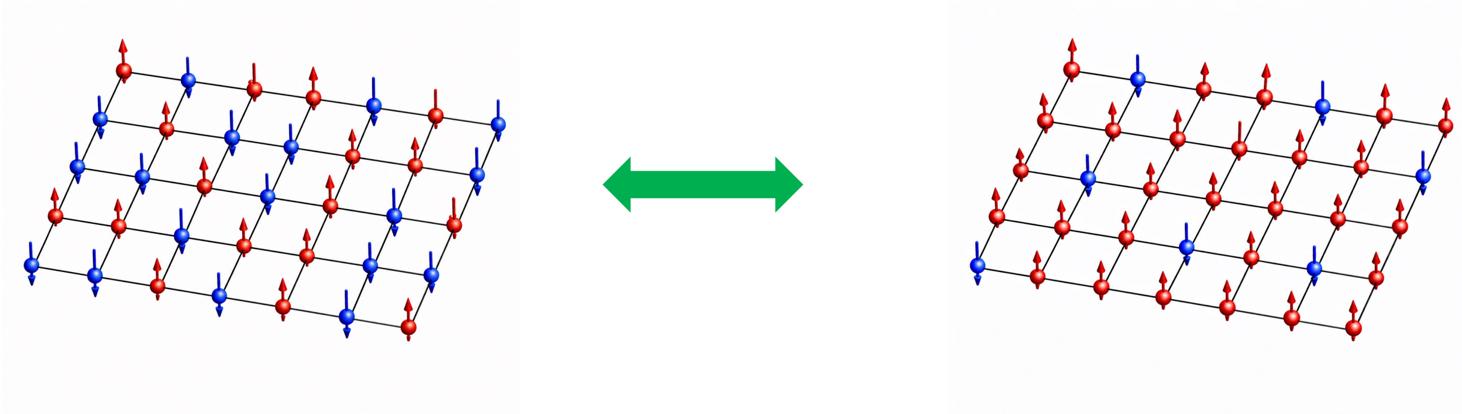
Kramers-Wannier duality, non-invertible symmetries and All That



Hua-Chen Zhang (Asia Pacific Center for Theoretical Physics, Pohang , Korea)

German Sierra (IFT UAM-CSIC, QuARC, Madrid, Spain)

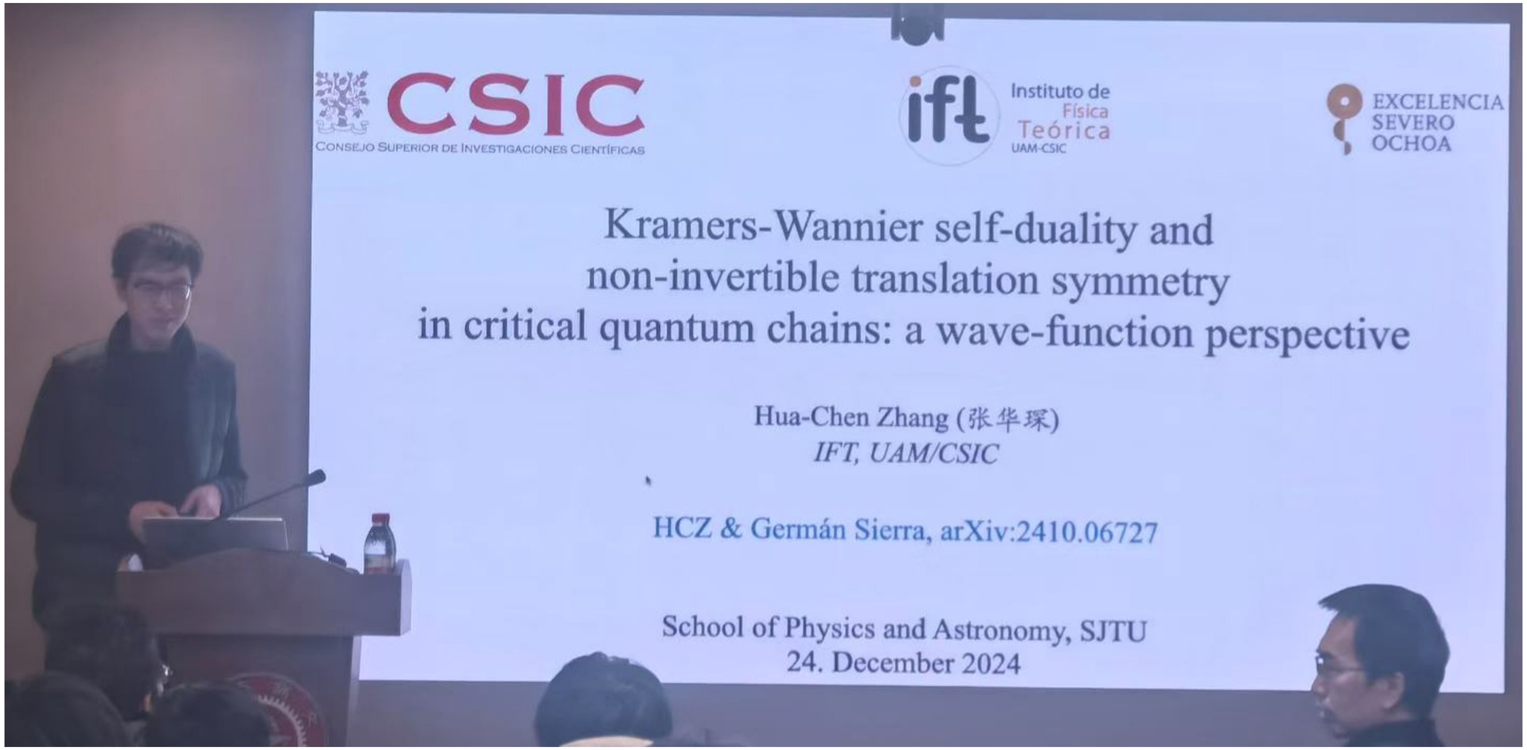
Developments on Generalized Symmetries of Quantum Systems,

Universidad de Murcia, Spain, 13 May, 2026

Talk based on

Kramers-Wannier self-duality and non-invertible translation symmetry in quantum chains: a wave-function perspective

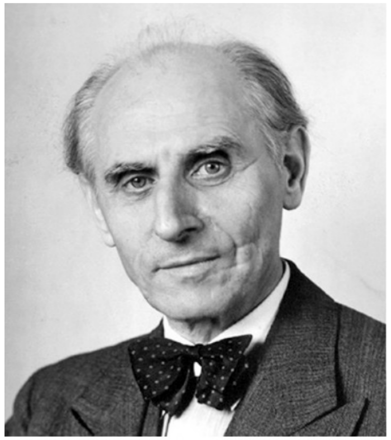
J. High Energ. Phys. 2025, 157 (2025)
Arxiv: 2410.0672



Hua-Chen Zhang and GS

Outline:

- Historical background
- Ising in a transverse field
- KW as a non-invertible symmetry
- Ising CFT
- KW in the continuum



Wilhelm Lenz (1988-1957)

Ising model in 1D

Beitrag zur Theorie des Ferromagnetismus [1]).

Von **Ernst Ising** in Hamburg.

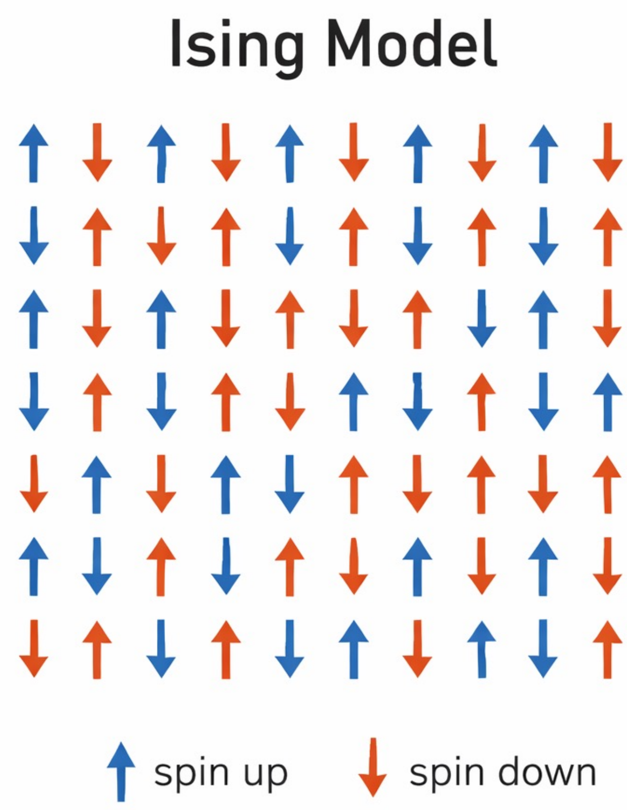
(Eingegangen am 9. Dezember 1924.)

Ernst Ising (1900-1998)

$$E = -J \sum_i S_i \, S_{i+1}, \qquad S_i = \pm 1$$

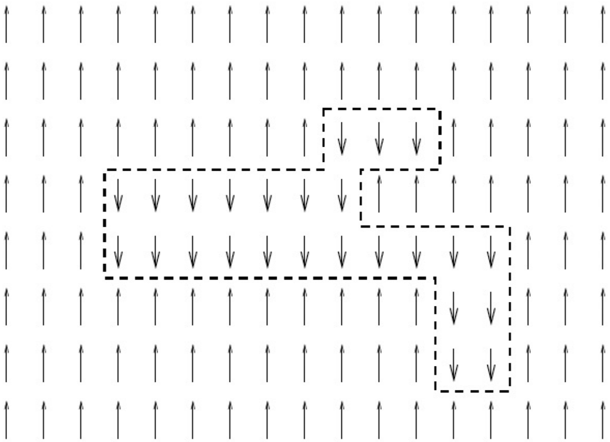
Conclusion: for all T >0 the system is disordered: No phase transition

Is there a pase transition in 2D ?



$$E = -\sum_{ij} J_{ij} S_i S_j$$

Rudolf Peierls
1907 - 1995



ON ISING'S MODEL OF FERROMAGNETISM

BY MR R. PEIERLS 1936

Energy cost to create a domain wall of length L

$$\Delta E = 2J\,L$$

Variation of entropy for creating the domain walls

$$\Delta S = k\,\ln(2^L)$$

Change of the free energy

$$\Delta F = \Delta E - T\Delta S = L(2J - kT\ln 2)$$

$$\Delta F \geq 0 \qquad T \leq T_c = \frac{2J}{k\ln 2} = 2.885\,\frac{J}{k}$$

H. A. Kramers
1911-1983

Statistics of the Two-Dimensional Ferromagnet. Part I

H. A. KRAMERS, *University of Leiden, Leiden, Holland*

AND

G. H. WANNIER, *University of Texas, Austin, Texas*[1]
(Received April 7, 1941)

$$H(\{\sigma\}) = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j, \qquad J > 0,$$

Duality between hight T and low T

$$\sinh(2K)\ \sinh(2K^*) = 1$$

$$K = \beta J = \frac{J}{k_B T}$$

Kramers-Wannier duality

$$Z(K) = \sum_{\{\sigma\}} \exp\left(K \sum_{\langle ij \rangle} \sigma_i \sigma_j \right)$$

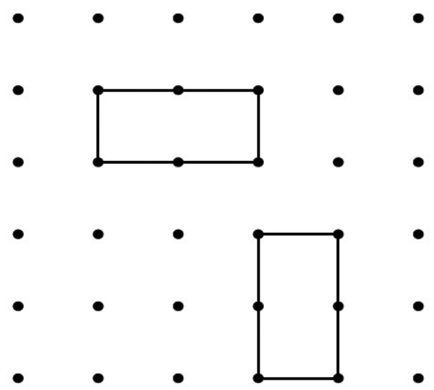
High-T expansion

$$Z(K) = 2^{N_s} (\cosh K)^{N_b} \sum_{\Gamma \text{ closed loops}} v^{|\Gamma|}$$

N_s # sites

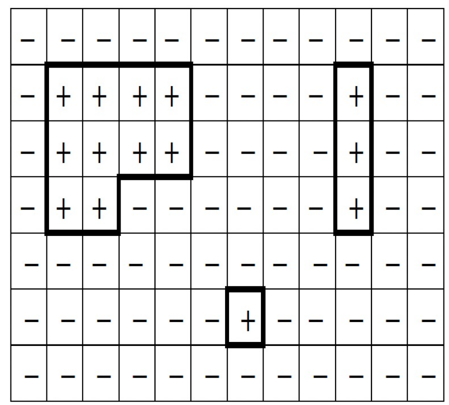
N_b # bonds

$v = \tanh K$



Low-T expansion

$$Z^*(K^*) = 2\, e^{K^* N_b^*} \sum_{\substack{\Gamma \text{ closed loops} \\ \text{In dual lattice}}} e^{-2K^*|\Gamma|}.$$



K-W duality

$$\tanh K = e^{-2K^*} \quad \Longleftrightarrow \quad \sinh(2K)\,\sinh(2K^*) = 1$$

$$\frac{Z(K)}{2^{N_s}\,(\cosh K)^{N_b}} = \frac{Z^*(K^*)}{2e^{K^* N_b}}$$

Self dual point $\quad K_c = K = K^* \qquad \sinh(2K_c) = 1$

$$K_c = \frac{1}{2}\ln(1+\sqrt{2})$$

$$T_c = \frac{1}{K_c}\frac{J}{k_B} = 2.2691\frac{J}{k_B}$$

Peierls : $T_c = 2.885\dfrac{J}{k_B}$

Note.- $$2\,N_s = N_b$$

At the critical point $$\frac{Z(K)}{(2\,cosh^2K)^{N_b/2}} = \frac{Z^*(K)}{2(e^{2K})^{N_b/2}}$$

$$2\,cosh^2K = e^{2K} = 1 + \sqrt{2}$$

Extra factor 2

Disorder phase is not degenerate

Order phase is doubly degenerate

Stat Mech signature of a non-invertible symmetry

Lars Onsager
1903 - 1976

PHYSICAL REVIEW VOLUME 65, NUMBERS 3 AND 4 FEBRUARY 1 AND 15, 1944

Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition

LARS ONSAGER
Sterling Chemistry Laboratory, Yale University, New Haven, Connecticut

(Received October 4, 1943)

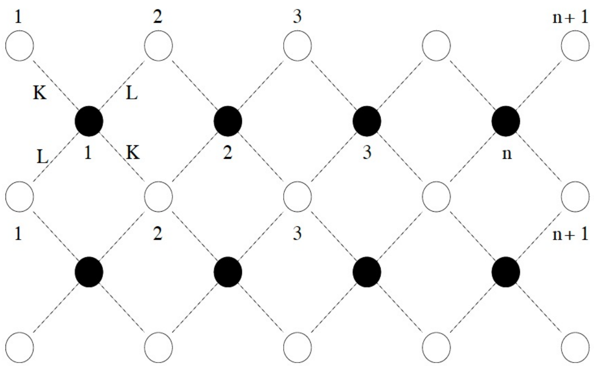
The partition function of a two-dimensional "ferromagnetic" with scalar "spins" (Ising model) is computed rigorously for the case of vanishing field. The eigenwert problem involved in the corresponding computation for a long strip crystal of finite width (n atoms), joined straight to itself around a cylinder, is solved by direct product decomposition; in the special case $n=\infty$ an integral replaces a sum. The choice of different interaction energies ($\pm J, \pm J'$) in the (0 1) and (1 0) directions does not complicate the problem. The two-way infinite crystal has an order-disorder transition at a temperature $T=T_c$ given by the condition

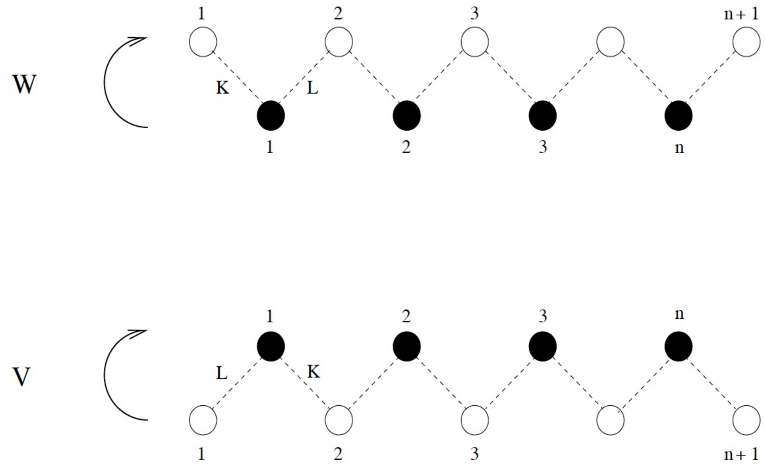
$$\sinh(2J/kT_c)\sinh(2J'/kT_c)=1.$$

The energy is a continuous function of T; but the specific heat becomes infinite as $-\log|T-T_c|$. For strips of finite width, the maximum of the specific heat increases linearly with $\log n$. The order-converting dual transformation invented by Kramers and Wannier effects a simple automorphism of the basis of the quaternion algebra which is natural to the problem in hand. In addition to the thermodynamic properties of the massive crystal, the free energy of a (0 1) boundary between areas of opposite order is computed; on this basis the mean ordered length of a strip crystal is

$$(\exp(2J/kT)\tanh(2J'/kT))^n.$$

Exact solution à la Baxter // integrability



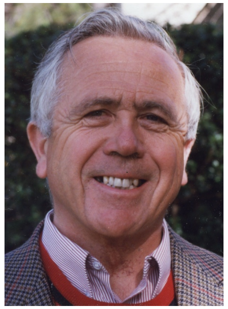


$$V_{\mu,\mu'}(K,L) \;=\; \exp\left[\sum_{i=1}^{n}(K\,\sigma_{i+1}\,\sigma'_i + L\,\sigma_i\,\sigma'_i)\right]$$

$$Z_N(K,L) \;=\; \mathrm{Tr}\,(VWVW\ldots VW) \;=\; \mathrm{Tr}\,(VW)^{m/2}$$

$$W_{\mu,\mu'}(K,L) \;=\; \exp\left[\sum_{i=1}^{n}(K\,\sigma_i\,\sigma'_i + L\,\sigma_i\,\sigma'_{i+1})\right]$$

C N Yang
1922-2025

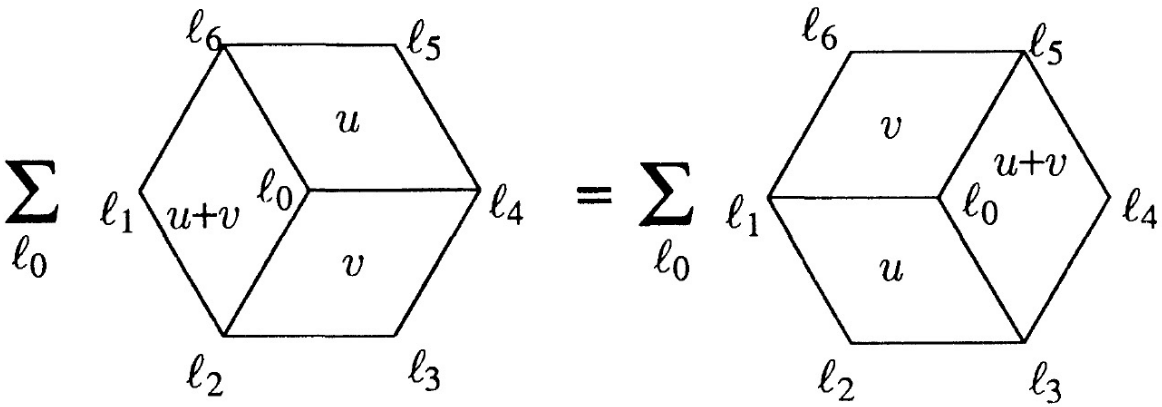


R J Baxter
1940-2025

$$[T_D(K,L), T_D(K',L')] = 0$$

$$\frac{1}{k} = \sinh 2K \, \sinh 2L = \sinh 2K' \, \sinh 2L'$$

Star triangle relation (Yang-Baxter eq)



Parametrization of the Boltzmann weights

$$e^{\pm 2K} = cn(iu) \mp isn(iu)$$

$$e^{\pm 2L} = i\, k^{-1}(dn(iu) \pm 1)/sn(iu)$$

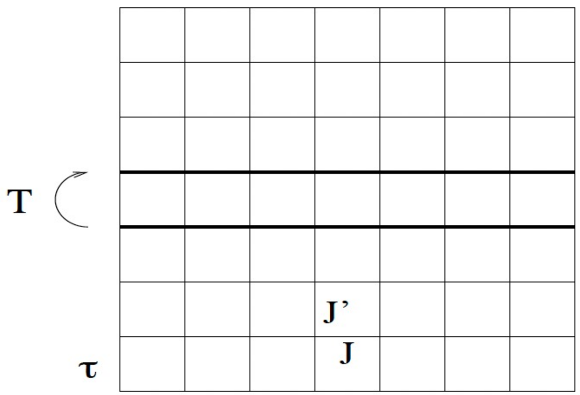
$$k < 1 \quad T < T_c$$
$$k = 1 \quad T = T_c$$
$$k > 1 \quad T > T_c$$

$$\frac{1}{k} = \sinh 2K \, \sinh 2L$$

In the critical case

$$\begin{aligned}
\exp(2K) &= (1 + \sin u)/\cos u \\
\exp(-2K) &= (1 - \sin u)/\cos u \\
\exp(2L) &= (1 + \cos u)/\sin u \\
\exp(2L) &= (1 - \cos u)/\sin u
\end{aligned}$$

Row-to-row transfer matrix and the Hamiltonian



$$T = \prod_{n=1}^{L} e^{K\,\sigma_n^z\,\sigma_{n+1}^z}\, e^{L\,\sigma_n^x}$$

Anisotropic limit $\tau \to 0$ corresponds to $K \gg L$

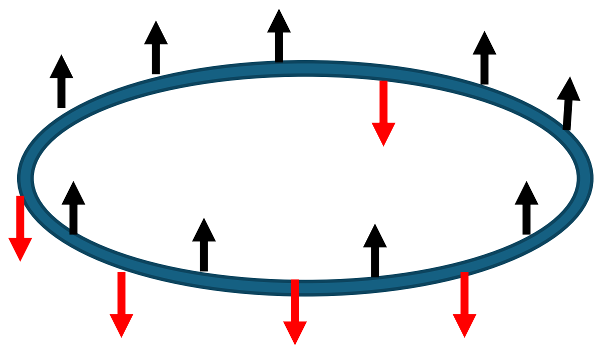
$$T \sim e^{-\tau H} \qquad H = -\sum_{n=1}^{L}(J\sigma_n^z\,\sigma_{n+1}^z + h\sigma_n^x)$$

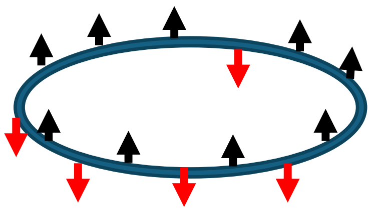
$$K \sim \tau J$$

$$e^{-2L} \sim \tau h$$

$$Ke^{2L} = \frac{J}{h} = \frac{1}{k}$$

Ising model in a tranverse field





Ising model in a tranverse field

$$H = -\sum_{j=1}^{N}(Z_j Z_{j+1} + g\,X_j),\qquad Z_j = Z_{j+N}\qquad X = X_{j+N}\qquad g = \frac{h}{J} = k$$

Symmetries:

Z_2 spin flip $\qquad \eta = \prod_{j=1}^{N} X_j \qquad [H,\eta] = 0 \qquad \eta^2 = 1 \qquad \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

Z_N Translation by one-site $\qquad T_{Ising} = \prod_{j=1}^{N-1} t_j \qquad [H, T_{Ising}] = 0 \qquad {T_{Ising}}^N = 1 \qquad \mathcal{H} = \oplus\, \mathcal{H}_k$

Permutation operator $\qquad t_j = \frac{1}{2}\left(X_j X_{j+1} + Y_j Y_{j+1} + Z_j Z_{j+1} + 1\right)$

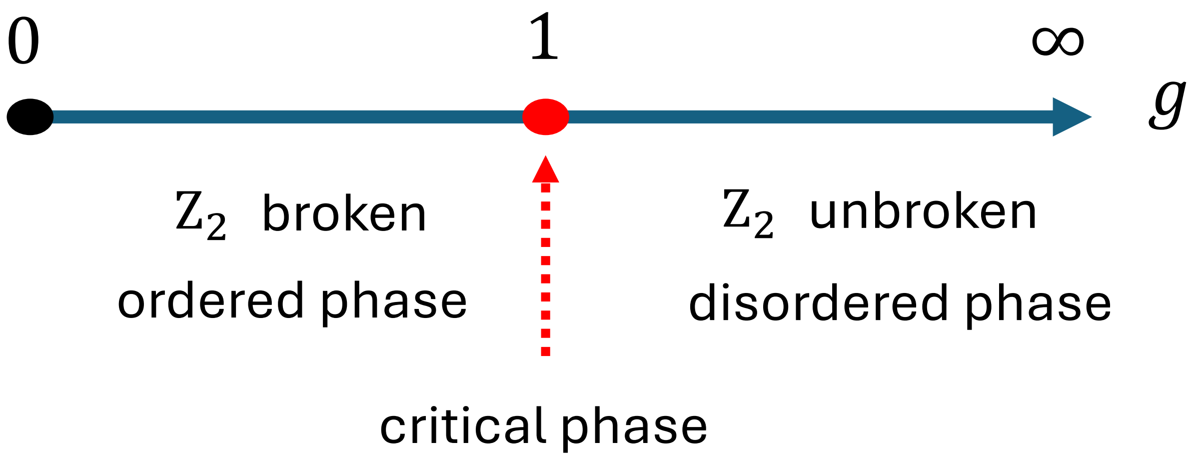
Phase diagram

Limit $g \to \infty$ $\quad |GS\rangle \to |+,+,\cdots,+\rangle,\quad |+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \quad X\,|\pm\rangle = \pm\,|\pm\rangle$

Limit $g \to 0$

$|GS_0\rangle \to |0,0,\cdots,0\rangle \quad Z\,|0\rangle = |0\rangle$

$|GS_1\rangle \to |1,1,\cdots,1\rangle \quad Z\,|1\rangle = -|1\rangle$



Kramers-Wannier transformation

$$X_j \to Z_j Z_{j+1}$$
$$Z_j Z_{j+1} \to X_{j+1}$$

$$H(g) = -\sum_{j=1}^{N} \left(Z_j Z_{j+1} + g\, X_j\right) \to \sum_{j=1}^{N} \left(X_{j+1} + g Z_j Z_{j+1}\right) = g\, H(1/g)$$

$H(g)$ is invariant at $g = 1$

Criticality : order $\Longleftrightarrow$ disorder

Kramers-Wannier transformation

$$X_j \to Z_j Z_{j+1}$$
$$Z_j Z_{j+1} \to X_{j+1}$$

$$\cancel{X_j \to Z_j Z_{j+1}}$$
$$\cancel{Z_j Z_{j+1} \to X_j}$$

$$H(g) = -\sum_{j=1}^{N} \left(Z_j Z_{j+1} + g\, X_j\right) \to -\sum_{j=1}^{N} \left(X_{j+1} + g Z_j Z_{j+1}\right) = g\, H(1/g)$$

$H(g)$ is invariant at $g = 1$

Criticality : order $\longleftrightarrow$ disorder

Is the KW transformation an invertible symmetry ?

Suppose $\exists\, U \; s.t. \quad U\, X_j\, U^{-1} = Z_j\, Z_{j+1}$

$$U\,\eta\,U^{-1} = U \prod_{j=1}^{N} X_j\, U^{-1} = \prod_{j=1}^{N} Z_j\, Z_{j+1} = 1 \quad \Rightarrow \quad \eta = 1$$

Refinement of the connection:

order $\Leftrightarrow$ disorder

$$D : \mathcal{H}_+ \leftrightarrow \mathcal{H}_+$$

$$D : \mathcal{H}_- \rightarrow 0$$

The KW transformation as a non-invertible symmetry ?

Symmetry à la Wigner $U\,H\,U^{\dagger} = H \xrightarrow{U \text{ unitary}} U\,H = H\,U$

Relaxing unitarity/invertibility $VH = H\,V \quad \nexists\, V^{-1}$

$$\begin{aligned} X_j &\to Z_j Z_{j+1} \\ Z_j Z_{j+1} &\to X_{j+1} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} D X_j &= Z_j Z_{j+1} D \\ D\, Z_j Z_{j+1} &= X_{j+1}\, D \end{aligned} \quad \Longrightarrow \quad D\,H = H\,D$$

The KW transformation as a half-step translation

$$D^2 X_j = D\, Z_j Z_{j+1} D = X_{j+1} D^2$$

$$D^2\, Z_j Z_{j+1} = D X_{j+1}\, D = Z_{j+1}\, Z_{j+2}\, D^2$$

$$\Rightarrow$$

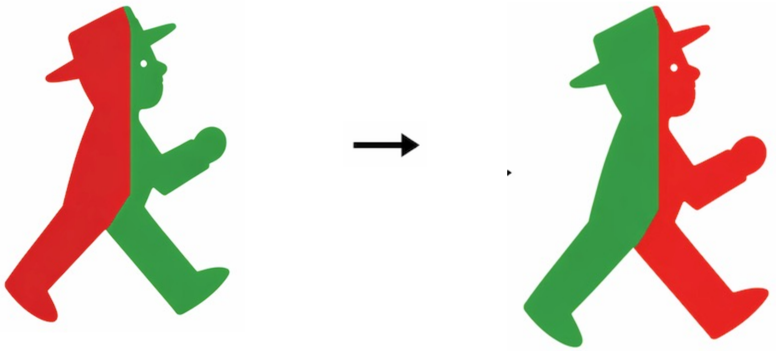
$$D^2 = \frac{1}{2}\,(1+\eta)\, T_{Ising}$$

$$D^2 = T_{Ising} \qquad \eta = 1$$

$$D^2 = 0 \qquad \eta = -1$$

Ampelmannchen step and half-step

Berlin’s traffic light



Ampelmannchen step and half-step

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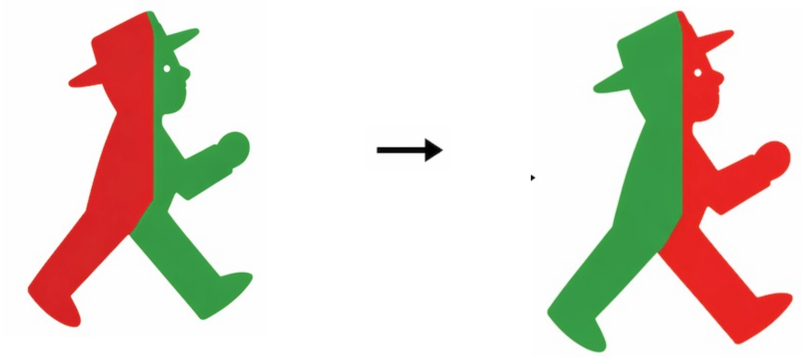
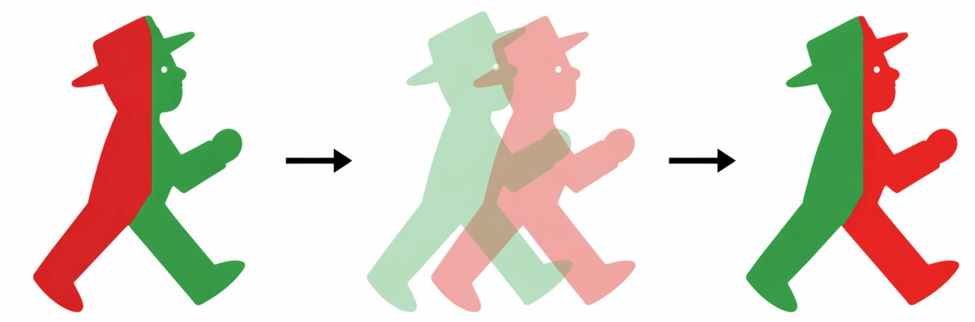
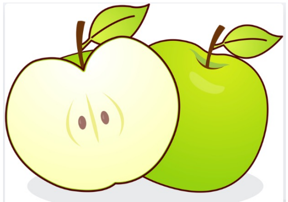
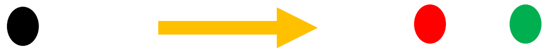




Fractionalization/unfolding of the spin degree of freedom



$$|\uparrow\rangle = |0\rangle$$
$$|\downarrow\rangle = |1\rangle$$



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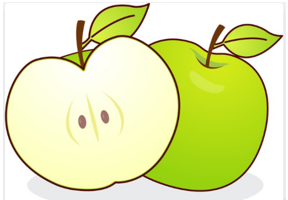

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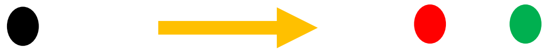

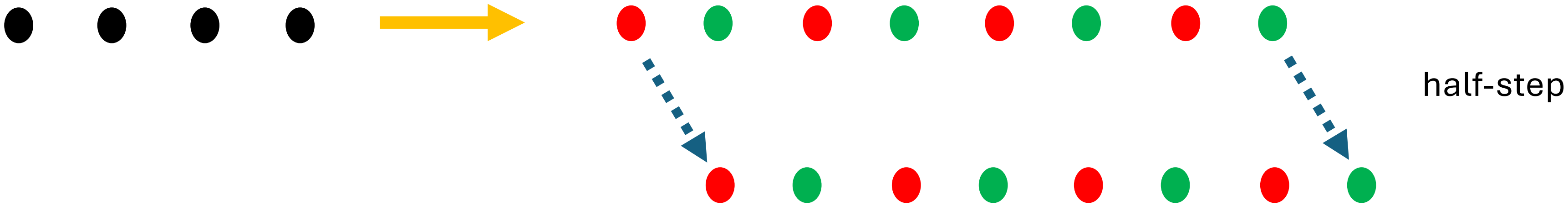

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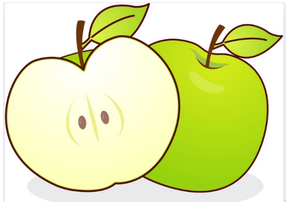

$$|\uparrow\rangle = |0\rangle$$
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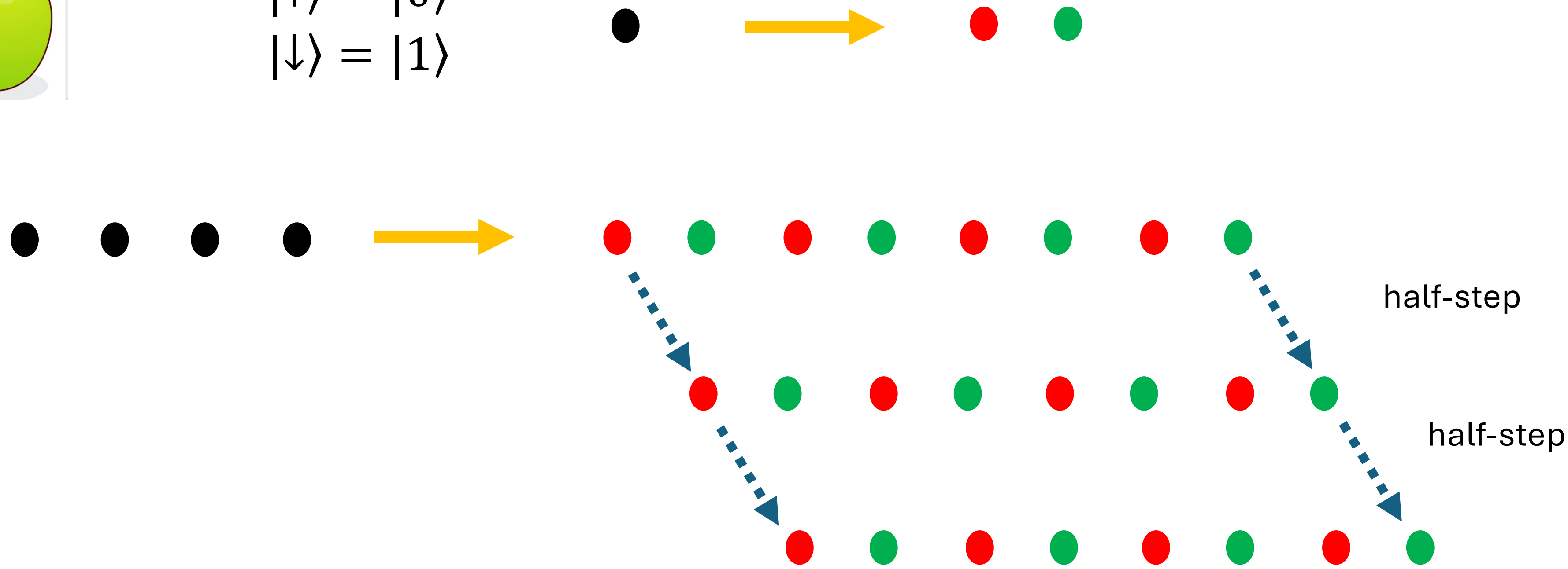
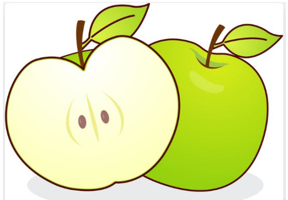
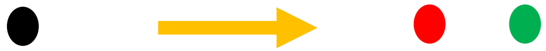
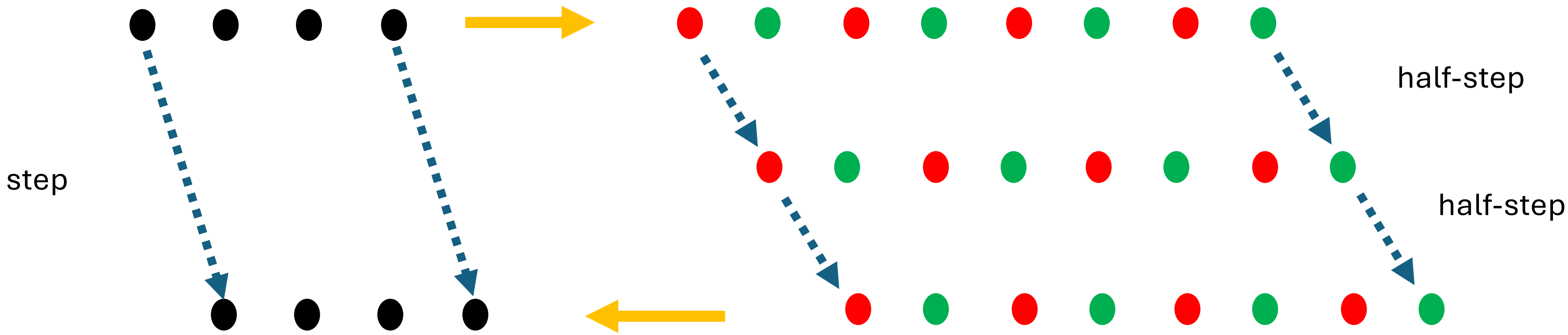


Fractionalization/unfolding of the spin degree of freedom



$$|\uparrow\rangle = |0\rangle$$
$$|\downarrow\rangle = |1\rangle$$





(

Affleck-Kennedy-Lieb-Tasaki spin 1 chain (AKLT, 1987)

$$H = \sum_{n=1}^{N} \vec{S}_n \cdot \vec{S}_{n+1} + \frac{1}{3}(\vec{S}_n \cdot \vec{S}_{n+1})^2$$

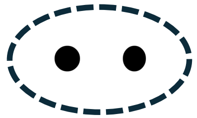
Clebsch-Gordon $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

Spinor basis $\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\psi_{\alpha\beta} = \frac{1}{\sqrt{2}}\left(\psi_\alpha \otimes \psi_\beta + \psi_\beta \otimes \psi_\alpha\right)$$

$$\psi_{11} = \sqrt{2}\,|s = +1\rangle,\ \psi_{12} = \psi_{21} = |s = 0\rangle,\ \psi_{22} = \sqrt{2}\,|s = -1\rangle$$

Ian Affleck
1952-2024

$s = +1, 0, -1 \quad |s\rangle$  $\longrightarrow$  $|\alpha, \beta\rangle \quad \alpha, \beta = 1,2$

$$\Omega_{\alpha\beta} = \psi_{\alpha\beta_1} \otimes \psi_{\alpha_2\beta_2} \otimes \psi_{\alpha_3\beta_3} \otimes \cdots \otimes \psi_{\alpha_N\beta}\, \varepsilon^{\beta_1\alpha_2}\, \varepsilon^{\beta_2\alpha_3} \cdots \varepsilon^{\beta_{N-1}\alpha_N}$$

edge spin 1/2

edge spin 1/2

First Matrix Product State (MPS)

Ideal prototype of Symmetry Protected Topological Phase (SPT):

Haldane pase

Duncan Haldane
1951

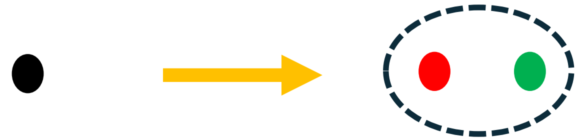
Spin-spin correlator $\left\langle S_0^a S_r^b \right\rangle = \delta^{ab} (-1)^r \frac{4}{3} 3^{-r}$

Finite spin gap $\Delta = E_1 - E_0 > 0$

)

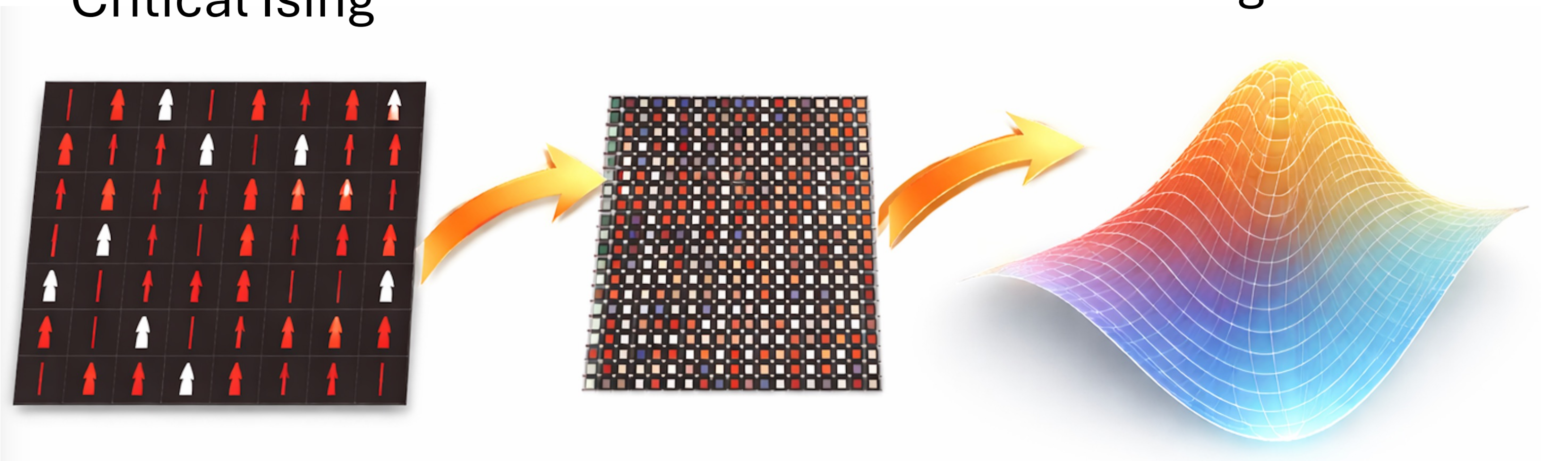
Fractionalization of the Ising spin?

$$|\uparrow\rangle = |0\rangle$$
$$|\downarrow\rangle = |1\rangle$$



Critical Ising

Ising CFT



Exploit the anyonic degrees of freedom of the critical Ising model:

$$\begin{array}{ccc} 1 & \to & \uparrow \\ \psi & \to & \downarrow \\ \sigma & \to & \sigma \end{array}$$

or

In CFT the anyons are in one-to-one correspondence with the primary fields

$$1, \psi(z), \sigma(z)$$

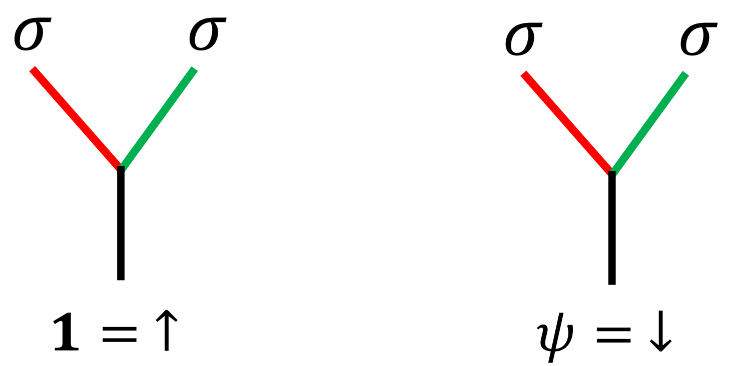
Conformal weights : $O_h(z) \sim z^{-h}$

$$h_{\mathbf{1}} = 0, h_\psi = \frac{1}{2}, h_\sigma = \frac{1}{16}$$

Fusion rules

$$\sigma \times \sigma = \mathbf{1} + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = \mathbf{1}$$

Fusion dictionary



Operator product expansion

$$\lim_{w_1 \to w_2} \sigma(w_1)\sigma(w_2) \sim \mathbf{1}/(w_1 - w_2)^{1/8} + (w_1 - w_2)^{3/8}\psi$$

$$\frac{1}{16} + \frac{1}{16} \qquad \frac{1}{16} + \frac{1}{16} - \frac{1}{2}$$

GS of the quantum Ising model for L=2 sites

$$\widehat{H}_{L=2} = -2X_1X_2 - Z_1 - Z_2,$$

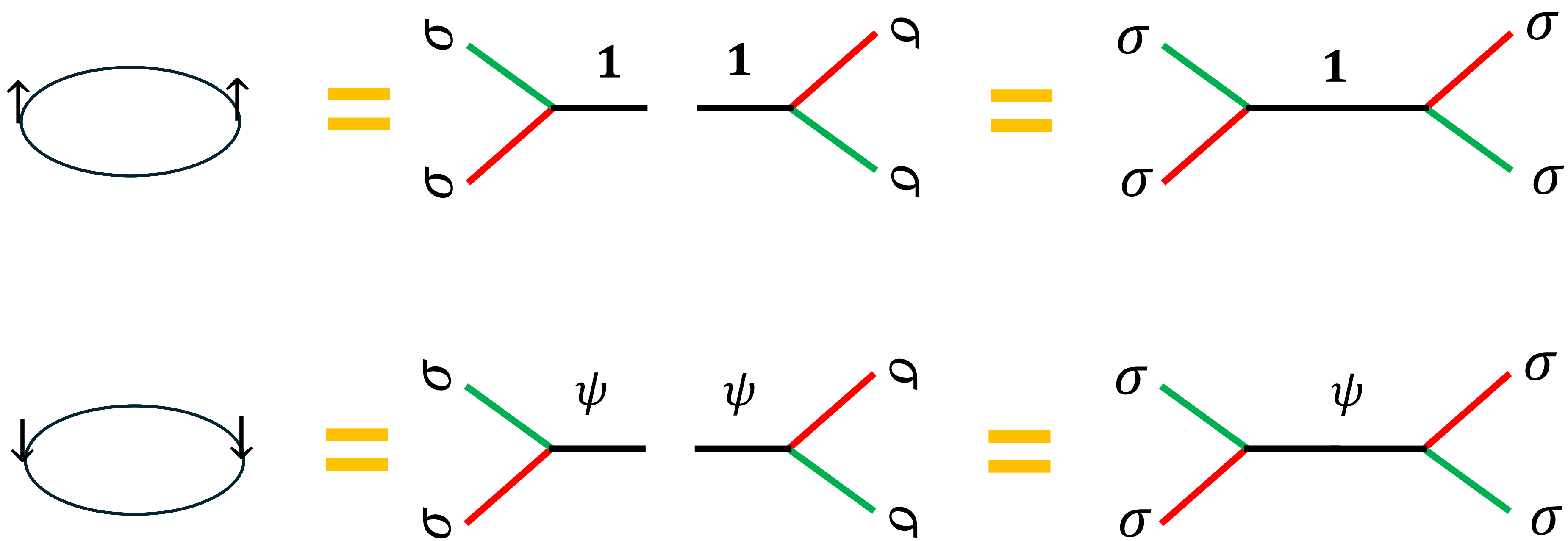
Even sector $\quad |\psi\rangle = \psi_{00}|00\rangle + \psi_{11}|11\rangle$

Schroedinger equation

$$-2\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \psi_{00} \\ \psi_{11} \end{pmatrix} = E\begin{pmatrix} \psi_{00} \\ \psi_{11} \end{pmatrix}$$

Ground state $\quad E_{\rm GS} = -2\sqrt{2}, \qquad (\psi_{\rm GS})_{11}/(\psi_{\rm GS})_{00} = \sqrt{2} - 1$

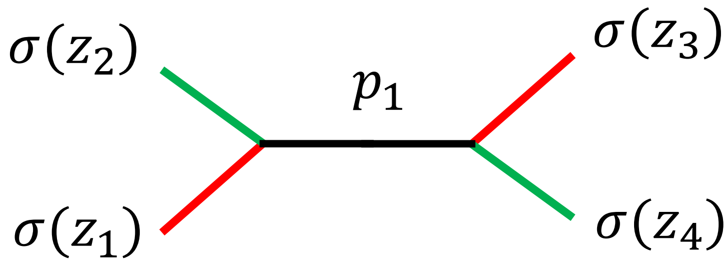
Building up the wave function



The wave function as a chiral correlator (conformal block)

Belavin, Polyakov, Zamolodchikov 1984

$$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle_{p_1} =$$



$$p_1 = 0, 1\ (\mathbf{1}, \psi)$$

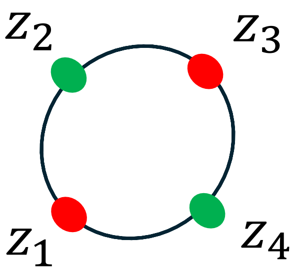
Decoupling of null vectors in the Ising CFT

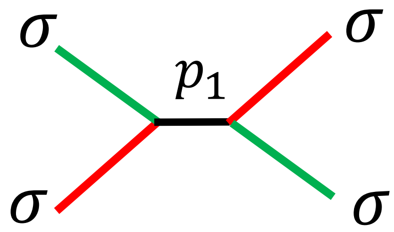
$$\left[\frac{3}{2(2h_\sigma + 1)}\frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i}\frac{h_\sigma}{\left(z_i - z_j\right)^2} - \sum_{j\neq i}\frac{1}{z_i - z_j}\frac{\partial}{\partial z_j}\right]\mathcal{F}_{p_1}(z_1, z_2, z_3, z_4) = 0$$

$$\mathcal{F}^4_{\mathbf{0},\mathbf{1}} = \langle \sigma(v_1)\, \sigma(v_2)\, \sigma(v_3)\, \sigma(v_3) \rangle_{\mathbf{0},\mathbf{1}} = 2^{-1/2} \prod_{a<b} v_{ab}^{-1/8} \left(\sqrt{v_{13}\, v_{24}} \pm \sqrt{v_{14}\, v_{23}}\right)^{1/2}$$

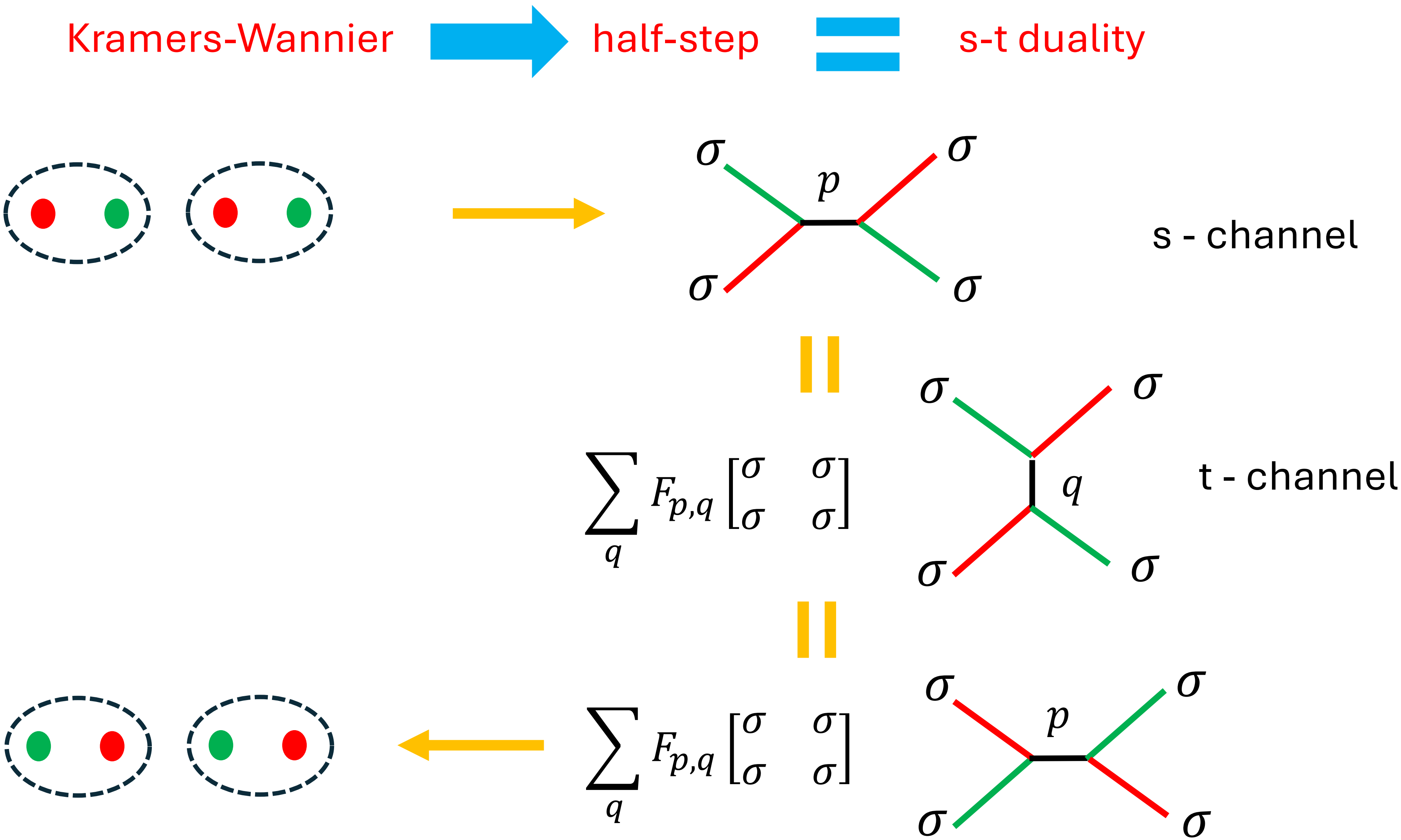
$$v_j = z_j$$
$$v_{ij} = v_i - v_j$$

$$z_j = \pm e^{i\pi/4}, \pm\, e^{-i\pi/4}$$





$$\frac{\mathcal{F}_1(z_1, z_2, z_3, z_4)}{\mathcal{F}_0(z_1, z_2, z_3, z_4)} = \sqrt{2} - 1 = \frac{(\psi_{\mathrm{GS}})_{11}}{(\psi_{\mathrm{GS}})_{00}}$$


Kramers-Wannier
half-step
s-t duality
σ
σ
p
σ
σ
s - channel
$\sum_q F_{p,q}\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}$
σ
σ
q
σ
σ
t - channel
$\sum_q F_{p,q}\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}$
σ
σ
p
σ
σ

Fusion matrix vs Hadamard gate

$$F\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \mathrm{H}$$

$$\left(1-\sqrt{1-x}\right)^{1/2} \pm \left(1-\sqrt{1-x}\right)^{1/2} = \pm\sqrt{2}\left(1\pm\sqrt{x}\right)^{1/2}$$

$$-2\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\psi_{00}\\ \psi_{11}\end{pmatrix} = E\begin{pmatrix}\psi_{00}\\ \psi_{11}\end{pmatrix} \quad \longrightarrow \quad \mathrm{H}|\psi_{\mathrm{GS}}\rangle = |\psi_{\mathrm{GS}}\rangle$$

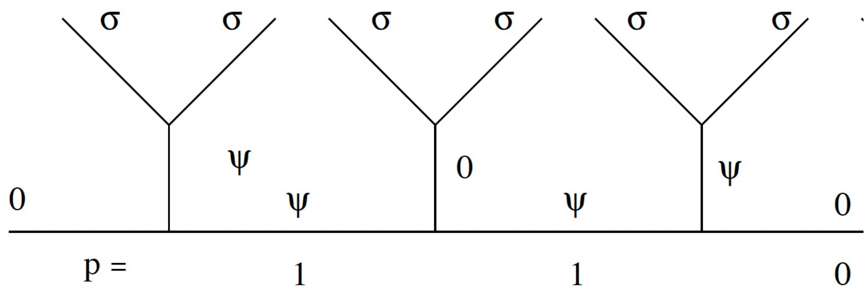
For L =2 sites the Kramers-Wannier is the Hadamard gate !!

General Conformal blocks

Nayak, Wilczek 1996;
Ardonne, GS 2010

$$\mathcal{F}^{6}_{\mathbf{p}} = 2^{-1} \prod_{a<b} v_{ab}^{-1/8} \left(\epsilon_{\mathbf{p0}}\sqrt{v_{135}\, v_{246}} + \epsilon_{\mathbf{p1}}\sqrt{v_{136}\, v_{245}} + \epsilon_{\mathbf{p2}}\sqrt{v_{146}\, v_{235}} + \epsilon_{\mathbf{p3}}\sqrt{v_{145}\, v_{236}}\right)^{1/2}$$

$$v_{abc} = v_{ab}\, v_{ac}\, v_{bc}$$

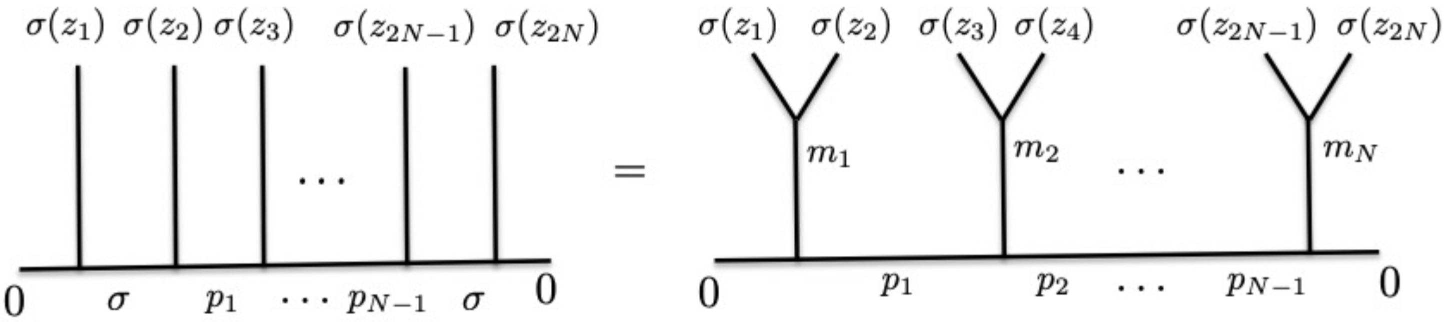


$$\langle \sigma(v_1, \overline{v}_1) \ldots \sigma(v_{2n}, \overline{v}_{2n}) \rangle^2 = \left(\sum_{\mathbf{p}} \mathcal{F}^{2n,0}_{\mathbf{p}}\, \overline{\mathcal{F}}^{2n,0}_{\mathbf{p}} \right)^2$$

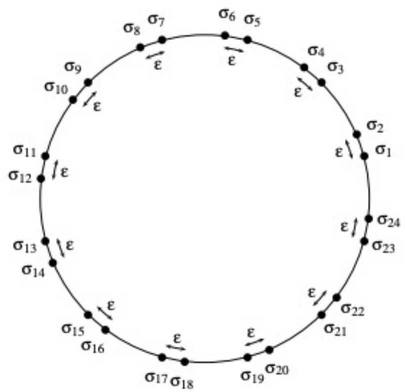
Many-body Lattice Wavefunctions From Conformal Blocks

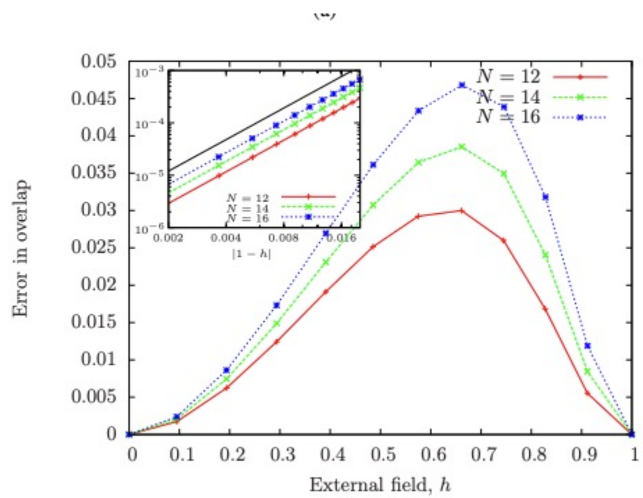
S. Montes, J. Rodriguez-Laguna, H.H. Tu, , GS. 2016
S. Montes, J. Rodriguez-Laguna, GS. 2017

$$|\psi\rangle = \sum_{\mathbf{p}} \mathcal{F}_{\mathbf{p}} \, |\mathbf{p}\rangle$$



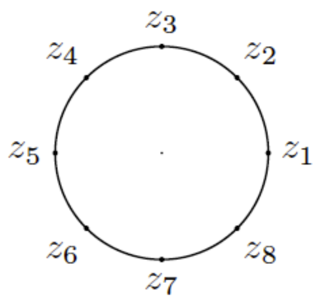
$$H(h) = -\sum_{n=1}^{N} \sigma_n^x \sigma_{n+1}^x - h \sum_{n=1}^{N} \sigma_n^z,$$



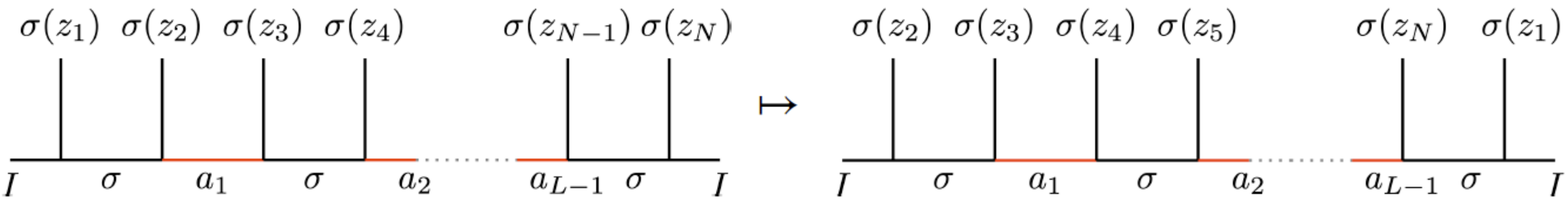


General half-step translation

H-C. Zhang, GS. 2024

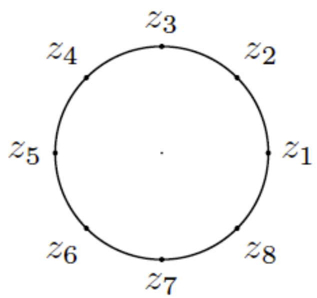


Intuition: upon the half-step translation,

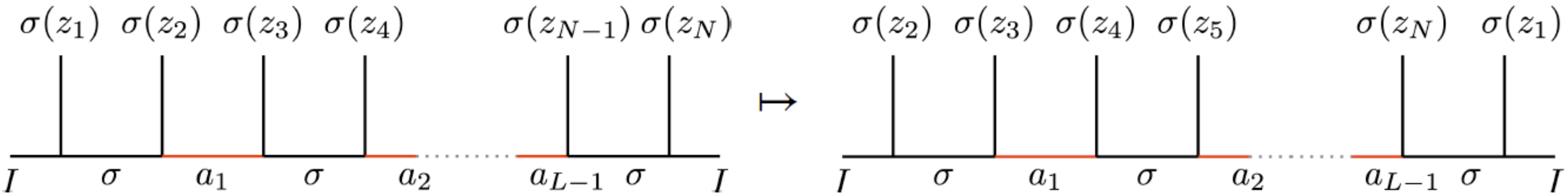


General half-step translation

H-C. Zhang, GS. 2024



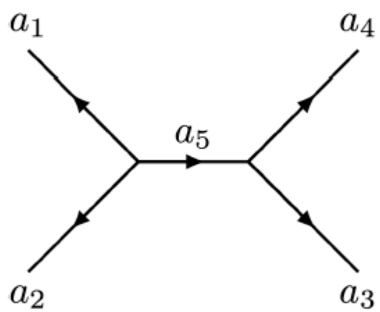
Intuition: upon the half-step translation,



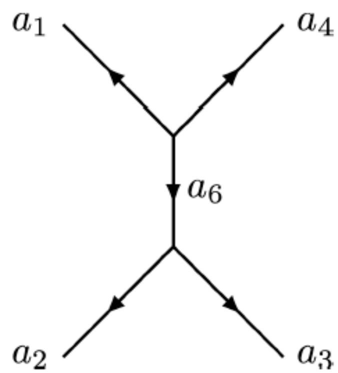
How to connect these two? F-moves!

Conformal blocks and F-moves

F-moves: basis transformations (‘*s*-*t* duality’) for the 4-point conformal blocks

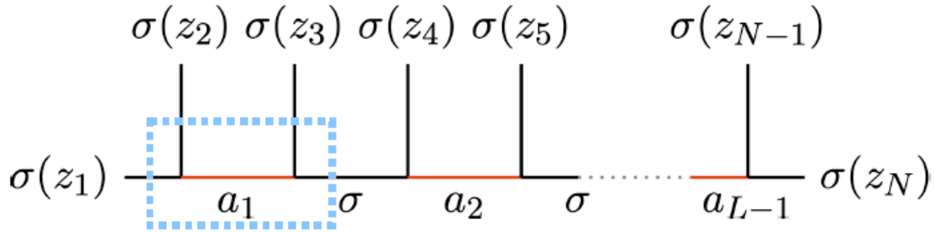


$$= \sum_{a_6} \left(F^{a_1 a_4}_{a_2 a_3} \right)_{a_5, a_6}$$

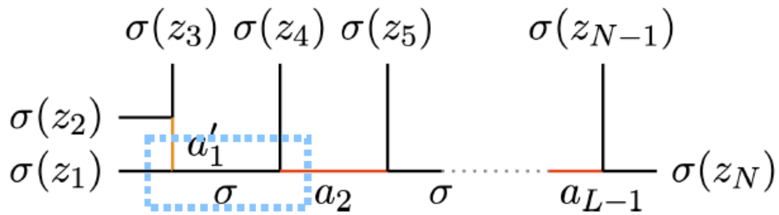


Transformation matrices: *F*-symbols

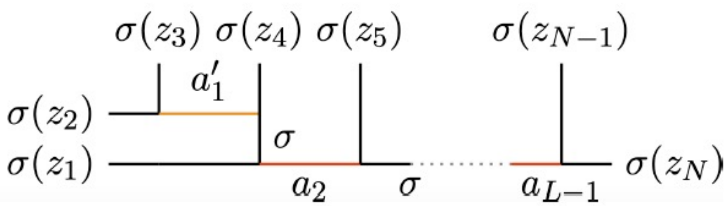
Half-step translation symmetry as KW self-duality

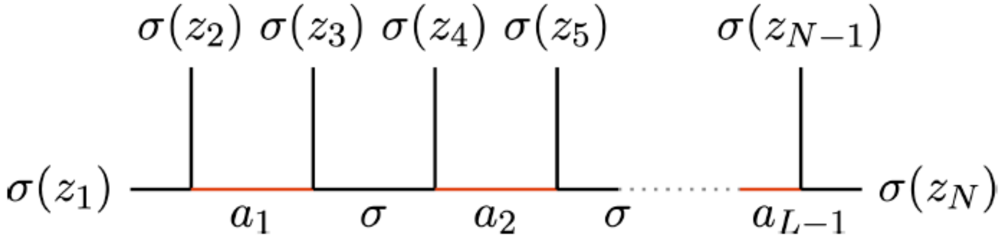


$$= \sum_{a_1'} (F^{\sigma\sigma}_{\sigma\sigma})_{a_1,a_1'}$$



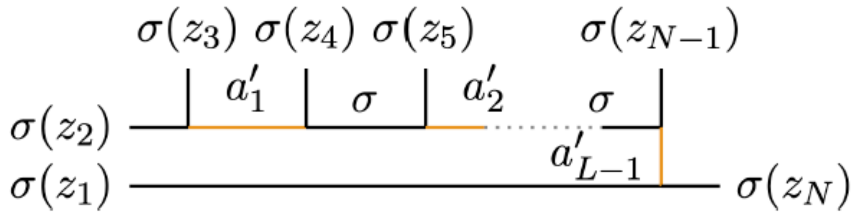
$$= \sum_{a_1'} (F^{\sigma\sigma}_{\sigma\sigma})_{a_1,a_1'} \left(F^{a_1'\sigma}_{\sigma a_2}\right)_{\sigma,\sigma}$$

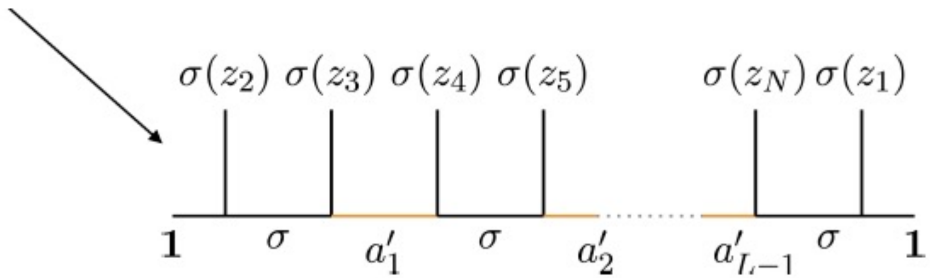


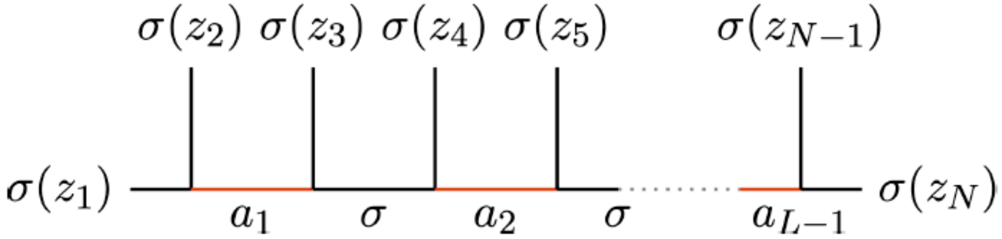


$$= \ldots\ldots$$

$$= \sum_{a'_1\ldots a'_{L-1}} (F^{\sigma\sigma}_{\sigma\sigma})_{a_1,a'_1} \left(F^{a'_1\sigma}_{\sigma a_2}\right)_{\sigma,\sigma} (F^{\sigma\sigma}_{\sigma\sigma})_{a_2,a'_2} \cdots \left(F^{a'_{L-1}\sigma}_{\sigma a_{L-1}}\right)_{\sigma,\sigma} (F^{\sigma\sigma}_{\sigma\sigma})_{a_{L-1},a'_{L-1}}$$

$\times$ 

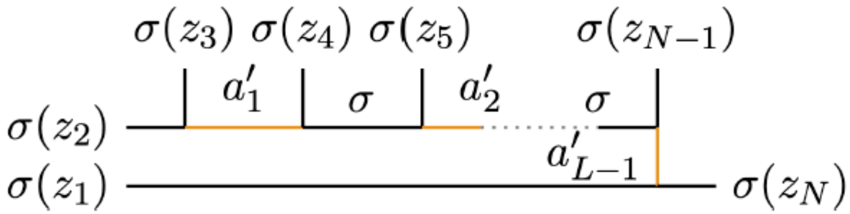




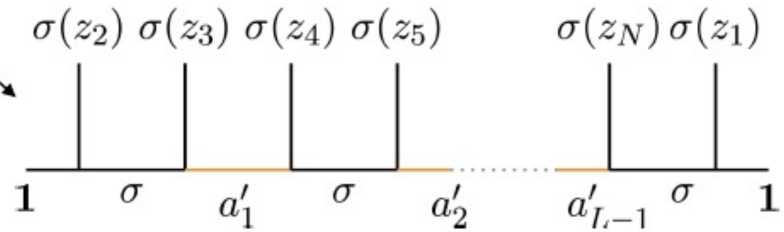
$$= \cdots\cdots$$

$$= \sum_{a_1'\ldots a_{L-1}'} (F^{\sigma\sigma}_{\sigma\sigma})_{a_1,a_1'} \left(F^{a_1'\sigma}_{\sigma a_2}\right)_{\sigma,\sigma} (F^{\sigma\sigma}_{\sigma\sigma})_{a_2,a_2'} \cdots \left(F^{a_{L-1}'\sigma}_{\sigma a_{L-1}}\right)_{\sigma,\sigma} (F^{\sigma\sigma}_{\sigma\sigma})_{a_{L-1},a_{L-1}'}$$

$\times$



$\tilde{\mathsf{D}}_{a_1\ldots a_{L-1},a_1'\ldots a_{L-1}'}$, elements of a $2^{L-1}\times 2^{L-1}$ matrix



$$\tilde{\mathsf{D}}_{a_1\ldots a_{L-1},a'_1\ldots a'_{L-1}} = \left(F^{\sigma\sigma}_{\sigma\sigma}\right)_{a_1,a'_1} \left(F^{a'_1\sigma}_{\sigma a_2}\right)_{\sigma,\sigma} \left(F^{\sigma\sigma}_{\sigma\sigma}\right)_{a_2,a'_2} \cdots \left(F^{a'_{L-1}\sigma}_{\sigma a_{L-1}}\right)_{\sigma,\sigma} \left(F^{\sigma\sigma}_{\sigma\sigma}\right)_{a_{L-1},a'_{L-1}}$$

Non-trivial F-symbols of the Ising CFT:

$$\left(F^{\chi\sigma}_{\sigma\chi}\right)_{\sigma,\sigma} = -1, \quad F^{\sigma\sigma}_{\sigma\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \mathsf{H} \text{ (Hadamard gate)}$$

$$\tilde{\mathsf{D}}_{a_1 \dots a_{L-1}, a'_1 \dots a'_{L-1}} = \left(F^{\sigma\sigma}_{\sigma\sigma}\right)_{a_1, a'_1} \left(F^{a'_1 \sigma}_{\sigma a_2}\right)_{\sigma,\sigma} \left(F^{\sigma\sigma}_{\sigma\sigma}\right)_{a_2, a'_2} \cdots \left(F^{a'_{L-1}\sigma}_{\sigma a_{L-1}}\right)_{\sigma,\sigma} \left(F^{\sigma\sigma}_{\sigma\sigma}\right)_{a_{L-1}, a'_{L-1}}$$

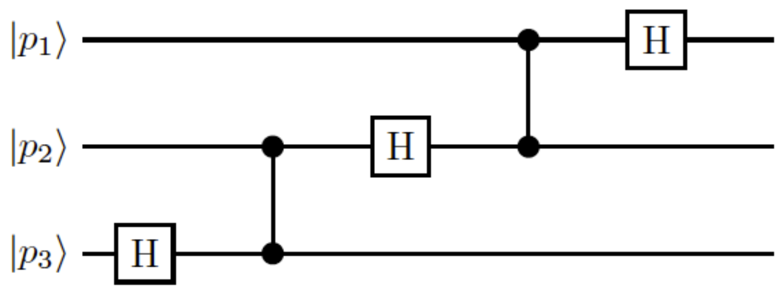
Non-trivial F-symbols of the Ising CFT:

$$\left(F^{\chi\sigma}_{\sigma\chi}\right)_{\sigma,\sigma} = -1, \quad F^{\sigma\sigma}_{\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \mathsf{H} \text{ (Hadamard gate)}$$

$\Rightarrow \tilde{\mathsf{D}}$ can be expressed as a Clifford circuit:

$$\tilde{\mathsf{D}} = \mathsf{H}_1 \mathsf{CZ}_{1,2} \mathsf{H}_2 \mathsf{CZ}_{2,3} \cdots \mathsf{H}_{L-2} \mathsf{CZ}_{L-2,L-1} \mathsf{H}_{L-1}$$

$$\mathsf{CZ}_{l,l+1} = \frac{1}{2}\left(1 + Z_l + Z_{l+1} - Z_l Z_{l+1}\right) \text{ (controlled } Z \text{ gate)}$$



Projecting into the even-parity sector:

$$\mathsf{D} = \sqrt{2}\,\frac{1+\eta}{2}\,\tilde{\mathsf{D}}\,\frac{1+\eta}{2}$$

where $\eta = \prod_{l=1}^{L} X_l$ — $\mathbb{Z}_2$ parity

$$H = -\sum_{l=1}^{L}\left(JZ_l Z_{l+1} + hX_l\right)$$

It can be proven:

$$\mathsf{D}X_l = Z_l Z_{l+1 \pmod L}\mathsf{D}, \quad \mathsf{D}Z_l Z_{l+1 \pmod L} = X_{l+1 \pmod L}\mathsf{D}$$

$\Rightarrow$ D implements the KW transformation, $[H_{\text{crit.}}, \mathsf{D}] = 0$;

$\mathsf{D}^2 = \mathsf{T}\cdot\dfrac{1+\eta}{2}$ with T the ‘ordinary’ translation

KW duality : lattice vs continuum

Ising model → Majorana model

Jordan-Wigner transformation

Field theory:

'speed of sound/light' Majorana mass

$$\mathscr{H} \sim -\mathrm{i}\left[\frac{v}{2}\left(\chi_{\mathrm{R}}\partial_x\chi_{\mathrm{R}} - \chi_{\mathrm{L}}\partial_x\chi_{\mathrm{L}}\right) + m\chi_{\mathrm{R}}\chi_{\mathrm{L}}\right]$$

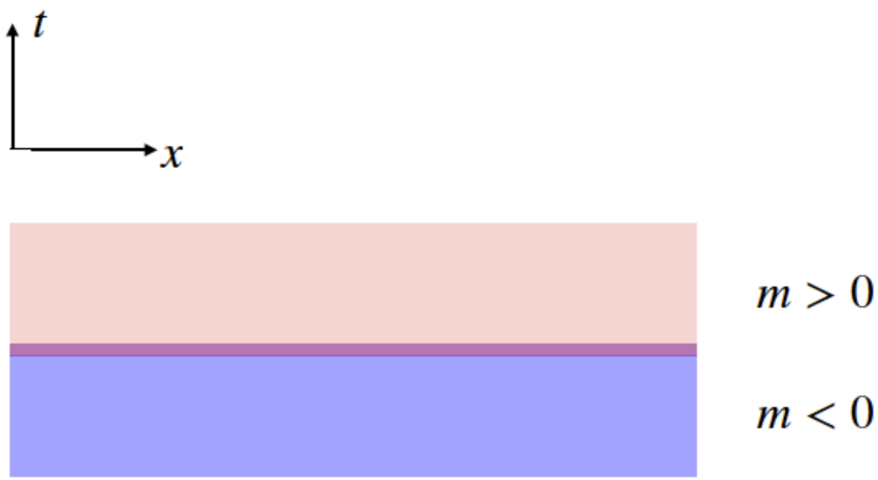
$\chi_{\mathrm{R/L}}$: right/left-moving component of the Majorana field

$J > h$, $m < 0$, ordered phase;

$J < h$, $m > 0$, disordered phase;

$J = h$, $m = 0$, critical, Ising conformal field theory (CFT), chiral decomposition

Order-disorder transition: sign change of m



KW transformation as a *defect line*

KW *self*-duality as a symmetry
of the Ising CFT

KW duality : lattice vs continuum

- Lattice vs. continuum, spacetime vs. internal
- Connection between these viewpoints?

'... *The non-invertible Kramers-Wannier duality operator of the continuum Ising CFT emanates from this non-invertible lattice translation of the transverse-field Ising model.*' — Nathan Seiberg and Shu-Heng Shao. SciPost Phys. 16, 064 (2024)

emanant symmetry UV $\rightarrow$ IR

- Is there a more direct connection? The answer is *yes*, according to our findings — the *wave-function* perspective

Summary & what wasn't mentioned

- Takeaway message:

i) For the critical Ising chain, the KW self-duality follows directly from the half-step translation symmetry of its ground-state wave function;

ii) The symmetry operator implementing the above transformation (UV) can be expressed in terms of F-symbols from the Ising CFT (IR/'topological')

- The approach generalises straightforwardly to other critical quantum chains, e.g. the 3-state Potts model
- Non-invertible fusion rules (e.g. $\mathsf{D}^2 = \mathsf{T} \cdot \frac{1+\eta}{2}$) mixed with lattice translation
- The symmetry operator can be brought into the form of a matrix product operator (MPO) with periodic boundary condition

Thanks for your attention