

- Reference [3] explored the in-medium quark condensates in isospin-asymmetric nuclear matter using the correlation functions.

[1] Jido, Hatsuda, Kunihiro, Phys. Lett. B 670, 109 ( 2008)

[2] Goda, Jido, PTEP 033D03, 26 ( 2014)

[3] Hübsch, Jido, PRC 104, 015202 ( 2021)

# Introduction

- In this research, the in-medium mass, wave function renormalization, and decay constant of each pion in nuclear matter with finite isospin are explored.

- The in-medium Gell-Mann-Oakes-Renner relation : ${m_\pi^*}^2 {f_\pi^*}^2 = -m_q \langle \bar{q}q \rangle^*$

- In this work, we assume that all in-vacuum calculations are done. Thus, we use experimental values for LECs.

# The chiral symmetry in QCD

- Quantum Chromodynamics describes dynamics among quarks, gluons, and hadrons, whose Lagrangian is given from the $SU(3)$ gauge symmetry by

$$\mathcal{L}_{QCD} = \sum_{i,j=1}^{3} \sum_{f=1}^{6} \sum_{a=1}^{8} \sum_{\alpha,\beta=1}^{4} \bar{q}_{\alpha,f,i} \left(\mathrm{i}\gamma^{\mu} D_{\mu} - m_f\right)_{\alpha\beta}^{ij} q_{\beta,f,j} - \frac{1}{4} G_{\mu\nu\mathrm{a}} G_{a}^{\mu\nu}.$$

$\mu$: the spacetime index

$f$: the flavor index

$i$: the color index

$\alpha$: the spinor index

- At low-energy scales, one can only consider the two light quarks: up- and down-quarks.
- The low-energy QCD Lagrangian exhibits the global $SU(2)_L \times SU(2)_R \times \mathrm{U}(1)_V$ symmetry in the chiral limit, where $m_u = m_d = 0$.

# The chiral symmetry in QCD

- For the Noether currents,

$$V_\mu^a \equiv R_\mu^a + L_\mu^a = \bar{q}\frac{\tau_a}{2}\gamma^\mu q, \qquad A_\mu^a \equiv R_\mu^a - L_\mu^a = \bar{q}\frac{\tau_a}{2}\gamma^\mu\gamma_5 q.$$

- $\int d^3x\, A(V)_0^a \equiv Q_{A(V)}^a$: the generators of the $SU(2)_{A(V)}$ transformations.

- For completeness, the scalar and pseudoscalar densities are introduced:

$$S_b = \bar{q}\lambda_b q, \qquad P_b = \mathrm{i}\bar{q}\gamma_5\lambda_b q, \qquad b = 0,1,2,3.$$

- $\mathrm{i}\langle 0|[Q_A^a, P_a]|0\rangle = \left\langle 0\middle|\bar{u}u + \bar{d}d\middle|0\right\rangle \equiv \langle \bar{\mathrm{q}}\mathrm{q}\rangle \neq 0$ : the sufficient condition of the spontaneous chiral symmetry breaking

# The chiral perturbation theory

- Due to the spontaneous chiral symmetry breaking, one obtains the Nambu-Goldstone bosons.

- In the $SU(2)$ case, the three pions are the Nambu-Goldstone bosons.

- From the transformation behaviors, the chiral Lagrangian is written as [8,9]

$$\mathcal{L}_{\pi}^{(2)} = \frac{f^2}{4} Tr\left\{D_\mu U^\dagger D^\mu U + \chi^\dagger U + \chi U^\dagger\right\},$$

$$\mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} = \bar{N}(i\gamma^\mu \Gamma_\mu + i g_A \gamma^\mu \gamma^5 \Delta_\mu + c_1 Tr\{\chi_+\}\bar{N}N + \frac{c_2}{2m_N^2} Tr\left\{u_\mu u_\nu\right\}\bar{N}D^\mu D^\nu N$$
$$- \frac{c_3}{2} Tr\left(u^\mu u_\mu\right)\bar{N}N + \cdots .$$

[8] Gasser, Sainio, Švarc, Nucl. Phys. B 250, 779 ( 1988)

[9] Gasser, Juerg, Leutwyler, Nucl. Phys. B 250, 465 ( 1985)

# The chiral perturbation theory

where

$$\chi_{\pm} = u^{\dagger}\chi\, u^{\dagger} \pm\; u\chi^{\dagger}u, \qquad\qquad u_{\mu} = 2\mathrm{i}\Delta_{\mu},$$

$$\Delta_{\mu} = \frac{1}{2}\{u^{\dagger}[\partial_{\mu} - \mathrm{i}r_{\mu}]u - u[\partial_{\mu} - \mathrm{i}l_{\mu}]u^{\dagger}\},$$

$$\Gamma_{\mu} = \frac{1}{2}\,[u^{\dagger}, \partial_{\mu}\, u] - \frac{\mathrm{i}}{2}u^{\dagger}r_{\mu}u - \frac{\mathrm{i}}{2}ul_{\mu}u^{\dagger},$$

$$U = \exp\left(\mathrm{i}\pi^{a}\tau^{a}\,\frac{y(\pi^{2})}{2\sqrt{\pi^{2}}}\right),$$

$$f_{R\mu\nu} = \partial_{\mu}\, r_{\nu} - \partial_{\nu}\, r_{\mu} - \mathrm{i}[r_{\mu}, r_{\nu}]. \quad f_{L\mu\nu} = \partial_{\mu}\, l_{\nu} - \partial_{\nu}\, l_{\mu} - \mathrm{i}[l_{\mu}, l_{\nu}],$$

$$f_{\mu\nu}^{\pm} = uf_{L\mu\nu}u^{\dagger} \pm u^{\dagger}f_{R\mu\nu}u,$$

# In-medium pion properties

- The normalization of the in-medium pion states

$$\left\langle \pi^{*\pm,0}\left(k_{\pi^{\pm,0}}\right)\middle|\pi^{*\pm,0}\left(p_{\pi^{\pm,0}}\right)\right\rangle = (2\pi)^3 2\omega_{\pi^{\pm,0}}(\vec{p})\delta^{(3)}\left(\vec{p}-\vec{k}\right).$$

$p_{\pi^{\pm,0}}$ : an on-shell momentum of the pions

$\omega_{\pi^{\pm,0}}$ : an in-medium pion energy

- The in-medium pion mass $\omega_{\pi^{\pm,0}}\left(\vec{0}\right) = m_\pi^*$.

- The in-medium pion mass is obtained by solving the mass equation

$$m_\pi^* = \sqrt{m_\pi^2 + \Sigma_\pi(m_\pi^*, \vec{0})}\ .$$

$\Sigma_\pi$ : the in-medium pion self-energy

# In-medium pion properties

- The matrix elements of the axial-vector current $A_\mu^{\pm,0}$ [1] :

$$\left\langle \Omega \middle| A_\mu^{\pm,0} \middle| \pi^{*\pm,0}(p_{\pi^{\pm,0}}) \right\rangle = \mathrm{i}\left[n_\mu N^*_{\pi^{\pm,0}}(p_{\pi^{\pm,0}}) + p_\mu^{\pi^{\pm,0}} F^*_{\pi^{\pm,0}}(p_{\pi^{\pm,0}})\right] e^{-\mathrm{i}p_{\pi^{\pm,0}}\cdot x},$$

$n_\mu$ : a vector of the Lorentz frame of the nuclear matter

In the rest frame, $n_\mu = (1,0,0,0)$.

- The pure vacuum state : $|0\rangle$
- The asymptotic state of nuclear matter : $|\Omega\rangle$.

- The pion decay constants in nuclear matter :

$$f_\pi^{*(t)} = F_\pi^* + \frac{N_\pi^*}{m_\pi^*}, \qquad f_\pi^{*(s)} = F_\pi^*$$

$f_\pi^{*(t)}$ : Temporal decay constant

$f_\pi^{*(s)}$ : Spatial decay constant

# Correlation function approach

- The correlation function of the two axial-vector currents:

$$\Pi_{\mu\nu}^{\pm\mp,00}(p) \equiv \int d^4x \, e^{\mathrm{i}p\cdot x}\left\langle \Omega \middle| \mathrm{T} A_\mu^{\pm,0}(x) A_\nu^{\mp,0}(0) \middle| \Omega \right\rangle.$$

- This can be expanded as

$$\begin{aligned}\Pi_{\mu\nu}^{\pm\mp,00}(p) &= \frac{\mathrm{i}\left(n_\mu N_{\pi^{\pm,0}}^* + p_\mu^{\pi^{\pm,0}}(\vec{p}) F_{\pi^{\pm,0}}^*\right)\left(n_\nu \bar{N}_{\pi^{\pm,0}}^* + p_\nu^{\pi^{\pm,0}}(\vec{p}) \bar{F}_{\pi^{\pm,0}}^*\right)}{2\omega_{\pi^{\pm,0}}(\vec{p})\left(p^0 - \omega_\pi^{\pm,0}(\vec{p}) + \mathrm{i}\epsilon\right)} + \cdots \\ &= \left(\mathrm{i}\hat{\Gamma}_\mu^{\pi^{\pm,0}}\right) \frac{\mathrm{i}}{\left(p^2 - m_\pi^2 - \Sigma_\pi^{\pm,0}(p) + i\epsilon\right)} \left(-\mathrm{i}\bar{\hat{\Gamma}}_\nu^{\pi^{\pm,0}}\right) + \cdots\end{aligned}$$

$\hat{\Gamma}_\mu^{\pi^{\pm,0}}$: The vertex correction for the axial vector current

$$\left\langle \Omega \middle| A_\mu^{\pm,0} \middle| \pi^{*\pm,0}\left(p_{\pi^{\pm,0}}\right) \right\rangle_{1PI} = \mathrm{i}\hat{\Gamma}_\mu^{\pi^{\pm,0}}$$

# Correlation function approach

- The wave function renormalization:

$$Z_\pi = \left(1 - \frac{1}{2\omega_\pi(\vec{p})} \frac{\partial \Sigma_\pi(p)}{\partial p_0}\bigg|_{p_0=\omega_\pi(\vec{p})}\right)^{-1}$$

$$\Pi_{\mu\nu}^{\pm\mp,00}(p) = \hat{\Gamma}_\mu^{\pi^{\pm,0}} \frac{\mathrm{i}Z_{\pi^{\pm,0}}}{2\omega_{\pi^{\pm,0}}(\vec{p})\left(p_0 - m^*_{\pi^{\pm,0}} - \mathcal{K}_{\pi^{\pm,0}}(\vec{p}) + i\epsilon\right)} \bar{\hat{\Gamma}}_\nu^{\pi^{\pm},0} + \cdots,$$

$\mathcal{K}_{\pi^{\pm,0}}(\vec{p}) = \omega_{\pi^{\pm,0}}(\vec{p}) - m^*_{\pi^{\pm,0}}$ : an in-medium kinetic energy.

- Effective vertex correction :

$$p_\mu \hat{f}_\pi \equiv \hat{\Gamma}_\mu^\pi.$$

- The calculation of the temporal decay constant:

$$f_\pi^{*(t)} = \sqrt{Z_\pi}\hat{f}_\pi.$$

# Correlation function approach

- In a similar way, the two-point function of the pseudoscalar currents:

$$\Pi^{\pm,\mp,00}(p) \equiv \int d^4x\, e^{\mathrm{i}p\cdot x}\langle\Omega\big|TP^{\pm,0}(x)P^{\mp,0}(0)\big|\Omega\rangle$$
$$= \frac{\mathrm{i}G^*_{\pi^{\pm,0}}\bar{G}^*_{\pi^{\pm,0}}}{2\omega_{\pi^{\pm,0}}(\vec{p})(p^0 - \omega_\pi^{\pm,0}(\vec{p}) + \mathrm{i}\epsilon)} + \cdots$$
$$= \left(\mathrm{i}\hat{G}_{\pi^{\pm,0}}\right)\frac{\mathrm{i}}{\left(p^2 - m_\pi^2 - \Sigma_\pi^{\pm,0}(p) + i\epsilon\right)}\left(-\mathrm{i}\bar{\hat{G}}_{\pi^{\pm,0}}\right) + \cdots.$$

- The pseudoscalar form factor $G_\pi^*$ :

$$G_\pi^* = \sqrt{Z_\pi}\hat{G}_\pi,$$

where

$$\langle\Omega\big|P^{\pm,0}(x)\big|\pi^{\pm,0}\rangle_{1PI} = \widehat{\mathrm{G}}_{\pi^{\pm,0}}\, e^{-\mathrm{i}p\cdot x}.$$

- The AP correlation function:

$$\Pi_{5\mu}^{\pm,\mp,00}(p) \equiv \int d^4x\, e^{\mathrm{i}p\cdot x}\langle\Omega\big|TA_\mu^{\pm,0}(x)P^{\mp,0}(0)\big|\Omega\rangle.$$

# Operator relations

- We start with the following correlation function:

$$\int d^4x\, e^{\mathrm{i}p\cdot x}\partial^\mu\langle\Omega|TA_\mu^{\pm,0}(x)P^{\mp,0}|\Omega\rangle.$$

The operator relations: $\left[Q_A^{\pm,0}, P^{\mp,0}\right] = -\mathrm{i}\bar{q}q,$

The partially conserved axial vector current (PCAC) relation: $\partial^\mu A_\mu^{\pm,0} = m_q P^{\pm,0}$

- The Chiral Ward identity:

$$\mathrm{i}\lim_{\vec{p}\to 0}\left[p^\mu\Pi_{5\mu}^{\pm\mp,00}(p) + m_q\Pi^{\pm\mp,00}(p)\right] = \mathrm{i}\langle\bar{q}q\rangle^*,$$

$m_q = (m_u + m_d)/2$: The averaged quark mass

$\langle\Omega|\bar{q}q|\Omega\rangle \equiv \langle\bar{q}q\rangle^*$: In-medium quark condensate

# Operator relations

- The correlation functions:

$$\mathrm{i}\,\Pi_{5\mu}^{\pm\mp,00}(p) = -\mathrm{i}p^{\mu}\left(\sum_{n^{\pm,0}} \frac{\left(n_\mu N^*_{\pi_n^{\pm,0}} + p_\mu^{n^{\pm,0}} F^*_{\pi_n^{\pm,0}}\right)\bar{G}^*_{\pi_n^{\pm,0}}}{2\omega_{n^{\pm,0}}(\vec{p})(p^0 - \omega_{n^{\pm,0}}(\vec{p}) + i\epsilon)} + \sum_{n^{\mp,0}} \frac{\left(n_\mu \bar{N}^*_{\pi_n^{\mp,0}} + p_\mu^{n^{\mp,0}} \bar{F}^*_{\pi_n^{\pm,0}}\right) G^*_{\pi_n^{\mp,0}}}{2\omega_{n^{\mp,0}}(\vec{p})\left(p^0 + \omega_{n^{\mp,0}}(\vec{p}) - i\epsilon\right)}\right),$$

$$\mathrm{i}\,m_q\Pi^{\pm\mp,00}\,(p) = \mathrm{i}m_q\left(\sum_{n^{\pm,0}} \frac{G^*_{\pi_n^{\pm,0}}\bar{G}^*_{\pi_n^{\pm,0}}}{2\omega_{n^{\pm,0}}(\vec{p})(p^0 - \omega_{n^{\pm,0}}(p) + i\epsilon)} - \sum_{n^{\mp,0}} \frac{\bar{G}^*{}_{\pi_n^{\mp,0}} G^*_{\pi_n^{\mp,0}}}{2\omega_{n^{\mp,0}}(\vec{p})\left(q^0 + \omega_{n^{\mp,0}}(\vec{p}) - i\epsilon\right)}\right).$$

- The PCAC relation in terms of pion states:

$$\left(\omega_{n^{\pm,0}}(\vec{p})^2 - |\vec{p}|^2\right)F^*_{\pi_n^{\pm,0}} + \omega_{n^{\pm,0}} N^*_{\pi_n^{\pm,0}} = m_q G^*_{\pi_n^{\pm,0}}.$$

- The in-medium Glashow-Weinberg relation:

$$\frac{1}{2}Re\left(\sum_{n^+}\left[\left(F^*_{\pi_n^+} + \frac{N^*_{\pi_n^+}}{\omega_{n^+}}\right)\bar{G}^*_{\pi^+}\right] + \sum_{n^-}\left[\left(F^*_{\pi_n^-} + \frac{N^*_{\pi_n^-}}{\omega_{n^-}}\right)\bar{G}^*_{\pi^-}\right]\right) = -\langle\bar{q}q\rangle^*,$$

$$Re\left(\sum_{n^0}\left[\left(F^*_{\pi_n^0} + \frac{N^*_{\pi_n^0}}{\omega_{n^0}}\right)\bar{G}^*_{\pi^0}\right]\right) = -\langle\bar{q}q\rangle^*.$$

# Operator relations

- The in-medium Gell-Mann-Oakes-Renner relation:

$$\frac{1}{2}\sum_{n^+}\left[\left|m^*_{\pi_n^+}F^*_{\pi_n^+}+N^*_{\pi_n^+}\right|^2\right]+\frac{1}{2}\sum_{n^-}\left[\left|m^*_{\pi_n^-}F^*_{\pi_n^-}+N^*_{\pi_n^-}\right|^2\right]=-m_q\langle\bar{q}q\rangle^*,$$

$$\sum_{n^0}\left[\left|m^*_{\pi_n^0}F^*_{\pi_n^0}+N^*_{\pi_n^0}\right|^2\right]=-m_q\langle\bar{q}q\rangle^*,$$

- Due to the power counting, we take only the lowest states in this work.

- The Gell-Mann-Oakes-Renner relation in terms of the temporal decay constants :

$$\frac{1}{2}\left(m^{*2}_{\pi^+}\left|f^{*(t)}_{\pi^+}\right|^2+m^{*2}_{\pi^-}\left|f^{*(t)}_{\pi^-}\right|^2\right)=-m_q\langle\bar{q}q\rangle^*,$$

$$m^{*2}_{\pi^0}\left|f^{*(t)}_{\pi^0}\right|^2=-m_q\langle\bar{q}q\rangle^*.$$

# In-medium chiral perturbation theory

- To derive the in-medium chiral perturbation theory, we develop the generating functional with the chiral Lagrangian between two asymptotic states of nuclear matter [4,5]:

$$Z[J] = \langle \Omega_{\mathrm{out}} | \Omega_{in} \rangle$$

[4] Oller, José A., *PRC* 65,(2002): 025204.

[5] Meißner, Ulf-G., Jose A. Oller, and Andreas Wirzba., *Annals of Phys*. 297 (2002): 27-66.

- Asymptotic states:

$$|\Omega\rangle = \prod_{i}^{|p_i| \leq k_F^{p,n}} a^{\dagger}(p_i)|0\rangle,$$

$i$ : the isospin and spin of the nucleon

Fermi momenta : $k_{\mathrm{F}}^{\mathrm{p,n}} = \left(3\pi^2 \rho_{p,n}\right)^{1/3}$.

# In-medium chiral perturbation theory

- The in-medium effects from the $\mathcal{O}(\rho^2)$ in the density expansion are neglected.

- The generating functional:

$$Z[J] = \int \mathcal{D}U\mathcal{D}\bar{N}\mathcal{D}N\langle\Omega_{\text{out}}|N_{t\to+\infty}\rangle \times e^{i\int d^4x(\mathcal{L}_\pi+\mathcal{L}_N+\mathcal{L}_{\pi N})}\langle N_{t\to-\infty}|\Omega_{in}\rangle$$
$$= \int \mathcal{D}U \det(D) \exp\Big\{ i\int d^4x\mathcal{L}_\pi$$
$$- \mathrm{i}\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \int d^4x d^4y\, e^{\mathrm{i}p(x-y)} \mathrm{Tr}\left(A[I_4 - D_0^{-1}A]^{-1}(x,y)(p^\mu\gamma_u + m_N)n(\vec{p})\right)$$
$$+ \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\int \frac{d^3q}{(2\pi)^3 2E_q}\int d^4xd^4yd^4x'd^4y'^{e^{\mathrm{i}p(x-y)}}e^{-\mathrm{i}q(x'-y')} \times Tr\Big(A[I_4$$
$$- D_0^{-1}A]^{-1}(y',y)\big(p^\mu\gamma_\mu + m_N\big)n(\vec{p}\,)A[I_4 - D_0^{-1}A]^{-1}(x,x')\big(q^\mu\gamma_\mu + m_N\big)n(\vec{q})\Big) + \cdots \Big\}.$$

# In-medium chiral perturbation theory

$A$: nucleon interaction terms

$D = D_0 - A$: the covariant derivative $D_0$+ interaction $A$.

- The loop integral for the in-medium nucleon propagator is performed up to the Fermi momenta with the Heaviside step functions,

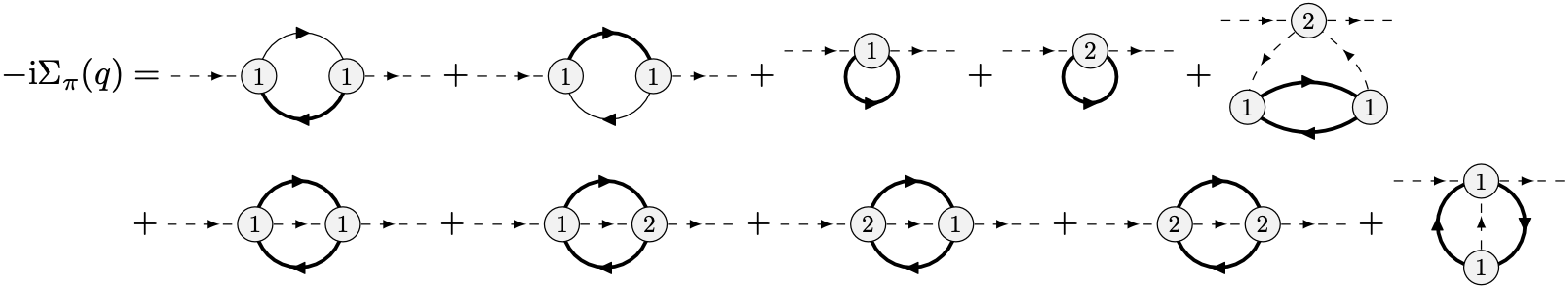
$$\Theta^{p,n}_{|\vec{p}|} \equiv \Theta\left(k_F^{p,n} - |\vec{p}|\right),$$

$$n(\vec{p}) = \begin{pmatrix} \Theta^{p}_{|\vec{p}|} & 0 \\ 0 & \Theta^{n}_{|\vec{p}|} \end{pmatrix}.$$

- One can perform the explicit perturbative calculations using the generating functional.

# Self-energies

- Relevant Feynman diagrams :




Dashed line : pion propagators,

Thin line : the in-vacuum nucleon propagator

Thick line : the in-medium nucleon propagator.

- The in-medium pion masses are calculated by solving the mass equations:

$$m_\pi^* = \sqrt{m_\pi^2 + \Sigma_\pi(m_\pi^*, \vec{0})}.$$

# Self-energies

- The leading order :

$$
\begin{aligned}
\Sigma^{+}_{1\text{-loop}}(q_0) &= \int \mathrm{d}|\boldsymbol{p}| \frac{\boldsymbol{p}^2}{4\pi^2 p_0 f^2}\Bigg[\frac{2q_0 g_A^2(q_0p_0+2\boldsymbol{p}^2)}{q_0+2p_0}\Theta^n_{\boldsymbol{p}} - \frac{2q_0 g_A^2(q_0p_0-2\boldsymbol{p}^2)}{q_0-2p_0}\Theta^p_{\boldsymbol{p}} \\
&\quad + 4m_N\left(4c_1m_\pi^2 - \frac{2c_2}{m_N^2}q_0^2p_0^2 - 2c_3q_0^2\right)\left(\Theta^p_{\boldsymbol{p}}+\Theta^n_{\boldsymbol{p}}\right) + 2q_0p_0\left(\Theta^p_{\boldsymbol{p}}-\Theta^n_{\boldsymbol{p}}\right)\Bigg] \\
&\approx \frac{q_0^2 g_A^2 m_N \rho}{f^2\left(4m_N^2-q_0^2\right)} + \frac{q_0^3 g_A^2(1-r)\rho}{2f^2(1+r)\left(4m_N^2-q_0^2\right)} + \frac{[4c_1m_\pi^2 - 2q_0^2(c_2+c_3)]\rho}{f^2} + \frac{q_0(1-r)\rho}{2f^2(1+r)}, \\
\Sigma^{-}_{1\text{-loop}}(q_0) &= \int \mathrm{d}|\boldsymbol{p}| \frac{\boldsymbol{p}^2}{4\pi^2 p_0 f^2}\Bigg[\frac{2q_0 g_A^2(q_0p_0+2\boldsymbol{p}^2)}{q_0+2p_0}\Theta^p_{\boldsymbol{p}} - \frac{2q_0 g_A^2(q_0p_0-2\boldsymbol{p}^2)}{q_0-2p_0}\Theta^n_{\boldsymbol{p}} \\
&\quad + 4m_N\left(4c_1m_\pi^2 - \frac{2c_2}{m_N^2}q_0^2p_0^2 - 2c_3q_0^2\right)\left(\Theta^p_{\boldsymbol{p}}+\Theta^n_{\boldsymbol{p}}\right) - 2q_0p_0\left(\Theta^p_{\boldsymbol{p}}-\Theta^n_{\boldsymbol{p}}\right)\Bigg] \\
&\approx \frac{q_0^2 g_A^2 m_N \rho}{f^2\left(4m_N^2-q_0^2\right)} - \frac{q_0^3 g_A^2(1-r)\rho}{2f^2(1+r)\left(4m_N^2-q_0^2\right)} + \frac{\left[4c_1m_\pi^2 - 2q_0^2(c_2+c_3)\right]\rho}{f^2} - \frac{q_0(1-r)\rho}{2f^2(1+r)}, \\
\Sigma^{0}_{1\text{-loop}}(q_0) &= \int \mathrm{d}|\boldsymbol{p}| \frac{\boldsymbol{p}^2}{8\pi^2 p_0 f^2}\Bigg[\frac{2q_0 g_A^2(q_0p_0+2\boldsymbol{p}^2)}{q_0+2p_0}\left(\Theta^p_{\boldsymbol{p}}+\Theta^n_{\boldsymbol{p}}\right) - \frac{2q_0 g_A^2(q_0p_0-2\boldsymbol{p}^2)}{q_0-2p_0}\left(\Theta^n_{\boldsymbol{p}}+\Theta^p_{\boldsymbol{p}}\right) \\
&\quad + 8m_N\left(4c_1m_\pi^2 - \frac{2c_2}{m_N^2}q_0^2p_0^2 - 2c_3q_0^2\right)\left(\Theta^p_{\boldsymbol{p}}+\Theta^n_{\boldsymbol{p}}\right)\Bigg] \\
&\approx \frac{q_0^2 g_A^2 m_N \rho}{f^2\left(4m_N^2-q_0^2\right)} + \frac{\left[4c_1m_\pi^2 - 2q_0^2(c_2+c_3)\right]\rho}{f^2},
\end{aligned}
$$

# Self-energies

- The next-to-leading order :

$$
\begin{aligned}
\Sigma^{+}_{\text{2-loop}}(q_0) = {} & \frac{m_N^2 g_A^2}{10(2\pi f)^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{p_0 k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta - m_N^2}{\left[(p-k)^2 - m_\pi^2\right]^2} \\
& \times \Big[ \big(20 q_0 p_0 - 20 q_0 k_0 - 4 p_0 k_0 + 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta + 4m_N^2 + 2q_0^2 - 4m_\pi^2\big)\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} \\
& + \big( - 20 q_0 p_0 + 20 q_0 k_0 - 4 p_0 k_0 + 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta + 4m_N^2 + 2q_0^2 - 4m_\pi^2\big)\Theta^n_{\boldsymbol{k}}\Theta^p_{\boldsymbol{p}} \\
& + \big(4 p_0 k_0 - 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta - 4m_N^2 - 2q_0^2 - m_\pi^2\big)\big(\Theta^p_{\boldsymbol{k}}\Theta^p_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big)\Big] + \frac{m_N^2 g_A^2}{10(2\pi f)^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \\
& \times \frac{1}{(p-k)^2 - m_\pi^2} \Big[ \big(5 q_0 k_0 - 5 q_0 p_0 + 2m_N^2 - 2 p_0 k_0 + 2|\boldsymbol{p}||\boldsymbol{k}|\cos\theta\big)\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}} \\
& + \big( - 5 q_0 k_0 + 5 q_0 p_0 + 2m_N^2 - 2 p_0 k_0 + 2|\boldsymbol{p}||\boldsymbol{k}|\cos\theta\big)\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} + \big(2 p_0 k_0 - 2|\boldsymbol{p}||\boldsymbol{k}|\cos\theta - 2m_N^2\big)\big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\big)\Big] \\
& - \frac{q_0^2}{4(2\pi)^4 f^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{p_0 k_0 + |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2}{(q+p-k)^2 - m_\pi^2} \big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} + 2\Theta^n_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}\big) \\
& - \frac{1}{(2\pi)^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{C^s_L C^s_R \big(p_0 k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2\big)}{(q+p-k)^2 - m_\pi^2} \big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big) \\
& - \frac{m_N q_0}{2(2\pi)^4 f^2} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{(p_0 + k_0)}{(q+p-k)^2 - m_\pi^2} \big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} - \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big)\big(C^s_L + C^s_R\big),
\end{aligned}
$$

# Self-energies

- The next-to-leading order :

$$
\begin{aligned}
\Sigma^{-}_{\text{2-loop}}(q_0) =& \frac{m_N^2 g_A^2}{10(2\pi f)^4} \int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{p_0 k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta - m_N^2}{\left((p-k)^2 - m_\pi^2\right)^2} \\
&\times \Big[\big(-20 q_0 p_0 + 20 q_0 k_0 - 4 p_0 k_0 + 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta + 4m_N^2 + 2q_0^2 - 4m_\pi^2\big)\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} \\
&+ \big(20 q_0 p_0 - 20 q_0 k_0 - 4 p_0 k_0 + 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta + 4m_N^2 + 2q_0^2 - 4m_\pi^2\big)\Theta^n_{\boldsymbol{k}}\Theta^p_{\boldsymbol{p}} \\
&+ \big(4 p_0 k_0 - 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta - 4m_N^2 - 2q_0^2 - m_\pi^2\big)\big(\Theta^p_{\boldsymbol{k}}\Theta^p_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big)\Big] + \frac{m_N^2 g_A^2}{10(2\pi f)^4} \int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \\
&\times \frac{1}{(p-k)^2 - m_\pi^2}\Big[\big(-5 q_0 k_0 + 5 q_0 p_0 + 2m_N^2 - 2 p_0 k_0 + 2|\boldsymbol{p}||\boldsymbol{k}|\cos\theta\big)\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}} \\
&+ \big(5 q_0 k_0 - 5 q_0 p_0 + 2m_N^2 - 2 p_0 k_0 + 2|\boldsymbol{p}||\boldsymbol{k}|\cos\theta\big)\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} + \big(2 p_0 k_0 - 2|\boldsymbol{p}||\boldsymbol{k}|\cos\theta - 2m_N^2\big)\big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\big)\Big] \\
&- \frac{q_0^2}{4(2\pi)^4 f^4} \int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{p_0 k_0 + |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2}{(q+p-k)^2 - m_\pi^2}\big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}} + 2\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\big) \\
&- \frac{1}{(2\pi)^4} \int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{C_L^s C_R^s\big(p_0 k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2\big)}{(q+p-k)^2 - m_\pi^2}\big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big) \\
&+ \frac{m_N q_0}{2(2\pi)^4 f^2} \int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{(p_0 + k_0)}{(q+p-k)^2 - m_\pi^2}\big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} - \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big)\big(C_L^s + C_R^s\big),
\end{aligned}
$$

# Self-energies

- The next-to-leading order :

$$
\begin{aligned}
\Sigma^0_{\text{2-loop}}(q_0) = {} & \frac{m_N^2 g_A^2}{10(2\pi f)^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0} \frac{p_0 k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta - m_N^2}{\left((p-k)^2 - m_\pi^2\right)^2} \\
& \times \Big[\big(-8m_N^2 + 8p_0k_0 - 8|\boldsymbol{p}||\boldsymbol{k}|\cos\theta - 4q_0^2 - 2m_\pi^2\big)\big(\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} + \Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\big) \\
& + \big(-8p_0k_0 + 8|\boldsymbol{p}||\boldsymbol{k}|\cos\theta + 8m_N^2 + 4q_0^2 - 3m_\pi^2\big)\big(\Theta^p_{\boldsymbol{k}}\Theta^p_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big)\Big] + \frac{m_N^2 g_A^2}{10(2\pi f)^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0} \\
& \times \frac{1}{(p-k)^2 - m_\pi^2}\Big[\big(4p_0k_0 - 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta - 4m_N^2\big)\big(\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{k}}\Theta^p_{\boldsymbol{p}}\big) \\
& + \big(-4p_0k_0 + 4|\boldsymbol{p}||\boldsymbol{k}|\cos\theta + 4m_N^2\big)\big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\big)\Big] \\
& - \frac{q_0^2}{2(2\pi)^4 f^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0} \frac{p_0k_0 + |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2}{(q+p-k)^2 - m_\pi^2}\big(\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}} + \Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big) \\
& - \frac{1}{(2\pi)^4} \int \mathrm{d}|\boldsymbol{p}|\, \mathrm{d}|\boldsymbol{k}|\, \mathrm{d}\cos\theta \frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0} \frac{C_L^s C_R^s\big(p_0k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2\big)}{(q+p-k)^2 - m_\pi^2}\big(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\big),
\end{aligned}
$$

where $C_L^s$ and $C_R^s$ are defined as

$$
\begin{aligned}
C_L^s &= \frac{8B_0 c_1 m_q}{f^2} - \frac{2c_2}{f^2 m_N^2} q_0 p_0 (q_0 p_0 + p_\mu p^\mu - k_\mu p^\mu) - \frac{2c_3}{f^2} q_0 (q_0 + p_0 - k_0), \\
C_R^s &= \frac{8B_0 c_1 m_q}{f^2} - \frac{2c_2}{f^2 m_N^2} q_0 k_0 (q_0 k_0 + k_\mu p^\mu - k_\mu k^\mu) - \frac{2c_3}{f^2} q_0 (q_0 + p_0 - k_0).
\end{aligned}
$$

# Masses

| | | $m^*_{\pi^-}/m_\pi$ | $m^*_{\pi^0}/m_\pi$ | $m^*_{\pi^+}/m_\pi$ | $m^*_\pi/m_\pi(r=1)$ |
|---|---|---|---|---|---|
| | This work (perturbative) | | | | |
| | Set 1 | 1.02 | 0.97 | 0.92 | 0.97 |
| | Set 2 | 1.07 | 1.02 | 0.97 | 1.02 |
| | Set 3 | 1.06 | 1.01 | 0.96 | 1.01 |
| | This work (partially perturbative) | | | | |
| | Set 1 | 1.05 | 0.94 | 0.84 | 0.94 |
| | Set 2 | 1.13 | 1.04 | 0.95 | 1.04 |
| | Set 3 | 1.10 | 1.02 | 0.93 | 1.02 |
| Theor. | Ref. [49] | 1.10 | 1.04 | 0.99 | |
| | Refs. [50,51] | 1.06 | – | – | – |
| | Ref. [47] | 1.13 | – | – | – |
| | Ref. [53] | – | – | – | 0.94 |
| | Ref. [89] | – | – | – | 1.03 |
| Expt. | Ref. [7] | 1.16–1.19 | – | – | – |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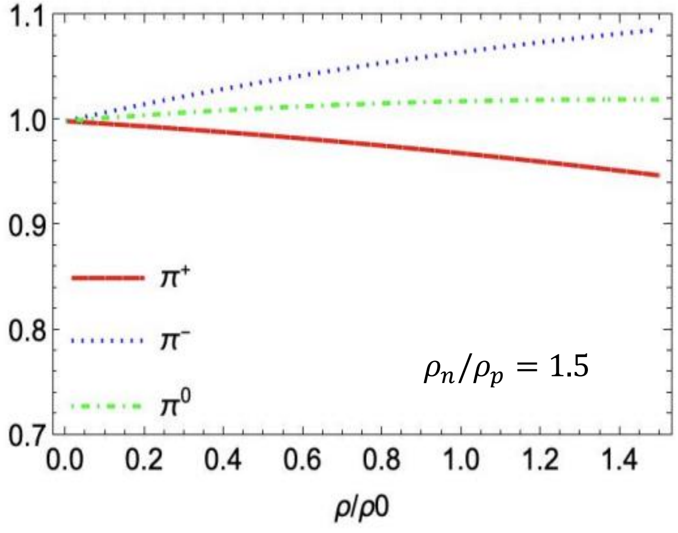
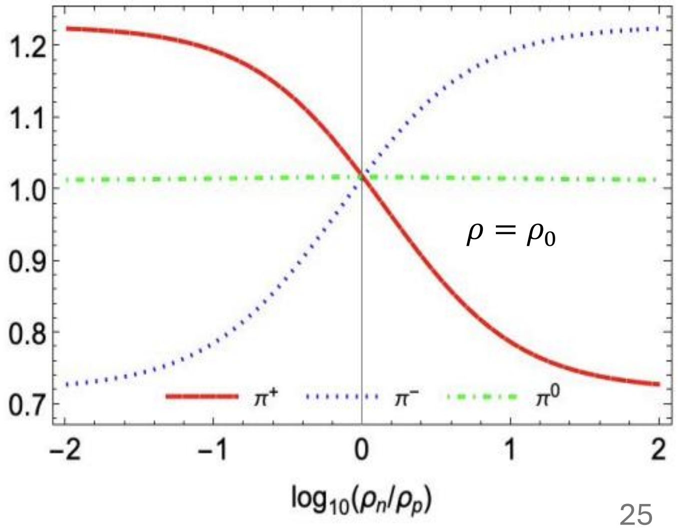



(b)




(d)

# Wave function renormalizations

| | | $Z_{\pi^-}$ | $Z_{\pi^0}$ | $Z_{\pi^+}$ | $Z_\pi(r=1)$ |
|---|---|---|---|---|---|
| | This work | | | | |
| | Set 1 | 2.1 | 1.85 | 1.68 | 1.86 |
| | Set 2 | 1.81 | 1.63 | 1.49 | 1.63 |
| | Set 3 | 1.77 | 1.61 | 1.47 | 1.67 |
| Theor. | Ref. [48] | | | | 1.51–1.53 |
| | Ref. [45] | | | | 1.40 |

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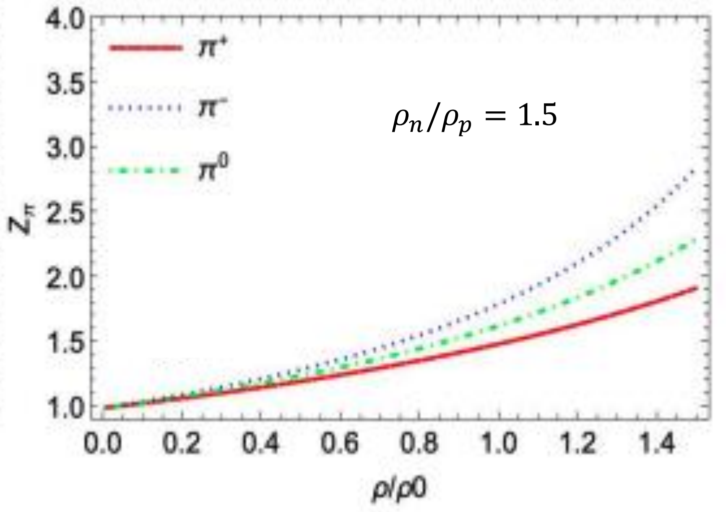
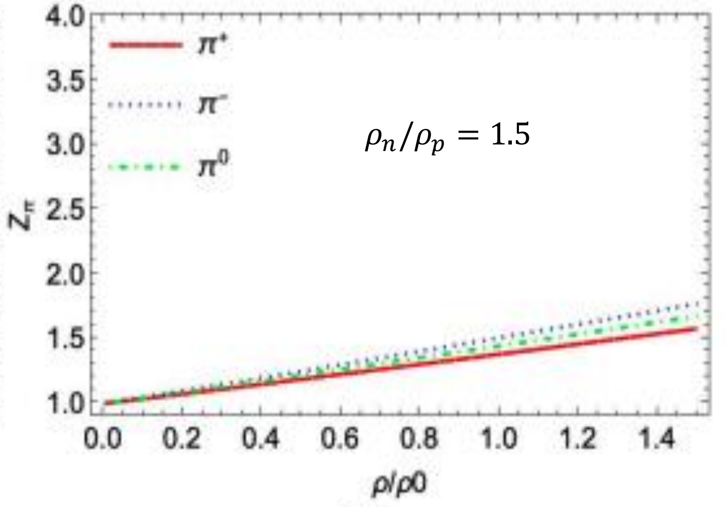


(b)

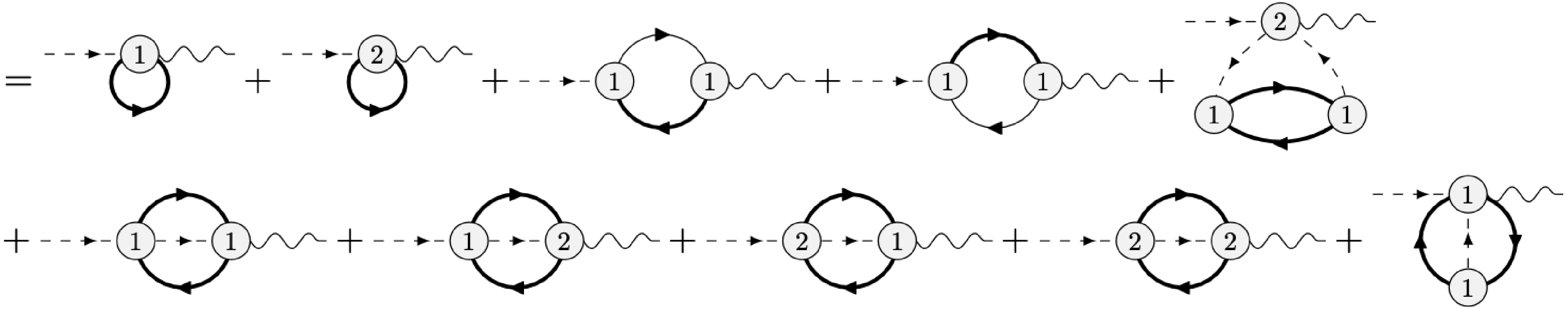


(d)

# Pion decay constants

- The effective vertex corrections:

$$\mathrm{i}q_0 \hat{f}_\pi =$$



- The wavy line : axial vector currents.

- With the wave function renormalization, the temporal pion decay constant is calculated as

$$f_\pi^{*(t)} = \sqrt{Z_\pi} \hat{f}_\pi.$$

# Pion decay constants

- The leading order:

$$\hat{f}^{+}_{\text{1-loop}}(q_0) = -\int \mathrm{d}|\boldsymbol{p}| \frac{g_A^2 \boldsymbol{p}^2}{2\pi^2 f q_0 p_0} \left( \frac{q_0 p_0 + 2\boldsymbol{p}^2}{2p_0 + q_0} \Theta^n_{\boldsymbol{p}} + \frac{q_0 p_0 - 2\boldsymbol{p}^2}{2p_0 - q_0} \Theta^p_{\boldsymbol{p}} \right) - \int \mathrm{d}|\boldsymbol{p}| \frac{\boldsymbol{p}^2}{2\pi^2 f q_0} \left( \Theta^p_{\boldsymbol{p}} - \Theta^n_{\boldsymbol{p}} \right)$$

$$+ \int \mathrm{d}|\boldsymbol{p}| \frac{\boldsymbol{p}^2}{\pi^2 p_0 f} \left( \frac{2c_2}{m_N} p_0^2 + 2m_N c_3 \right) \left( \Theta^p_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{p}} \right)$$

$$\approx -\frac{m_N g_A^2}{f\left(4m_N^2 - q_0^2\right)} \rho + \frac{g_A^2 q_0 (r-1)}{2f(1+r)\left(4m_N^2 - q_0^2\right)} \rho + \frac{r-1}{2f q_0 (r+1)} \rho + \frac{2(c_2 + c_3)}{f} \rho,$$

$$\hat{f}^{-}_{\text{1-loop}}(q_0) = -\int \mathrm{d}|\boldsymbol{p}| \frac{g_A^2 \boldsymbol{p}^2}{2\pi^2 f q_0 p_0} \left( \frac{q_0 p_0 + 2\boldsymbol{p}^2}{2p_0 + q_0} \Theta^p_{\boldsymbol{p}} + \frac{q_0 p_0 - 2\boldsymbol{p}^2}{2p_0 - q_0} \Theta^n_{\boldsymbol{p}} \right) + \int \mathrm{d}|\boldsymbol{p}| \frac{\boldsymbol{p}^2}{2\pi^2 f q_0} \left( \Theta^p_{\boldsymbol{p}} - \Theta^n_{\boldsymbol{p}} \right)$$

$$+ \int \mathrm{d}|\boldsymbol{p}| \frac{\boldsymbol{p}^2}{\pi^2 p_0 f} \left( \frac{2c_2}{m_N} p_0^2 + 2m_N c_3 \right) \left( \Theta^p_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{p}} \right)$$

$$\approx -\frac{m_N g_A^2}{f\left(4m_N^2 - q_0^2\right)} \rho + \frac{g_A^2 q_0 (1-r)}{2f(1+r)\left(4m_N^2 - q_0^2\right)} \rho + \frac{1-r}{2f q_0 (1+r)} \rho + \frac{2(c_2 + c_3)}{f} \rho,$$

$$\hat{f}^{0}_{\text{1-loop}}(q_0) = -\int \mathrm{d}|\boldsymbol{p}| \frac{g_A^2 \boldsymbol{p}^2}{4\pi^2 f q_0 p_0} \frac{4p_0^2 q_0 - 4\boldsymbol{p}^2 q_0}{4p_0^2 - q_0^2} \left( \Theta^p_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{p}} \right) + \int \mathrm{d}|\boldsymbol{p}| \frac{2m_N \boldsymbol{p}^2}{\pi^2 p_0 f} \left( c_2 \frac{p_0^2}{m_N^2} + c_3 \right) \left( \Theta^p_{\boldsymbol{p}} + \Theta^n_{\boldsymbol{p}} \right)$$

$$\approx -\frac{m_N g_A^2}{f\left(4m_N^2 - q_0^2\right)} \rho + \frac{2(c_2 + c_3)}{f} \rho,$$

# Pion decay constants

- The next-to-leading order:

$$
\begin{aligned}
\hat{f}^{+}_{\text{2-loop}}(q_0) = &-\frac{g_A^2 m_N^2}{5(2\pi)^4 f^3 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\,\frac{p_0k_0-|\boldsymbol{p}||\boldsymbol{k}|\cos\theta-m_N^2}{\left((p-k)^2-m_\pi^2\right)^2}\\
&\times\left((-2q_0+10p_0-10k_0)\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}+(-2q_0-10p_0+10k_0)\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}-3q_0\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}+\Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\right)\right)\\
&+\frac{g_A^2 m_N^2}{4(2\pi)^4 f^3 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\,\frac{k_0-p_0}{(p-k)^2-m_\pi^2}\left(-4\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}+2\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right)\\
&+\frac{1}{4(2\pi)^4 f^3}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\,\frac{p_0k_0+|\boldsymbol{p}||\boldsymbol{k}|\cos\theta+m_N^2}{(q+p-k)^2-m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}+\Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}+2\Theta^n_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}\right)\\
&-\frac{1}{(2\pi)^4 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\,\frac{C_L^fC_R^f\left(p_0k_0-|\boldsymbol{p}||\boldsymbol{k}|\cos\theta+m_N^2\right)}{(q+p-k)^2-m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}+\Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right)\\
&-\frac{m_N}{2(2\pi)^4 f^2}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\,\frac{(p_0+k_0)}{(q+p-k)^2-m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}-\Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right)C_R^f\\
&+\frac{m_N}{2(2\pi)^4 f q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\,\frac{(p_0+k_0)}{(q+p-k)^2-m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}-\Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\right)C_L^f,
\end{aligned}
$$

# Pion decay constants

- The next-to-leading order:

$$
\begin{aligned}
\hat{f}^{-}_{2\text{-loop}}(q_0) = &-\frac{g_A^2 m_N^2}{5(2\pi)^4 f^3 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0}\,\frac{p_0k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta - m_N^2}{\left((p-k)^2 - m_\pi^2\right)^2} \\
&\times\left((-2q_0 - 10p_0 + 10k_0)\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} + (-2q_0 + 10p_0 - 10k_0)\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}} - 3q_0\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\right)\right) \\
&+\frac{g_A^2 m_N^2}{4(2\pi)^4 f^3 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0}\,\frac{k_0 - p_0}{(p-k)^2 - m_\pi^2}\left(2\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}} - 4\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right) \\
&+\frac{1}{4(2\pi)^4 f^3}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0}\,\frac{p_0k_0 + |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2}{(q+p-k)^2 - m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}} + 2\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\right) \\
&-\frac{1}{(2\pi)^4 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0}\,\frac{C_L^f C_R^f\left(p_0k_0 - |\boldsymbol{p}||\boldsymbol{k}|\cos\theta + m_N^2\right)}{(q+p-k)^2 - m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} + \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right) \\
&+\frac{m_N}{2(2\pi)^4 f^2}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0}\,\frac{(p_0 + k_0)}{(q+p-k)^2 - m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} - \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right)C_R^f \\
&-\frac{m_N}{2(2\pi)^4 f q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0 k_0}\,\frac{(p_0 + k_0)}{(q+p-k)^2 - m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}} - \Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right)C_L^f,
\end{aligned}
$$

# Pion decay constants

- The next-to-leading order:

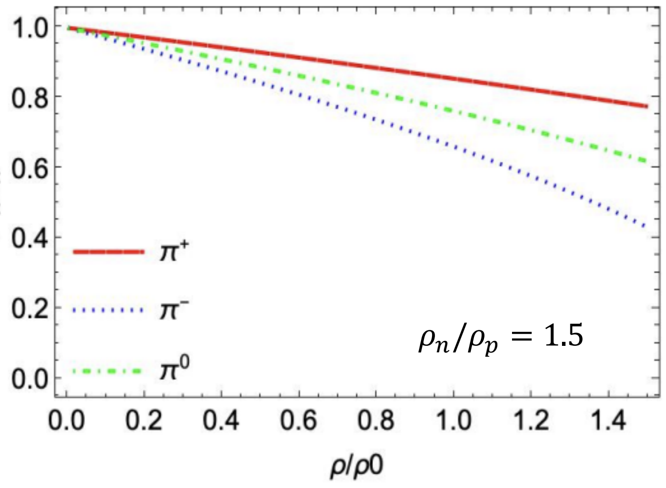
$$
\begin{aligned}
\tilde{f}^0_{2\text{-loop}}(q_0) = &-\frac{g_A^2 m_N^2}{5(2\pi)^4 f^3 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\frac{p_0k_0-|\boldsymbol{p}||\boldsymbol{k}|\cos\theta-m_N^2}{\left((p-k)^2-m_\pi^2\right)^2}\\
&\times\left(-6q_0\Theta^p_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}-6q_0\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}+q_0\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}+\Theta^n_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}\right)\right)\\
&+\frac{1}{2(2\pi)^4 f^3}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\frac{p_0k_0+|\boldsymbol{p}||\boldsymbol{k}|\cos\theta+m_N^2}{(q+p-k)^2-m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^n_{\boldsymbol{k}}+\Theta^n_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}\right)\\
&-\frac{1}{(2\pi)^4 q_0}\int \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{k}|\,\mathrm{d}\cos\theta\,\frac{\boldsymbol{p}^2\boldsymbol{k}^2}{p_0k_0}\frac{C_L^f C_R^f\left(p_0k_0-|\boldsymbol{p}||\boldsymbol{k}|\cos\theta+m_N^2\right)}{(q+p-k)^2-m_\pi^2}\left(\Theta^p_{\boldsymbol{p}}\Theta^p_{\boldsymbol{k}}+\Theta^n_{\boldsymbol{k}}\Theta^n_{\boldsymbol{p}}\right),
\end{aligned}
$$

where $C_L^f$ and $C_R^f$ are written as

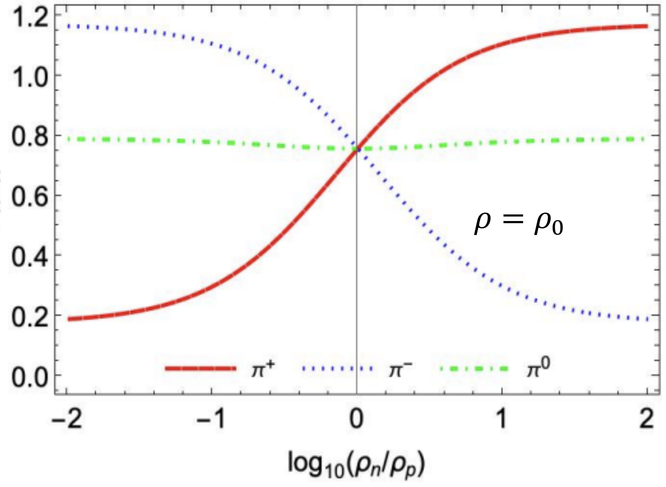
$$
\begin{aligned}
C_L^f &= \frac{8B_0c_1m_q}{f^2}-\frac{2c_2}{f^2m_N^2}q_0p_0(q_0p_0+p_\mu p^\mu-k_\mu p^\mu)-\frac{2c_3}{f^2}q_0(q_0+p_0-k_0),\\
C_R^f &= \frac{2c_2}{fm_N^2}k_0(q_0k_0+k_\mu p^\mu-k_\mu k^\mu)+\frac{2c_3}{f}(q_0+p_0-k_0).
\end{aligned}
$$

# Pion decay constants

| | | $f^*_{\pi^-}/f_\pi$ | $f^*_{\pi^0}/f_\pi$ | $f^*_{\pi^+}/f_\pi$ | $f^*_\pi/f_\pi(r=1)$ |
|---|---|---|---|---|---|
| | This work | | | | |
| | Set 1 | 0.80 | 0.84 | 0.89 | 0.84 |
| | Set 2 | 0.83 | 0.87 | 0.92 | 0.87 |
| | Set 3 | 0.83 | 0.88 | 0.92 | 0.88 |
| Theor. | Ref. [23] | | | | 0.89 |
| Expt. | Ref. [54] | 0.85–0.91 | | | |
| | Ref. [3] | 0.84–0.89 | | | |



(b)



(d)

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[23] D. Jido, T. Hatsuda, and T. Kunihiro, Phys. Lett. B 670, 109 (2008)
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# In-medium Gell-Mann-Oakes-Renner relation

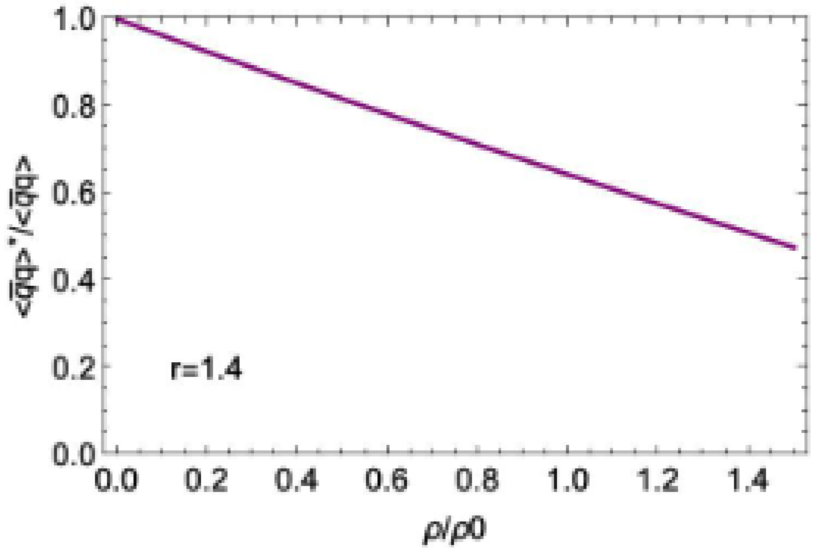
- The in-medium Gell-Mann-Oakes-Renner relation:

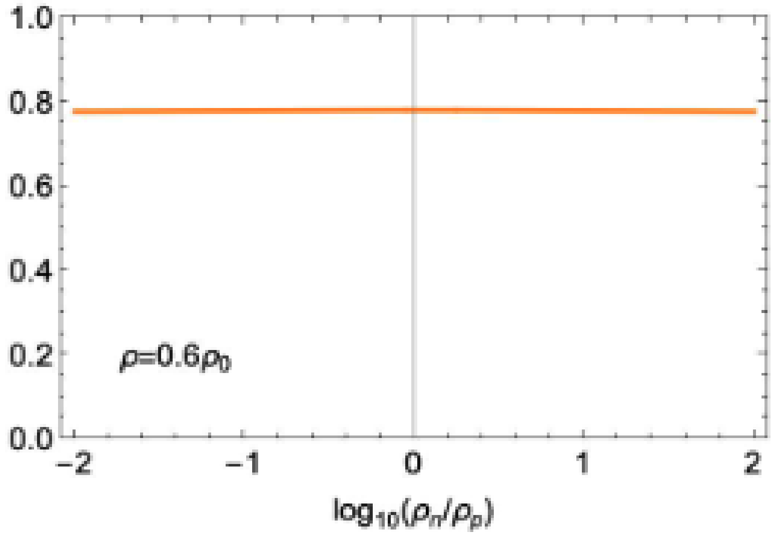
$$\frac{1}{2}\left(m_{\pi^+}^{*2}\left|f_{\pi^+}^{*(t)}\right|^2 + m_{\pi^-}^{*2}\left|f_{\pi^-}^{*(t)}\right|^2\right) = -m_q\langle\bar{q}q\rangle^*,$$

$$m_{\pi^0}^{*2}\left|f_{\pi^0}^{*(t)}\right|^2 = -m_q\langle\bar{q}q\rangle^*.$$

- For $^{121}Sn$ : about 23% reduction.

- This result is in agreement with Ref. [7], which showed the 21-25% reduction.




[7] Nishi, et al., Nature Phys. 19, 788 (2023)

# Summary

- We have calculated the in-medium pion properties using in-medium chiral perturbation theory in the physical basis.
- The different densities of proton and neutron are compensated by the different Fermi momenta in actual calculations.
- Using the in-medium Gell-Mann-Oakes-Renner relations, we can estimate the partial restoration of the chiral symmetry in nuclear matter.
- The overall results are symmetric around the isospin symmetric case.

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