Gauge theories for fractons and Carroll symmetries

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Developments on Generalized Symmetries of Quantum Systems

Universidad de Murcia, May 14, 2026

Based on:

- **E. A. Bergshoeff, A. Głodkowski, F. Peña-Benítez, PSR - In progress**

Outline

- Introduction to NL symmetries (Galilean, Carrollian, Aristotelian)
- Carroll gravity actions
- Gauge theory for fractons
- Relation with Carroll symmetry/gravity
- 2-form extension
- Future directions

Non-Lorentzian Symmetry

Non-Lorentzian limits of Poincaré

BATTELLE RENCONTRES

1967 LECTURES IN MATHEMATICS AND PHYSICS

Structure of Space-Time*

R. PENROSE

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 9, NUMBER 10 OCTOBER 1968

Possible Kinematics

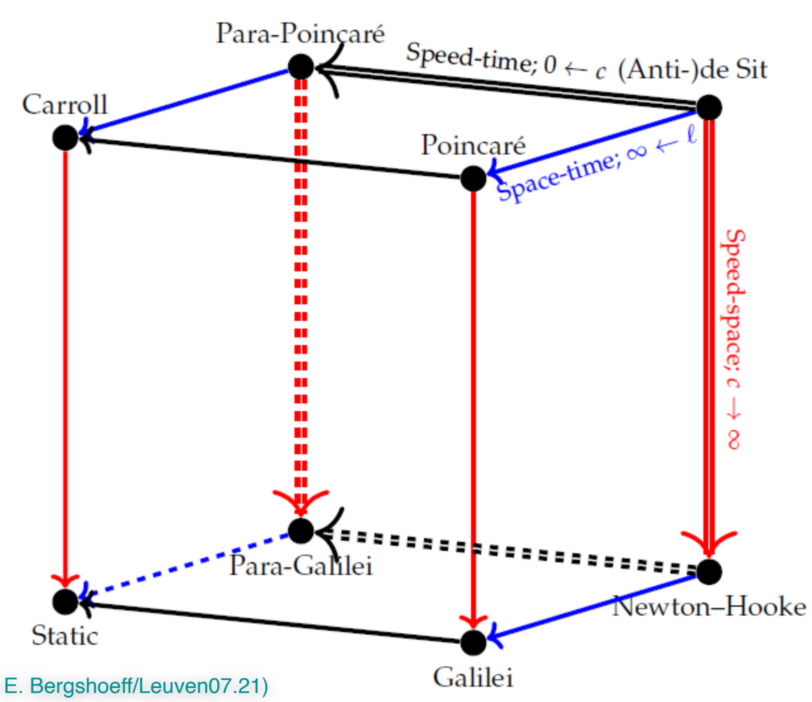
Henri Bacry*
The Institute for Advanced Study, Princeton, New Jersey

AND

Jean-Marc Lévy-Leblond
Physique Théorique, Faculté des Sciences, 06-Nice, France

(Received 27 May 1967)

The kinematical groups are classified; they include, besides space-time translations and spatial rotations, "inertial transformations" connecting different inertial frames of reference. When parity and time-reversal are required to be automorphisms of the groups, and when a weak hypothesis on causality is made, the only possible groups are found to consist of the de Sitter groups and their rotation-invariant contractions. The scheme of the contractions connecting these groups enables one to discuss their physical meaning. Beside the de Sitter, Poincaré, and Galilei groups, two types of groups are found to present some interest. The "static group" applies to the static models, with infinitely massive particles. The other type, halfway between the de Sitter and the Galilei groups, contains two nonrelativistic cosmological groups describing a nonrelativistic curved space-time.



(source: E. Bergshoeff/Leuven07.21)

THROUGH THE LOOKING GLASS

LEWIS CARROLL

Red Queen Effect

"Now, here, you see, it takes all the running you can do just to keep in the same place. If you want to get somewhere else, you must run at least twice as fast!"
- Red Queen from *Through the Looking Glass*

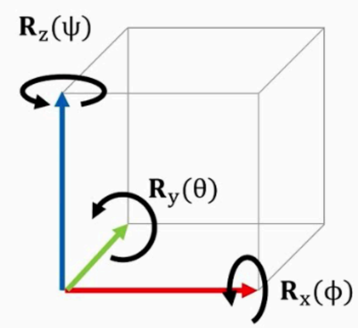
Non-Lorentzian Symmetry

$SO(3) \rightarrow$ Rotations in 3D

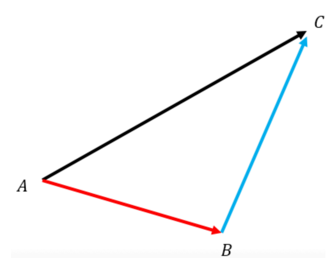
$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Translations



- **Poincaré group**

$$x' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt)$$
$$t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{xv}{c^2}\right)$$

Lorentz boosts

- **Galilei group**

$$x' = x - vt$$
$$t' = t$$

Galilei boosts

- **Carroll group**

$$x' = x$$
$$t' = t - vx$$

Carroll boosts

- **Aristotelian group**

No boosts

- **Aristotelian algebra**

$$[J_{ab}, J_{cd}] = \delta_{ac}J_{db} - \delta_{ad}J_{cb} - \delta_{bc}J_{da} + \delta_{bd}J_{ca}$$

$$[J_{ab}, P_c] = \delta_{cb}P_a - \delta_{ca}P_b$$

$$[J_{ab}, B_c] = \delta_{cb}B_a - \delta_{ca}B_b$$

Galilei

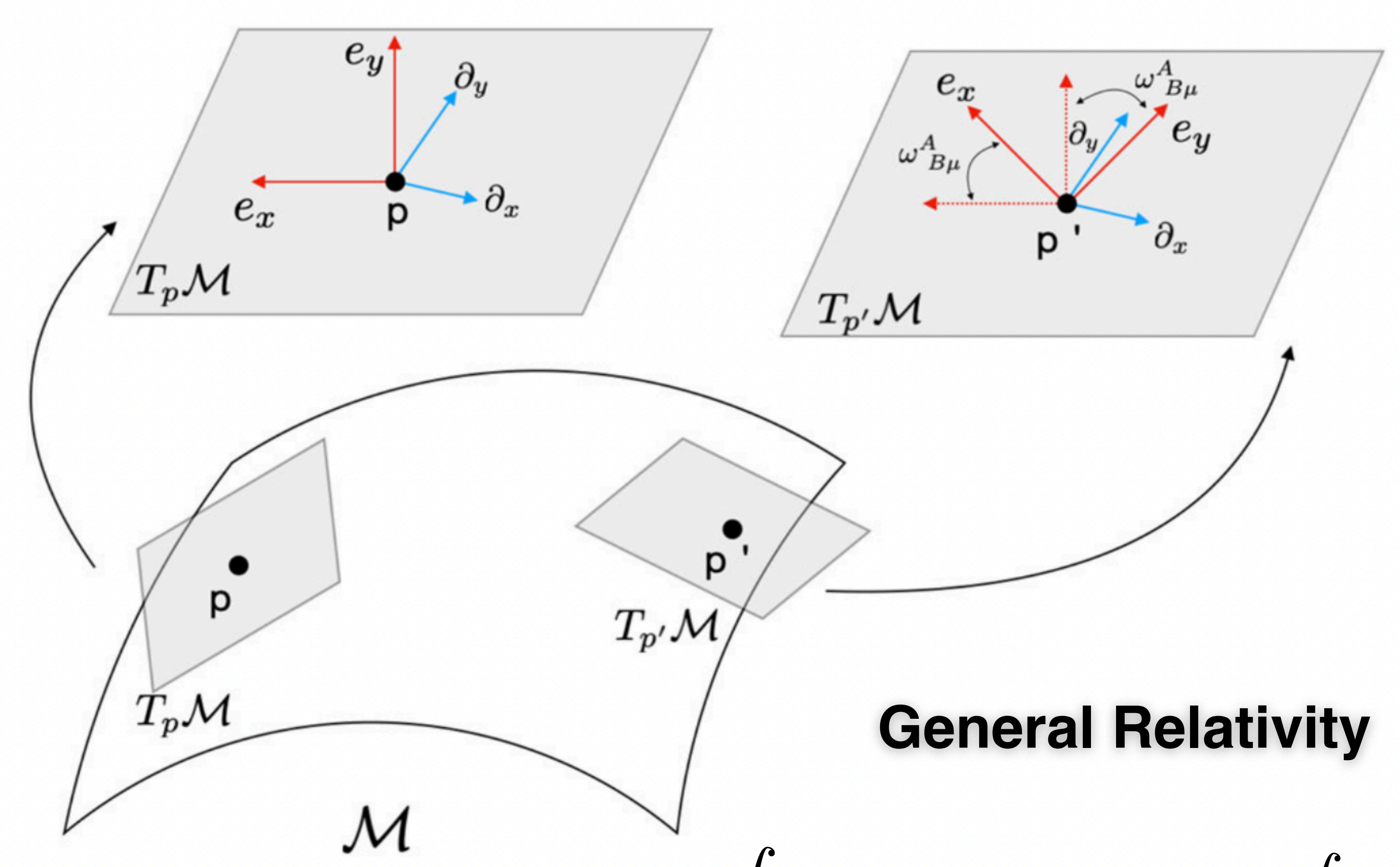
$$[B_a, H] = P_a$$

Carroll

$$[B_a, P_b] = \delta_{ab}H$$

Geometry from Symmetry

	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
Lorentzian	anti de Sitter $SO(3,2)/SO(3,1)$	Minkowski $ISO(3,1)/SO(3,1)$	de Sitter $SO(4,1)/SO(3,1)$
Riemannian	hyperbolic $SO(4,1)/SO(4)$	Euclidean $ISO(4)/SO(4)$	spherical $SO(5)/SO(4)$



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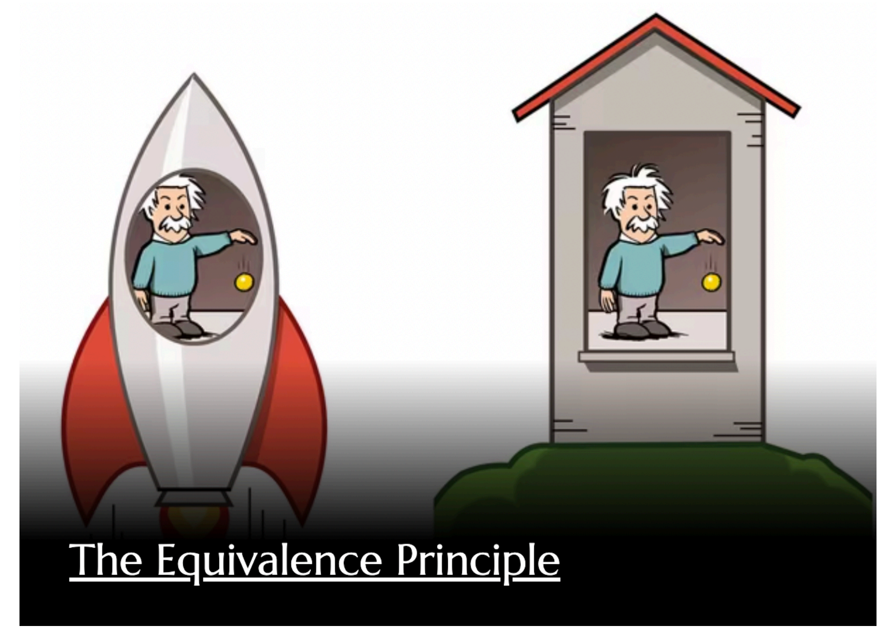


(source: whatsreal.it)

General Relativity

$$\int d^4x\,\sqrt{|g|}\,R \quad \sim \int \epsilon_{ABCD}(d\omega^{AB} + \omega^A{}_N \wedge \omega^{NB}) \wedge e^C \wedge e^D$$

Gravity as a gauge theory

- **The (2+1)-dimensional case**

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}R \sim \int \left\langle A \wedge dA + \frac{2}{3}A^3 \right\rangle$$

$$A_\mu = e_\mu^A P_A + \frac{1}{2}\omega_\mu^{AB} J_{AB}$$
$$F_{\mu\nu} = T_{\mu\nu}^A P_A + \frac{1}{2}R_{\mu\nu}^{AB} J_{AB}$$

$$T^A = de^A + \omega^{AB} \wedge e_B \qquad R^{AB} = d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB}$$

- **In the AdS case, black hole solutions (BTZ) and infinite dimensional conformal symmetry as asymptotic symmetry** [Bañados, Teitelboim, Zanelli (1992)], [Brown-Henneaux (1986)]

- **In 3+1 dimensions the best that can be done is to introduce a cosmological constant and define a gauge theory for the Lorentz group**

$$S = \frac{1}{32\pi G\Lambda}\int \langle F \wedge F \rangle$$

- **In arbitrary dimensions one can always write a Lorentz invariant theory as**

$$S = \frac{1}{16\pi G}\int d^D x\, det(e)\left(R^{AB} e_A^\mu e_B^\nu - 2\Lambda\right)$$

- **Einstein gravity can be recovered when solving the Torsionless constraint**

$$T^A = 0 \quad \Rightarrow \omega^{AB} = \omega^{AB}(e, \partial e)$$

- **The procedure of gauging a symmetry can be extended to the non-Lorentzian case**

Non-Lorentzian Gravity

- **Gauge theories**

$$A_\mu^{rel} = e_\mu^A P_A + \frac{1}{2}\omega_\mu^{AB} J_{AB} \qquad\qquad A = \{0, a\}$$

$$A_\mu^{Gal/Carr} = \tau_\mu H + e_\mu^a P_a + \frac{1}{2}\omega_\mu^{ab} J_{ab} + \omega_\mu^a B_a$$

$$A_\mu^{Ar} = \tau_\mu H + e_\mu^a P_a + \frac{1}{2}\omega_\mu^{ab} J_{ab}$$

- **Chern-Simons theories** [Papageorgiou, Schroers (2010)], [Hartong (2015)], [Bergshoeff, Rosseel (2016)], [Hartong, Lei, Obers (2016)], [Gomis, Kleinschmidt, Palmkvist, Salgado-Rebolledo (2021)], [Concha, Ravera, Rodríguez (2021)]

$$\int \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle$$

- **MM-like theories** [Campoleoni, Henneaux, Pekar, Perez, Salgado-Rebolledo (2021)], [Concha, Rodriguez, Rubio (2022)] [Ecker, Grumiller, Salgado-Rebolledo (2025)]

$$\int \langle F \wedge F \rangle$$

Recent applications

GALILEI

- **Non-AdS extensions of Holography** [Taylor (2008)], [Son (2008)], [Guica, Skenderis, Taylor, van Rees (2011)], [Bagchi, Fareghbal (2012)], [Hartong, Kiristis, Obers (2015)]
- **Non-relativistic string theory** [Danielsson, Guijosa, Kruczensk(2000)], [Gomis, Ooguri (2001)], [Bagchi (2013)], [Bergshoeff,Gomis, Yan (2018)], [Hallmark, Hartong, Menculini, Obers, Oling (2019)]
- **Newton-Cartan gravity** [Cartan (1923)], [Kuchar (1980)], [Andringa, Bergshoeff, Panda, de Roo (2011)], [Christensen, Hartong, Obers, Rollier (2014)]
- **Hydrodynamics** [Geracie Prabhu, Roberts (2015)], [Hartong, Armas, Have, Nielsen, Obers (2019)]
- **Condensed matter** [Greiter, Witten, Wilczek (1989)], [Carter, Khalatnikov (1994)], [Son, Wingate (2006)]
- **Fractional quantum Hall effect** [Bradlyn, Read (2012)], [Son (2013)], [Gromov, Abanov (2014)]

CARROLL

- **Flat Holography** [Donnay, Fiorucci, Herfray, Ruzziconi (2022)], [Bagchi, Dhivakar, Dutta (2023)]
- **Black hole physics** [Donnay, Marteau (2019)], [Oling, Pedraza (2024)]
- **Cosmology** [de Boer, Hartong, Obers, Sybesma, Vandoren (2021)]
- **Hydrodynamics** [Fridel, Jai-akson (2022)], [Ciambelli, Marteau, Petkou, Petropoulos, Siampos (2018)]
- **Fractons** [Figueroa-O’Farrill, Peréz, Prohazka (2023)]
- **Flat band systems** [Bagchi, Banerjee, Basu, Islam, Mondal (2022)]

ARISTOTELIAN

- **Lifshitz field theory** [Kachru, Liu, Mulligan (2018)]
- **Bimetric theory for the fractional quantum Hall effect** [Gromov, Son (2017)]
- **Hydrodynamics** [Armas, Jain (2021)], [Pinzani-Fokeeva, Yarom (2022)]
- **Field theories for fractons** [Pretko (2016)], [Seiberg, Shao (2020)], [Bidussi, Hartong, Have, Musaeus, Prohazka (2021)], [Grosvenor, Hoyos, Peña-Benítez, Surówka (2021)], [Angus, Kim, Park (2021)]

Carroll Gravity

- **Carroll (AdS) algebra**

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]} \qquad [J_{ab}, B_c] = 2\delta_{c[b}B_{a]} \qquad [P_a, H] = -\Lambda B_a$$
$$[J_{ab}, P_c] = 2\delta_{c[b}P_{a]} \qquad [P_a, B_b] = \delta_{ab}H \qquad [P_a, P_b] = -\Lambda J_{ab}$$

- **Carroll connection and curvature**

$$A^{Carroll} = \tau H + e^a P_a + \omega^a B_a + \frac{1}{2}\omega^{ab}J_{ab}$$
$$F^{Carroll} = dA + A \wedge A = T\,H + T^a P_a + R^a B_a + \frac{1}{2}R^{ab}J_{ab}$$

$$R^a = d\omega^a + \omega^a{}_b \wedge \omega^b - \Lambda\,\tau \wedge e^a \qquad T^a = de^a + \omega^a{}_b \wedge e^b$$
$$T = d\tau + e^a \wedge \omega_a \qquad R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \; -\Lambda\, e^a \wedge e^b$$

- **Magnetic Carroll gravity action** [Bergshoeff, Gomis, Rollier, Rosseel, ter Veldhuis (2017)]

$$-\frac{1}{16\pi G_M \Lambda}\int \langle F \wedge F\rangle \quad \rightarrow \frac{1}{16\pi G_M}\int dt d^D x\, \Omega\left(2\,\tau^\mu e^\nu{}_a R^a_{\mu\nu} + e^\mu{}_a e^\nu{}_b R^{ab}_{\mu\nu} - 2\Lambda\right)$$

$$\Omega = \det\left(\tau_\mu, e_\mu{}^a\right) \qquad \begin{aligned} &e_\mu{}^a e^\mu{}_b = \delta^a_b\,, \quad \tau_\mu\tau^\mu = 1\,, \quad \tau^\mu e_\mu{}^a = 0\,, \\ &\tau_\mu e^\mu{}_a = 0\,, \quad e_\mu{}^a e^\nu{}_a + \tau_\mu \tau^\nu = \delta^\nu_\mu \end{aligned}$$

Carroll Gravity

- **Magnetic and Electric Carroll limits of GR** [Henneaux, Salgado-Rebolledo (2021)]

$$S = \int dt\, d^D x \left(\pi^{ij}\dot{g}_{ij} - N\mathcal{H}_\perp - N^i\mathcal{H}_i\right) \quad \to \quad \int dt\, d^D x \left(\pi^{ij}\dot{g}_{ij} - N\mathcal{H}_{M,E} - N^i\mathcal{H}_i\right)$$

$$\mathcal{H}_\perp = \mathcal{H}_M + \mathcal{H}_E \qquad\qquad \mathcal{H}_i = -\,2\,\nabla_j {\pi_i}^j$$

$$\mathcal{H}_M = -\frac{\sqrt{g}}{16\pi G_M}\,(R - 2\Lambda) \qquad \mathcal{H}_E = \frac{16\pi G_M c^2}{\sqrt{g}}\left(\pi^{ij}\pi_{ij} - \frac{1}{D-1}\,\pi^2\right)$$

- **Magnetic action is equivalent to the action found when gauging the Carroll algebra after solving the torsion constraints**

$$\begin{aligned} T &= d\tau + e^a \wedge \omega_a = 0 \\ T^a &= de^a + {\omega^a}_b \wedge e^b = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} {\omega_\mu}^a &= -\,\tau_\mu \tau^\nu e^{\rho\, a}\partial_{[\nu}\tau_{\rho]} - e^{\nu\, a}\partial_{[\mu}\tau_{\nu]} + S^{ab}e_{\mu\, b} \\ {\omega_\mu}^{ab} &= 2\, e^{\rho[a}\partial_{[\mu}{e_{\rho]}}^{b]} - e_{\mu\, c}e^{\rho\, a}e^{\nu\, b}\partial_{[\rho}{e_{\nu]}}^c \end{aligned}$$

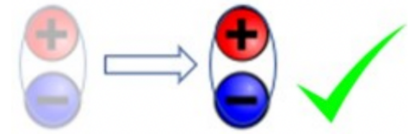
- **Using the identification** [Campoleoni, Henneaux, Pekar, Perez, Salgado-Rebolledo (2022)]

$$\pi^{ij} = \frac{\sqrt{g}}{16\pi G_M}\,\left(S^{ij} - g^{ij}S\right) \qquad \begin{aligned} &\tau^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right), \qquad e^\mu_a = \left(0, e^i_a\right), \\ &\tau_\mu = (N, 0)\,, \qquad e^a_\mu = \left(e^a_i N^i, e^a_i\right) \qquad g_{ij} = \delta_{ab}e^a_i e^b_j \end{aligned}$$

- **In relation to fractons one could consider more general actions. For example**

$$S = \int \left(T^A \star T_A + \frac{1}{2}R^{AB} \star R_{AB}\right) \to \int \left(T \star T + T^a \star T_a + R^a \star R_a + \frac{1}{2}R^{ab} \star R_{ab}\right)$$

Fractons



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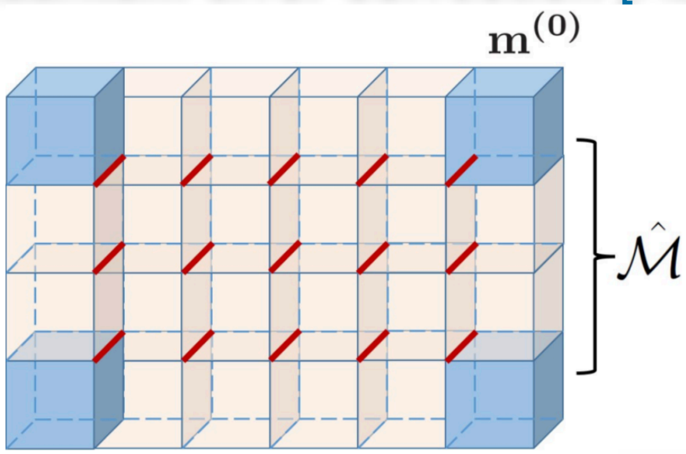
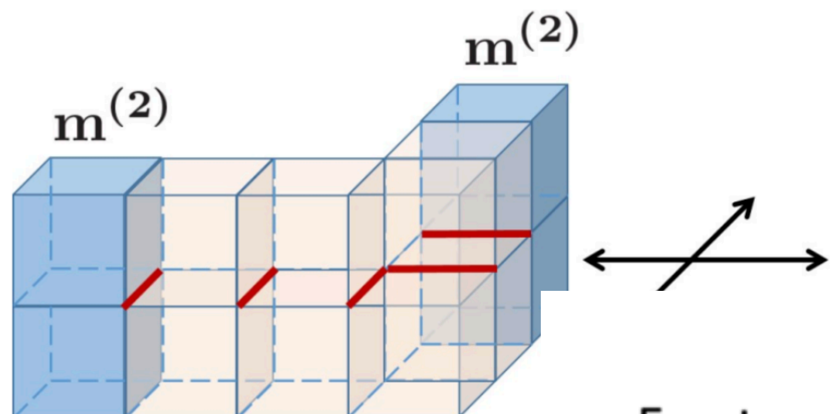
- **Quantum error correction** [Haah (2011)], [Vijay, Haah, Fu (2016)], [Shirley, Slagle, Chen (2018)]





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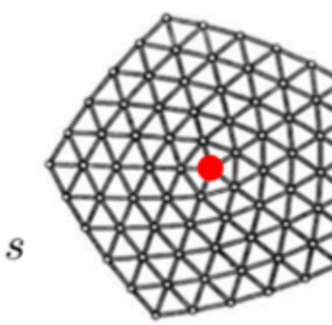
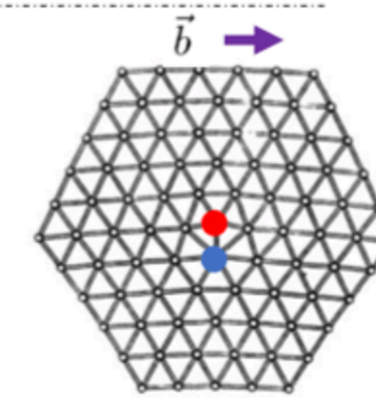
- **Higher-rank gauge theory** [Pretko (2016)]
- **Duality with Carroll symmetry** [Figueroa-O'Farrill, Peréz, Prohazka (2023)], [Marsot, Zhang, Chrenodub, Horvathy (2022)]
- **Fracton-elasticity duality** [Pretko, Radzihovski (2017)]
- **FQHE** [Du, Mehta, Nguyen, Son (2024)]

Fracton $\partial_i\partial_j E^{ij} = \rho$	Disclination $\epsilon^{ik}\epsilon^{j\ell}\partial_i\partial_j u_{k\ell} = s$ 
Dipole	Dislocation 
Gauge Modes	Phonons
Electric Field E_{ij}	Strain Tensor u_{ij}

(source: 2001.01722)

Fracton gauge theory

- **Simplest case: conservation of charge and dipole moment**

$$Q = \int d^D x \rho \qquad D^a = \int d^D x x^a \rho$$

- **Scalar charge theory**

$$S_{SC} = \frac{1}{2} \int d^{D+1} x \left(F_{0ab} F_{0ab} - \frac{1}{2} F_{abc} F^{abc} \right)$$

$$F_{0ab} = \partial_0 A_{ab} - \partial_a \partial_b A_0 \qquad \delta A_0 = \partial_0 \lambda_{U(1)}$$

$$F_{ab,c} = \partial_a A_{bc} - \partial_b A_{ac} \qquad \delta A_{ab} = \partial_a \partial_b \lambda_{U(1)}$$

- **Analogy with electrodynamics**

$$S_{Maxwell} = -\frac{1}{4} \int d^{D+1} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \int d^{D+1} \left(F_{0a} F_{0a} - \frac{1}{2} F_{ab} F^{ab} \right), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{0a} = \partial_0 A_a - \partial_a A_0 \qquad F_{ab} = \partial_a A_b - \partial_b A_a \qquad \delta A_\mu = \partial_\mu \lambda_{U(1)}$$

- **Adding sources generalizes the continuity equation accordingly**

$$\int d^D x A_\mu J^\mu \to \int d^D x (A_0 \rho + A_{ab} J^{ab})$$

$$\partial_\mu J^\mu = \partial_0 \rho + \partial_a J^a = 0 \quad \longrightarrow \quad \partial_0 \rho + \partial_a \partial_b J^{ab} = 0$$

Fracton gauge theory from elasticity

- **Elasticity Lagrangian in 2+1 dimensions**

$$S=\int d^2x dt\,\frac{1}{2}\left[(\partial_t u^i)^2 - C^{ijk\ell}u_{ij}u_{k\ell}\right] \qquad u_{ij}=\tfrac{1}{2}(\partial_i u_j+\partial_j u_i)$$

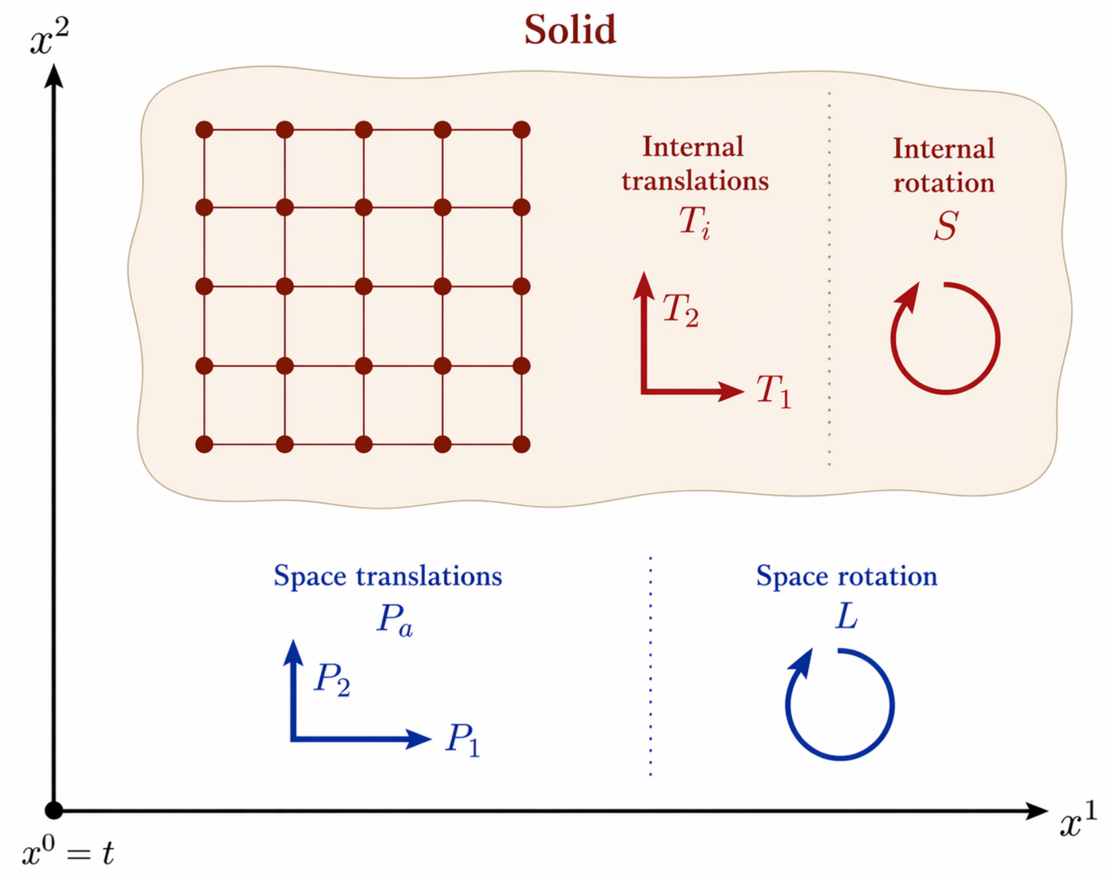
- **Action for the Goldstone boson associated to the breaking of translational symmetry**

- **Aristotelian symmetry**

$$[L,P_a]=\epsilon_a{}^b P_b$$
$$[S,T_i]=\epsilon_i{}^j T_j$$

- **Unbroken generators**

$$R_a=P_a+\delta_a{}^i T_i$$
$$J=L+S$$



- **Lagrangian for Goldstone bosons**

$$\Omega=g^{-1}dg \qquad g(x,u,\theta)=e^{x^a P_a}e^{\phi^i(x)T_i}e^{\theta(x)S}=e^{x^a R_a}e^{u^i(x)T_i}e^{\theta(x)S} \qquad \phi^i(x)=x^i+u^i(x)$$

$$\mathcal{L}=\frac{1}{2}A_{ij}\Omega_t{}^i\Omega_t{}^j-\frac{1}{2}K^{ai,bj}\Omega_{ai}\Omega_{bj}-\frac{1}{2}I(\Omega_t^S)^2-\frac{1}{2}\kappa^{ab}\Omega_a^S\Omega_b^S$$

Fracton gauge theory from elasticity

- **Elasticity Lagrangian in 2+1 dimensions**

$$S = \int d^2x dt \, \frac{1}{2} \left[(\partial_t u^i)^2 - C^{ijk\ell} u_{ij} u_{k\ell} \right]$$

$$u_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i)$$
$$u^i = \tilde{u}^i + u^{i(s)}$$

- **Hubbard-Stratonovich transformation**

$$S = \int d^2x dt \left[\frac{1}{2} C^{-1}_{ijk\ell} \sigma^{ij} \sigma^{k\ell} - \frac{1}{2} \pi^i \pi_i \quad - \sigma^{ij} (\partial_i \tilde{u}_j + u^{(s)}_{ij}) + \pi^i \partial_t (\tilde{u}_i + u^{(s)}_i) \right]$$

- **Dualization**

$$E^{ij} = \epsilon^{ik} \epsilon^{j\ell} \sigma_{k\ell} \qquad B^i = \epsilon^{ij} \pi_j \qquad B^i = \epsilon_{jk} \partial^j A^{ki} \qquad E^{ij} = \partial_t A^{ij} + \partial_i \partial_j \phi$$

$$S = \int d^2x \, dt \left[\frac{1}{2} \widetilde{C}^{-1}_{ijkl} E^{ij} E^{kl} - \frac{1}{2} B^i B_i - \rho \phi - J^{ij} A_{ij} \right]$$

$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha \, , \qquad \phi \to \phi + \partial_t \alpha$$

- **This type of theories will be considered only in flat space**

Fracton gauge fields and Aristotelian geometry

- **The fracton field theories should naturally couple to Aristotelian geometry**

$$A^{Ar} = \tau H + e^a P_a + \frac{1}{2}\omega^{ab} J_{ab}$$

- **Aristotelian field strength encodes torsion and curvature**

$$F^{Ar} = d\tau H + T^a P_a + \frac{1}{2} R^{ab} J_{ab}$$

$$T^a = de^a + \omega^a{}_b \wedge e^b \qquad R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$$

- **One can introduce an affine connection through a vielbein postulate**

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^a{}_{b\mu} e^b_\nu - \Gamma^\rho{}_{\mu\nu} e^a_\rho = 0 \qquad \mu = 0, \ldots, D-1$$

$$\nabla_\mu \tau_\nu = \partial_\mu \tau_\nu - \Gamma^\rho{}_{\mu\nu} \tau_\rho = 0 \qquad a = 1, \ldots, D-1$$

$$e^a_\mu e^\mu_b = \delta^a_b, \quad \tau_\mu e^\mu_a = 0 = \tau^\mu e^a_\mu$$

$$\tau_\mu \tau^\mu = 1, \quad \tau_\mu \tau^\nu + e^a_\mu e^\nu_a = \delta^\nu_\mu$$

$$\Gamma^\rho{}_{\mu\nu} = \tau^\rho \partial_\mu \tau_\nu + e^\rho_a (\partial_\mu e^a_\nu + \omega^a{}_{b\mu} e^b_\nu)$$

- **In terms of a spatial metric** $h_{\mu\nu} = \delta_{ab}\, e^a{}_\mu e^b{}_\nu \qquad h_{\mu\nu}\tau^\nu = 0 \qquad g_{\mu\nu} = \tau_\mu \tau_\nu + h_{\mu\nu}$

$$\nabla_\mu \tau_\nu = 0 = \nabla_\mu h_{\nu\rho} \qquad \Gamma^\mu{}_{\nu\rho} = \tau^\mu \partial_\nu \tau_\rho + \frac{1}{2} h^{\mu\lambda} \left(\partial_\nu h_{\lambda\rho} + \partial_\rho h_{\nu\lambda} - \partial_\lambda h_{\nu\rho}\right)$$

Fracton gauge fields and Aristotelian geometry

- **Consider again the fracton gauge theory**

$$S = \frac{1}{2}\int d^{d+1}x \left(F_{0ij}F_{0ij} - \frac{1}{2}F_{ijk}F_{ijk}\right) \qquad \begin{aligned} F_{ijk} &= \partial_i A_{jk} - \partial_j A_{ik} \\ F_{0ij} &= \dot{A}_{ij} + \partial_i\partial_j\phi \end{aligned}$$

- **Can be rewritten as**

$$S = -\frac{1}{4}\int d^D x\, F_{\mu\nu\rho}F^{\mu\nu\rho}$$

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} - \partial_\nu A_{\mu\rho} \qquad \delta A_{\mu\nu} = \partial_\mu\partial_\nu\varepsilon \qquad A_{0\mu} = -\partial_\mu\phi$$

- **Curved space generalization**

$$S = -\frac{1}{4}\int d^D x\, \sqrt{|g|}\, F_{\mu\nu\rho}F^{\mu\nu\rho}$$

$$F_{\mu\nu\lambda} = \nabla_\mu A_{\nu\lambda} - \nabla_\nu A_{\mu\lambda} \qquad \delta A_{\mu\nu} = \nabla_\mu\nabla_\nu\epsilon \qquad \tau^\nu A_{\mu\nu} = -\partial_\mu\phi$$

- **Curvature spoils gauge invariance**

$$\delta F_{\mu\nu\rho} = [\nabla_\mu, \nabla_\nu]\partial_\rho\epsilon = -R^\alpha{}_{\rho\mu\nu}\,\partial_\alpha\epsilon$$

Fracton gauge fields and Aristotelian geometry

- **One can propose the modified action** [Peña-Benitez, Salgado-Rebolledo (2023)]

$$S = -\int d^{d+1}x \sqrt{|g|} \left(\frac{1}{2}F_{\mu\nu\lambda}F^{\mu\nu\lambda} - R^{\mu\nu\rho\sigma}A_{\mu\rho}A_{\nu\sigma}\right)$$

- **Where gauge invariance requires**

$$\nabla_\mu R^{\mu\nu\rho\sigma} = 0$$

- **Other approaches require different constraints on the geometry** [Slagle, Pretko (2016)], [Bidussi, Hartong, Have, Musaeus, Prohazka (2021)]

- **In 2+1 dimensions there is a Chern-Simons formulation** [Hartong, Palumbo, Pekar, Pérez, Prohazka (2024)]

- **Fracton gauge fields can be coupled to a linearized gravity** [Tsaloukidis, Fernandez-Melgarejo, Molina-Vilaplana, Surówka (2023)]

- **Gauge invariant actions can be formulated by introducing extra fields in the theory** [Bidussi, Hartong, Have, Musaeus, Prohazka (2021)], [Peña-Benitez (2021)], [Jain, Jensen (2021)], [Peña-Benitez, Salgado-Rebolledo (2023)], [Bergshoeff, Giorgi, Rosseel, Salgado-Rebolledo (2025)]

From Carroll algebra

- **Carroll algebra**

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]} \qquad [J_{ab}, B_c] = 2\delta_{c[b}B_{a]}$$
$$[J_{ab}, P_c] = 2\delta_{c[b}P_{a]} \qquad [P_a, B_b] = \delta_{ab}H$$

- **Add a U(1) charge and obtain the dipole preserving algebra by the following redefinition**

$$Q \leftrightarrow H \qquad B_a \to D_a$$

- **Dipole preserving algebra or fracton algebra**

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]} \qquad [J_{ab}, D_c] = 2\delta_{c[b}D_{a]}$$
$$[J_{ab}, P_c] = 2\delta_{c[b}P_{a]} \qquad [P_a, D_b] = \delta_{ab}Q$$

- **Compatible with the fracton charges**

$$Q = \int d^D x \rho \qquad D^a = \int d^D x x^a \rho$$

- **Extension to higher moment charges: multipole algebra** [Gromov (2018)]

- **Use tools from Carroll gravity**

$$A^{Carroll} = \tau H + e^a P_a + \omega^a B_a + \frac{1}{2}\omega^{ab}J_{ab}$$
$$\to \; A = a\,Q + e^a P_a + A^a D_a + \frac{1}{2}\omega^{ab}J_{ab} + \tau H$$

From Carroll algebra

- **Curvature**

$$F = dA + A \wedge A = T^a P_a + \frac{1}{2} R^{ab} J_{ab} + f\, Q + F^a D_a + d\tau H$$

$$T^a = de^a + \omega^a{}_b \wedge e^b \qquad F^a = dA^a + \omega^a{}_b \wedge A^b$$

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \qquad f = da + e^a \wedge A_a$$

- **Gauge transformations**

$$\delta A = d\epsilon + [A, \epsilon], \qquad \epsilon = \eta Q + \rho^a P_a + \frac{1}{2}\theta^{ab} J_{ab} + \beta^a D_a + \gamma H$$

- **Under U(1) and dipole gauge transformations**

$$\delta a = d\eta + e^a \beta_a \qquad \delta A^a = d\beta^a + \omega^a{}_b \beta^b$$

- **Consider flat space**

$$e^a = \delta^a_\mu dx^\mu \qquad \omega^{ab} = 0$$

- **The gauge fixing condition** $a_a = 0$ **leads to the fracton gauge transformation** $\delta A^a_\mu = -\partial_\mu \partial^a \eta$

From Carroll algebra

- **Torsion free condition in Carroll gravity leads to a symmetric gauge field**
  [Bergshoeff, Gomis, Rollier, Rosseel, ter Veldhuis (2017)]

$$\omega_\mu{}^a \quad = \quad \tau_\mu \tau^\nu e^{\rho a} \partial_{[\nu} \tau_{\rho]} + e^{\nu a} \partial_{[\mu} \tau_{\nu]} + S^{ab} e^b_\mu$$

- **It can be identified with the symmetric rank-2 tensor in the scalar charge theory**

$$\omega^a_\mu \to A^a_\mu \qquad S_{ab} \to A_{ab}$$

- **Carroll gravity action can be transformed into an action for fracton gauge fields**

$$\begin{aligned} S_{Carroll} &= \int \left(T \star T + T^a \star T_a + R^a \star R_a + \frac{1}{2} R^{ab} \star R_{ab} \right) \\ &\to \int \left(f \star f + T^a \star T_a + F^a \star F_a + \frac{1}{2} R^{ab} \star R_{ab} \right) \end{aligned}$$

- **However this type of action is not U(1) invariant in the presence of curvature. Thus we set**

$$e^a = \delta^a_\mu dx^\mu \qquad \omega^{ab} = 0$$

From Carroll algebra

- **We find an alternative formulation of Pretko's action**

$$S = -\frac{1}{2}\int (F^a \wedge *F_a + f \wedge *f) = -\frac{1}{4}\int d^{D+1}x\,(F_{\mu\nu}{}^a F^{\mu\nu}{}_a + f_{\mu\nu} f^{\mu\nu})$$

$$F^a = dA^a, \qquad f = da + dx^a \wedge A_a, \qquad \delta A^a = d\beta^a, \qquad \delta a = d\eta + dx^a \beta_a$$

- **This action can be found from Cosserat elasticity (keeping θ independent) upon dualization** [Gromov, Surówka (2020)]

- **Coupling to currents leads to the fraction continuity equation**

$$\int (A^a \wedge *J_a + a \wedge *j) = \int d^{D+1}x\,(A_\mu{}^a J^\mu{}_a + a^\mu j_\mu)$$

$$\partial_0 \rho + \partial_a \partial_b J^{ab} = 0 \qquad \rho = j^0 + \partial_a J^{0a}$$

- **Gauge fixing and imposing a constraint leads to Pretko's theory**

$$a_a = 0 \qquad f = 0 \qquad a_0 = -\phi$$

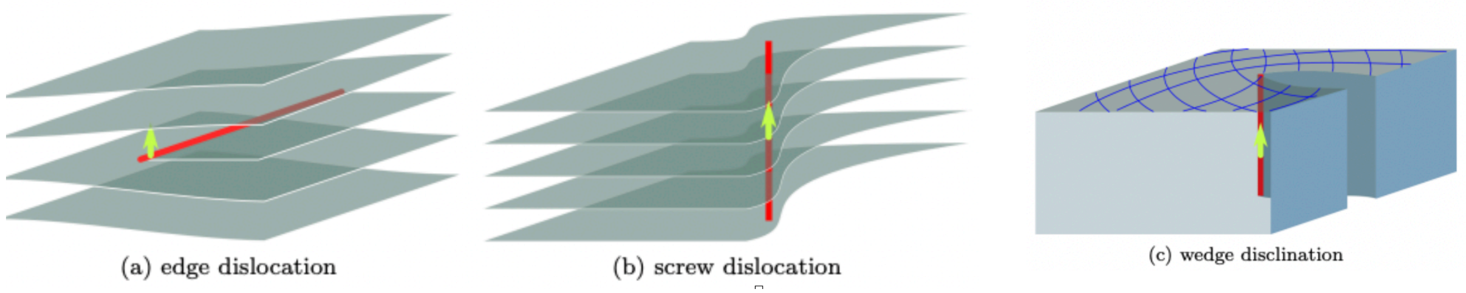
$$S_{SC} = \frac{1}{2}\int d^{D+1}x \left(F_{0ab}F_{0ab} - \frac{1}{2} F_{abc}F^{abc} \right)$$

$$F_{0ab} = \dot{A}_{ab} + \partial_a \partial_b \phi$$

$$F_{abc} = \partial_{[b} A_{c]a}$$

Two-form extension

- In 3+1 dimensions disclinations and dislocations are line-like defects (strings) [Beekman, Nissinen, Wu, Zaanen (2017)], [Pai, Pretko (2018)]



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- Dualization of the elasticity action leads to a higher-rank gauge theory

$$S = \int d^3x\, dt\, \frac{1}{2}\left[(\partial_t u^i)^2 - C^{ijkl} u_{ij} u_{kl}\right]$$
$$\downarrow$$
$$S = \int d^3x\, dt \left[\frac{1}{2}\widetilde{C}^{-1}_{ijkl\,pqrs} E^{ijkl} E^{pqrs} - \frac{1}{2} B^{ij} B_{ij} - \rho^{ij}\phi_{ij} - J^{ijkl} A_{ijkl}\right]$$

$$\begin{aligned} E_{abmn} &= \partial_m \partial_b \phi_{an} - \partial_m \partial_a \phi_{nb} + \partial_a \partial_n \phi_{bm} \\ &\quad - \partial_b \partial_n \phi_{ma} - \partial_t A_{abmn} \\ B_{ab} &= \epsilon^{jmn}\, \partial_j A_{abmn} \end{aligned}$$

$$\begin{aligned} A_{abmn} &\to A_{abmn} + (\partial_b \partial_m \alpha_{na} - \partial_m \partial_a \alpha_{nb} \\ &\qquad + \partial_n \partial_a \alpha_{bm} - \partial_b \partial_n \alpha_{ma}) \\ \phi_{ab} &\to \phi_{ab} + \partial_t \alpha_{ab} \end{aligned}$$

- Generalized continuity and charges

$$\partial_0 \rho^{il} + \partial_i \partial_k J^{ijkl} = 0 \qquad Q^i = \int_S dn_j \rho^{ij} \qquad D^{ij} = \int_S dn_k x^{[i} \rho^{j]k}$$

- Gauge fields that couple to extended object are Abelian p-forms [Henneaux, Teitelboim (1986)]

Two-form extension

- **How to define a two-form extension of our fracton gauge theory?**

- **It is possible to extend the dual formulation of a Lie algebra to include p-forms: Free differential algebra** [D'Auria, Fré, Regge (1980)]

$$dA^I + \frac{1}{2} C^I{}_{JK}\, A^J A^K = 0$$

$$\underbrace{dB^{\tilde I} + C^{\tilde I}{}_{J\tilde K}\, A^J B^{\tilde K}}_{\nabla B^{\tilde I}} + \underbrace{\frac{1}{(p+1)!} C^{\tilde I}{}_{I_1\cdots I_{p+1}}\, A^{I_1}\cdots A^{I_{p+1}}}_{\Omega^{\tilde I}(A)} = 0$$

- **Jacobi identities are obtained by acting on these equations with the exterior derivative**

$$C^I{}_{KM}\, C^M{}_{LJ} + C^I{}_{LM}\, C^M{}_{JK} + C^I{}_{JM}\, C^M{}_{KL} = 0$$

$$C^{\tilde I}{}_{J\tilde K}\, C^{\tilde K}{}_{L\tilde M} - C^{\tilde I}{}_{L\tilde K}\, C^{\tilde K}{}_{J\tilde M} = C^K{}_{JL}\, C^{\tilde I}{}_{K\tilde M}$$

$$\nabla \Omega^{\tilde I} = 0$$

- **One can define the Lie derivatives** [Castellani (2013)] $\ell_I = di_I + i_I d, \qquad \tilde\ell_{\tilde I} = di_{\tilde I} + i_{\tilde I} d$

- **Consider the case** $\Omega^{\tilde I} = 0,\ \tilde I \to I,\ p = 2$

- **They form a the semi direct sum algebra**

$$[\ell_I, \ell_J] = C^K{}_{IJ}\, \ell_K, \qquad [\ell_I, \tilde\ell_J] = C^K{}_{IJ}\, \tilde\ell_K, \qquad [\tilde\ell_I, \tilde\ell_J] = 0$$

Two-form extension

- **Stringy Carrollian algebra**

$$[J_{\alpha\beta}, H_\gamma] = 2\eta_{\gamma[\beta}H_{\alpha]} \qquad [J_{\alpha\beta}, J_{\gamma\delta}] = 4\eta_{\alpha[\gamma}J_{\delta]\beta]}$$
$$[J_{ab}, P_c] = 2\delta_{c[b}P_{a]} \qquad [J_{ab}, J_{cd}] = 4\delta_{a[c}J_{d]b]}$$

$$[J_{ab}, B_{c\gamma}] = 2\delta_{c[b}B_{a]\gamma}$$
$$[J_{\alpha\beta}, B_{c\gamma}] = 2\eta_{\gamma[\beta}B_{c|\alpha]}$$
$$[P_a, B_{b\beta}] = \delta_{ab}H_\beta$$

- **Identification** $\quad H_\alpha \leftrightarrow Q_a \qquad B_{a\alpha} \to D_{ab} \qquad \eta_{\alpha\beta} \to \delta_{ab}$

- **Proposal for the stringy dipole preserving symmetry**

$$[J_{ab}, J_{cd}] = 4\eta_{[a[c}J_{d]b]}$$
$$[J_{ab}, P_c] = 2\delta_{c[b}P_{a]}$$

$$[J_{ab}, Q_c] = 2\delta_{c[b}Q_{a]}$$
$$[J_{ab}, D_{cd}] = 4\delta_{[a[c}D_{d]b]}$$
$$[P_a, D_{bc}] = \delta_{a[b}Q_{c]}$$

- **This algebra satisfies a semi direct sum structure with** $\quad \ell_I \sim (P_a, J_a), \quad \tilde{\ell}_I \sim (Q_a, D_{ab})$

$$A = e^a P_a + \frac{1}{2}\omega^{ab}J_{ab} \qquad B = a^a Q_a + \frac{1}{2}A^{ab}D_{ab}$$

- **Curvatures**

$$F = dA + A \wedge A = T^a P_a + \frac{1}{2}R^{ab}J_{ab}$$

$$\mathcal{F} = dB + A \wedge B - B \wedge A = f^a Q_a + \frac{1}{2}F^{ab}D_{ab}$$

$$F^{ab} = dA^{ab} \qquad f^a = da^a + e_b \wedge A^{ab}$$

Two-form extension

- **Two-form extension** [Bergshoeff, Peña-Benitez, Głodkowski, Salgado-Rebolledo]

$$S = -\frac{1}{2}\int\left(\frac{1}{2}F^{ab}\wedge *F_{ab} + f^a \wedge *f_a\right) = -\frac{1}{12}\int d^{D+1}x\left(\frac{1}{2}F_{\mu\nu\rho}{}^{ab}F^{\mu\nu\rho}{}_{ab} + f_{\mu\nu\rho}{}^{a}f^{\mu\nu\rho}{}_{a}\right)$$

$$F^{ab} = dA^{ab} \qquad f^a = da^a + e_b \wedge A^{ab}$$

- **Gauge transformations**

$$\delta B^{ab} = d\beta^{ab}, \qquad \delta a^a = d\eta^a + e_b \wedge \beta^{ab}$$

- **Currents**

$$\int\left(\frac{1}{2}B^{ab}\wedge *J_{ab} + a^a \wedge *j_a\right) = \int d^Dx\left(\frac{1}{4}B_{\mu\nu}{}^{ab}J^{\mu\nu}{}_{ab} + \frac{1}{2}a_{\mu\nu}{}^{a}j^{\mu\nu}{}_{a}\right)$$

- **Same continuity equation as Pai and Pretko**

$$\partial_\alpha \rho^{\alpha ab} + \partial_c\partial_d \mathcal{J}^{acbd} = 0, \qquad \mu = (\alpha, a)$$

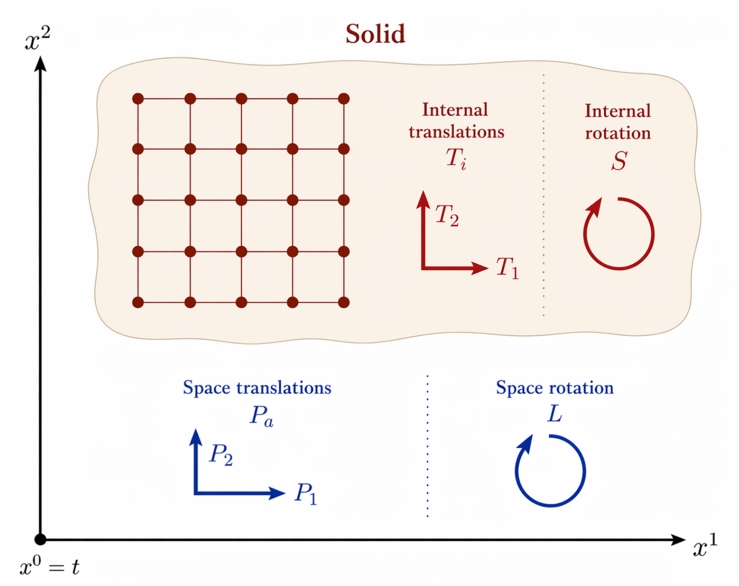
- **Currents are related as**

$$\rho^{\alpha ab} = j^{\alpha ab} + \frac{1}{2}\partial_c\left(J^{\alpha cab} - J^{\alpha acb} - J^{\alpha bca}\right)$$

$$\mathcal{J}^{abcd} = -\frac{1}{2}\left(J^{abcd} + J^{cdab}\right), \qquad \mathcal{J}^{abcd} = \mathcal{J}^{cdab}$$

Future directions

- **Generalization of the coset construction to obtain Cosserat elasticity**



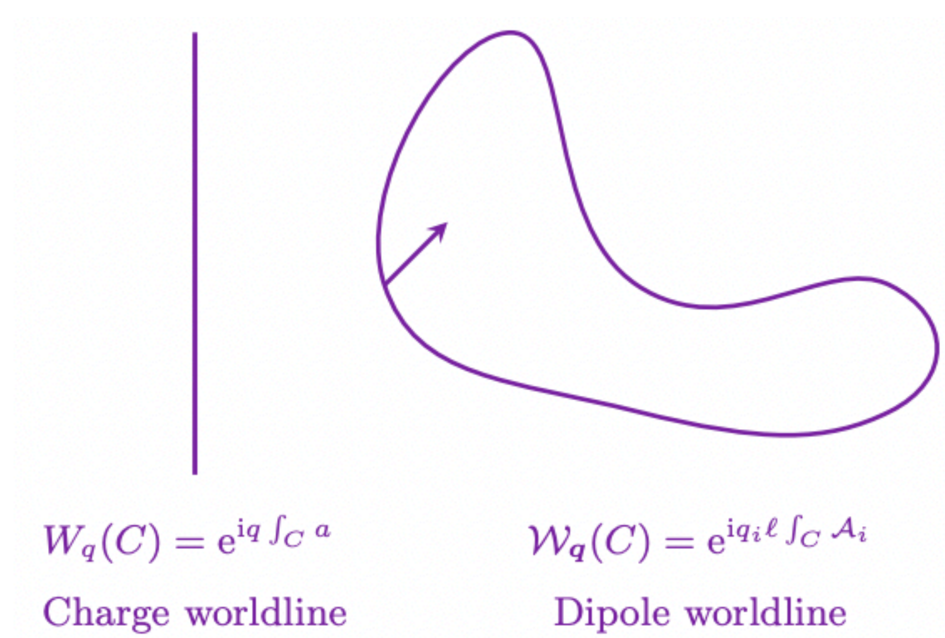
$$L \to L_{ab} \qquad [L_{ab}, L_{cd}] = 4\eta_{[a[c}L_{d]b]} \qquad [S_{ij}, S_{kl}] = 4\eta_{[i[k}S_{l]j]}$$

$$S \to L_{ij} \qquad [L_{ab}, P_c] = 2\delta_{c[b}P_{a]} \qquad [S_{ij}, T_k] = 2\delta_{k[j}T_{i]}$$

$$\Omega = g^{-1}dg \qquad g(x,u,\theta) = e^{x^a R_a} e^{u^i(x)T_i} e^{\theta^{ij}(x)S_{ij}}$$

$$S = \int dt\, d^3x \left[\frac{1}{2}A_{ij}\,\Omega_t{}^i \Omega_t{}^j - \frac{1}{2}K^{ai,bj}\,\Omega_{ai}\Omega_{bj} + \frac{1}{2}I_{ij,kl}\,\Omega_t{}^{ij}\Omega_t{}^{kl} - \frac{1}{2}\kappa^{aij,bkl}\,\Omega_a{}^{ij}\Omega_b{}^{kl} \right]$$

- **Higher-form symmetries**
  [Hirono, You, Angus, Cho (2022)]



- **Coupling to curved space (stringy Aristotelian geometry)**

- **Define string-like actions for line-like fractions that couple to the two-form gauge fields**

Summary

- Fracton are quasiparticles with restricted mobility that couple to higher-rank gauge theories
- The isomorphism between the Carroll algebra and the dipole preserving symmetry allows one to find a fracton gauge theory from an action for Carroll gravity
- The result is equivalent to the action for fractons that is obtained from elasticity in 2+1 dimensions
- In 3+1 dimensions elasticity leads to line-like fractons, which should be related to some sort of two-form Aristotelian gauge theory
- The Carroll/fracton isomorphism, can be extended to find an algebra for a vector charge and a dipole tensor that fits this description
- Gauging such symmetry in terms of a one-form and a two-form field allows to find a higher-rank gauge theory that reproduces the continuity equation associated to line-like fractons
- To be shown: By imposing a suitable constraint ad a gauge fixing condition, the action reduces exactly to the action by Pretko and Pai
- To be shown: The two-form gauge field action can be reproduced using the coset construction and dualization techniques

Thank You!