UNIVERSITÄT BONN
bctp Bethe Center for Theoretical Physics
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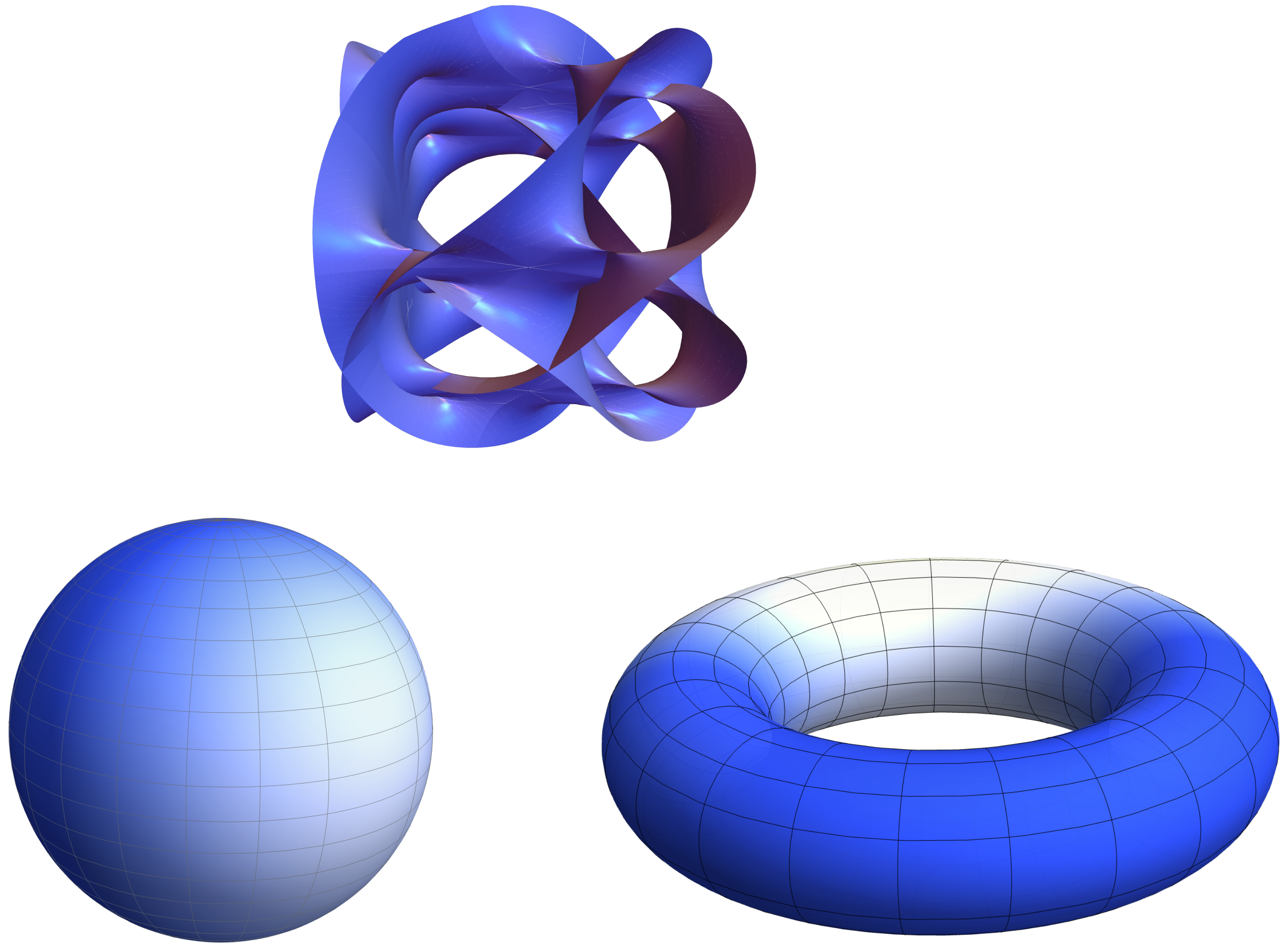
(Canonical) differential equations beyond polylogarithms

Based on: arXiv 2305.14090 (JHEP)
arXiv 2503.20655 (JHEP)



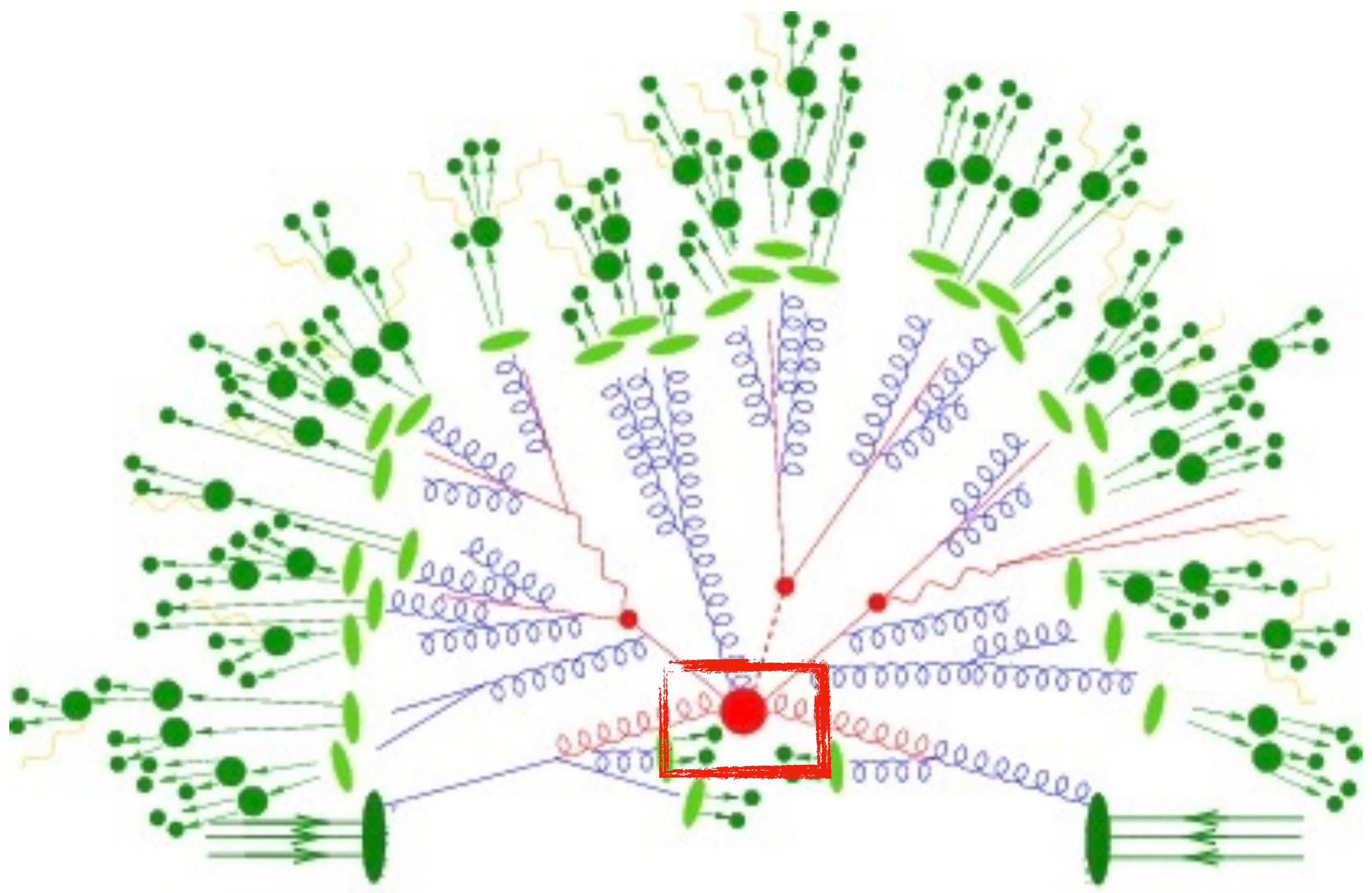
In collaboration with: *Claude Duhr, Christoph Nega, Benjamin Sauer, Lorenzo Tancredi, Fabian J. Wagner*

Sara Maggio | Phenomenology seminar, Torino | 23/04/2026

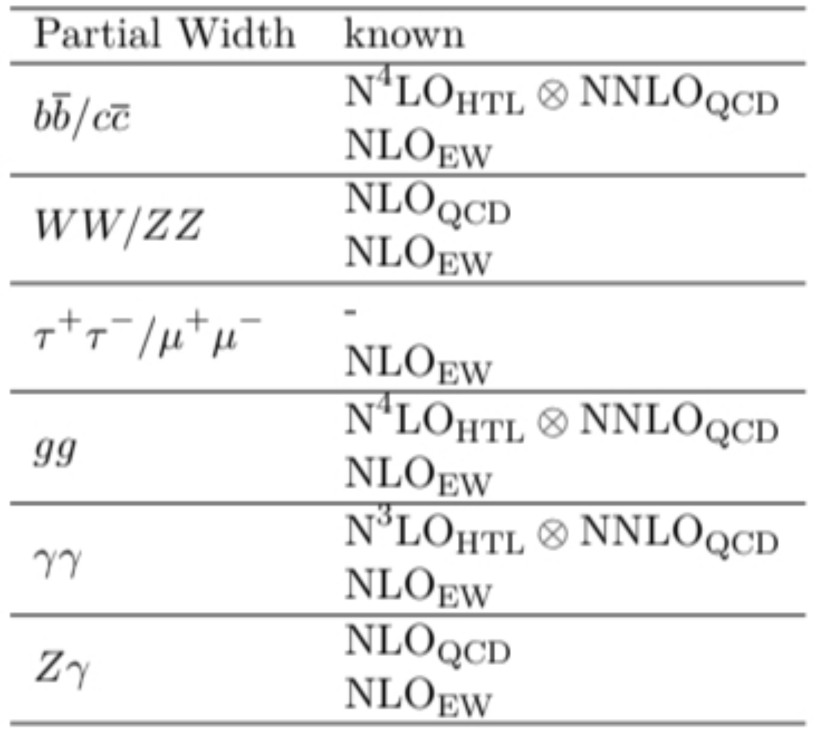
Motivation

Particle Physics

- The LHC will give us new data for ~ 2 decades;
- Precision measurements in the Higgs sector are fundamental for testing the Standard Model or for finding New Physics;



[Sherpa's artistic view of protons' collisions]



Partial Width	known
$b\bar{b}/c\bar{c}$	N^{4}LO$_{\mathrm{HTL}}$ ⊗ NNLO$_{\mathrm{QCD}}$ NLO$_{\mathrm{EW}}$
WW/ZZ	NLO$_{\mathrm{QCD}}$ NLO$_{\mathrm{EW}}$
$\tau^+\tau^-/\mu^+\mu^-$	- NLO$_{\mathrm{EW}}$
gg	N^{4}LO$_{\mathrm{HTL}}$ ⊗ NNLO$_{\mathrm{QCD}}$ NLO$_{\mathrm{EW}}$
$\gamma\gamma$	N^{3}LO$_{\mathrm{HTL}}$ ⊗ NNLO$_{\mathrm{QCD}}$ NLO$_{\mathrm{EW}}$
$Z\gamma$	NLO$_{\mathrm{QCD}}$ NLO$_{\mathrm{EW}}$

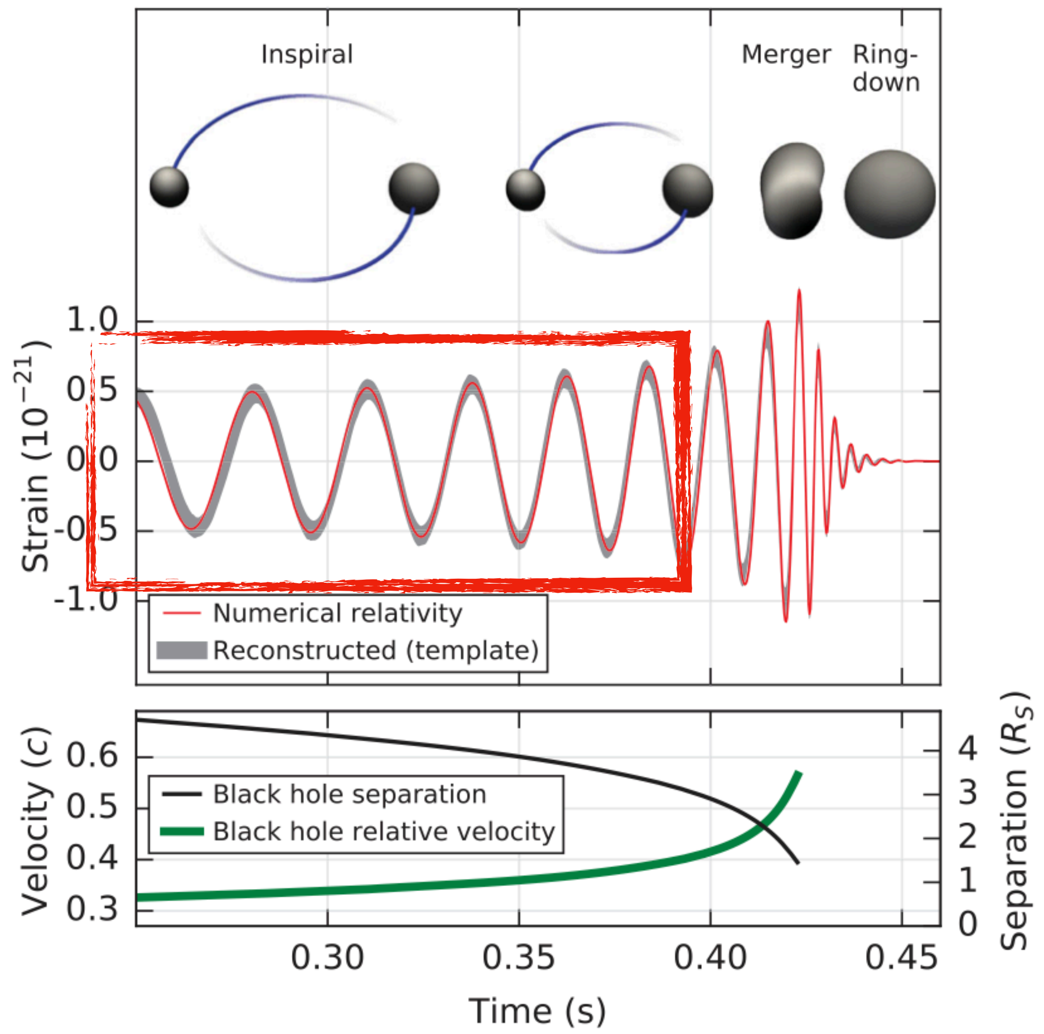
[Les Houches wishlist 2023]

This requires as accurate as possible theoretical predictions

Perturbation theory

GWs Physics

- LIGO, Virgo, Kagra & upcoming Einstein Telescope, Cosmic Explorer, LISA;
- Precision measurements for testing Einstein gravity, black holes and neutron stars properties or for finding New Physics;



[The LIGO and the Virgo Collaboration, '16]

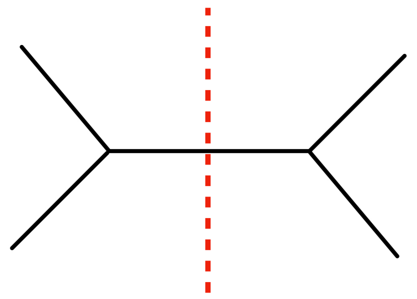
Amplitudes

Unitarity

Singularities

poles

single-particle states go on-shell

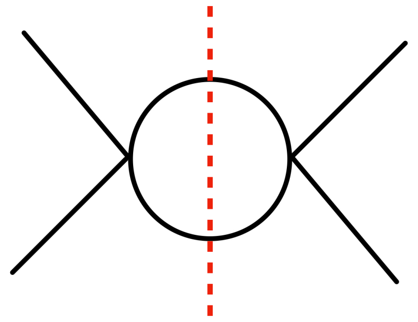


$\sim \frac{1}{s}$

"cutting" the propagator ~ the particle goes on shell

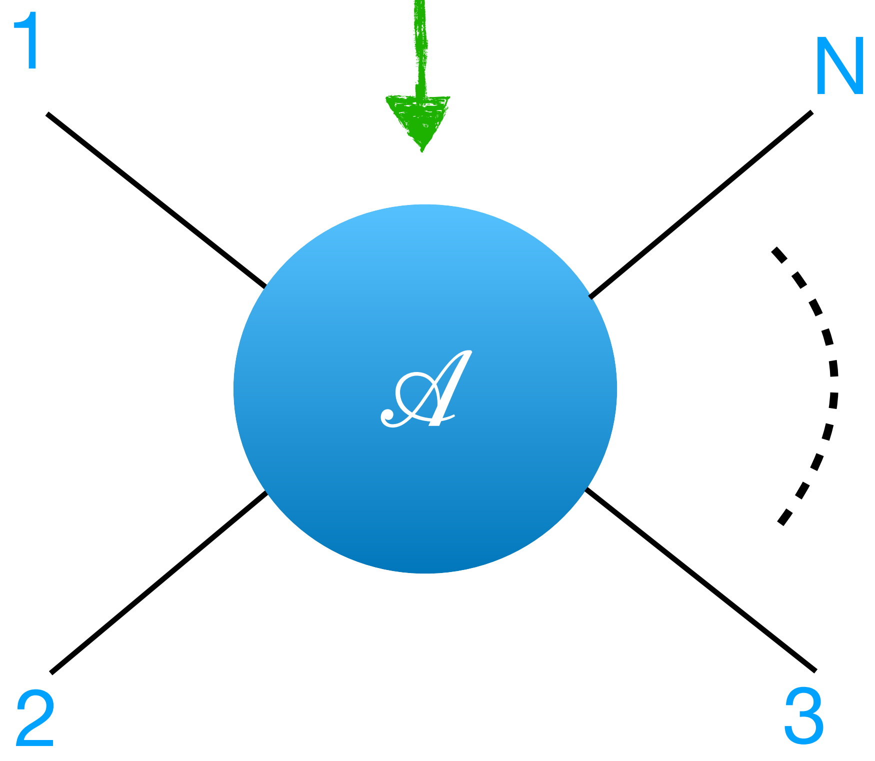
branch-cuts

multi-particle states go on-shell



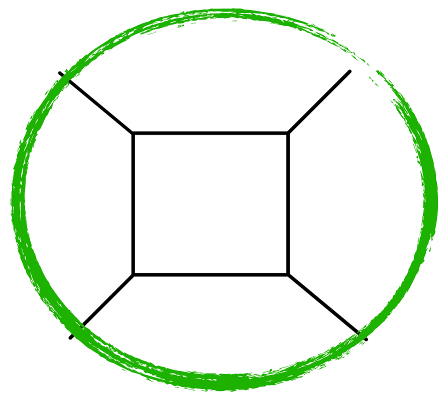
$\sim \sqrt{s}\,, \log(s)$

Analytic



Computation

$\mathscr{A} \sim \quad T^{\mu\nu}$



tensor structure

family of scalar Feynman integrals

one-loop

integration variables

kinematic variables

$$\sum_i A_i(\mathbf{s}) \int_\Omega \mathrm{d}\log f_n(\mathbf{x},\mathbf{s}) \wedge \ldots \wedge \mathrm{d}\log f_1(\mathbf{x},\mathbf{s})$$

Locality

algebraic functions

special functions

transcendental: poles and branch-cuts

evident with *pure* integrals

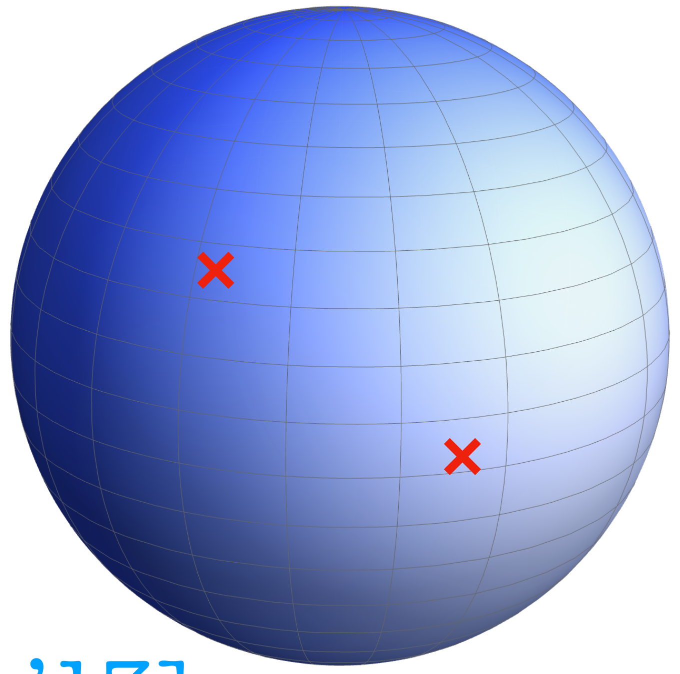
[Arkani et al., '10]

What stays beyond?

First of all… when are there only (poly)logs?

Evaluating (generalised) discontinuities → rational functions

The *integrand* lives on $\mathbb{C}^n$ with some marked points



Independent forms: $\dfrac{\mathrm{d}x}{x - a_i}$

$D = D_0 - 2\varepsilon$

dim-reg parameter

The *integrals* are expected $J \sim \int_\Omega \mathrm{d}\log f_n(\mathbf{x},\mathbf{s}) \wedge \ldots \wedge \mathrm{d}\log f_1(\mathbf{x},\mathbf{s})\, \mathscr{G}(\mathbf{x},\mathbf{s})^{k\varepsilon}$

canonical

[Henn, '13]

- $\varepsilon-$factorised differential equations: $\partial_{s_i}\mathbf{J} = \varepsilon\, \mathbf{B}_i(\mathbf{s})\, \mathbf{J}$
- The entries of $\mathbf{B}(\mathbf{s})$ have only simple poles ($\mathrm{d}\log f_i$)
- The differential forms in $\mathbf{B}(\mathbf{s})$ are independent

$$\sum_i c_i \omega_i = \mathrm{d}f \to c_i = 0$$

→

$$\mathbf{J} = \mathbb{P}\mathrm{exp}\left(\varepsilon \int_\gamma \mathbf{B}(\mathbf{s}') \right) \mathbf{J_0}$$

Iterated integrals

pure
~only logs

uniform transcendental weight
~ same transcendentality

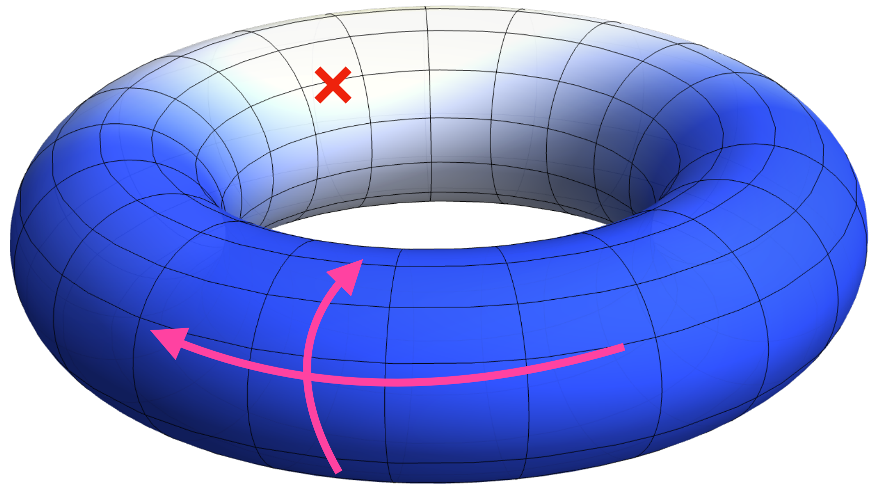
$$\sim \sum_k \varepsilon^k \log^k$$

And now beyond

Evaluating (generalised) discontinuities → trascendental functions

simplest generalisation: **(complex) elliptic curve**

$y = \sqrt{P_4(x)}$ elliptic curve



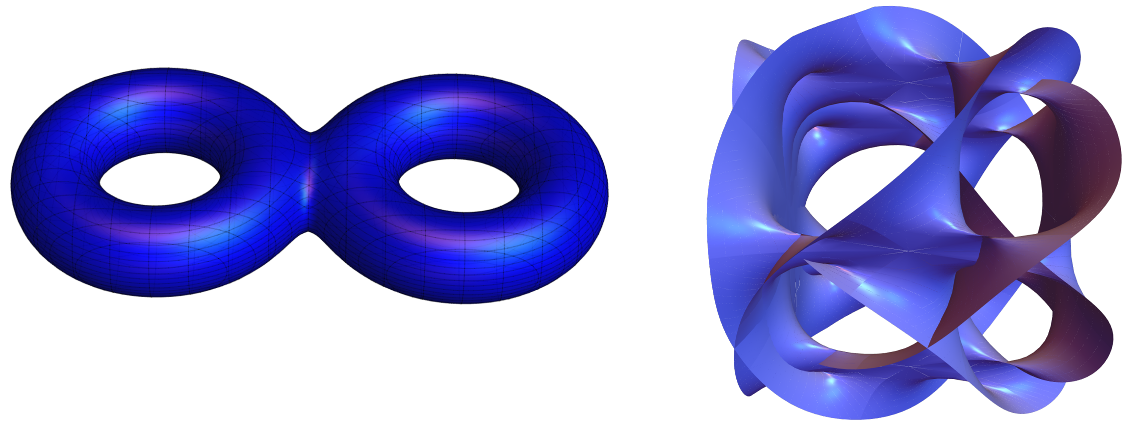
still forms that produce log singularities: $\frac{\mathrm{d}x}{(x-a_i)y}$ simple poles

and others that don't! $\frac{\mathrm{d}x}{y}$, no poles

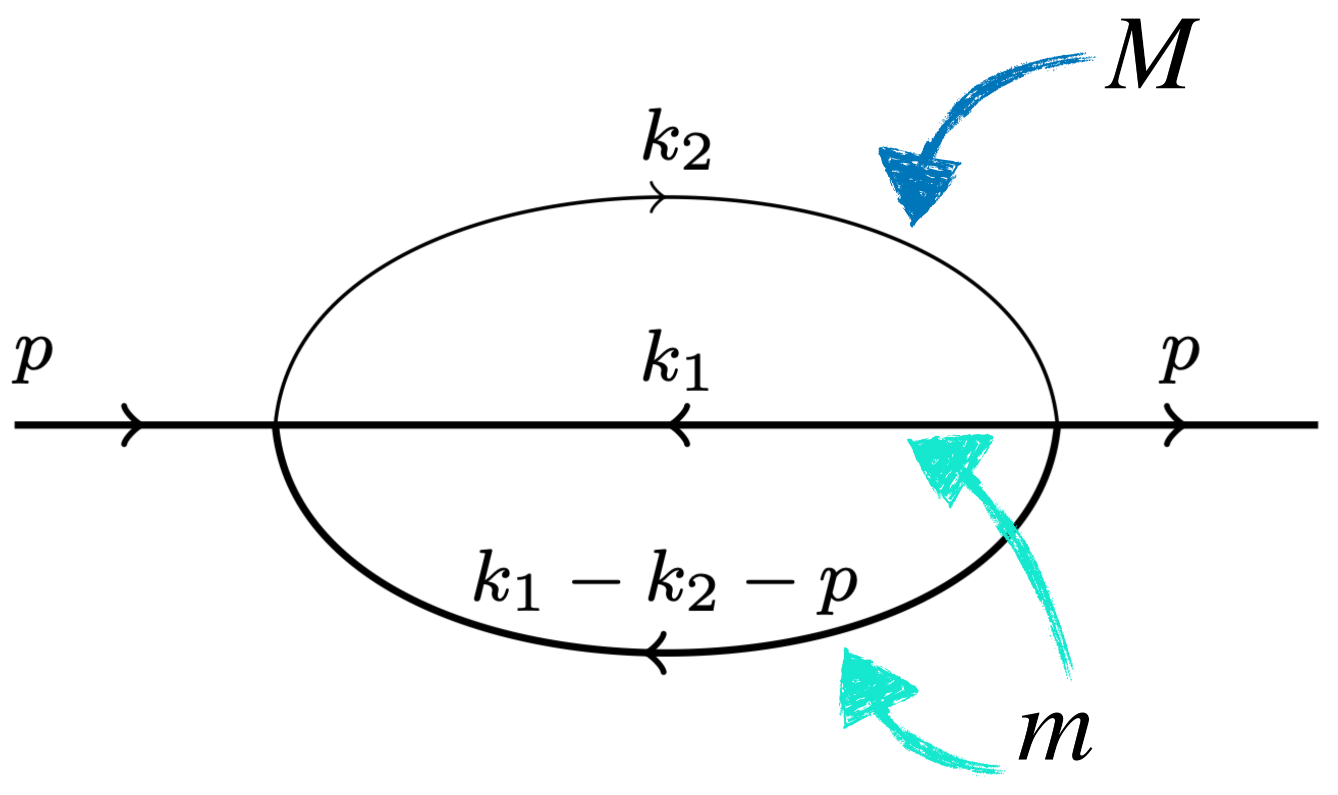
$\frac{\mathrm{d}x}{y}\left(x^2 - \frac{s_1 x}{2}\right)$ double pole without residues

Can we keep similar properties as before? → yes! **Today's talk!**

It works for the geometries we encountered so far!



Running example: two-loop sunrise



Two cases:

- $M = 0$ **polylogarithmic**
- $M \neq 0$ **elliptic**

$$I_{\nu_1,\ldots,\nu_5}(\mathbf{s};d) = \int \left(\prod_{j=1}^{2} \frac{\mathrm{d}^d k_j}{i\pi^{d/2}}\right) \frac{(k_1\cdot p)^{-\nu_4}(k_2\cdot p)^{-\nu_5}}{(k_1^2-m^2)^{\nu_1}(k_2^2-M^2)^{\nu_2}((k_1-k_2-p)^2-m^2)^{\nu_3}} = \int \left(\prod_{j=1}^{2} \frac{\mathrm{d}^d k_j}{i\pi^{d/2}}\right) \frac{1}{D_1^{\nu_1}D_2^{\nu_2}D_3^{\nu_3}D_4^{\nu_4}D_5^{\nu_5}}$$

- Integrals to make properties of the scattering amplitude manifest?
- Properties independent of the geometries?

Integrand analysis

- *Parametric* representation (**Baikov**):

$$I_{1,1,1,0,\nu_5} = s^{(2-D)/2}\int_\gamma \frac{\mathrm{d}z_1\ldots\mathrm{d}z_5}{z_1z_2z_3}\frac{z_5^{-\nu_5}}{B(\mathbf{z},\mathbf{s})^{(4-D)/2}}$$

- Fix integer number of dimensions ($D=2$):

$$I_{1,1,1,0,\nu_5} = \int_\gamma \frac{\mathrm{d}z_1\ldots\mathrm{d}z_5}{\boxed{z_1z_2z_3}}\frac{z_5^{-\nu_5}}{B(\mathbf{z},\mathbf{s})}$$

$$\rightarrow z_i = 0$$

- Simplify the analysis:

subset of the analytic structure:
(it works also on the full integral)

Maximal Cut ↓

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} = \int \mathrm{d}z_5 z_5^{-\nu_5}\int\frac{\mathrm{d}z_4}{\boxed{(z_4-A^+(z_5))}\boxed{(z_4-A^-(z_5))}}$$

two simple poles, same residue (up to sign)

↓

derivative of a log!

$$\Rightarrow\quad I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int\frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}}\int \mathrm{d}z_4\partial_{z_4}\boxed{\log f(z_4,z_5,\mathbf{s})}$$

↓

$\sqrt{P_4(z_5)}$: elliptic curve!

↓

$\mathrm{d}\log$ that we wanted

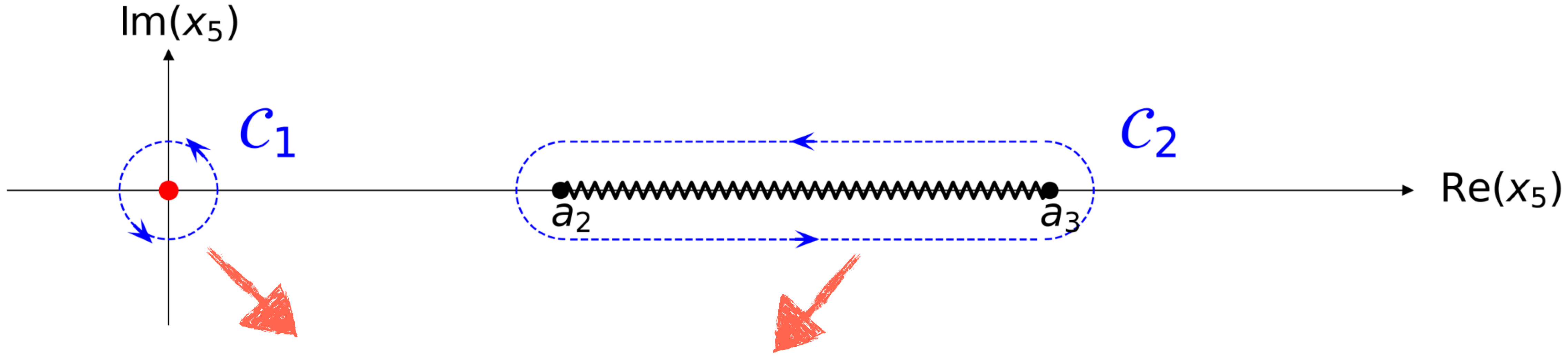
Case 1: $M = 0$

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{z_5}\sqrt{\boxed{(2z_5+s)(4m^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

simple pole in z_5

$\sqrt{P_2(z_5)}$

- $\nu_5 = 0$



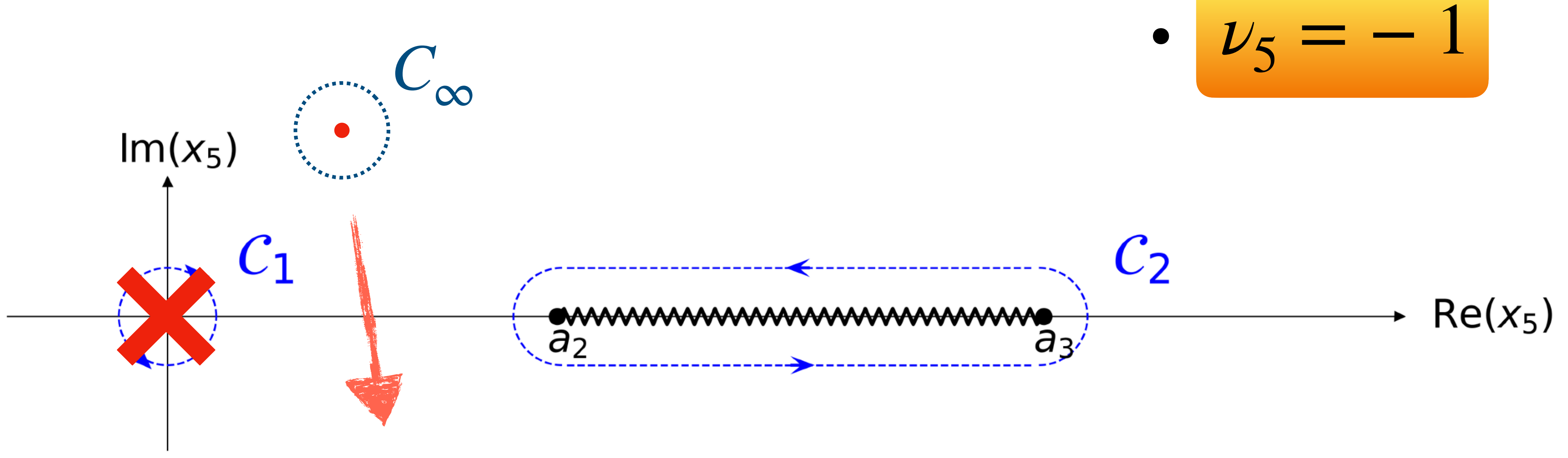
one independent cycle!

$$\int_{C_1} \frac{\mathrm{d}z_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{C_2} \frac{\mathrm{d}z_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \frac{1}{\sqrt{s(s-4m^2)}}$$

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \frac{1}{\sqrt{s(s-4m^2)}} \int \mathrm{d}\log g_1 \int \mathrm{d}\log f$$

LS

- $\nu_5 = -1$



pole at ∞ !

$$\int_{C_\infty} \frac{\mathrm{d}z_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{C_2} \frac{\mathrm{d}z_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto 1$$

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \int \mathrm{d}\log g_2 \int \mathrm{d}\log f$$

$$J_1 = \sqrt{s(s-4m^2)}\, I_{1,1,1,0,0}$$

$$\sum_i A_i(\mathbf{s}) \int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1$$

$$J_2 = I_{1,1,1,0,-1}$$

Case 1: $M = 0$: restoring the ε-dependence

easy for polylogs!

$$J_1\,|_{z_1=z_2=z_3=0} \sim \int \mathrm{d}\log g_1(z_5,\mathbf{s}) \int \mathrm{d}\log f(\mathbf{z},\mathbf{s})\, (\mathcal{G}(\mathbf{z},\mathbf{s}))^{k\,\varepsilon}$$

$$J_2\,|_{z_1=z_2=z_3=0} \sim \int \mathrm{d}\log g_2(z_5,\mathbf{s}) \int \mathrm{d}\log f(\mathbf{z},\mathbf{s})(\mathcal{G}(\mathbf{z},\mathbf{s}))^{k\,\varepsilon}$$

$$\boxed{(\mathcal{G}(\mathbf{z},\mathbf{s}))^{k\,\varepsilon} \sim 1 + k\,\varepsilon \log \mathcal{G} + \mathcal{O}(\varepsilon^2)}$$

only *logarithmic* singularities!

Moreover J_1, J_2 satisfy:

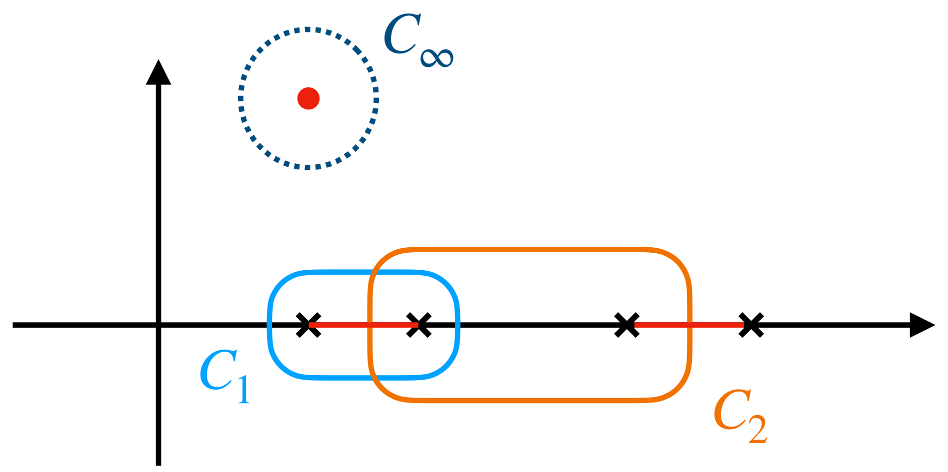
- $\varepsilon-$factorised differential equations: $\partial_{s_i}\mathbf{J} = \varepsilon\,\mathbf{B}_i(\mathbf{s})\,\mathbf{J}$
- The entries of $\mathbf{B}(\mathbf{s})$ have only simple poles $(\mathrm{d}\log f_i)$
- The differential forms in $\mathbf{B}(\mathbf{s})$ are independent

$$\mathbf{J} = \mathbb{P}\exp\left(\varepsilon \sum_i \mathbf{B}_i \int_\gamma \mathrm{d}\log f_i\right)\mathbf{J_0}$$

polylogarithms!

if f_i are rational functions

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{x_1=x_2=x_3=0} \sim \int \frac{\mathrm{d}x_5 x_5^{-\nu_5}}{\boxed{\sqrt{(2x_5+s+M^2)(M^2s-x_5^2)(4m^2-M^2-s-2x_5)}}} \int \mathrm{d}\log f(x_4,x_5,\mathbf{s})$$

$\boxed{:= y}$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{x_1=x_2=x_3=0} \sim \int \frac{\mathrm{d}x_5}{y} \int \mathrm{d}\log f(\mathbf{x},\mathbf{s})$$

$$\int_{C_2} \frac{\mathrm{d}x_5}{y} \propto \psi_0$$

holomorphic period of the elliptic curve!

$$\psi_0 = 1 + \sum_{j=0}^{\infty} c_j s^j$$

around the MUM point

$$J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{x_1=x_2=x_3=0} \sim \int \frac{\mathrm{d}x_5\, x_5}{y} \int \mathrm{d}\log f(\mathbf{x},\mathbf{s})$$

$$\int_{C_\infty} \frac{\mathrm{d}x_5\, x_5}{y} \propto 1$$

simple pole at ∞

$$J_2 = I_{1,1,1,0,-1}$$

- $\nu_5 = -2$ 2nd kind integral

$$I_{1,1,1,0,-2}\big|_{x_1=x_2=x_3=0} \sim \int \frac{\mathrm{d}x_5\, x_5^2}{y} \int \mathrm{d}\log f(\mathbf{x},\mathbf{s})$$

$$\int \frac{\mathrm{d}x_5\, x_5^2}{y}$$

double pole at ∞

no log singularities?!

$$\int \frac{\mathrm{d}x_5}{y}\left(x_5^2 - \frac{s_1\, x_5}{2}\right)$$

removing the simple pole

integrating → mix rational and log contributions!

$$J_1 \sim \frac{1}{\psi_0} \int \frac{\mathrm{d}x_5}{y} \int \mathrm{d}\log f(\mathbf{x},\mathbf{s}) (\mathcal{G}(\mathbf{x},\mathbf{s}))^{k\,\varepsilon}$$

$$J_2 \sim \int \frac{\mathrm{d}x_5\, x_5}{y} \int \mathrm{d}\log f(\mathbf{x},\mathbf{s}) (\mathcal{G}(\mathbf{x},\mathbf{s}))^{k\,\varepsilon}$$

No problem in the ε-expansion!

$$J_3 \sim \int \frac{\mathrm{d}x_5}{y} \left(x_5^2 - \frac{s_1\, x_5}{2}\right) \int \mathrm{d}\log f(\mathbf{x},\mathbf{s}) (\mathcal{G}(\mathbf{x},\mathbf{s}))^{k\,\varepsilon}$$

IBPs would produce undesired $\varepsilon-$powers

better differential form?

A derivative basis

derivative integrand is also an independent form!

pure

$$\boxed{J} = c^{(0)} + \varepsilon \sum_k c_k^{(1)} \mathscr{J}_k^{(1)} + \varepsilon^2 \sum_k c_k^{(2)} \mathscr{J}_k^{(2)} + \mathcal{O}(\varepsilon^3)$$

UT

canonical

$$\partial J \sim \sum_k A_k c_k^{(1)} + \varepsilon \sum_k c_k^{(2)} \sum_j A_j \mathscr{J}_{k,j}^{(1)} + \mathcal{O}(\varepsilon^2)$$

Not pure anymore, but UT (after multiplying by $1/\varepsilon$)

$$\mathbf{I}_i = \int_C \phi_i$$

$$\partial_{s_i} \mathbf{I}(\mathbf{s}, \varepsilon) = \mathbf{\Omega}_i(\mathbf{s}, \varepsilon)\, \mathbf{I}(\mathbf{s}, \varepsilon)$$

$$\partial_{s_i} \mathbf{W}(\mathbf{s}, \varepsilon) = \mathbf{\Omega}_i(\mathbf{s}, \varepsilon)\, \mathbf{W}(\mathbf{s}, \varepsilon)$$

$$\mathbf{W}_{i,j} = \int_{\gamma_j} \phi_i$$

$\mathbf{W}$: fundamental solution matrix (*period matrix*)

How to remove the algebraic contributions?

A look at the DEQ

How does the DEQ look like with this choice of basis?

- $\partial_{s_i} \boldsymbol{J} = \sum_{j=0}^{2} \varepsilon^j \, \boldsymbol{\Omega}^{(j)} \boldsymbol{J}$ with $\boldsymbol{\Omega}^{(j)}$ rational in the kinematics

$\simeq$ [Badger, Becchetti, Giraudo, Zoia,'24]

- The diagonal entry wrt J_2 is ε-factorised

To achieve this:

- Integrand analysis
- Basis aligned with the underlying geometry

Algorithm of

[Görges, Nega, Tancredi, Wagner '23]
[Duhr, **SM**, Nega, Sauer, Tancredi, Wagner '25]

$$\psi_1 = \psi_0 \log(s) + \sum_{j=0}^{\infty} d_j s^j$$

[Broedel, Duhr, Dulat, Penante, Tancredi,'18]

$$\psi_0 = 1 + \sum_{j=0}^{\infty} c_j s^j$$

$$\mathbf{W} = \begin{pmatrix} \psi_0 & \psi_1 \\ \partial_s \psi_0 & \partial_s \psi_1 \end{pmatrix} = \begin{pmatrix} \psi_0 & 0 \\ \partial_s \psi_0 & \frac{\Delta}{\psi_0} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{\psi_1}{\psi_0} (=\tau) \\ 0 & 1 \end{pmatrix}$$

derivative basis

$\mathbf{W}_{\mathrm{ss}}$ ~ LS $\det W$
algebraic

$\mathbf{W}_{\mathrm{u}}$ ~ logarithmic
(@ MUM point)

Known for all the geometries
that appeared so far!

- To remove the "impurity": $\mathbf{W}_{\mathrm{ss}}^{-1} . \mathbf{I}$
- Realign the transcendental weight ($\varepsilon^{-1}\tilde{I}_2$) **weight drop**
- $\varepsilon-$factorisation: further rotation $\mathbf{U}_{\mathrm{t}}$
  $\mathrm{d}\mathbf{J}(\mathbf{s},\varepsilon) = \varepsilon \mathbf{B}(\mathbf{s})\mathbf{J}(\mathbf{s},\varepsilon)$

$$\begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & t_3 & 1 \end{pmatrix}$$

- defined by first-order differential equations;
- solve them by series expansion;

- $\varepsilon-$factorised differential equations: $\partial_{s_i}\mathbf{J} = \varepsilon\,\mathbf{B}_i(\mathbf{s})\,\mathbf{J}$
- The entries of $\mathbf{B}(\mathbf{s})$ have only simple poles
- The differential forms in $\mathbf{B}(\mathbf{s})$ are independent

close to singular points
degenerate to the canonical
forms for $\mathrm{d}\log$!

Function space

What functions did we introduce in the canonical DEQ?

- transcendental from $\mathbf{W}_{\mathrm{ss}}^{-1}$: $\psi_0, \dots$

sunrise

$$\begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & t_3 & 1 \end{pmatrix}$$

- $\varepsilon-$functions from $\mathbf{U}_{\mathrm{t}}$, to $\varepsilon-$factorise the DEQ:

  $$\mathrm{d}\mathbf{J}(\mathbf{s}, \varepsilon) = \varepsilon \mathbf{B}(\mathbf{s}) \mathbf{J}(\mathbf{s}, \varepsilon)$$

  - defined by first-order differential equations: → integrals of elements in $\mathbf{W}_{\mathrm{ss}}^{-1}$; $\int \mathrm{d}s_i \, R(\mathbf{s}) \, f(\psi_0(\mathbf{s}))$
  - solve them by series expansion;

for **elliptic** integrals: t_1, t_2, t_3 are just the LS!

[Görges, Nega, Tancredi, Wagner ’23]
[Duhr, **SM**, Nega, Sauer, Tancredi, Wagner ’25]

…Beyond elliptic curves

- Good initial basis: *integrand analysis* in integer dimensions: choose integrals aligned with the geometry (mixed Hodge structure) associated to the maximal cuts at $\varepsilon = 0$.

- Rotate the initial basis $\mathbf{I}(\mathbf{s}, \varepsilon)$ by a sequence of rotations:

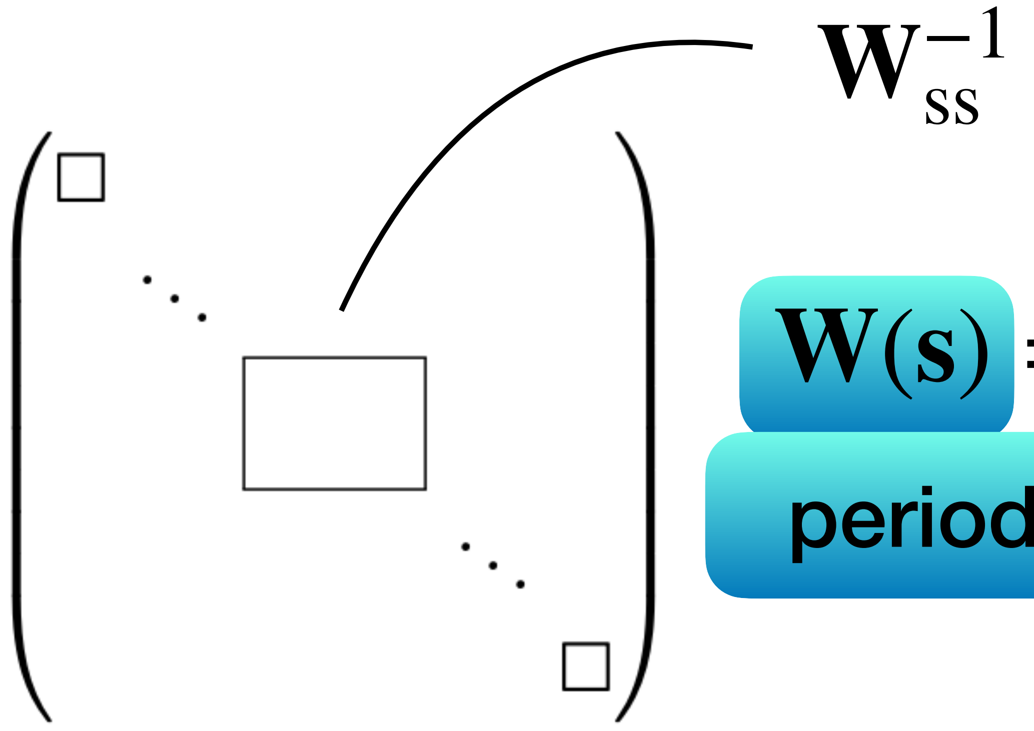
$$\mathbf{J}(\mathbf{s}, \varepsilon) = \mathbf{U}_{\mathrm{t}}(\mathbf{s}, \varepsilon)\mathbf{U}_{\varepsilon}(\varepsilon)\mathbf{U}_{\mathrm{ss}}(\mathbf{s})\mathbf{I}(\mathbf{s}, \varepsilon)$$

to get $\varepsilon-$form:

$$\mathrm{d}\mathbf{J}(x, \varepsilon) = \varepsilon\mathbf{B}(x)\mathbf{J}(x, \varepsilon)$$

$$\frac{1}{\varepsilon^{n-1}}\mathbf{U}_{\mathrm{t}}^{(1-n)}(\mathbf{s}) + \ldots + \frac{1}{\varepsilon}\mathbf{U}_{\mathrm{t}}^{(-1)}(\mathbf{s}) + \mathbf{U}_{\mathrm{t}}^{(0)}(\mathbf{s})$$

$\varepsilon-$scaling to realign the transcendental weight

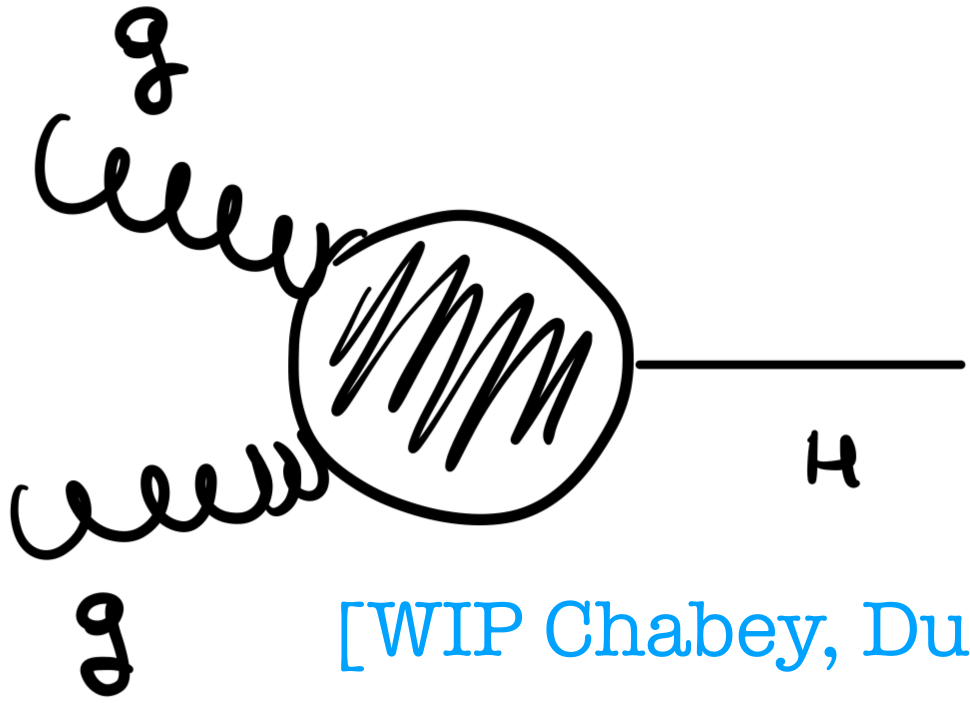


$$\mathbf{W}(\mathbf{s}) = \mathbf{W}_{\mathrm{ss}}(\mathbf{s})\mathbf{W}_{\mathrm{u}}(\mathbf{s})$$

period matrix

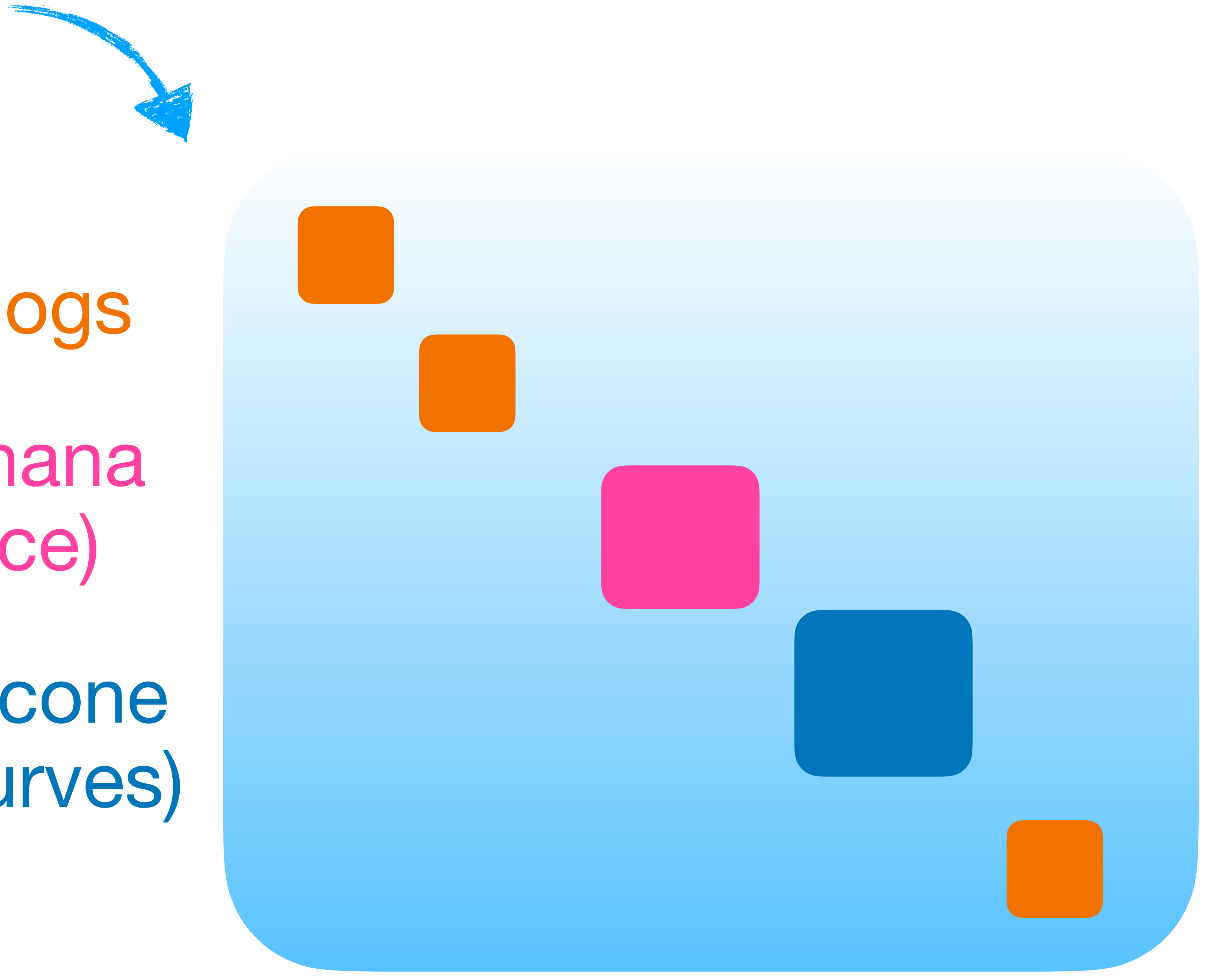
disentangles the logs

Hgg @ 3-loops



[WIP Chabey, Duhr, **SM**]

- 3-point 3-loop topology, @ N3LO corrections to the gluon form factor
- 12 families



polylogs

3-loop banana
(K3 surface)

3-loop ice-cone
(2 elliptic curves)

32 X 32

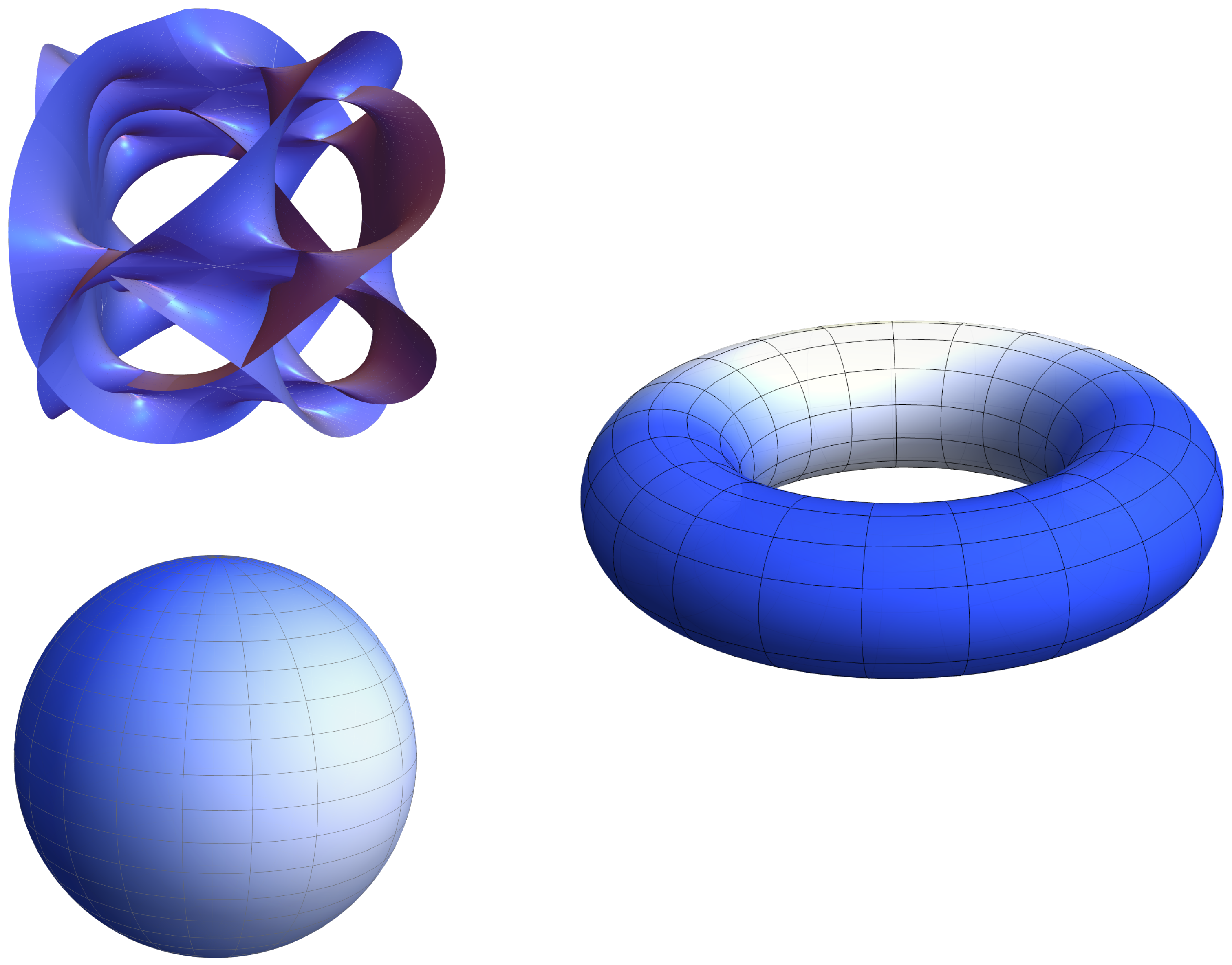
- [Czakon, Niggetiedt '20]
- Simpler 3-loop process (with internal masses)
- Techniques applicable to other processes

Summary

- Analytically control integrals beyond polylogs
- How elliptic integrals appear
- Integrand analysis to study elliptic integral (and beyond)
- Geometric choice of basis for better properties of the DEQ
- Canonical DEQs for elliptic integral (and beyond)

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}x}{(x-a_i)\,y} \\ \dfrac{\mathrm{d}x}{y} \\ \partial_s \dfrac{\mathrm{d}x}{y} \end{array} \right.$$

[Görges, Nega, Tancredi, Wagner '23]
[Duhr, **SM**, Nega, Sauer, Tancredi, Wagner '25]



Outlooks

- Quite **successful!**

elliptic amplitudes and correlators

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl, '24]
[Forner, Nega, Tancredi, '24]
[Marzucca, McLeod, Nega, '25]
[Becchetti, Coro, Nega, Tancredi, Wagner, '25]
[Coro, Nega, Tancredi, Wagner, '25]

application to higher genus [Duhr, Porkert, Stawinski, '24]

application to black holes scattering (CY)
[Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch, '24,'26]

integrals with a multivariate CY:
[**SM**, Sohnle, '25]
[Duhr, **SM**, Porkert, Semper, Stawinski, '25]
[Duhr, **SM**, '25]

application by other groups:
[Becchetti, Dlapa, Zoia, '25]

- Better understanding of the transcendental functions? Hints from twisted cohomology
- More applications to SM problems?
- Proof that it always delivers always a (canonical) form?
- How well it behaves with numerical evaluation?

Thank you for your attention!

Back-up

Some notes: in 2412.13876 they find another way to compute things with elliptic, in particular, their basis makes it manifest that the elliptic functions appear solely in the finite remainder

is it also for us?

It seem to depend on the choice of the basis, not obvious for me is the MI we choose that satisfy a DE with only positive eps powers will do the job

But I could say that, relating to 2404.12325, the way we choose our basis has the same characteristics as the one they have so the third kind is eps factorised in the max cut and the connection matrices doesn't have eps at the denominator

put more emphasis into this