Symmetry Tools for Soft Theorems

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Based on arXiv:2604.06088

Introduction

The understanding of symmetries in QFT has undergone many recent developments:

- symmetries in QFT $=$ extended topological operators [Gaiotto-Kapustin-Seiberg-Willett 2014]

$$U(\Sigma_{d-(p+1)}) \quad (\Sigma\text{-codimension } p+1 \text{ closed manifold})$$

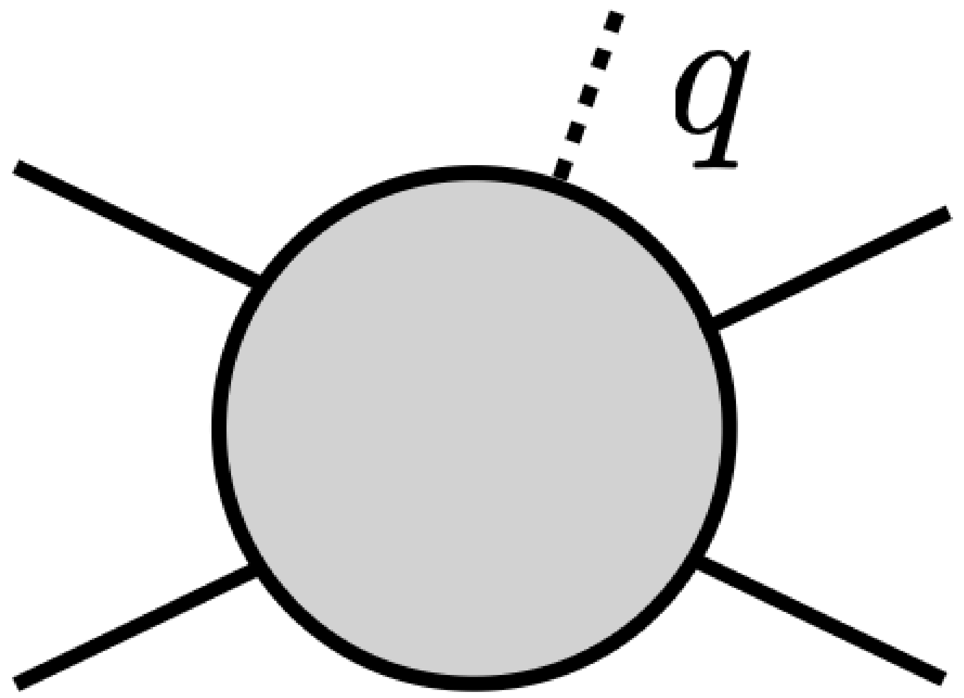
- $U(\Sigma)$ acts on extended objects of dimension p (like particle, worldlines,etc..)

- Fusion of multiple $U(\Sigma)$ can be richer than an ordinary group

However, these ideas remain poorly explored in Lorentzian signature, where many observables of interest, especially for particle physics, are defined.

Soft Theorems

The behavior of scattering amplitudes when the momentum of a particle is small is often universal:



$$\lim_{q \to 0} \mathscr{A}_{n+1} = S \mathscr{A}_n$$

$$\lim_{q_\pi \to 0} \mathscr{A}_{n+1} = 0$$

[Adler 60's]

$$\lim_{q \to 0} \mathscr{A}_{n+\gamma_\varepsilon(q)} = e \sum_{a=1}^{n} Q_a \frac{\varepsilon \cdot v_a}{q \cdot v_a} \mathscr{A}_n$$

[Low; Burnett-Kroll; Weinberg 60's]

Soft Theorems

Traditionally, these results are understood from many different perspectives:

- Diagrammatically
- EFTs of soft modes → Factorization
- Asymptotic + Non-linearly realized symmetries

Interesting to explore connections between these ideas, in particular:

What is the role of generalized global symmetries?

[Berean-Dutcher-Derda-Parra-Martinez 2025]

Setup

We will focus on QED:

$$\mathscr{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$$

Here Ψ is a charge Q Dirac fermion. We will consider both the massive and the massless limit. For any value of the mass the theory has a magnetic 1-form symmetry $U(1)^{(1)}_m$:

$$J^{(2)}_m = \frac{1}{2\pi} * F \ , \quad G = \int_{\Sigma_2} * J^{(2)}_m \in \mathbb{Z}$$

Only broken by dynamical magnetic monopoles. Charged objects are ’t Hooft lines.

When $m = 0$, there is an emergent non-invertible axial symmetry.

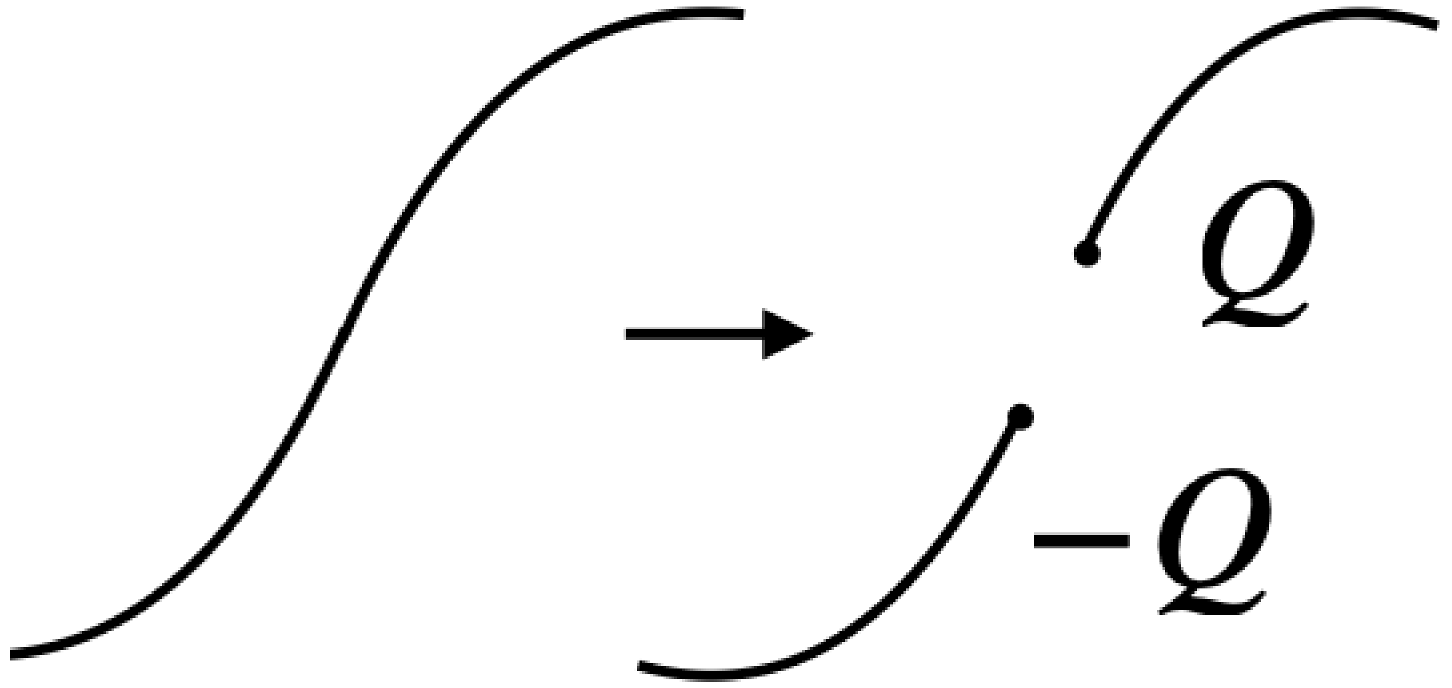
Pair Production

Natural question: What about the electric 1-form symmetry $U(1)_e^{(1)}$?

$$J_e^{(2)} = \frac{1}{e^2} F \ , \quad Q = \int_{\Sigma_2} * J_e^{(2)}$$

Unfortunately, in QED this symmetry is broken by charged matter:

$$\partial_\mu J^{\mu\nu} = j_M^\nu$$ $\leftrightarrow$ Wilson lines are screened



In QED pair production allows electric flux lines to end on charged particles.

HQET

To remedy for this problem, we use an EFT approach. In the massive case consider a regime where Ψ has momentum:

$$p^\mu = mv^\mu + k^\mu \ , \quad v^2 = 1 \ , \quad |k^\mu| \ll m$$

k^μ encodes the small off-shell fluctuations of the heavy particle around its mass shell.

The field Ψ can be decomposed as:

$$\Psi(x) = e^{-imv\cdot x}(h(x) + H(x))$$

$$h(x) \equiv P_+\Psi(x) \ , \qquad H(x) = P_-\Psi(x) \ , \quad P_\pm \equiv \frac{1}{2}(1 + \gamma^\mu v_\mu)$$

The heavy anti-particle field $H(x)$ has mass gap $2m$ and can be integrated out

HQET

This limit is described by the heavy quark effective theory: [Eichten-Hill;Georgi; Isgur-Wise 90]

$$\mathscr{L}_{\text{HQET}} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + i\bar{h}_v\,(v\cdot D)\,h_v + \frac{i}{2m}\bar{h}_v(D_\perp)^2 h_v + \frac{e}{4m}\bar{h}_v\sigma^{\mu\nu}F_{\mu\nu}h_v + \cdots$$

Soft photons see only a charge moving with fixed velocity v^μ while recoil and spin enter at $\mathcal{O}(1/m)$.

Side comment: A velocity change $v \to v'$ probe the cusp anomalous dimension

[Korchemsky-Radyushkin 92]

$$\Gamma_{cusp}(\gamma)\;, \quad \cosh\gamma = v\cdot v'$$

SCET

For the massless theory, we can repeat similar steps. We consider lightlike charged legs along trajectories n^μ , $n^2 = 0$

$$p^\mu = \frac{\bar{n}\cdot p}{2} n^\mu + \frac{n \cdot p}{2} \bar{n}^\mu + p_\perp^\mu \,, \qquad \begin{array}{l} n\cdot\bar{n} = 2 \\ n\cdot p_\perp = \bar{n}\cdot p_\perp = 0 \end{array}$$

Physics in this regime is captured by soft collinear effective theory. Momentum modes split into usoft and collinear

[Bauer–Fleming–Luke–Pirjol–Stewart 00/01]

$$p_c^\mu \sim \mathcal{Q}(1,\lambda^2,\lambda), \qquad p_{us}^\mu \sim \mathcal{Q}(\lambda^2,\lambda^2,\lambda^2), \qquad \lambda \sim |p_\perp|/\mathcal{Q}\,.$$

EFT power counting in λ, for $\lambda \ll 1$, collinear particles have large energy and move inside a very narrow cone around the lightlike direction n^μ.

SCET

For each energetic massless leg moving along n^μ:

$$\Psi = \xi_n + \eta_n \,, \qquad \xi_n \equiv P_n \Psi \,, \qquad P_{\bar{n}} \Psi = \eta_n \,.$$

The mode η_n is off-shell and can be integrated out to obtain the leading-order Lagrangian:

$$\mathcal{L}^{(0)}_{\text{SCET}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\tilde{\xi}}_n \frac{\gamma^\mu \bar{n}_\mu}{2} \left(i\, n \cdot D \right) \tilde{\xi}_n + \mathcal{L}^{(0)}_{\perp}$$

The covariant derivative D retains both collinear A^c_μ and ultrasoft gauge field A^{us}_μ.

There is also a usoft component ψ_{us} , more on it later.

Change of Variables

For both EFTs there is a useful change of variables:

$$h_v = W_Q(C)\tilde{h}_v = \exp\left(iQ\int_0^\infty v\cdot A\right)\tilde{h}_v \qquad \xi_n = Y_Q(C)\tilde{\xi}_n = \exp\left(iQ\int_0^\infty n\cdot A^{us}\right)\tilde{\xi}_n$$

The Wilson lines $W_Q(C)$ and $Y_Q(C)$ describe probe electric charges moving timelike/lightlike trajectories.

This change of variables implies that:

- HQET/massive: fully decouples A_μ from heavy fields at leading power.
- SCET/massless: decouples A_μ^{us} from collinear fields at leading power, A_μ^c remains.

Change of Variables

Resulting EFT is only coupled to $F_{\mu\nu}$ or its line integrals along heavy worldlines.

→ Emergent electric $U(1)_e^{(1)}$ 1-form symmetry

$$A^{(1)} \to A^{(1)} + \Lambda_e^{(1)} \,, \quad \int_{\Sigma_2} d\Lambda_e^{(1)} \in 2\pi\mathbb{Z}$$

Massless QED needs further caution since even restricting to usoft sector:

- Away from tree level, vacuum polarization by ψ_{us}, need to consider $J_{eff}^{(2)}$
- More dangerously, there can be connected ψ_{us} webs in which the emitted soft photon attaches directly to a virtual massless charged loop

[Ma-Sterman-Venkata 2023]

$$\Delta S \sim \alpha^3 S_{tree}$$

Factorization

This formulation illuminates the idea that the QED scattering amplitude at leading power schematically factorizes as

$$\mathscr{A}_n \sim \left\langle \prod_{i \in massless} Y_{Q_i}(n_i) \prod_{a \in massive} W_{Q_a}(v_a)\ \mathcal{O}_H \right\rangle$$

Where $Y_{Q_i}(n_i)$, $W_{Q_a}(v_a)$ lightlike/timelike Wilson lines and $\mathcal{O}_H$ is a local “hard” operator.

IR dependence is fully encoded in the soft function obtained as a correlation function of Wilson lines.

1-Form Ward Identity

A $U(1)^{(1)}$ global symmetry gives rise to a 1-form symmetry Ward identity:

$$\left\langle d * J^{(2)}(x) \prod_{i}^{n} L_{Q_i}(C_i) \mathcal{O}_H \right\rangle = \sum_{i}^{n} Q_i \delta^{(3)}(C_i) \left\langle \prod_{i} L_{Q_i}(C_i)\, \mathcal{O}_H \right\rangle$$

For an ordinary $U(1)^{(0)}$ global symmetry recall that:

$$\left\langle d * j^{(1)}(x) \prod_{i} \mathcal{O}_{q_i}(x_i)\, \mathcal{O}_H \right\rangle = \sum_{i} q_i\, \delta^{(4)}(x - x_i) \left\langle \prod_{i} \mathcal{O}_{q_i}(x_i)\, \mathcal{O}_H \right\rangle$$

Broken 1-Form Symmetry and Mixed Anomaly

As with any ordinary global symmetry, 1-form symmetries can be spontaneously broken.

The Goldstone theorem for $U(1)^{(1)}$ implies that:

$$\langle \gamma_\varepsilon(q) \,|\, J_{\mu\nu}(x) \rangle = \frac{i}{e}(q_\mu \varepsilon^*_\nu - q_\nu \varepsilon^*_\mu) e^{iq\cdot x}$$

Photon is the NG of $U(1)^{(1)}$ and its shift symmetry is $A^{(1)} \to A^{(1)} + \Lambda^{(1)}$

Moreover these EFTs have a mixed 't Hooft anomaly between $U(1)^{(1)}_e$ and $U(1)^{(1)}_m$:

$$B^{(2)}_{e,m} \to B^{(2)}_{e,m} + d\Lambda^{(1)}_{e,m} \qquad\qquad \delta S_{EFT}[A^{(1)}, B^{(2)}_e, B^{(2)}_m] = \frac{1}{2\pi}\int B^{(2)}_m \wedge d\Lambda^{(1)}_e$$

Infinite Dimensional Symmetry

The EFTs we described in fact have a larger set of unusual symmetries. To see this, we use the construction by [Hofman-Iqbal 2018]

$$P_\pm = \frac{1}{2}(1 \pm * \, i) \qquad \mathcal{J}^{(2)} \equiv \frac{1}{e^2}F^{(2)} + \frac{i}{2\pi} * F^{(2)} \qquad j_+^{(2)} \equiv P_+ \mathcal{J}^{(2)} \; , \quad j_-^{(2)} \equiv P_- \mathcal{J}^{(2)}$$

One further introduces chiral/anti-chiral 1-forms $\Xi_\pm^{(1)}$ such that:

$$P_+ d\Xi_-^{(1)} = 0 \; , \quad P_- d\Xi_+^{(1)} = 0$$

This allows us to define new conserved charges on codimension 1 hypersurfaces:

$$\bar{Q}_{\Xi_+} = \int_{\Sigma_3} j_- \wedge \Xi_+ \; , \quad Q_{\Xi_-} = \int_{\Sigma_3} j_+ \wedge \Xi_- \qquad [\bar{Q}, Q] = \frac{1}{2\pi e^2} \int_{\partial \Sigma_3} d\Xi_+ \wedge d\Xi_-$$

Infinite-dimensional symmetry, similar to $U(1)^{(0)}$ Kac-Moody but in d=4.

Recovering the Soft Theorem

We exploit the causal structure of Minkowski and consider again the 1-form Ward identity:

$$\int_{\mathcal{I}^+} \epsilon(z,\bar{z}) \left\langle d * J^{(2)}(x) \prod_i^n L_{Q_i}(C_i) \mathcal{O}_H \right\rangle = \int_{\mathcal{I}^+} \epsilon(z,\bar{z}) \sum_i^n Q_i \delta^{(3)}(C_i) \left\langle \prod_i L_{Q_i}(C_i)\, \mathcal{O}_H \right\rangle$$

On the LHS we split $J^{(2)} = \frac{1}{e^2} * F + J_H^{(2)}$ and obtain:

$$\frac{1}{e^2} \int_{\mathcal{I}^+} \left\langle d\epsilon(z,\bar{z}) \wedge {}^*F \prod_i^n L_{Q_i}(C_i) \mathcal{O}_H \right\rangle + \mathcal{R}_H = \langle Q_\epsilon^+ \prod_i L_{Q_i}(C_i) \mathcal{O}_H \rangle + \mathcal{R}_H$$

Choosing a suitable Ξ^+ it is easy to see that Q_ϵ^+ is a special case of the charges we just discussed. These are the charges discussed in asymptotic symmetry literature!

[Strominger 2013]

Recovering the Soft Theorem

On the RHS we just compute the contact term by evaluating the delta function:

$$\sum_i^n Q_i \int_{\mathcal{I}^+} \epsilon(z,\bar{z})\delta^{(3)}(C_i) = \sum_i^n Q_i \epsilon(z_i,\bar{z}_i)$$

We can also consider a more general hypersurface that includes timelike infinity i^+ to capture massive worldlines.

Putting everything together we have:

$$\langle Q_\epsilon^+ \prod_i L_{Q_i}(C_i)\mathcal{O}_H\rangle + \mathcal{R}_H = \sum_i^n Q_i \epsilon(z_i,\bar{z}_i)\left\langle \prod_i L_{Q_i}(C_i)\,\mathcal{O} \right\rangle$$

At leading order in the soft limit $q \to 0$, the contribution from $\mathcal{R}_H$ is $\mathcal{O}(q^0)$.

Recovering the Soft Theorem

It follows from factorization and a choice of $\epsilon(z, \bar{z})$ to isolate poles on the celestial sphere, that:

$$Q_\epsilon^+ \mathcal{A}_n = \left(\sum_{i \in \text{massless}} \frac{Q_i}{w - z_i} + \sum_{a \in \text{massive}} \frac{Q_a}{w - \widehat{v}_a} \right) \mathcal{A}_n$$

Writing $\mathscr{A}_n$ in terms of $\mathscr{S}$ matrix and in/out asymptotic states one has:

$$\langle \text{out}| \left(Q_\epsilon^+ \mathcal{S} - \mathcal{S} Q_\epsilon^- \right) |\text{in}\rangle =$$
$$\left[\sum_{i \in \text{massless,in}} \frac{Q_i^{\text{in}}}{w - z_i^{\text{in}}} + \sum_{a \in \text{massive,in}} \frac{Q_a^{\text{in}}}{w - \widehat{v}_a^{\text{in}}} - \sum_{j \in \text{massless,out}} \frac{Q_j^{\text{out}}}{w - z_j^{\text{out}}} - \sum_{b \in \text{massive,out}} \frac{Q_b^{\text{out}}}{w - \widehat{v}_b^{\text{out}}} \right] \langle \text{out}|\mathcal{S}|\text{in}\rangle$$

Here Q_ϵ^- is the charge obtained by repeating the same construction at $\mathscr{I}^-$.

Recovering the Soft Theorem

- With a suitable parametrization of a soft photon we can obtain the textbook formula:

$$\lim_{q\to 0} \mathscr{A}_{n+\gamma_\varepsilon(q)} = e\sum_{a=1}^{n} Q_a \frac{\varepsilon\cdot v_a}{q\cdot v_a}\mathscr{A}_n + \mathcal{O}(q^0)\,.$$

- Same Ward identity argument works for magnetic 1-form symmetry, leading to:

$$\lim_{q\to 0} \mathscr{A}_{n+\tilde{\gamma}_{\tilde{\varepsilon}(q)}} = \frac{2\pi}{e}\sum_{a=1}^{n} G_a \frac{\tilde{\varepsilon}\cdot v_a}{q\cdot v_a}\mathscr{A}_n + \mathcal{O}(q^0)\,.$$

- Electric and magnetic soft theorems are Ward identities of $U(1)^{(1)}_e, U(1)^{(1)}_m$

Mixed Anomaly Contact Term

Consider the ordered soft limit $q_1 \to 0$ followed by $q_2 \to 0$:

$$\mathscr{A}_{n+2} = S_e(q_1)S_m(q_2)\mathscr{A}_n + R(q_1, q_2)\mathscr{A}_n + \mathcal{O}(q_1^0, q_2^0)$$

$R(q_1, q_2)$ is scalar distribution in the soft variables such that:

- It is gauge invariant.
- It has no extra $(p_i \cdot q_a)^{-1}$ poles.
- It is supported where the charge algebra is anomalous: $[Q_e(w_1), Q_m(w_2)] = \frac{2\pi}{e^2}\delta^{(2)}(w_1 - w_2)$

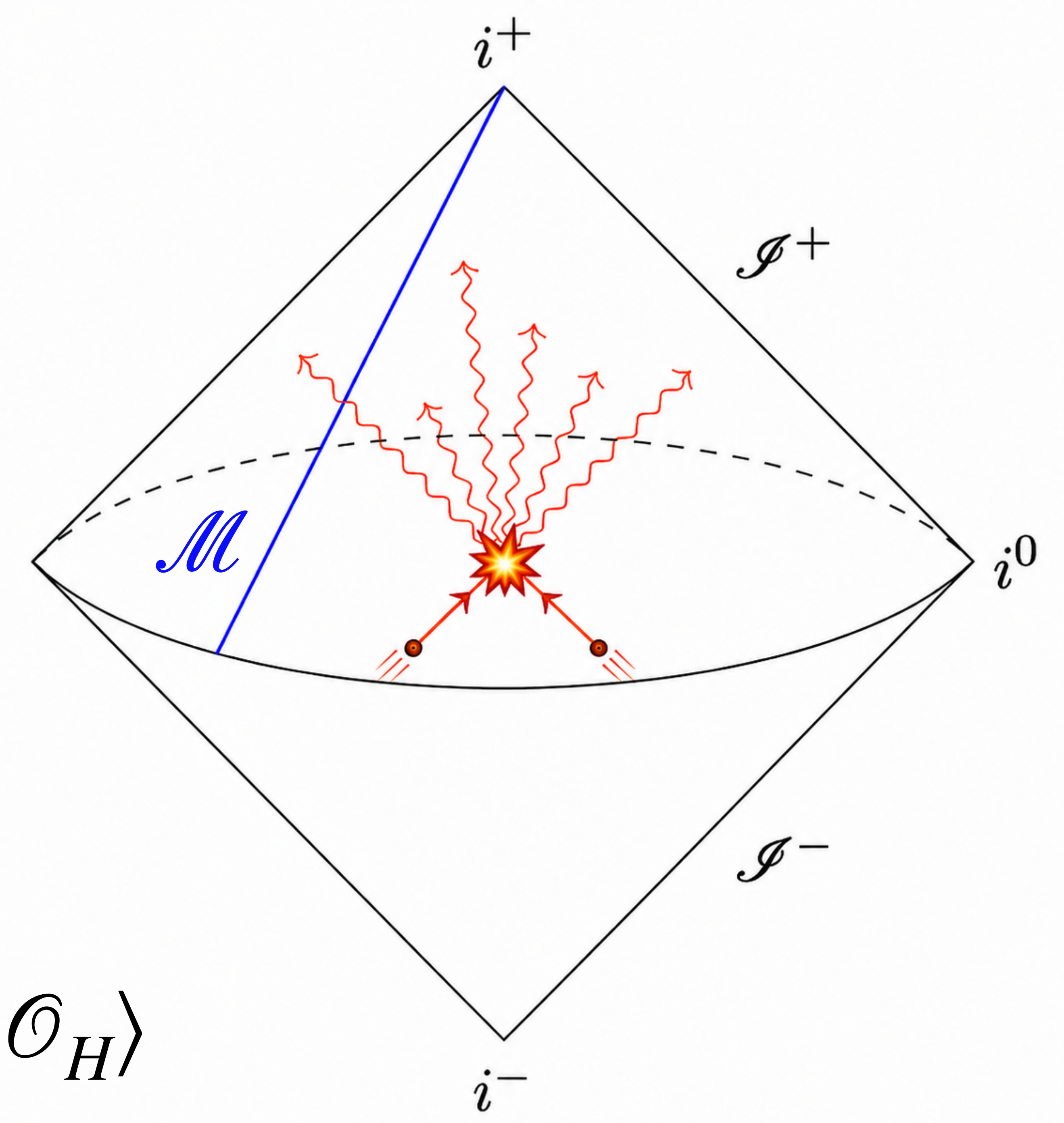
$$\Rightarrow R(q_1, q_2) = \frac{2\pi}{e^2}\varepsilon_1 \cdot \tilde{\varepsilon}_2 \, \delta^{(2)}(\hat{q}_1 - \hat{q}_2)$$

Mixed 1-form anomaly fixes the contact term appearing in the ordered limit.

Beyond Amplitudes: Inclusive Observables

Consider a measurement/"detector" operator $\mathscr{M}$ at $\mathscr{I}^+$:

$$\langle \mathscr{M} \rangle_{in} \equiv \langle in \,|\, \mathcal{S}^\dagger \mathscr{M} \mathcal{S} \,|\, in \rangle$$



We can similarly derive an "in-in" Ward identity, at leading order we have:

$$\langle in \,|\, \mathcal{S}^\dagger [Q_\epsilon^+, \mathscr{M}] \mathcal{S} \,|\, in \rangle = \langle \mathcal{O}_H^\dagger (\mathcal{O}_{soft}^\dagger \, [Q_\epsilon^+, \mathscr{M}] \mathcal{O}_{soft}) \mathcal{O}_H \rangle$$

The soft theorem now becomes a Ward identity for how the detector operator transforms under the asymptotic charge.

Beyond Amplitudes: Inclusive Observables

Two detector choices, their Ward identities imply:

$$\mathscr{M} = \mathscr{N}(\omega, z, \bar{z}) \equiv \sum_{h=\pm} a_h^\dagger(\omega, z, \bar{z}) a_h(\omega, z, \bar{z}) \quad \Rightarrow \quad \text{Soft theorem for cross sections.}$$

$$\mathscr{M} = \mathscr{D}_{J_L}(z, \bar{z}) \equiv \sum_{h=\pm} \int \frac{d\omega}{(2\pi)^3 2\omega} \omega^{-J_L} \, a_h^\dagger(\omega, z, \bar{z}) a_h(\omega, z, \bar{z}) \quad \Rightarrow \quad \text{Universal pole at } J_L = -2$$

Detector Ward identities lead to universal statements about inclusive observables.

See connection with recent analysis in: [Chang, Chen, Simmons-Duffin, Zhu 25] [Gonzalez, Salzer 25] [Moult, Narayan, Pasterski 25]

Summary

- The soft sector of QED has an emergent 1-form symmetry which enhances to an infinite set of conserved charges.
- The scattering amplitude is recast as a correlation function of Wilson lines along worldline trajectories. Their Ward identity imply the soft photon theorem.
- This derivation bridges ideas from EFT, asymptotic and generalized symmetries.
- Similar methods can be applied to more general Lorentzian observables.

Outlook

- Scattering in the presence of monopoles and related puzzles?
- Gravity? Soft-Collinear Gravity? Does it have bi-form symmetries?

[Hinterbichler, Hofman, Joyce, Mathys 22]

- QCD?
- S-Matrix, asymptotic states??

Thanks!

Backup: Minkowski Structure