Emergent Hamiltonians from boundary symmetries

Developments on Generalized Symmetries of Quantum Systems
12-14 May, Murcia, Spain

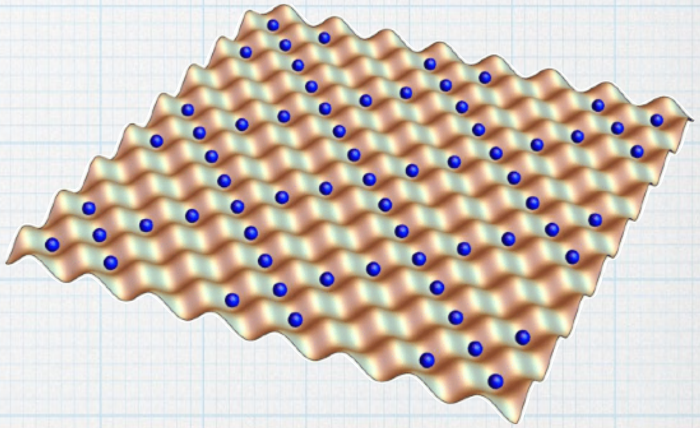
José Garre Rubio, Institute of Theoretical Physics (Madrid)

FWF Austrian Science Fund

ift Instituto de Física Teórica UAM-CSIC

What quantum phases are there?
and how to classify them

Quantum many-body systems



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- Difficulty: Knowledge on fundamental interactions is not enough to predict nature. We need to understand how **global phenomena emerge**.
- A very useful strategy are **solvable models** (Ising, Kitaev, AKLT…)
  - ✳ Capture global properties in a simple way and allow us to realize previously inconceivable phases, generalize and classify them.
  - ✳ It is a milestone to come up with them.

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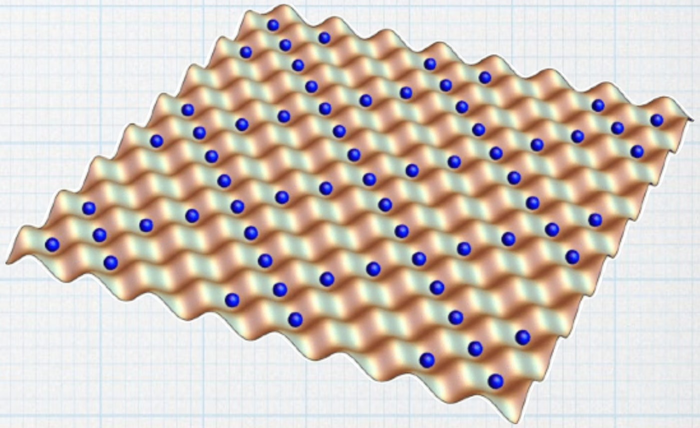

[Anderson, P. W., More is Different, Science, 1972]

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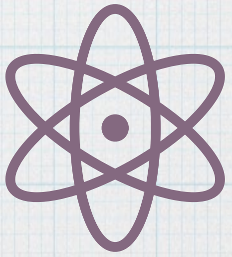

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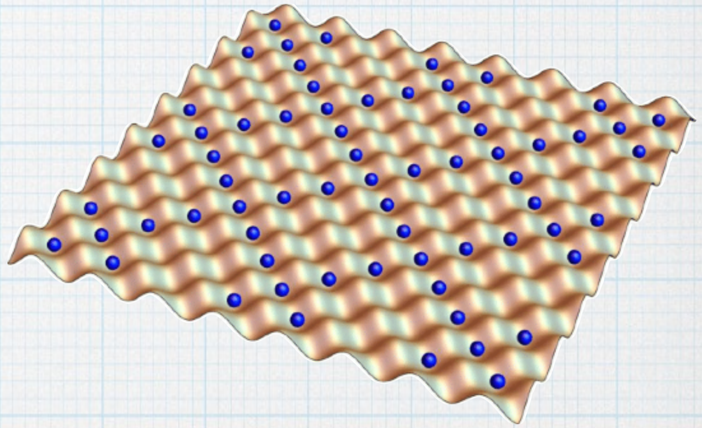

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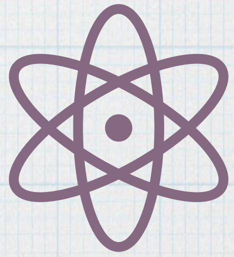
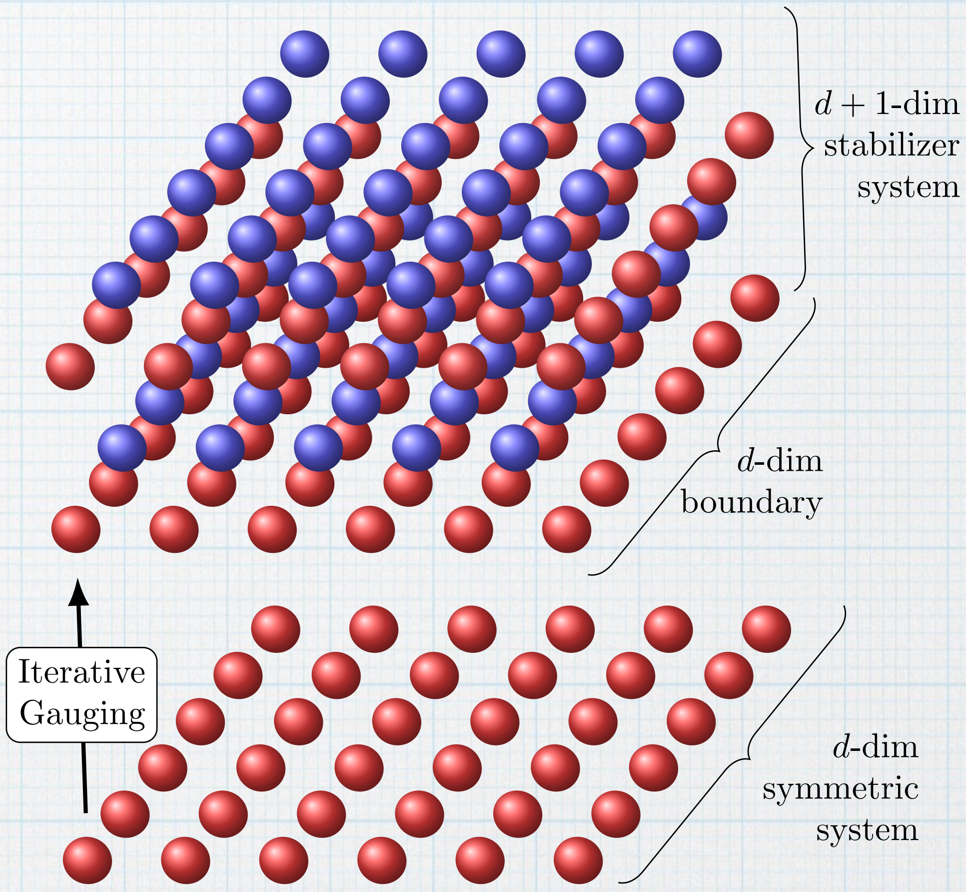


THIS TALK:

How to construct & order higher-dim (topological) model from lower (symmetric) ones

Outline

* **Main example:** how the 2D toric code comes from the global $\mathbb{Z}_2$-symmetry of the 1D Ising model. Also how its rough/smooth boundaries relate to ordered/disordered Ising phases



* **Generalizations:** Any non-anomalous higher-form and subsystem symmetry in d-dim creates a different (d+1)-dim model.

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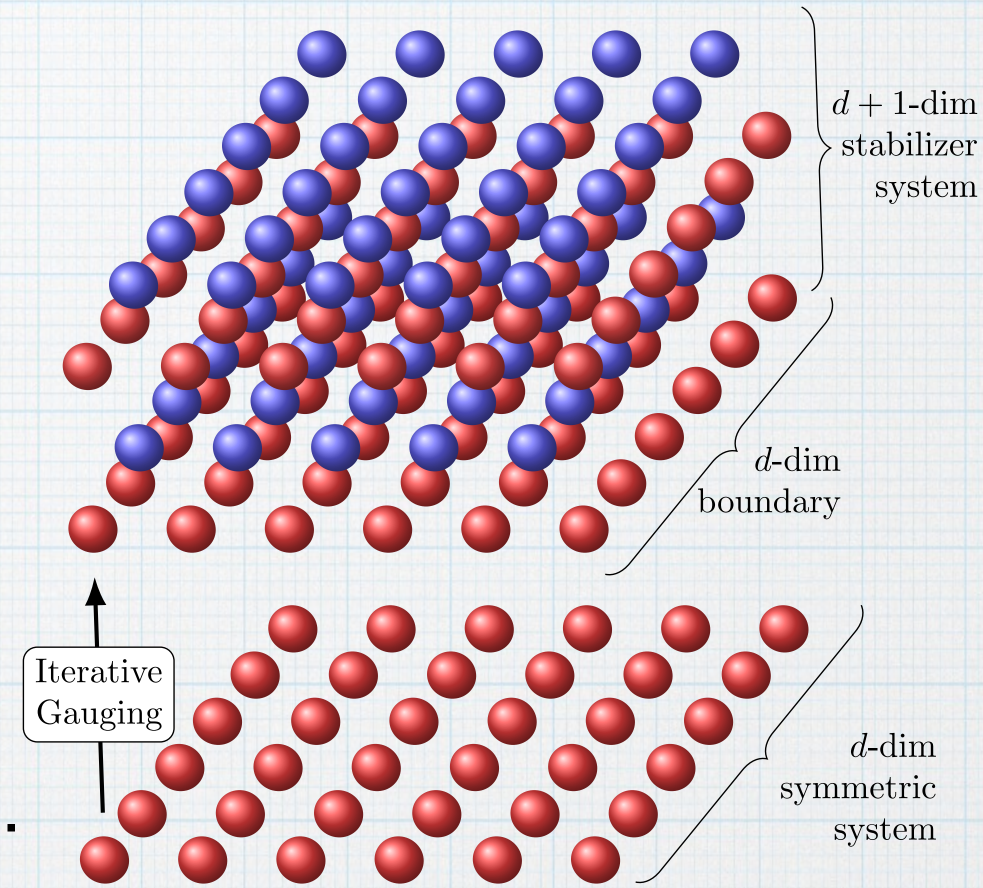

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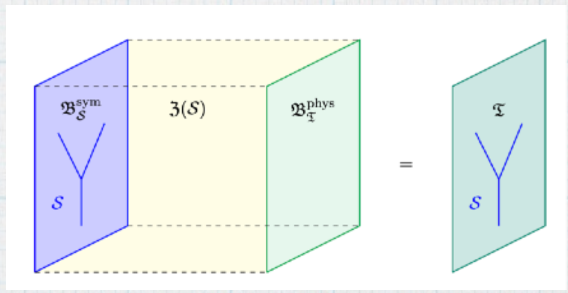

~ SymTFT

(Finite, lattice, Hamiltonians)

Dualities & gauging

1D Global (on-site) symmetry from $\mathbb{Z}_2$**:** $[H, X^{\otimes n}] = 0$

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Kramers-Wannier duality : $Z \otimes Z \longleftrightarrow X$

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TF Ising model: $H_{TFI} = -\sum_i (Z_i Z_{i+1} + \mu X_i) \longleftrightarrow \hat{H}_{TFI} = -\sum_i (Z_{\hat{i}} + \mu X_{\hat{i}} X_{\hat{i}+1})$

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- **Duality = Gauging + decoupling matter**

[B. Vancraeynest-De Cuiper, **JGR**, F. Verstraete, K. Vervoort, D. Williamson, L. Lootens. *SciPost Phys. 20, 113 (2026)*]

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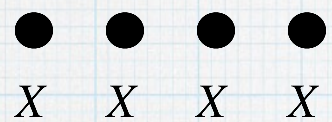
Local symmetry $X_{\hat{i}-1} X_i X_{\hat{i}}$ (Gauss Law) $\longrightarrow$ <u>**What happens if we gauge again?**</u>

Emergent symmetry: $[\tilde{H}_{\mathrm{TFI}}, Z^{\otimes n}] = 0$

- **Gauging: New gauge fields + Gauss Law + Emergent global symmetry**



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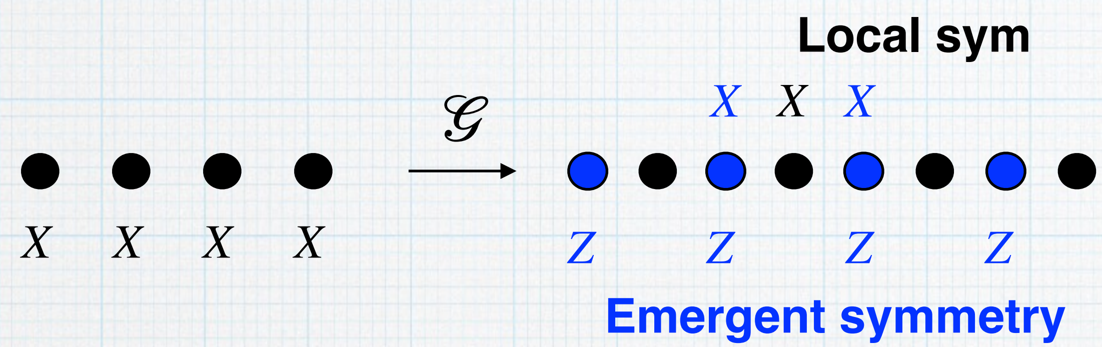

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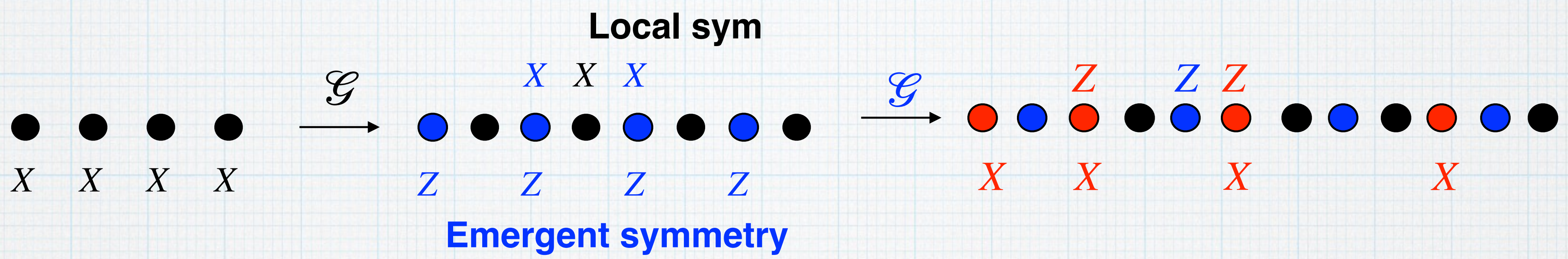

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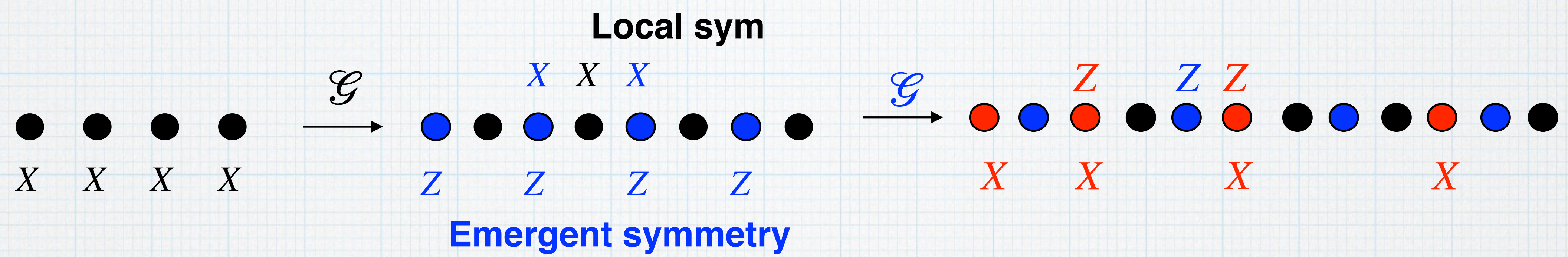

1) Ordering?

2) How does XXX change after $\mathscr{G}$?

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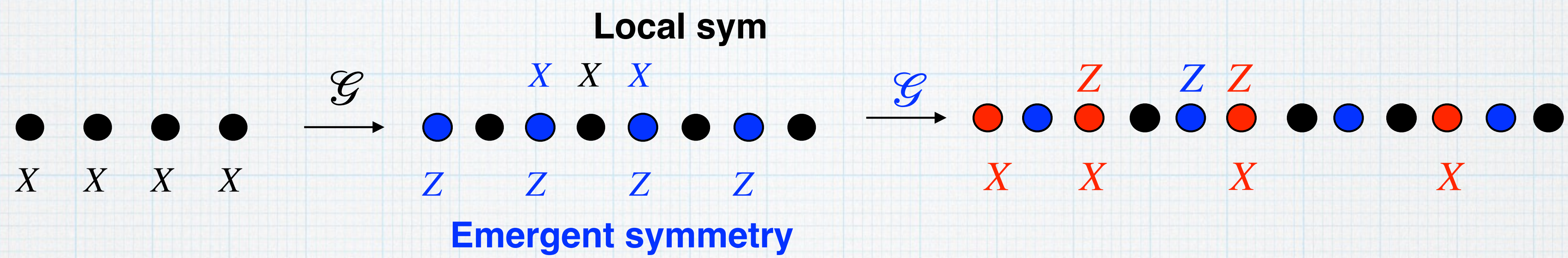

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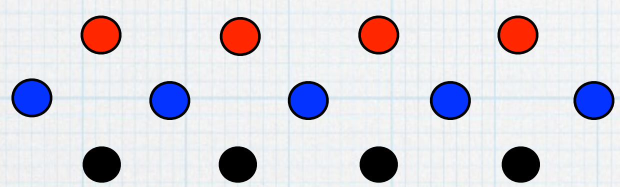

$$XXX \xrightarrow{\mathscr{G}} X \overset{X}{\underset{X}{\,}} X$$

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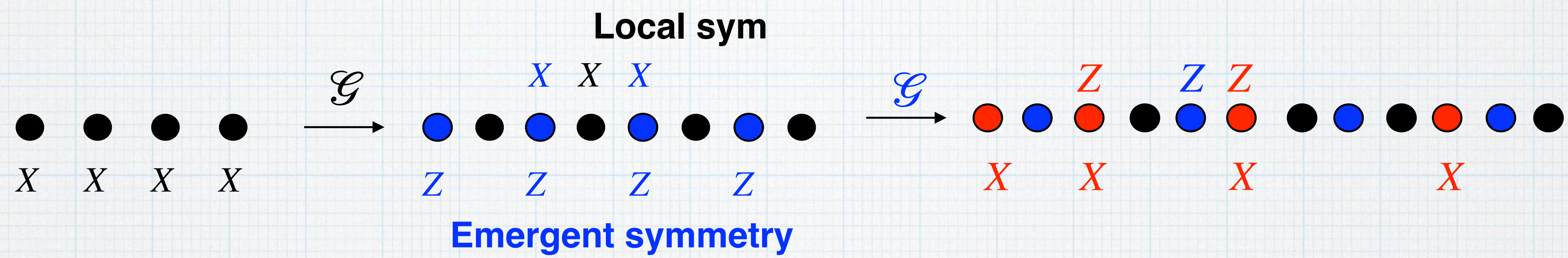

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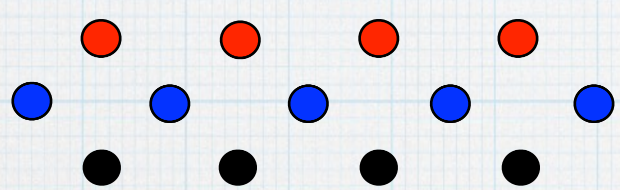

$$XXX \xrightarrow{\mathscr{G}} \begin{matrix} & X & \\ X & & X \\ & X & \end{matrix} \qquad ZZZ \xrightarrow{\mathscr{G}} \begin{matrix} & Z & \\ Z & & Z \\ & Z & \end{matrix}$$

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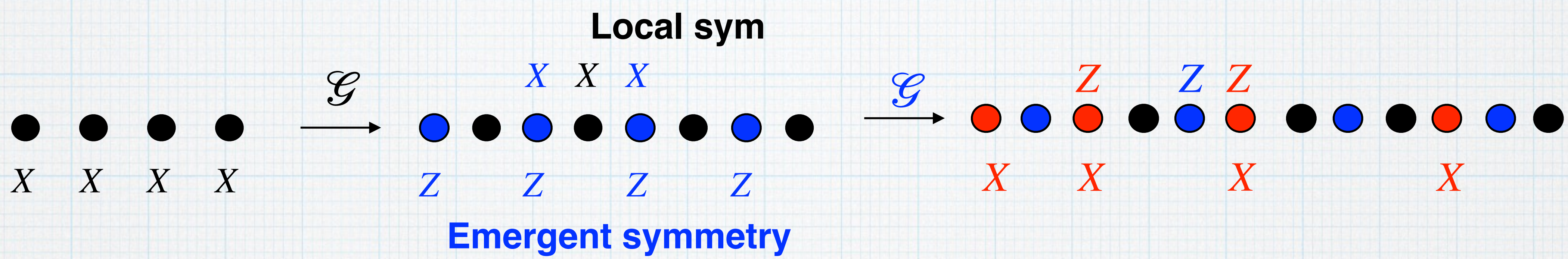

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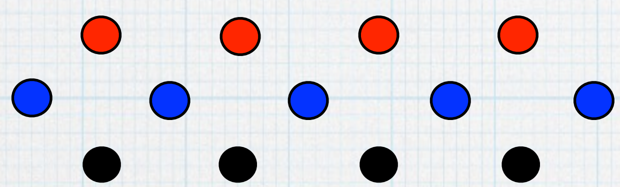
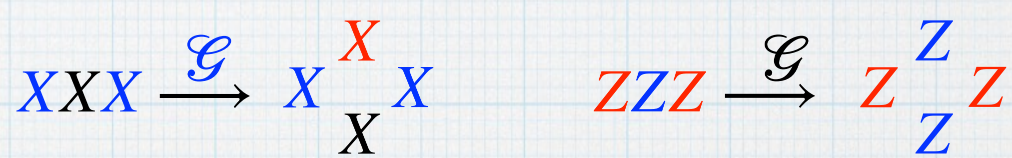
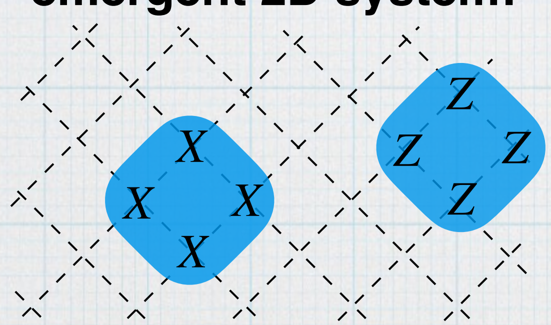




Local symmetries of the emergent 2D system:



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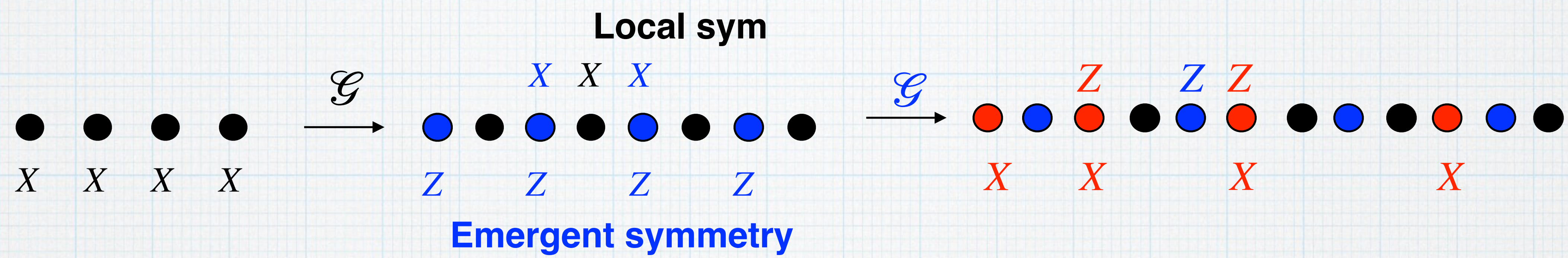

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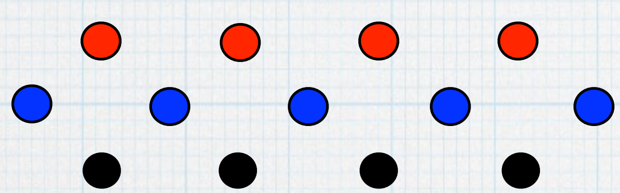

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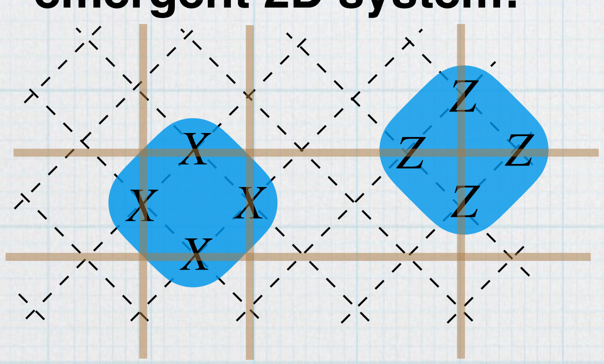
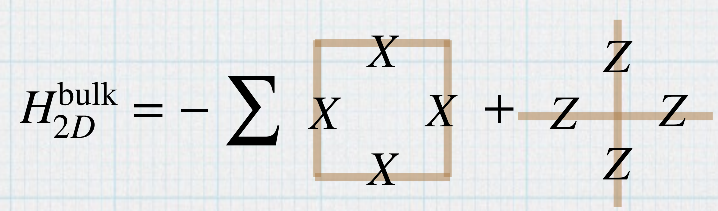


Stabilizer Hamiltonian:



Toric Code

[A. Kitaev, *Annals Phys. 303* (2003)]

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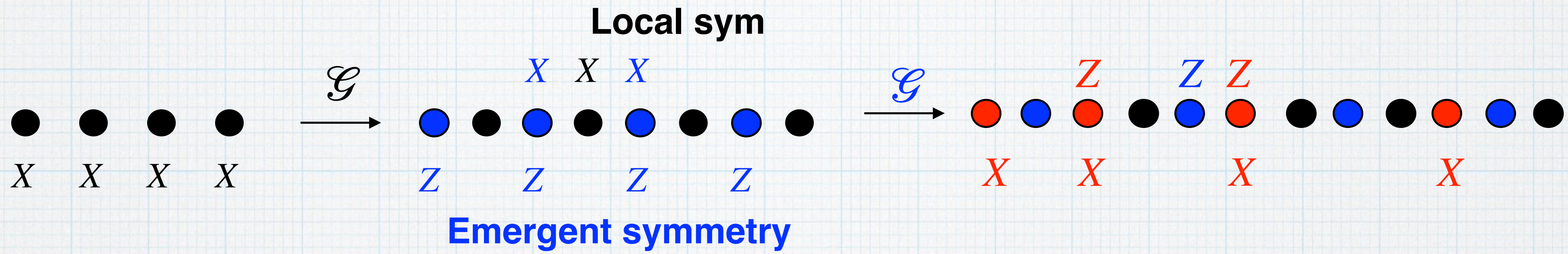

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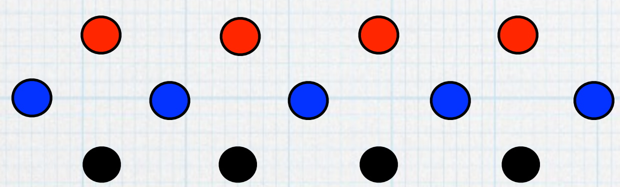

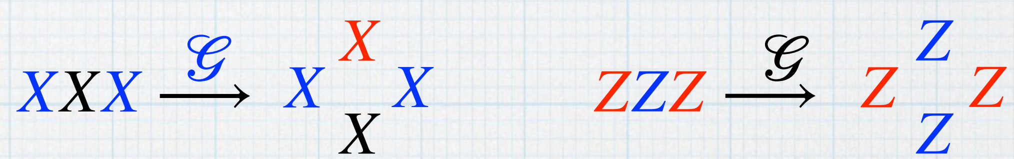

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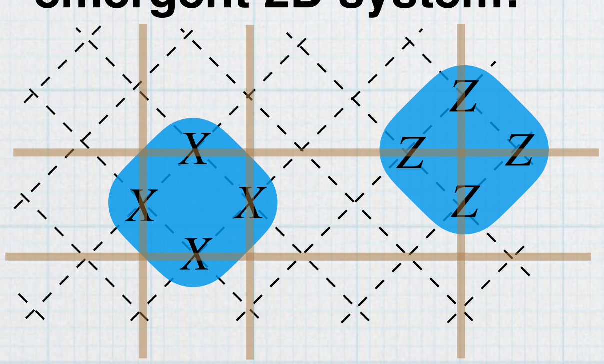

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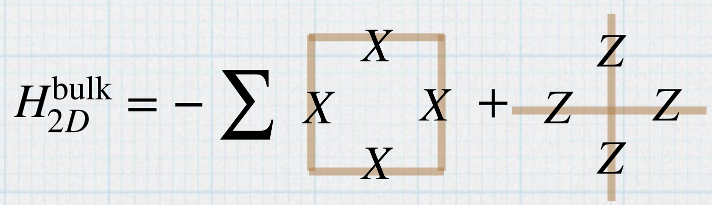

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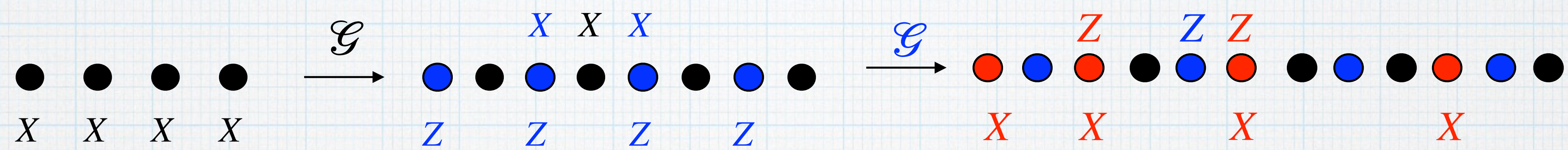
[D. Blanik & **JGR,** *ArXiv:2512.08749*]

☑ **Non-abelian groups**

Gauging: New gauge fields + Gauss Law + Emergent global symmetry

Local sym



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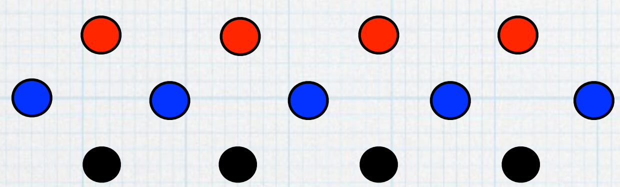

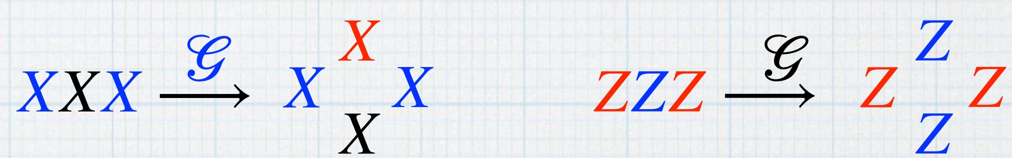

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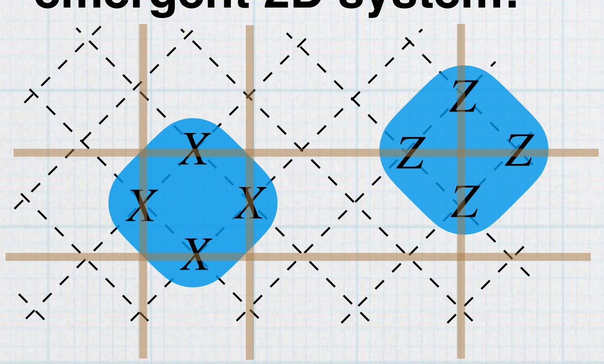

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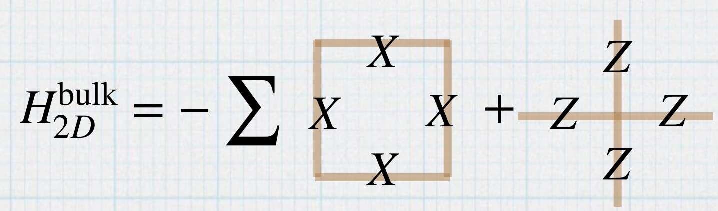

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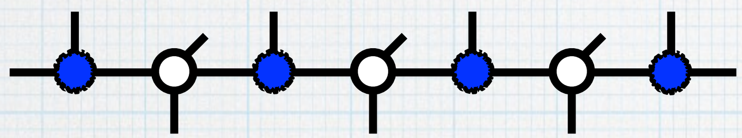
<u>What happens at the boundary?</u>

H_{2D}^{bulk} **only depends on the symmetry** $X^{\otimes n}$ **but the boundary terms depend on the phase of the initial Hamiltonian (the symmetries of the GS).** $H_{TFI} = -\sum_i (Z_i Z_{i+1} + \mu X_i)$

$\mathscr{G} =$ 

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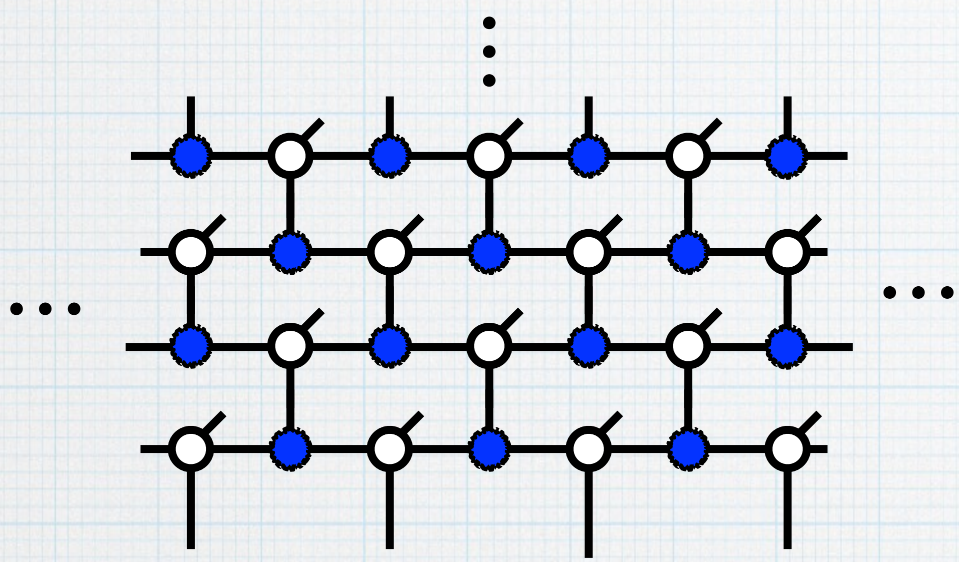

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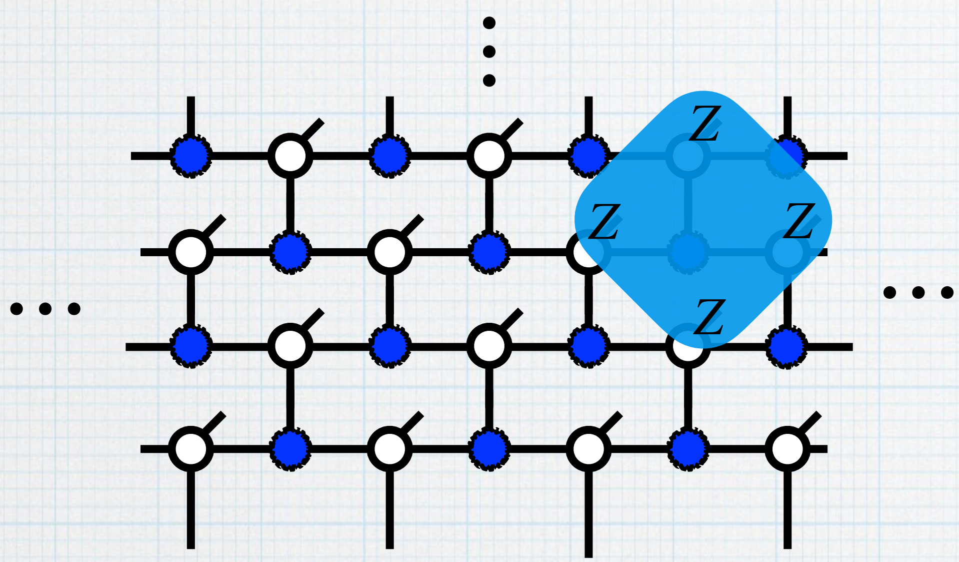

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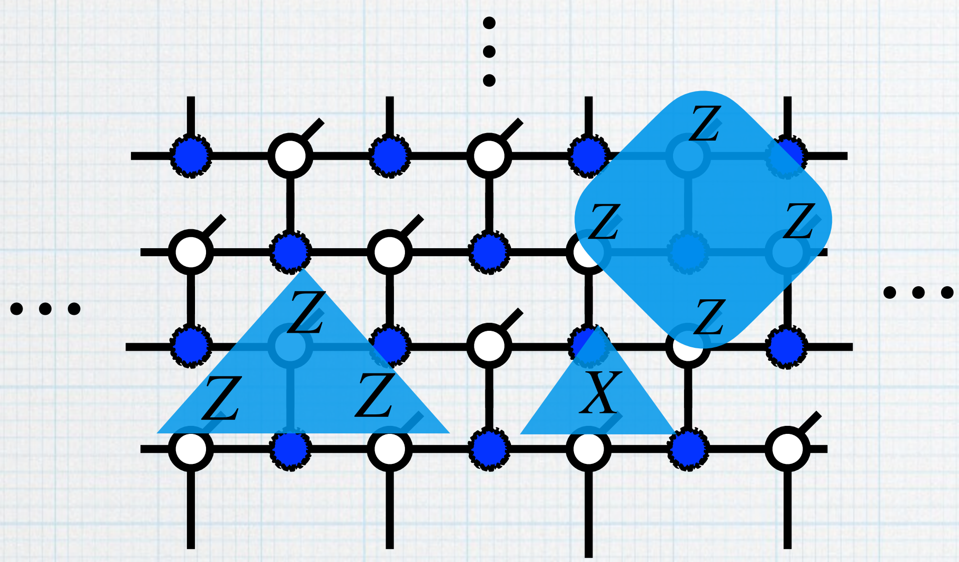

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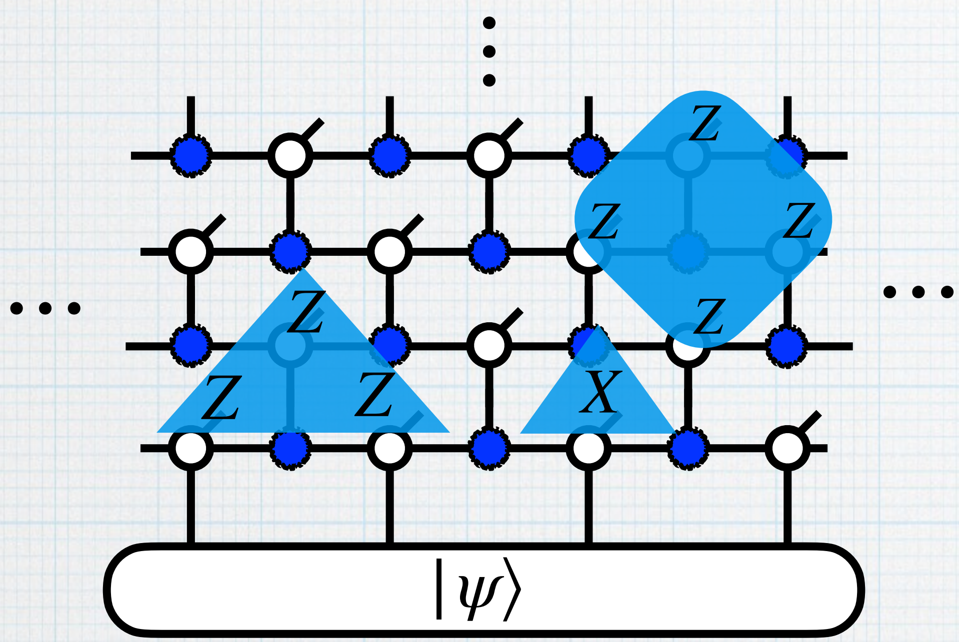

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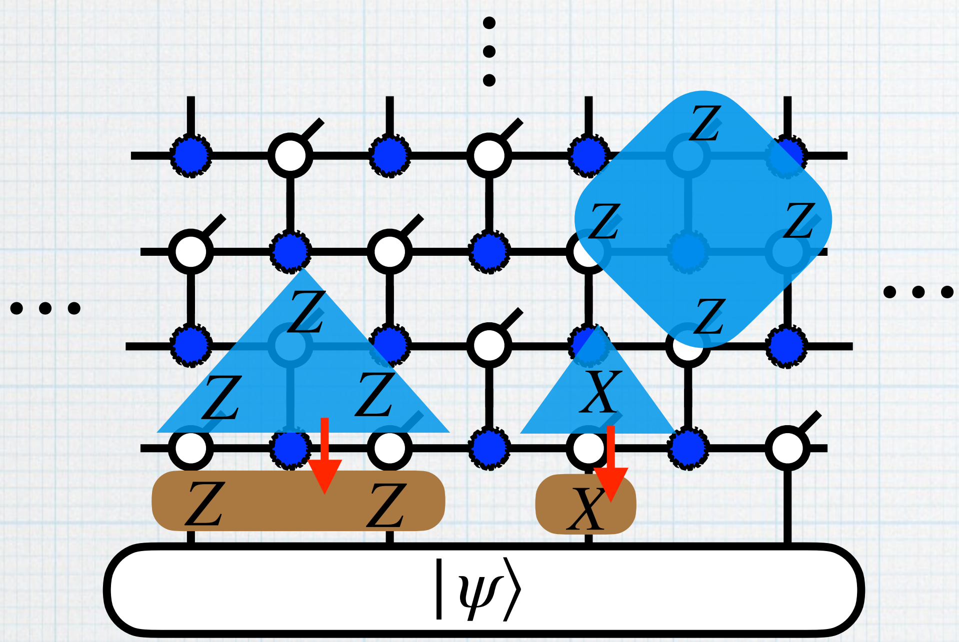

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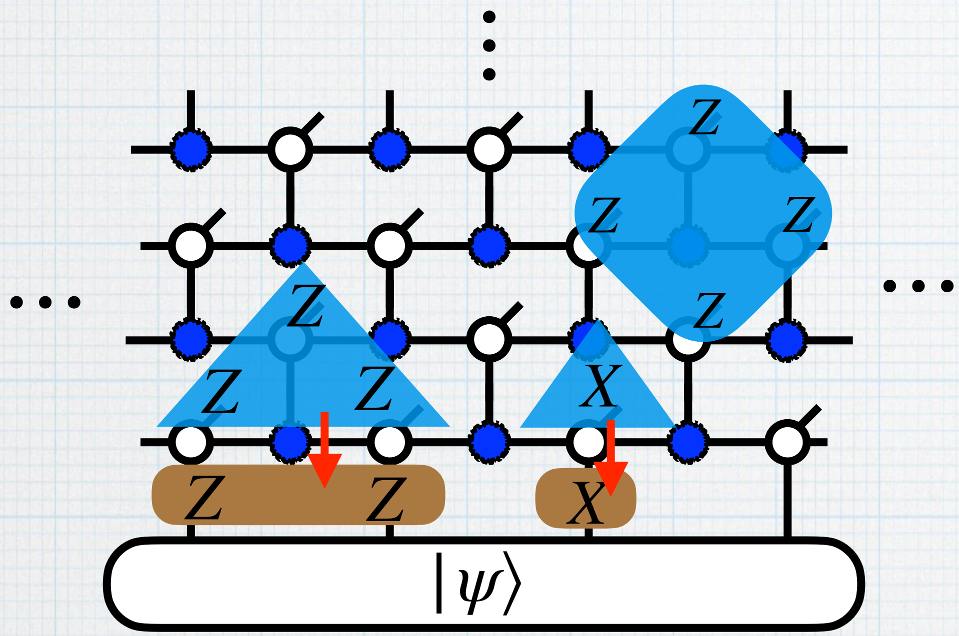

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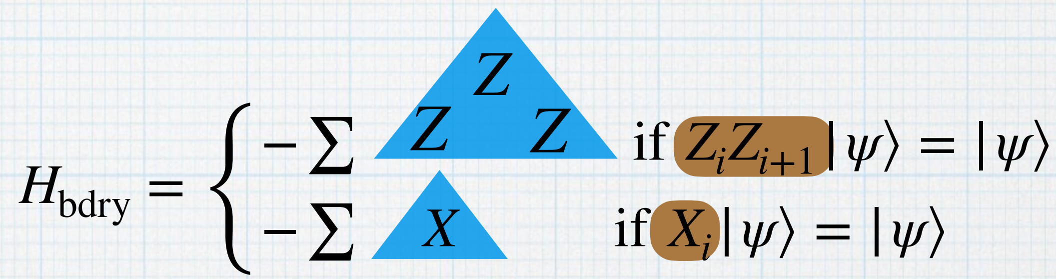
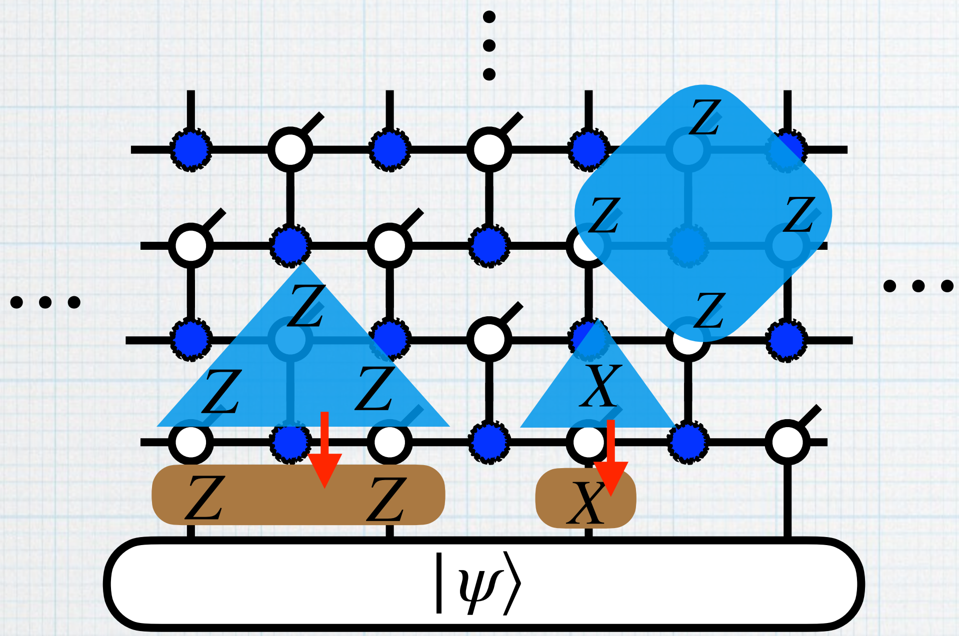
$$|\psi\rangle = \begin{cases} |+\rangle^{\otimes n}, & (\mu \to \infty) \Rightarrow \ X_i|\psi\rangle = |\psi\rangle \\ |0\rangle^{\otimes n} + |1\rangle^{\otimes n}, & (\mu = 0) \Rightarrow \ Z_i Z_{i+1}|\psi\rangle = |\psi\rangle \end{cases}$$



Gapped boundaries of TC: condensing e or m

H_{2D}^{bulk} **only depends on the symmetry** $X^{\otimes n}$ **but the boundary terms depend on the phase of the initial Hamiltonian (the symmetries of the GS).** $H_{TFI} = -\sum_i (Z_i Z_{i+1} + \mu X_i)$



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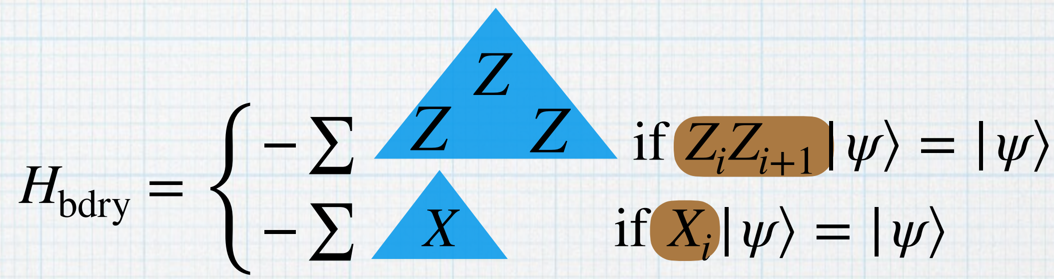

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☆ **Gapped boundaries of 2D** $\mathscr{D}(G)$ **= Quantum phases of global G symmetry = Module cat. over** Vec_G

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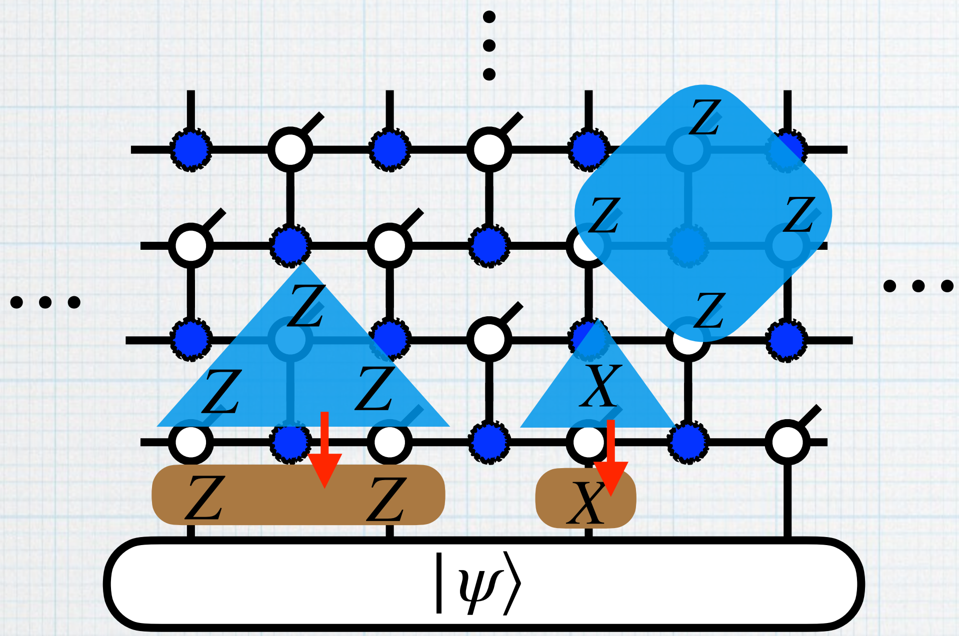

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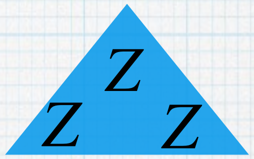
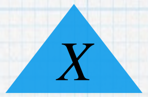
$H_{\text{bdry}} = \Big\{$ $-\sum$  if $Z_i Z_{i+1}|\psi\rangle = |\psi\rangle$

$-\sum$  if $X_i|\psi\rangle = |\psi\rangle$

Gapped boundaries of TC: condensing e or m

⭐ **Gapped boundaries of 2D $\mathscr{D}(G)$ = Quantum phases of global G symmetry = Module cat. over** Vec_G

$$H^{sym}_{1D} \longrightarrow \text{sym}\{\mathscr{G} \circ \cdots \mathscr{G} \circ \mathscr{G}|\psi\rangle\} \;=\; H^{2D}_{\text{Emerg.}} = H_{\text{bulk}} + H_{\text{bdry}}$$

Topological part: only G

Micro part: Q. Phase

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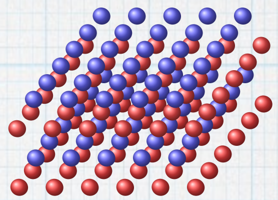
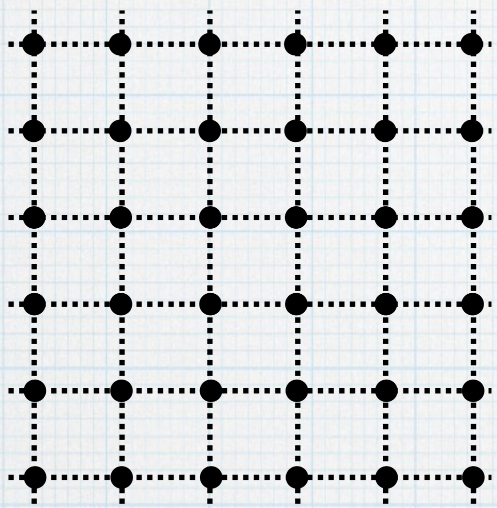
From 2D symmetries to 3D codes



Global: $X^{\otimes n}$



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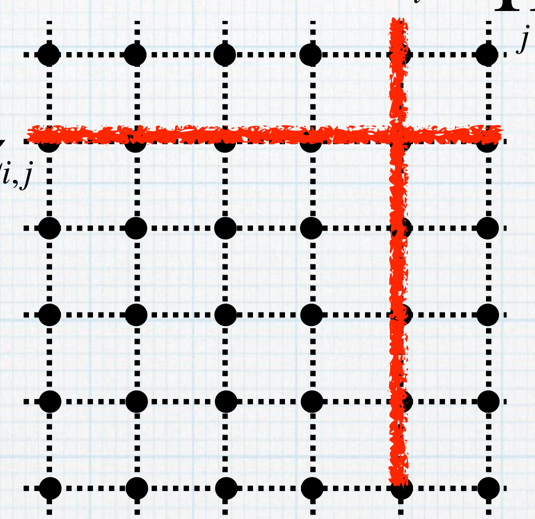

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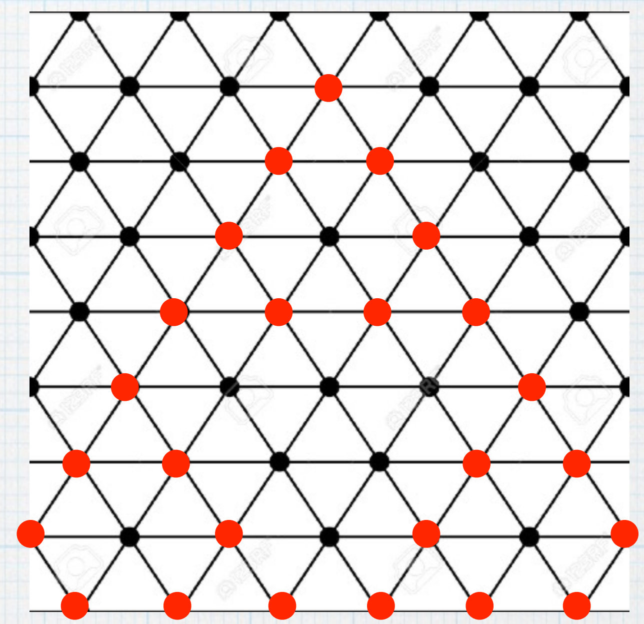

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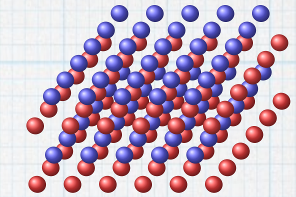

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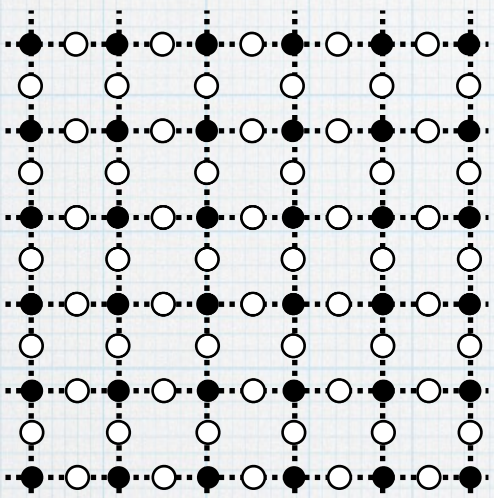

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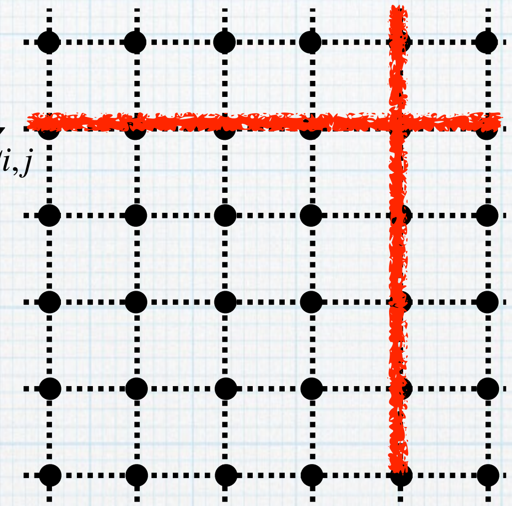

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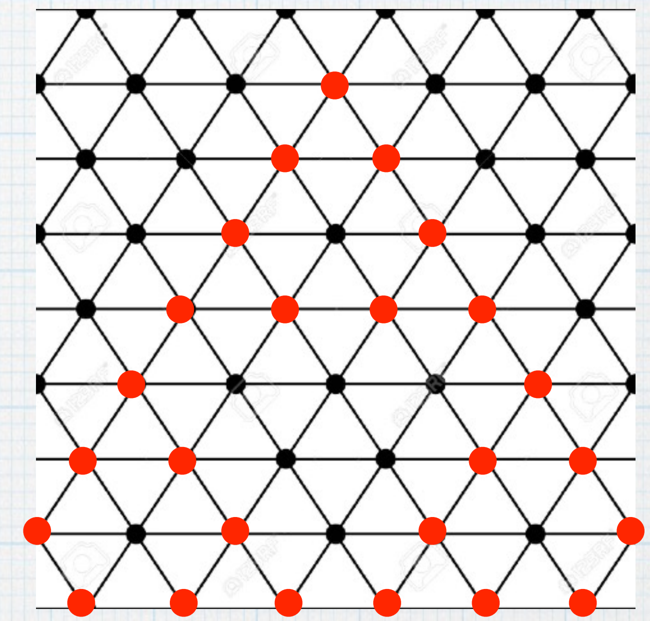

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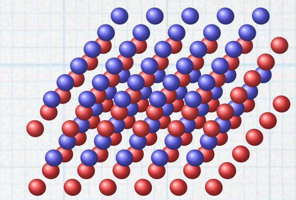

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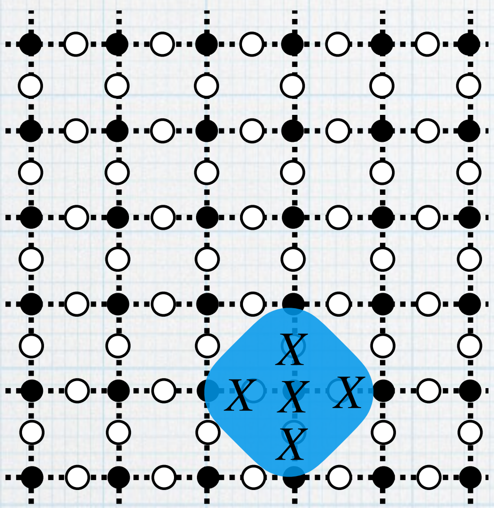

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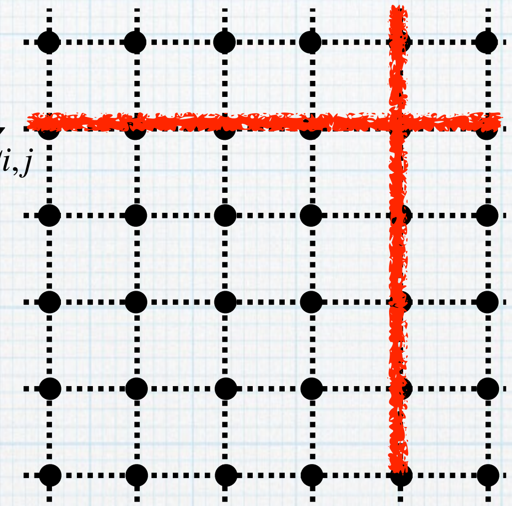

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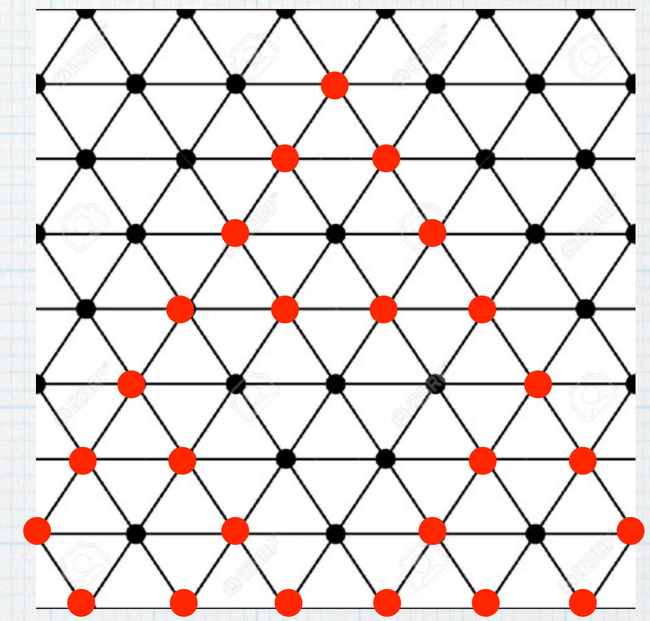

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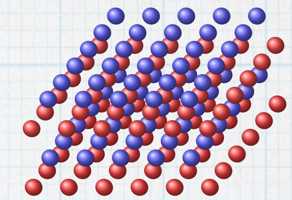

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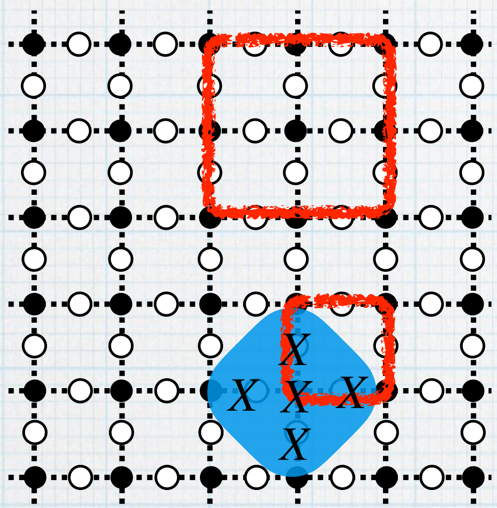

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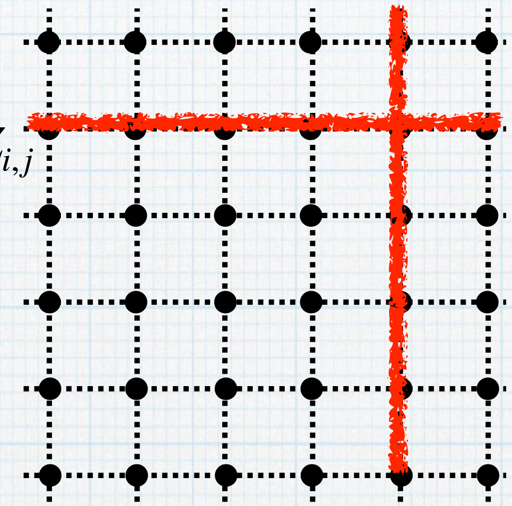

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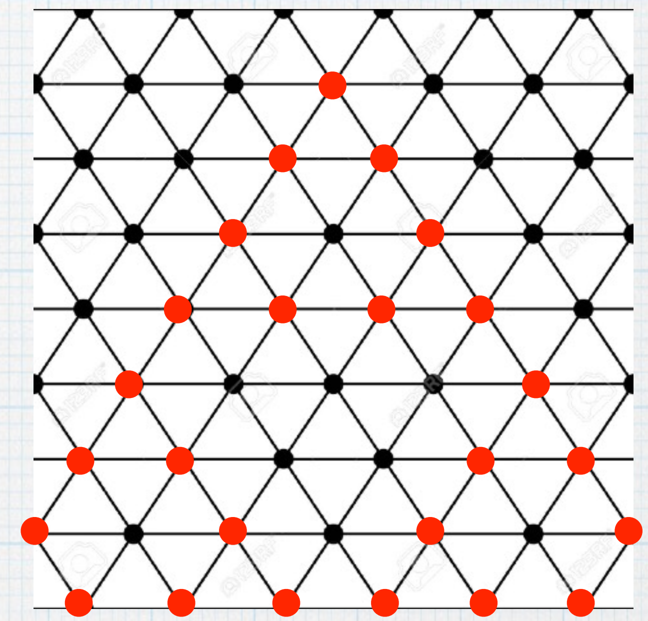

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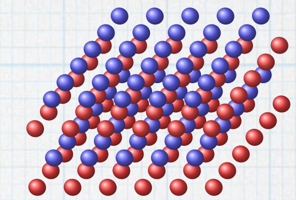

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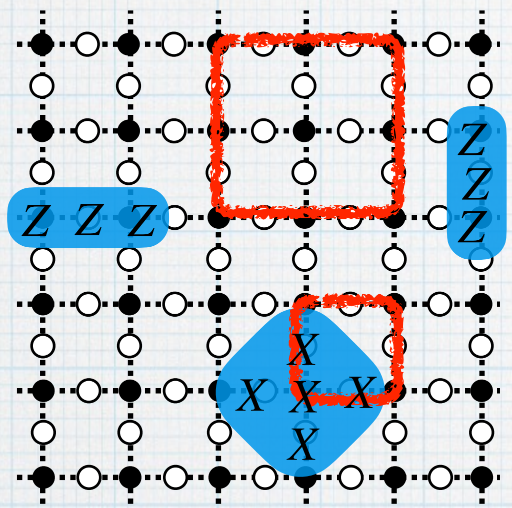

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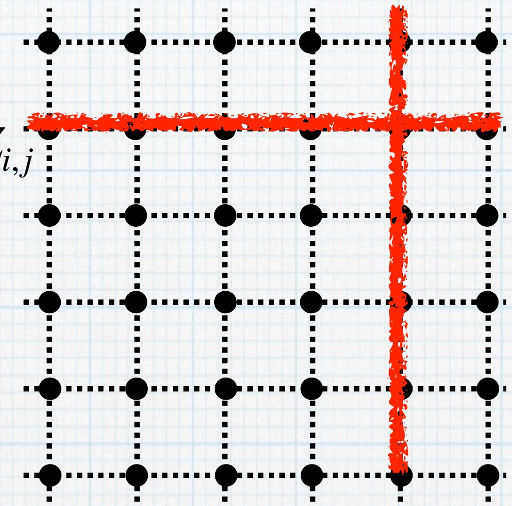

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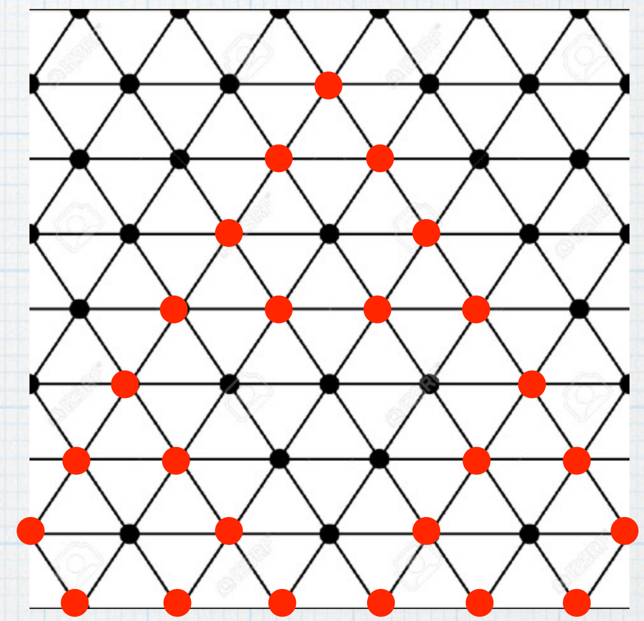

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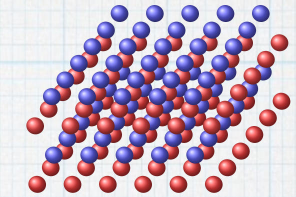

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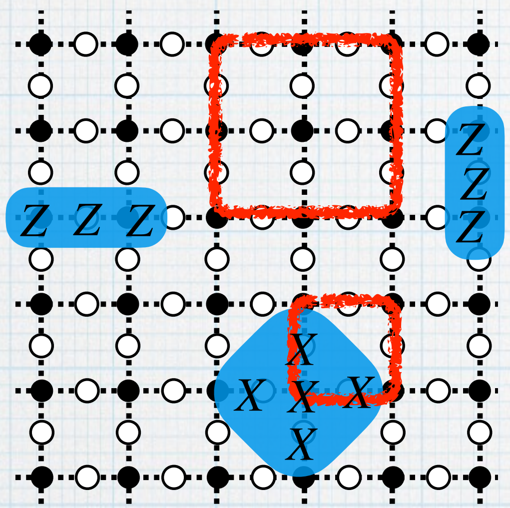
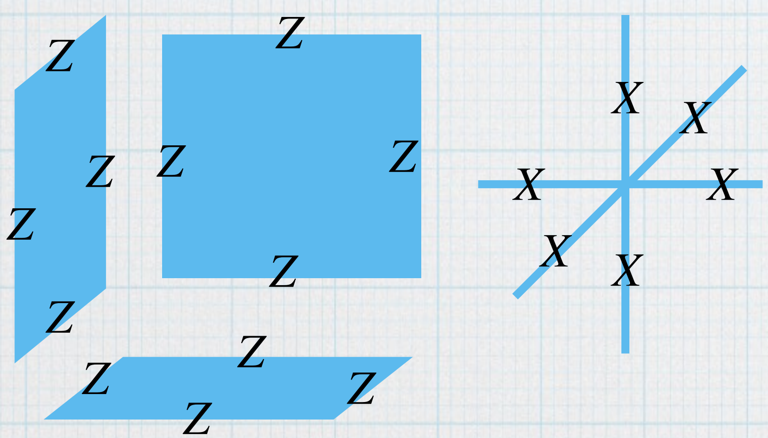


3D Toric code



[Dennis et.al. JMP. **43**, 4452 (2002)]

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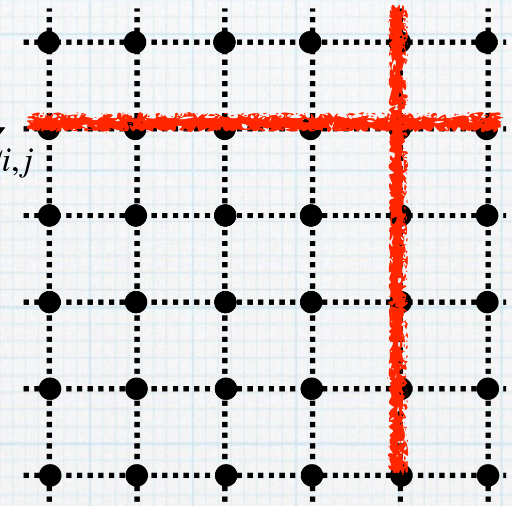

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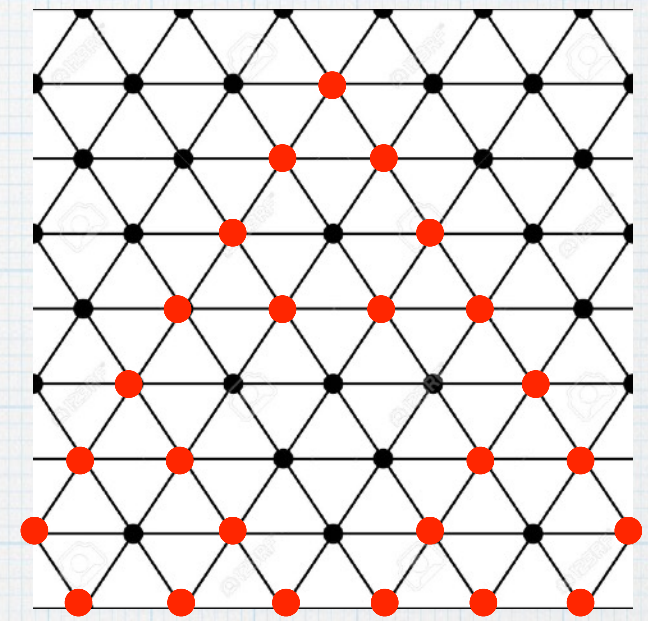

From 2D symmetries to 3D codes

Ising, plaquette Ising & CS and Newman-Moore

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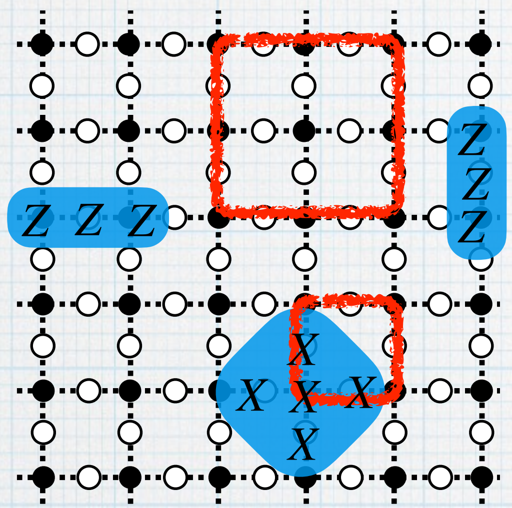

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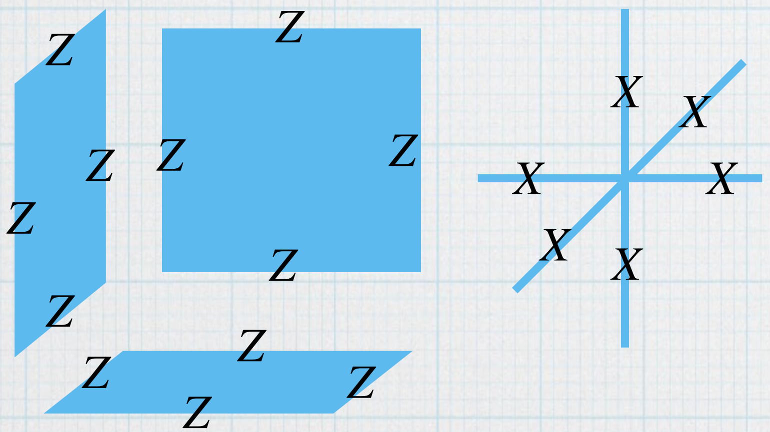

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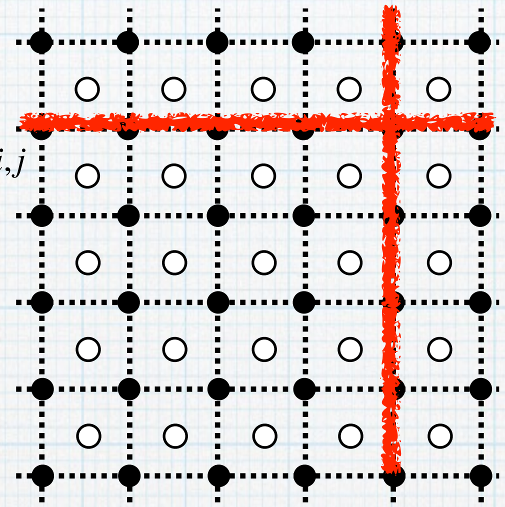

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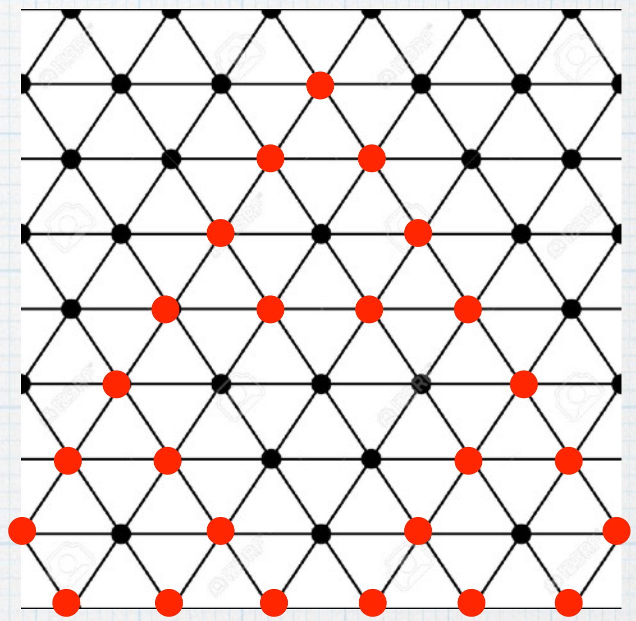

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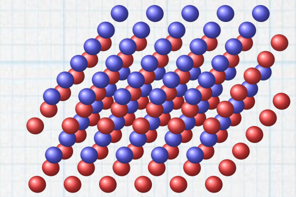

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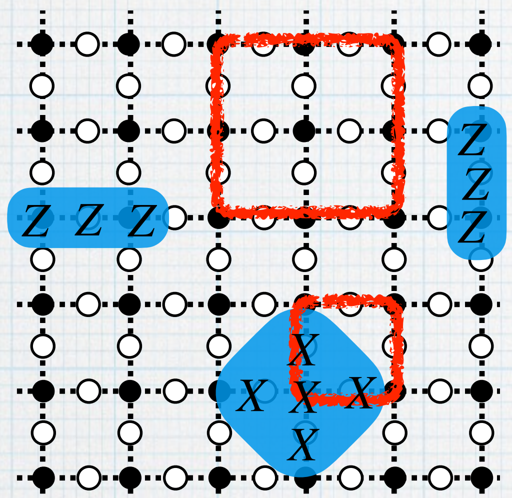

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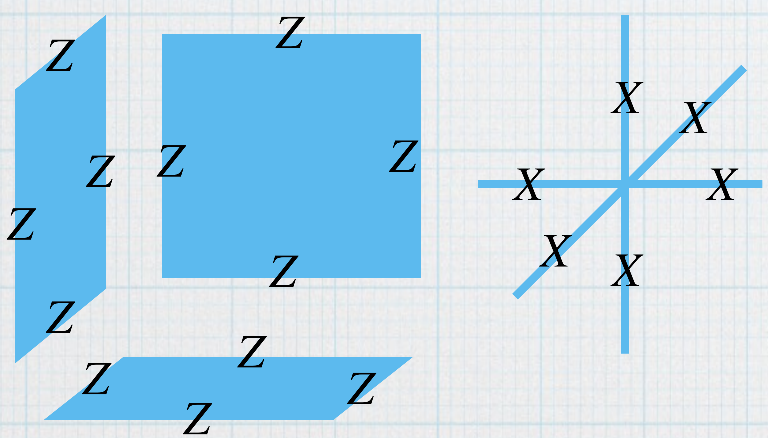

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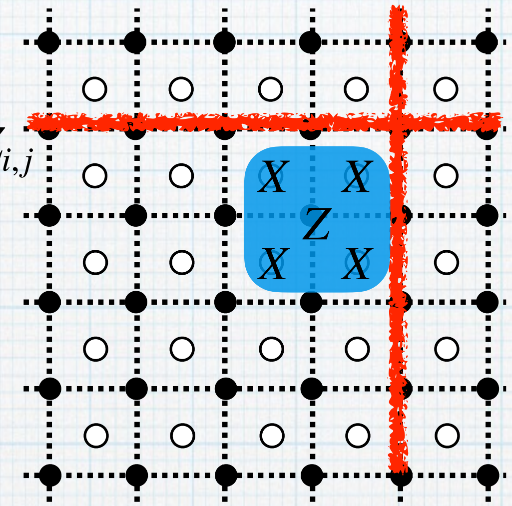

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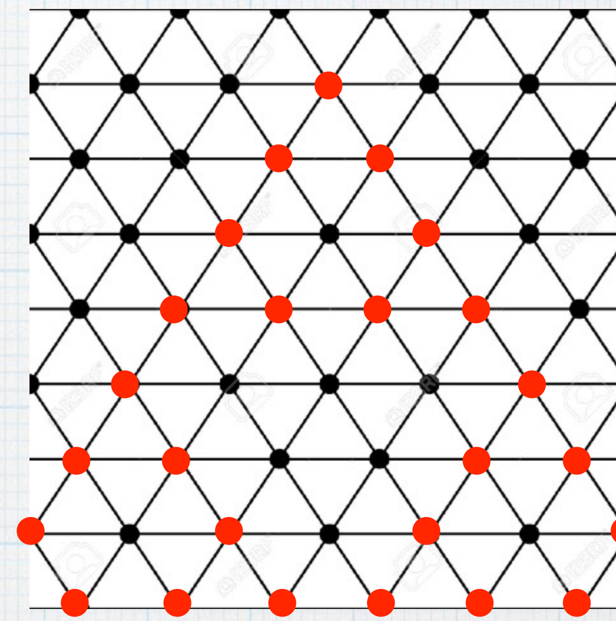

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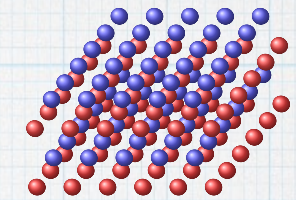
From 2D symmetries to 3D codes



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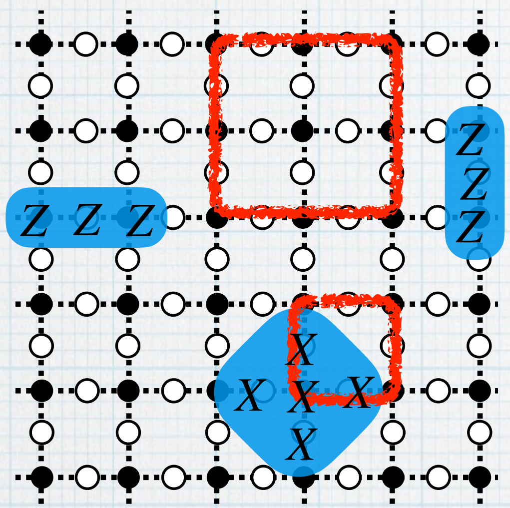

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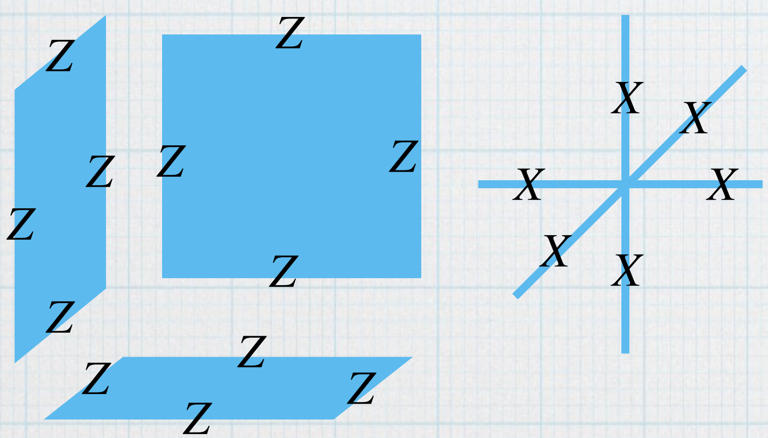

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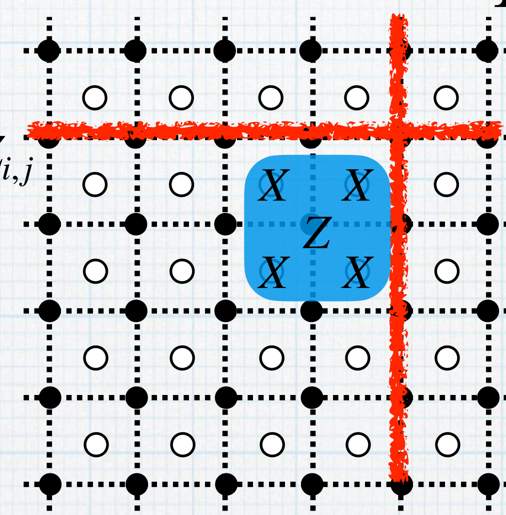
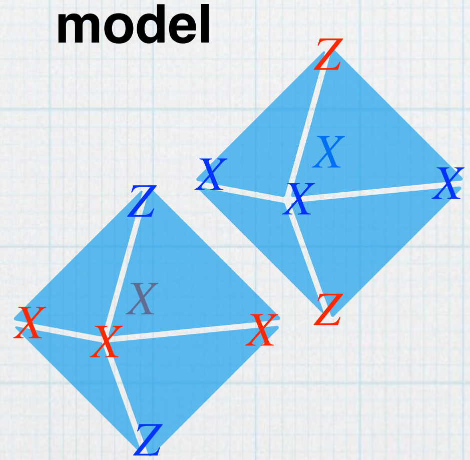


(Foliated type-I fracton)

Anisotropic model



[C. Chamon, PRL 94 (2005)]

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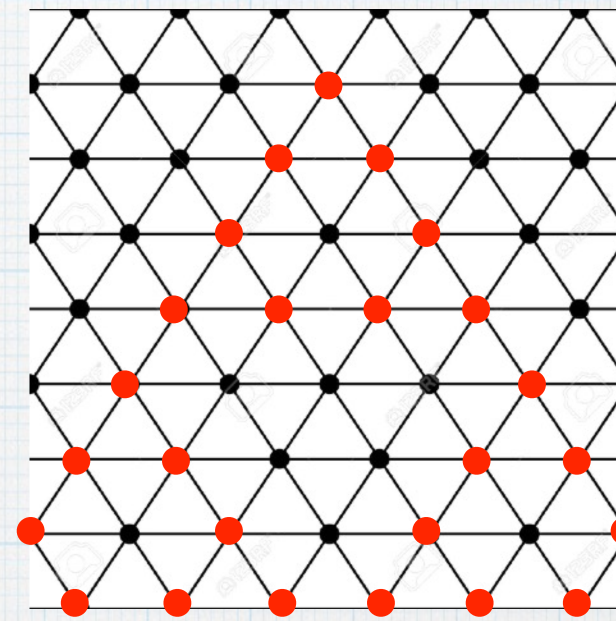

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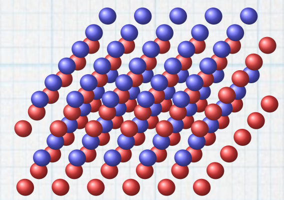

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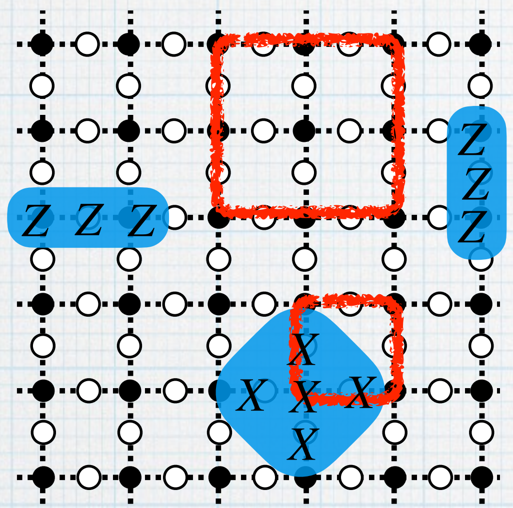

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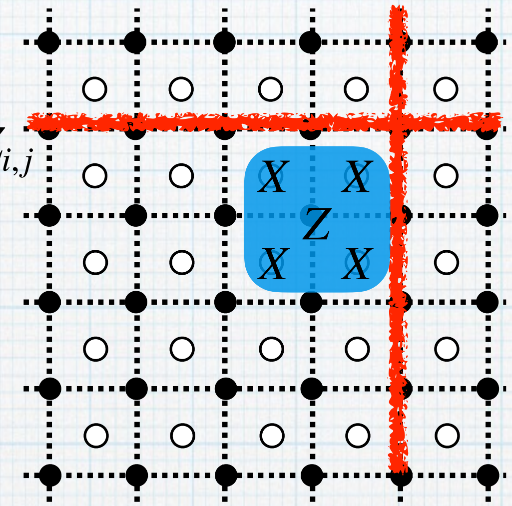

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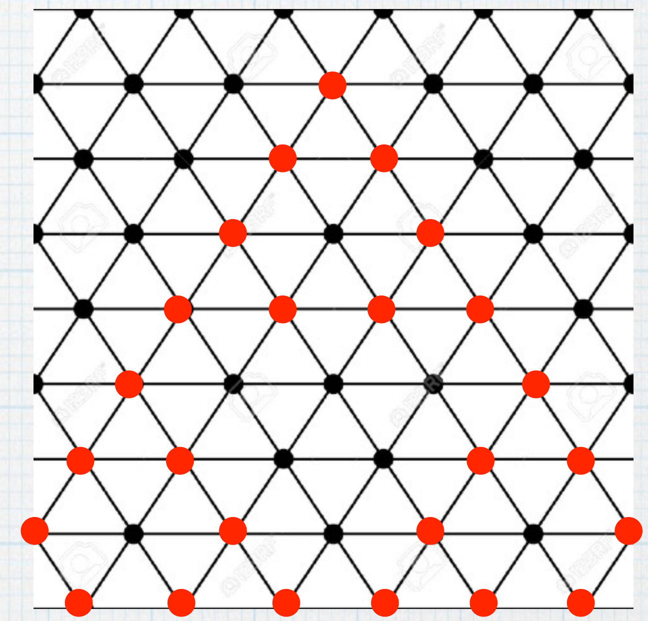

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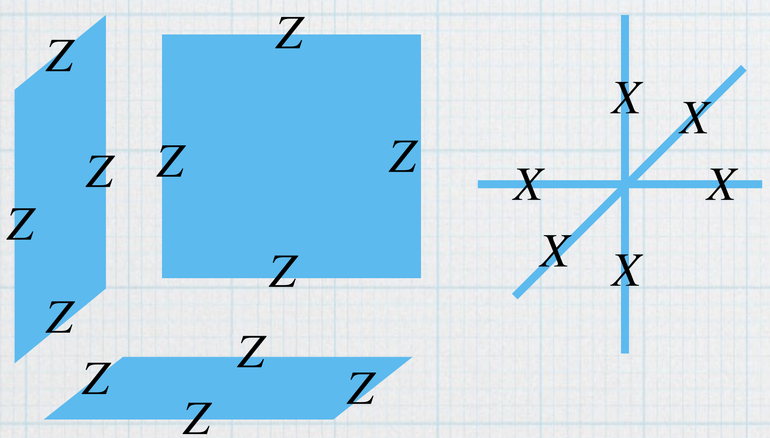

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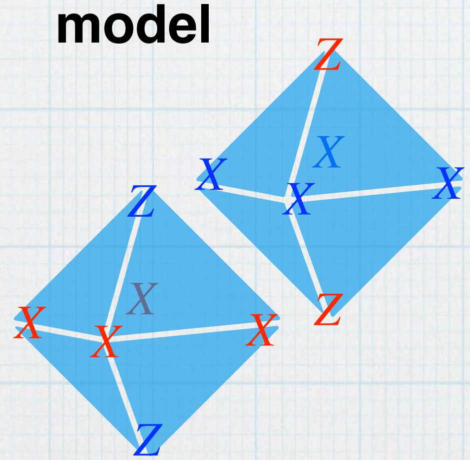

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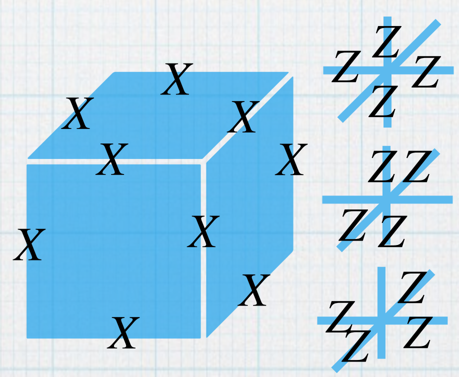
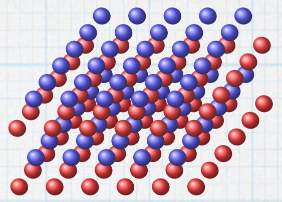
X-cube model



[Vijay et. al.PRB 94, 235157 (2016)]

Ising, plaquette Ising & CS and Newman-Moore

From 2D symmetries to 3D codes



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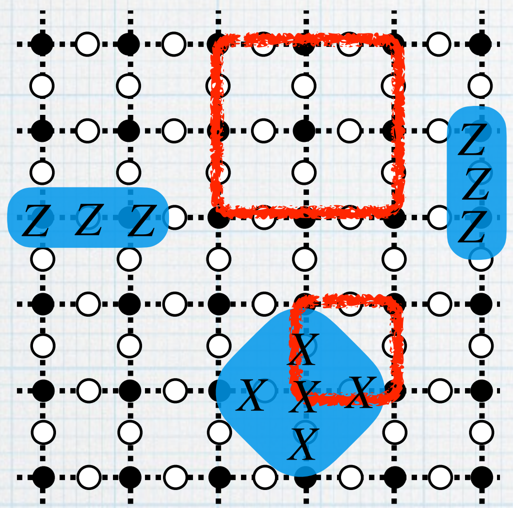

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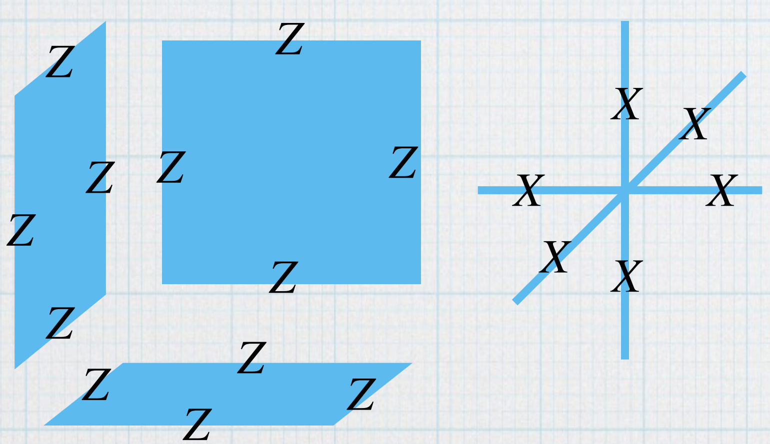

[Dennis et.al. JMP. **43**, 4452 (2002)]

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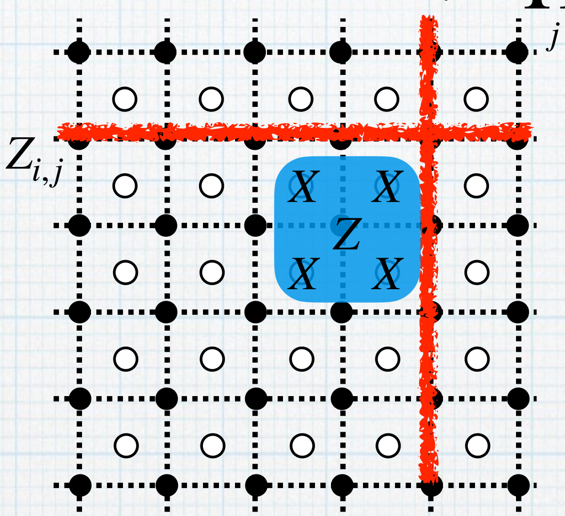

(Foliated type-I fracton)

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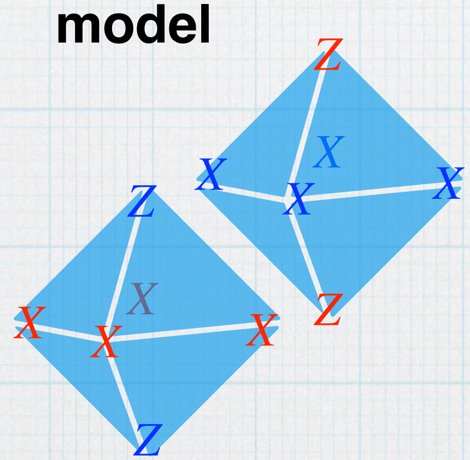

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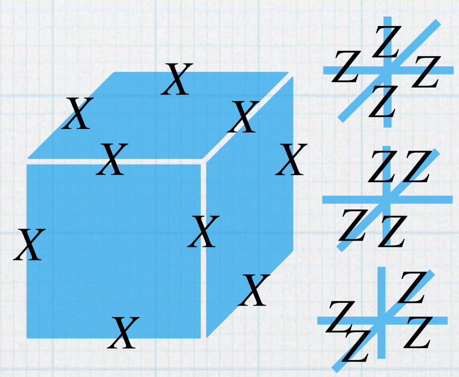

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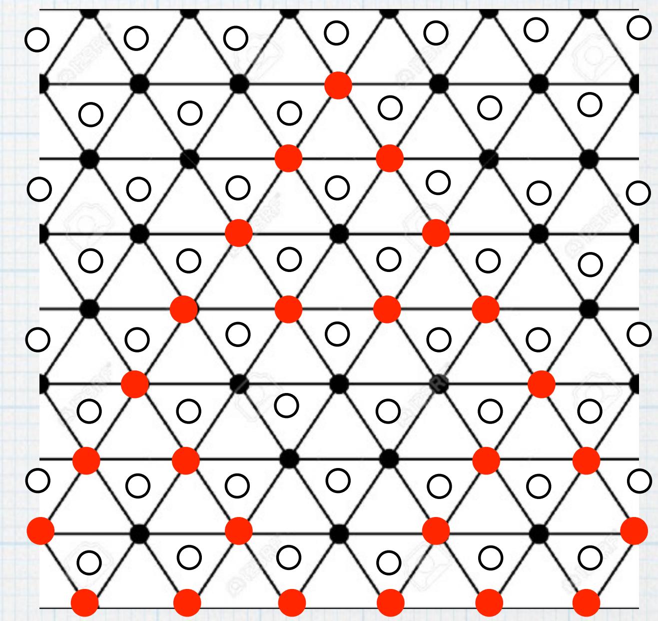

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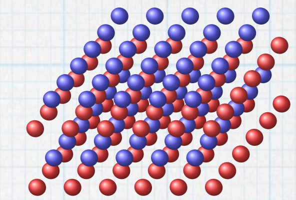

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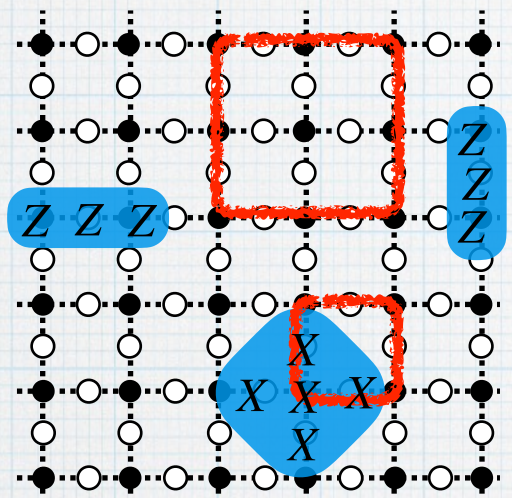

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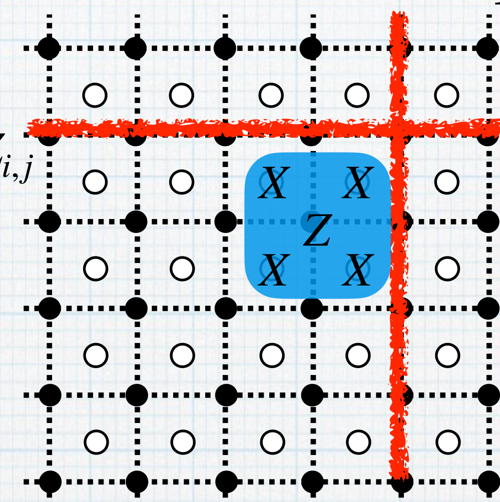

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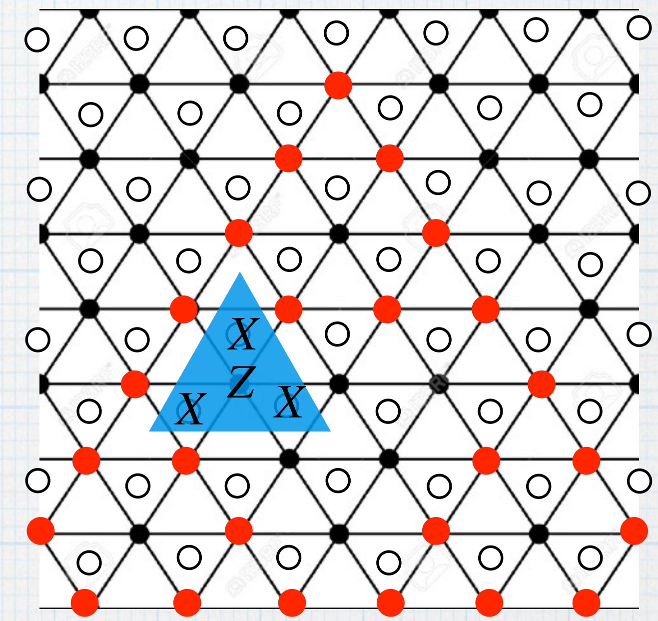

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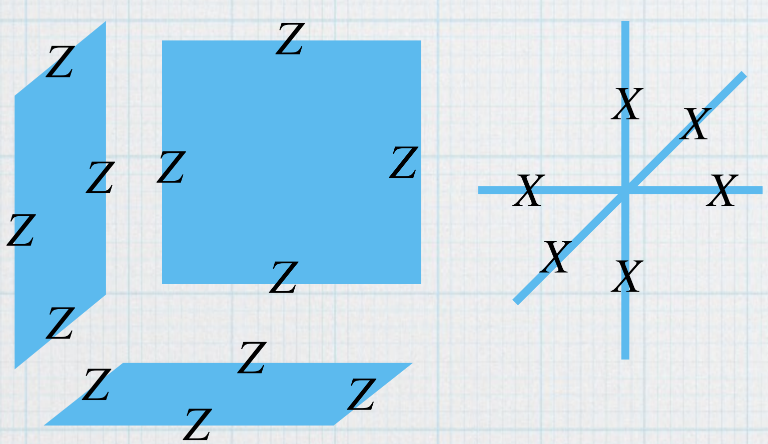

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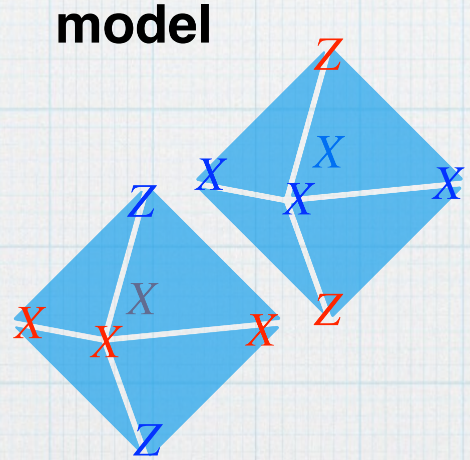

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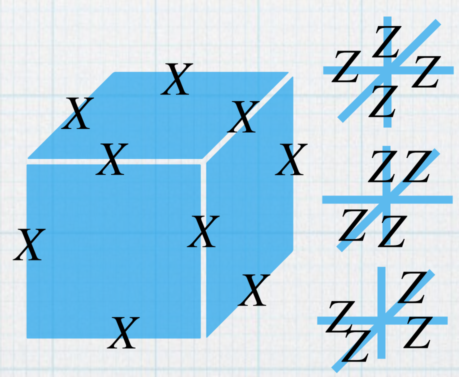

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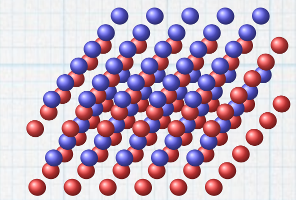

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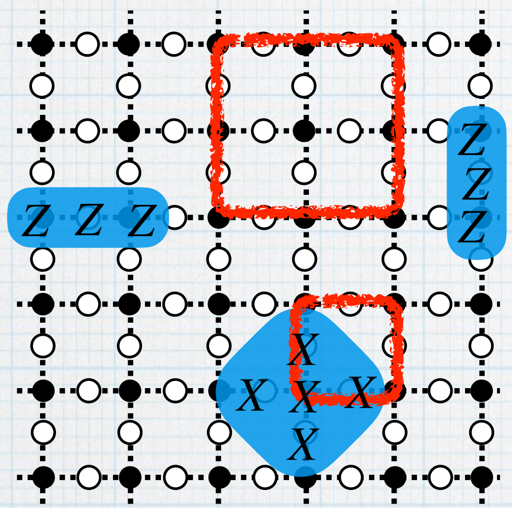

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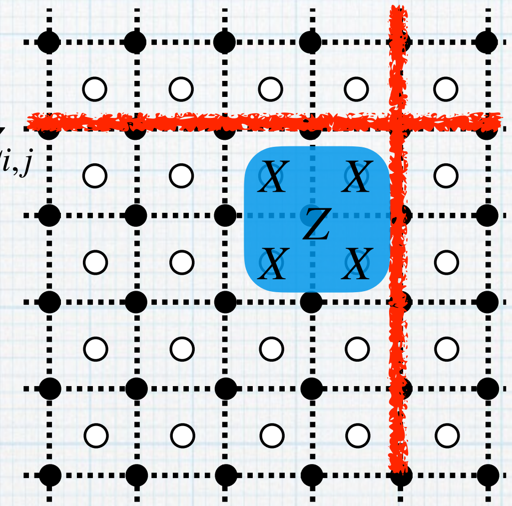

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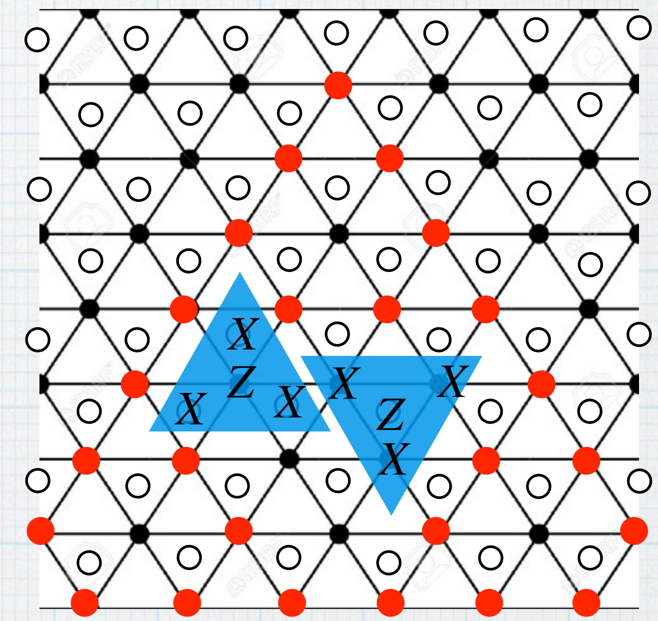

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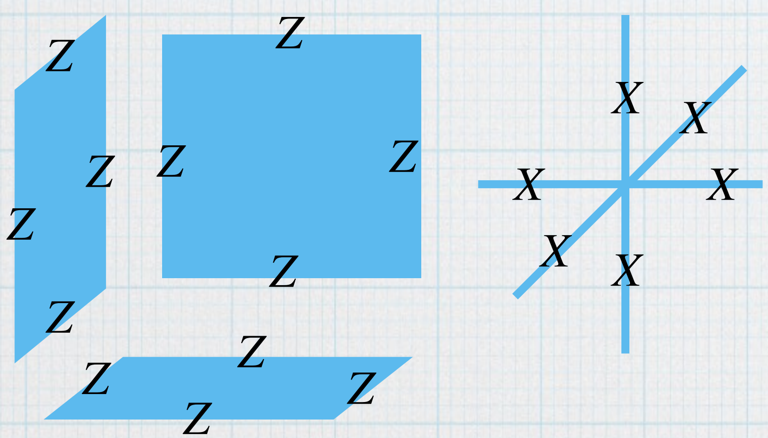

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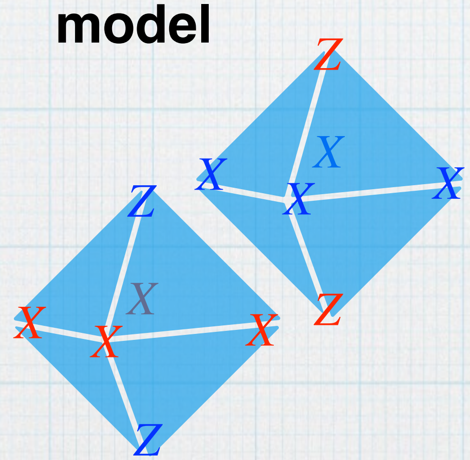

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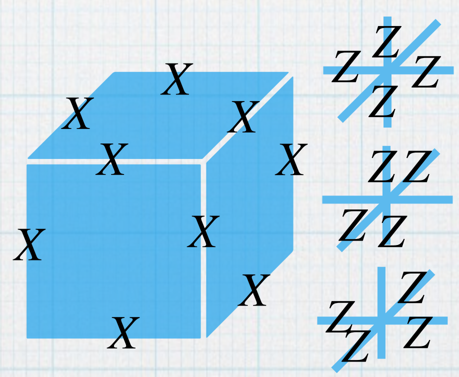

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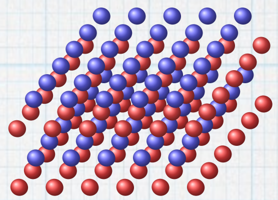

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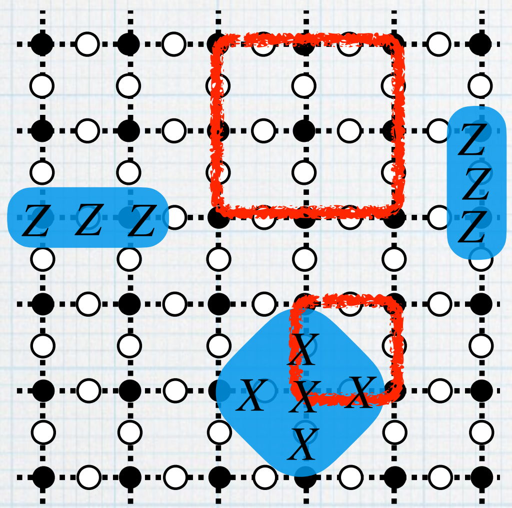

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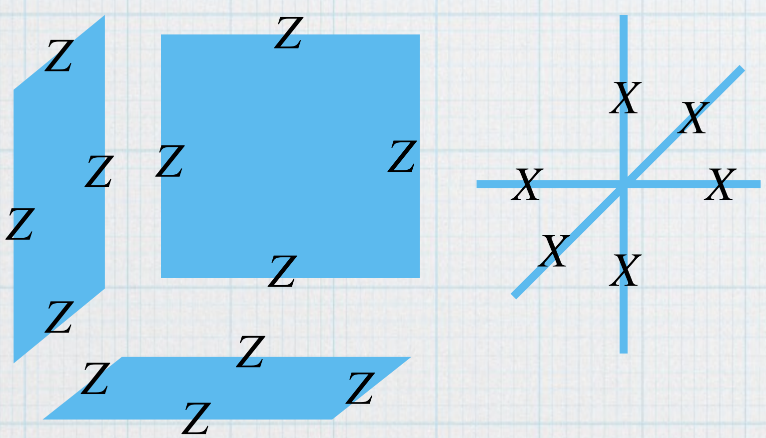

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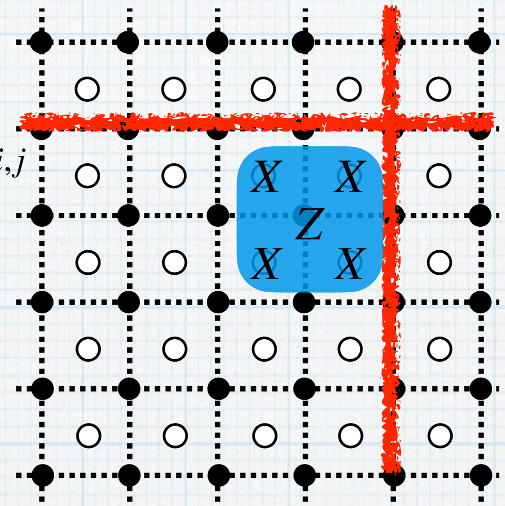

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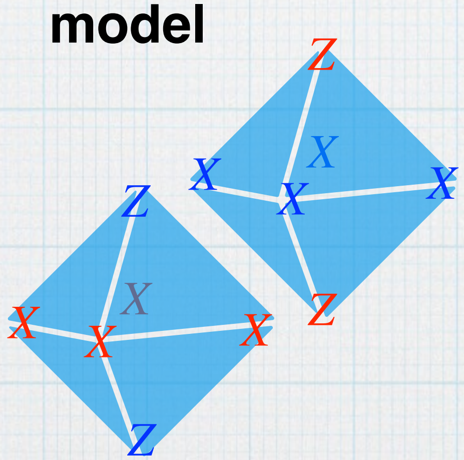

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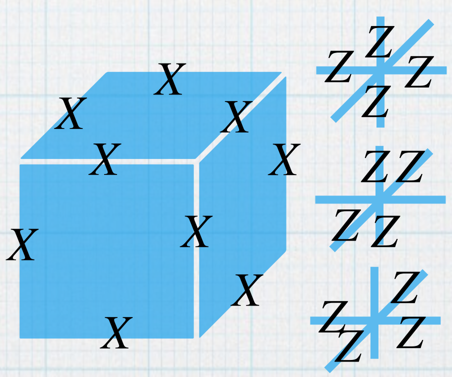

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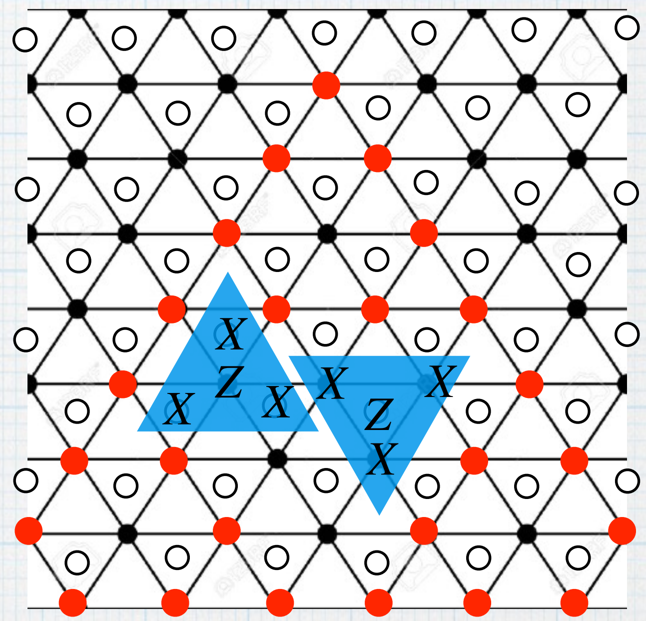

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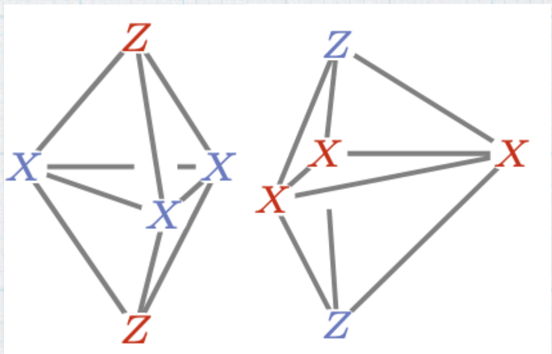

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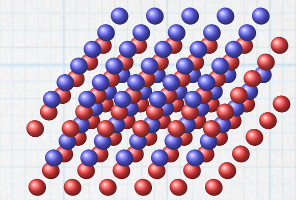

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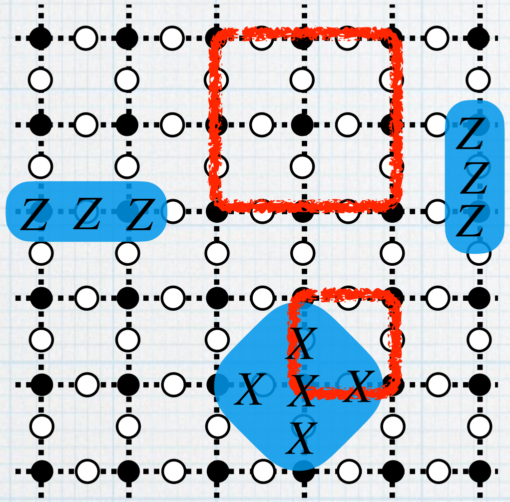

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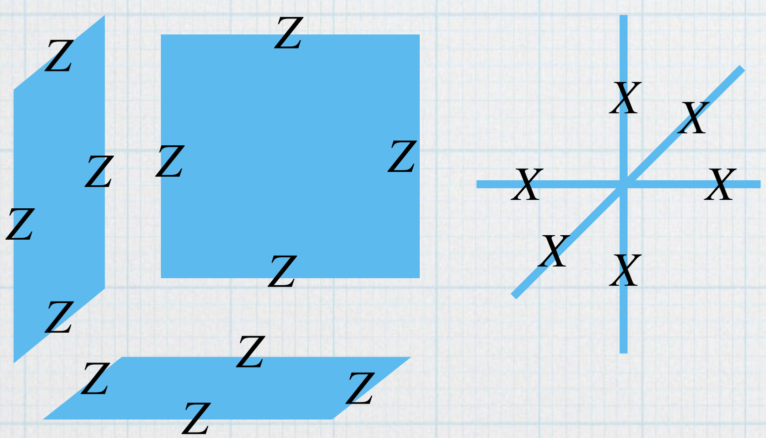

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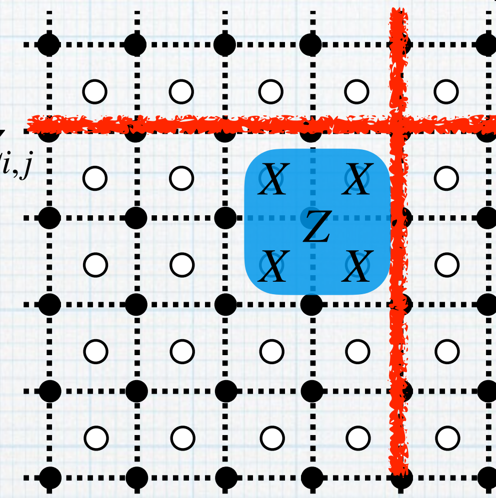

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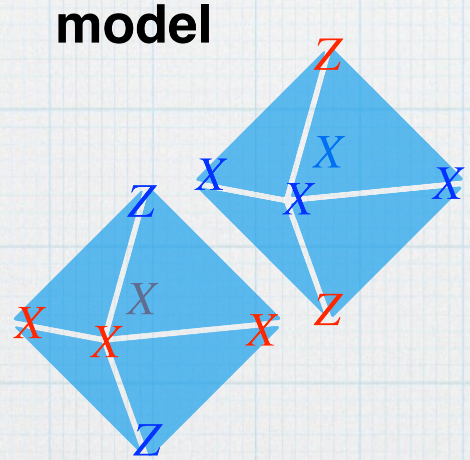

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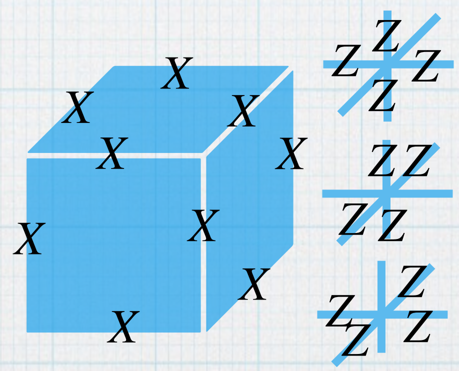

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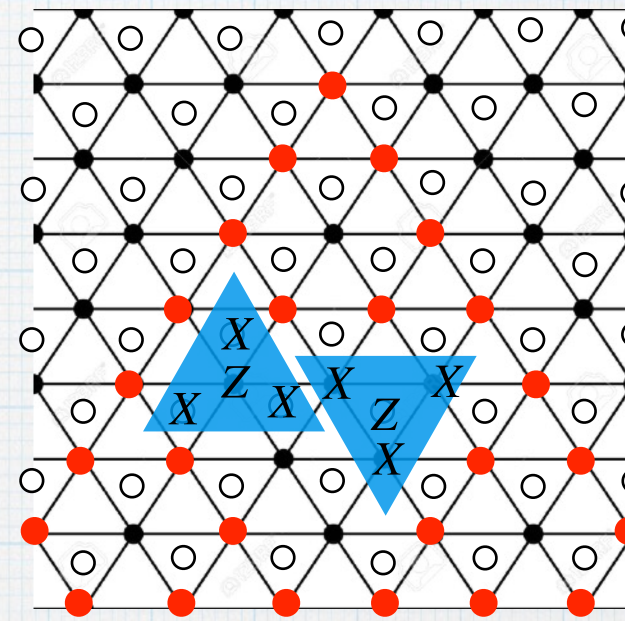

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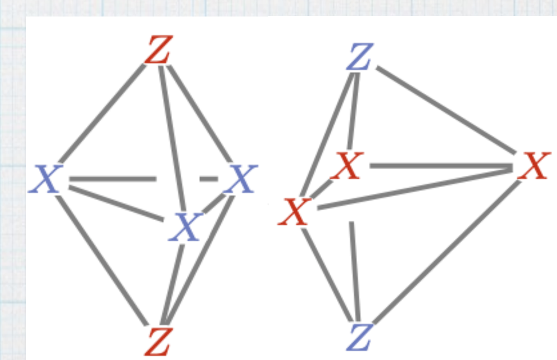

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☑ **Abelian groups, TN, symmetric phases & gapped boundaries…** *[B. Vancraeynest-De Cuiper & **JGR**, Quantum 9, 1852 (2025)]*

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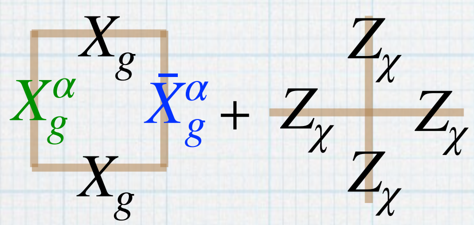

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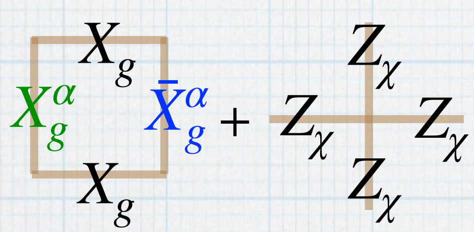
$H^{\text{bulk}}_{2D}(\alpha) = -\sum$ 

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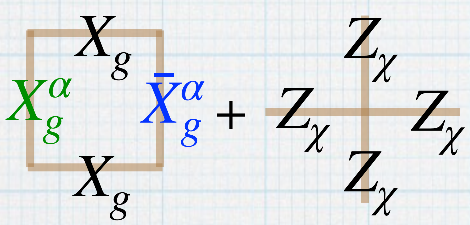
$H^{\text{bulk}}_{2D}(\alpha) = -\sum$ 

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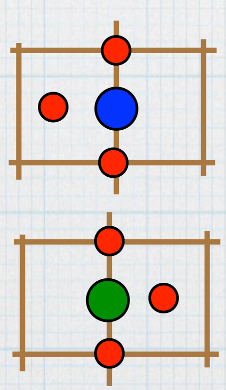
Excitation = ●



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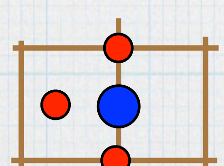
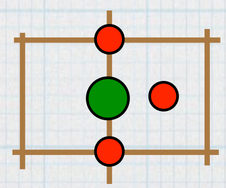
Reduce Topo. Order: $(\mathbb{Z}_2\times\mathbb{Z}_2,\alpha)\rightarrow TC+$**confinement +dipoles**

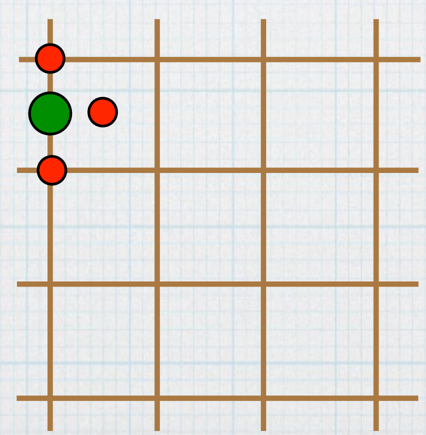
$\bar{X}_g^{\alpha}=$

$X_g^{\alpha}=$

Excitation =







[**JGR,** *PRB 112,125134* (2025)]

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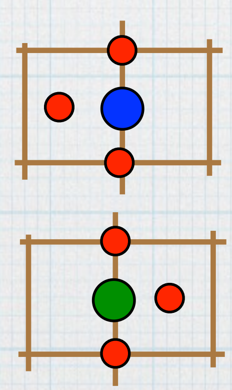

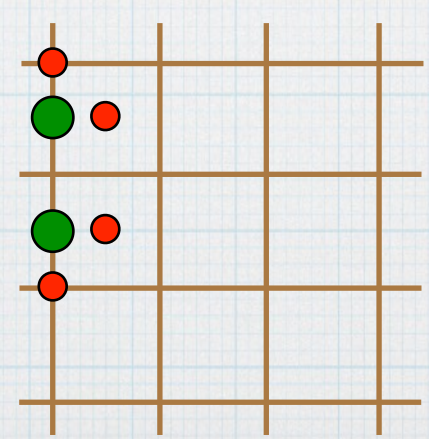

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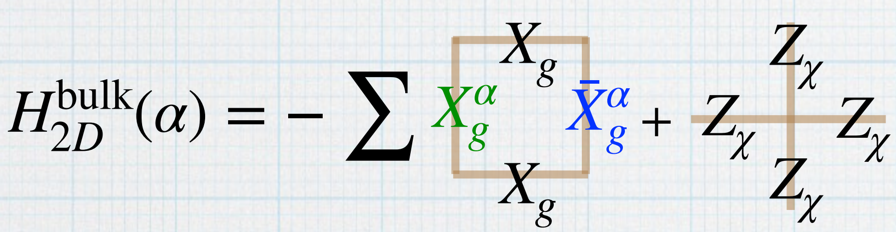

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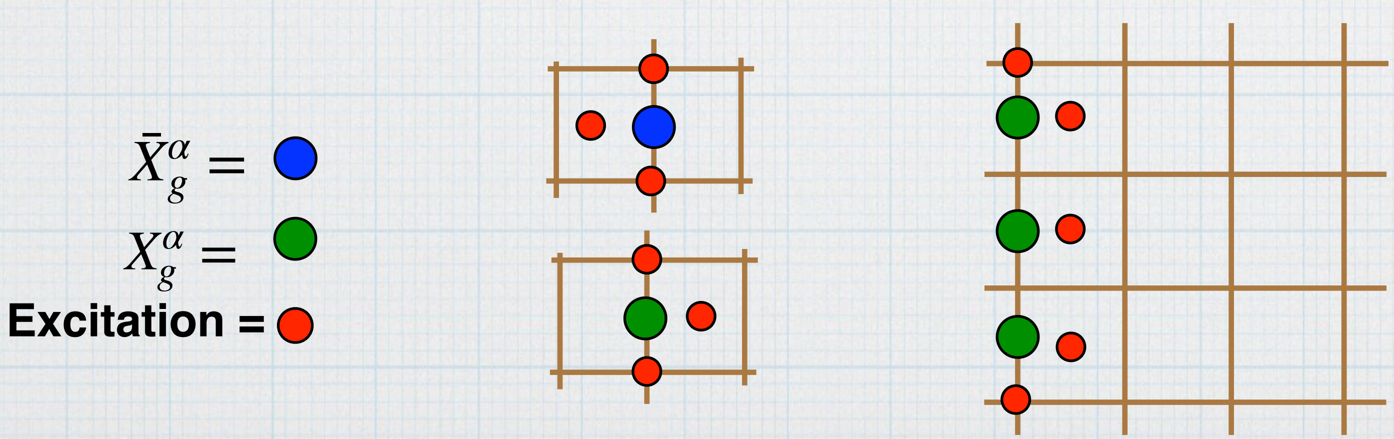

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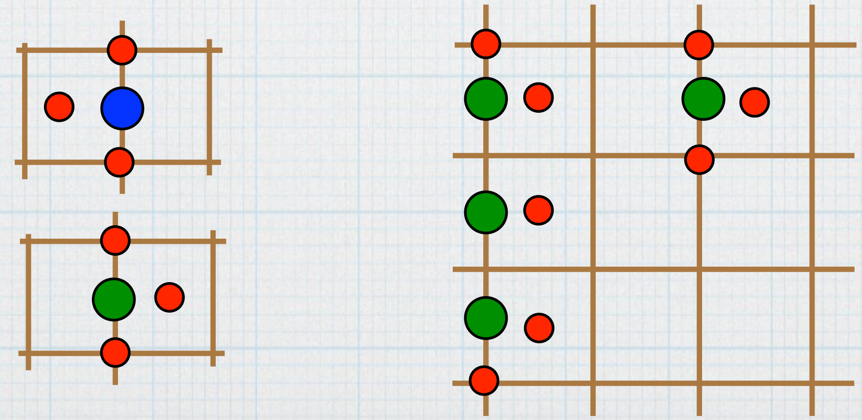

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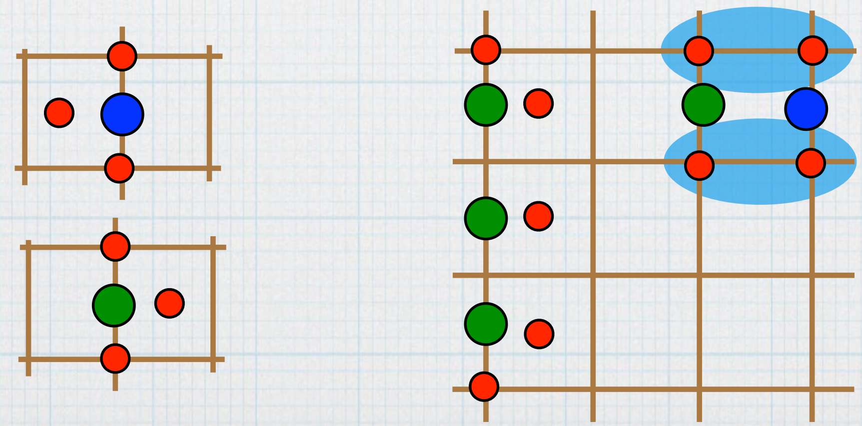
$\bar{X}_g^{\alpha} =$ (blue)

$X_g^{\alpha} =$ (green)

Excitation = (red)



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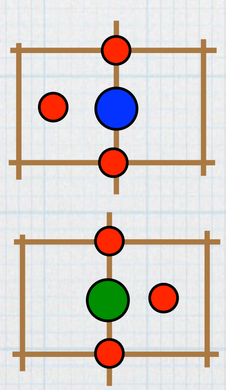

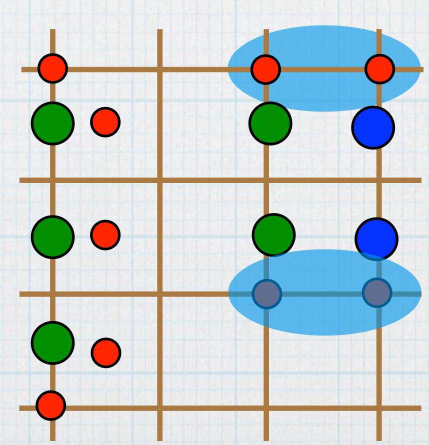

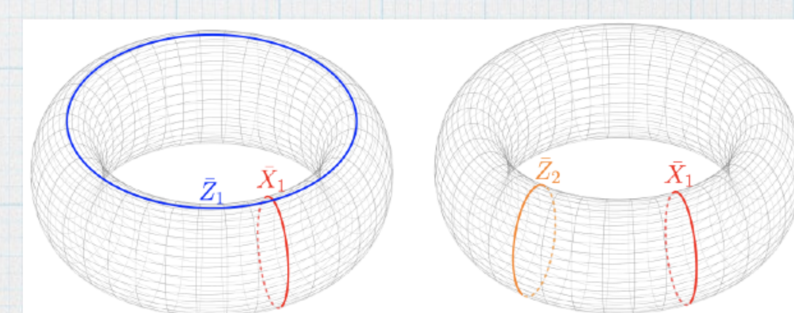

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Classifying phases protected by matrix product operator symmetries using matrix product states

José Garre-Rubio[1], Laurens Lootens[2], and András Molnár[1]

Accepted in Quantum 2023-02-10, click title to verify.

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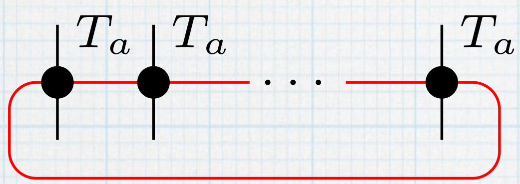
$O_a =$ 

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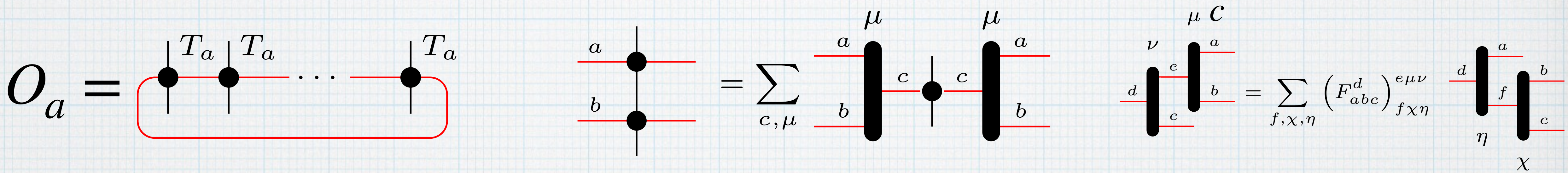

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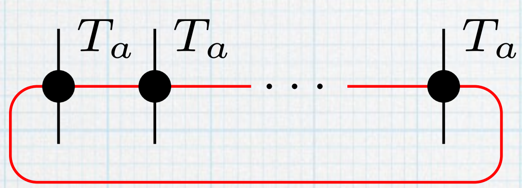
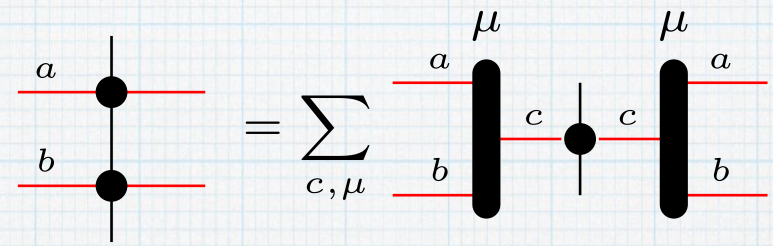
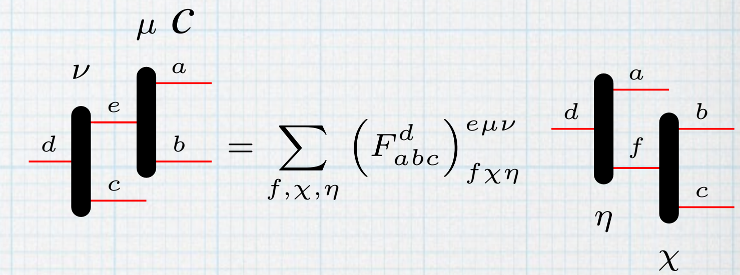
$O_a =$ 





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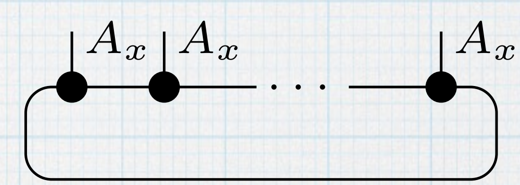
$\{\,|\psi_{A_x}\rangle =$ 

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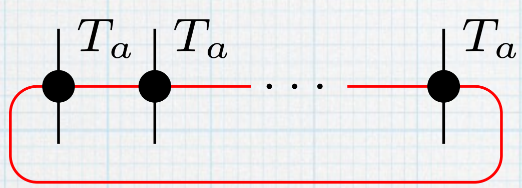
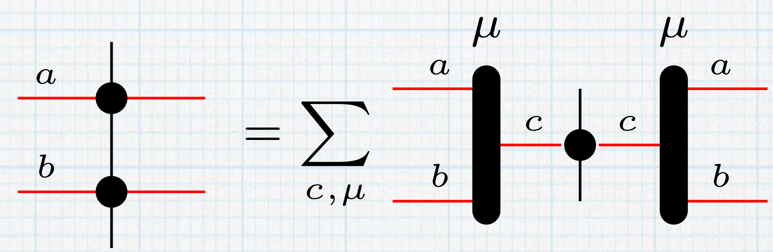
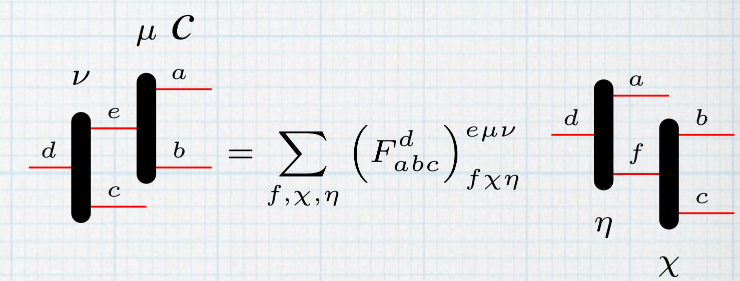
$O_a =$ 





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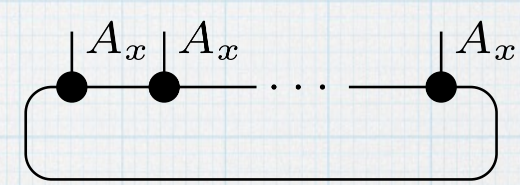
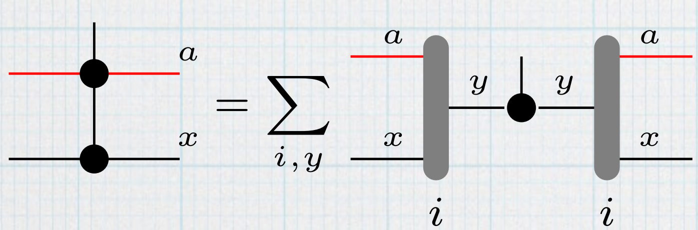
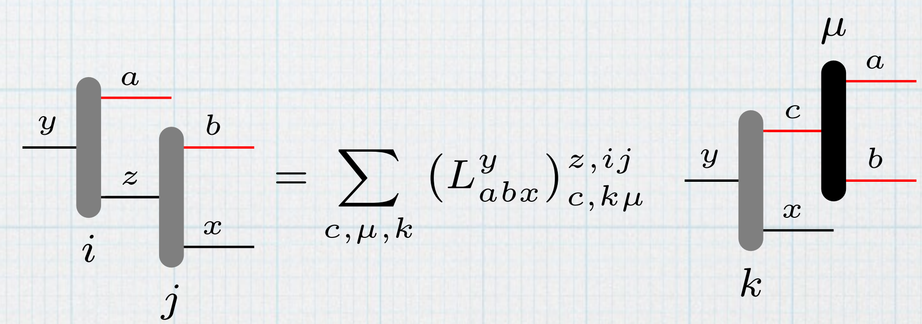
$\{|\psi_{A_x}\rangle\} =$ 





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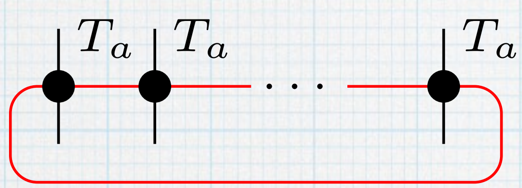
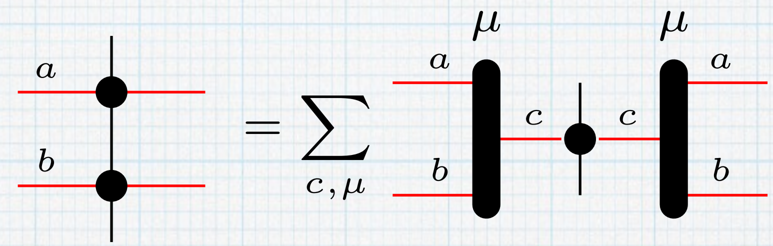
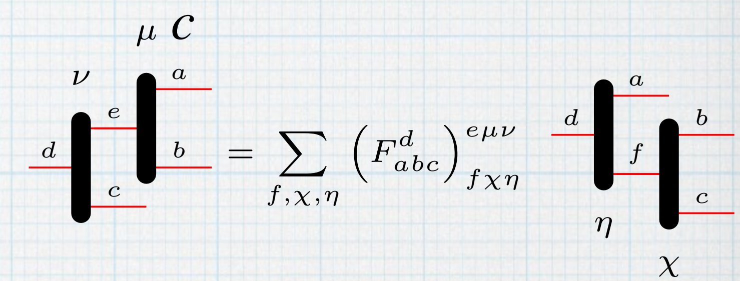
$O_a =$ 





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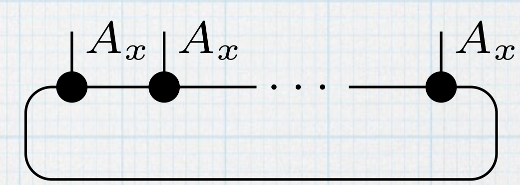
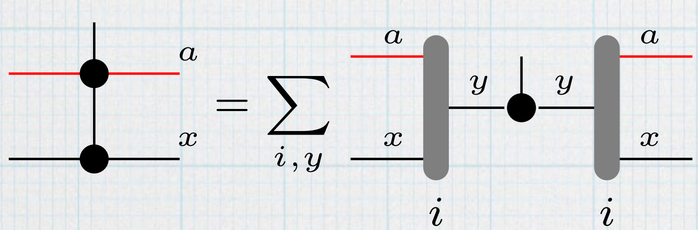
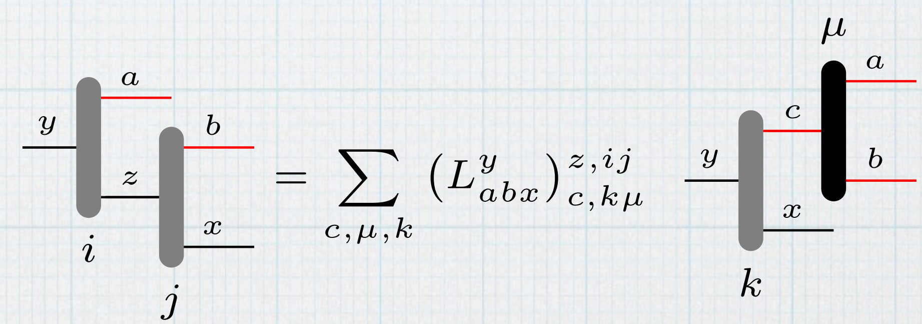
$\{ |\psi_{A_x}\rangle =$ 





$$\boxed{\sum_{d,n,\eta,\chi} \left(L^y_{abt}\right)^{z,ij}_{d,n\eta} \left(L^y_{dcx}\right)^{t,nk}_{e,m\chi} \left(F^e_{abc}\right)^{d,\chi\eta}_{f,\nu\mu} = \sum_l \left(L^z_{bcx}\right)^{t,jk}_{f,l\mu} \left(L^y_{afx}\right)^{z,il}_{e,m\nu}}$$

☑ Classification equivalent to modules of fusion categories

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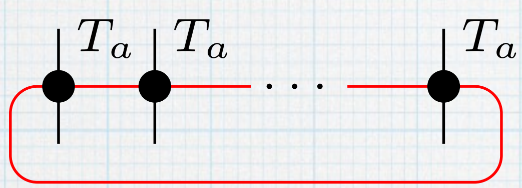
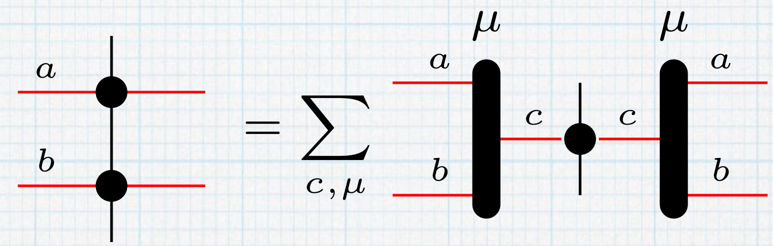
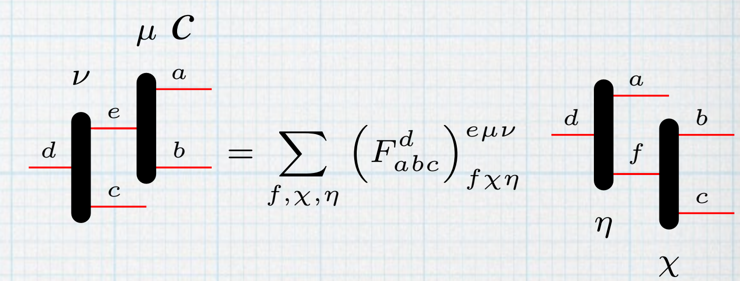
$O_a =$ 





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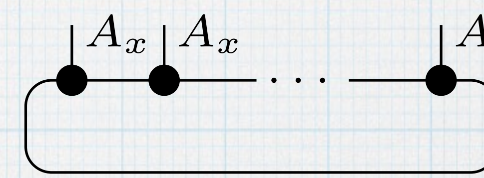
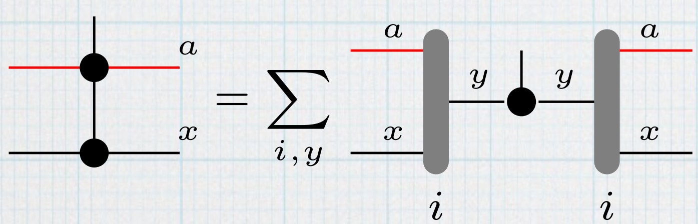
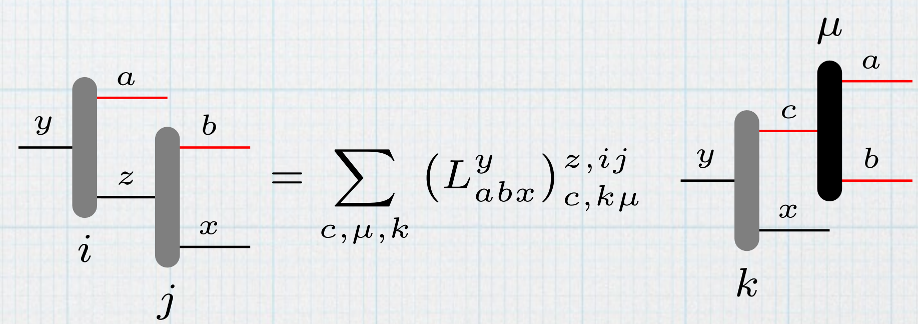
$\{ |\psi_{A_x}\rangle =$  $\}_{x \in \mathscr{I}}$





$$\sum_{d,n,\eta,\chi} \left(L_{abt}^{y}\right)_{d,n\eta}^{z,ij} \left(L_{dcx}^{y}\right)_{e,m\chi}^{t,nk} \left(F_{abc}^{e}\right)_{f,\nu\mu}^{d,\chi\eta} = \sum_l \left(L_{bcx}^{z}\right)_{f,l\mu}^{t,jk} \left(L_{afx}^{y}\right)_{e,m\nu}^{z,il}$$

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[**JGR,** A.Molnar. NJoP 27, 104510 (2025)]

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Classifying phases protected by matrix product operator symmetries using matrix product states

José Garre-Rubio[1], Laurens Lootens[2], and András Molnár[1]

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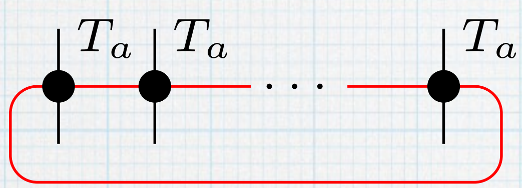
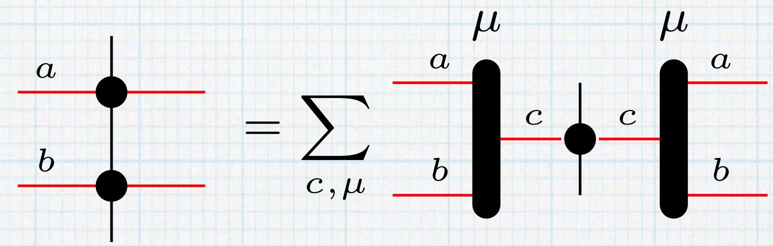
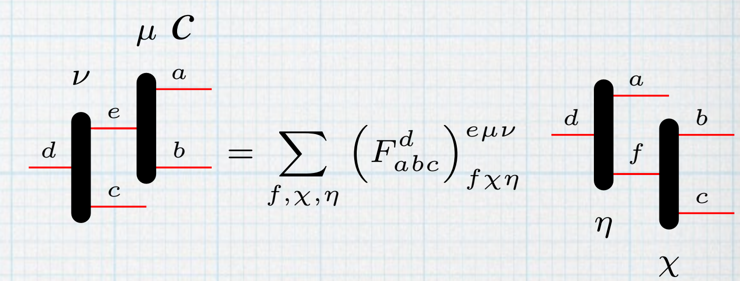
$O_a =$ 





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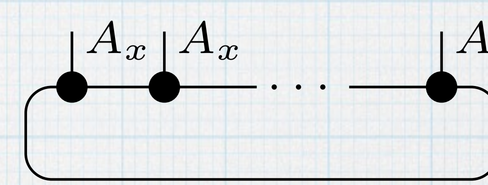
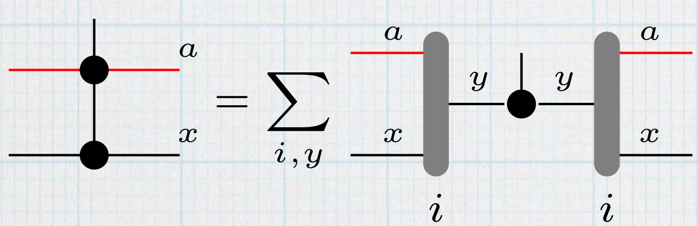
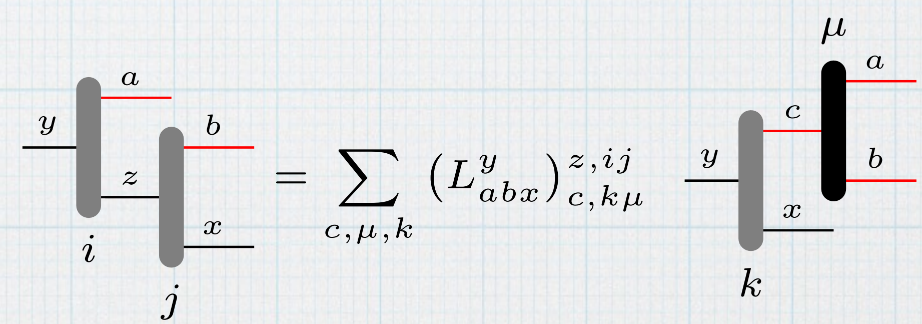
$\{ |\psi_{A_x}\rangle =$  $\}_{x\in\mathscr{I}}$





$$\boxed{\sum_{d,n,\eta,\chi} \left(L^y_{abt}\right)^{z,ij}_{d,n\eta} \left(L^y_{dcx}\right)^{t,nk}_{e,m\chi} \left(F^e_{abc}\right)^{d,\chi\eta}_{f,\nu\mu} = \sum_l \left(L^z_{bcx}\right)^{t,jk}_{f,l\mu} \left(L^y_{afx}\right)^{z,il}_{e,m\nu}}$$

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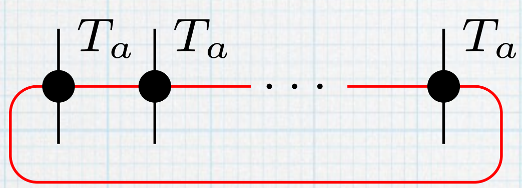
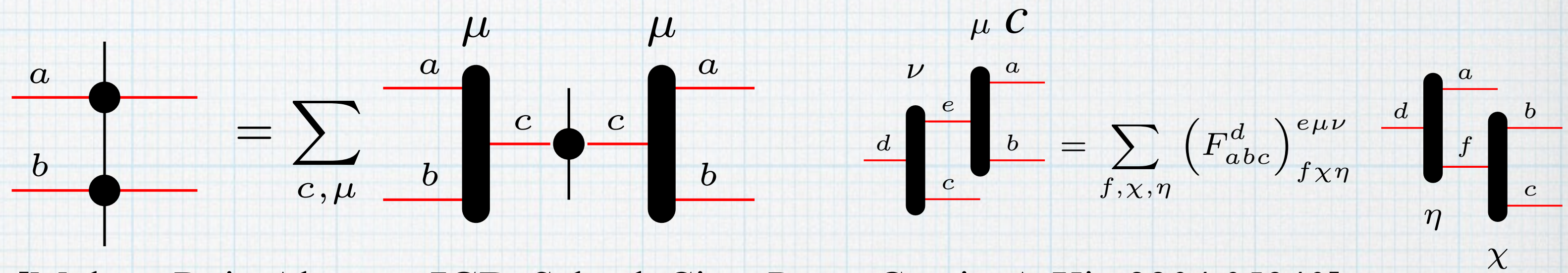
$O_a =$ 



[Molnar, Ruiz-Alarcon, **JGR,** Schuch,Cirac,Perez-Garcia, ArXiv:2204.05940]

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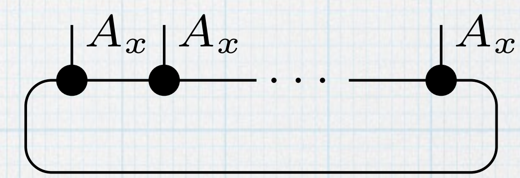
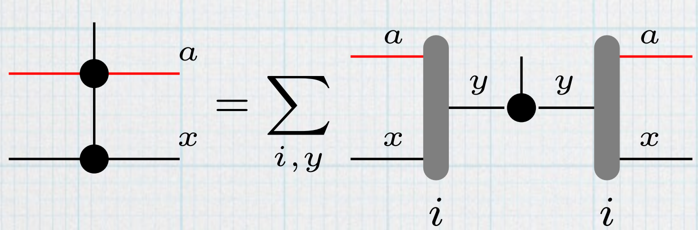
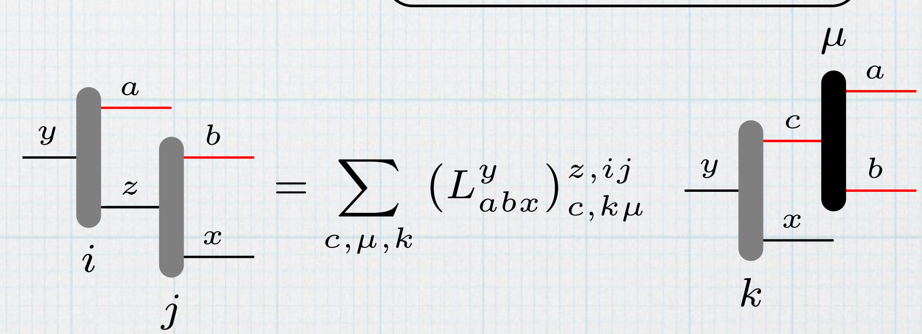
$\{ |\psi_{A_x}\rangle =$  $\}_{x \in \mathscr{I}}$





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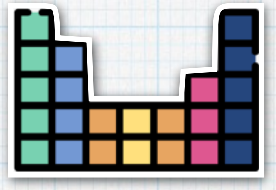
Outlook

* **Emergent principle to construct and order phases ~ Periodic table** 

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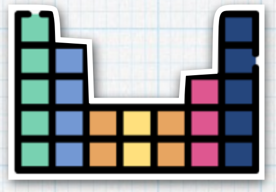

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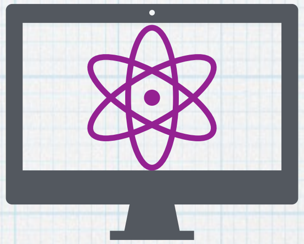
[D. Aasen et. al., *Microsoft Quantum,* arXiv:2506.15130]   [D. Hayes et. al., Quantinuum, *PRA 110,062413 (2024)*]



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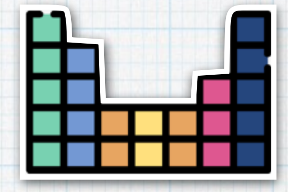

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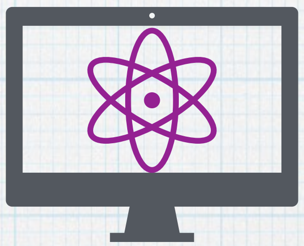

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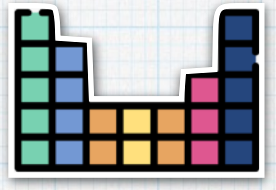

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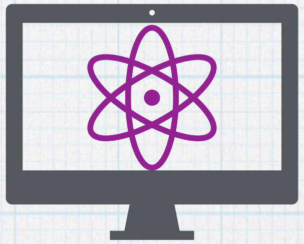

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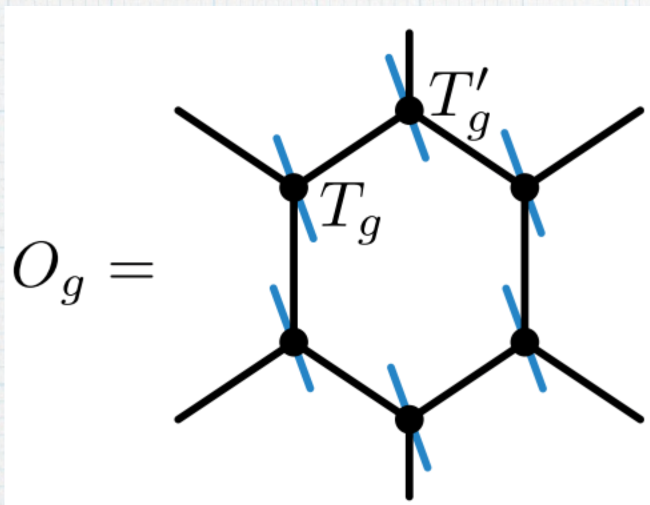
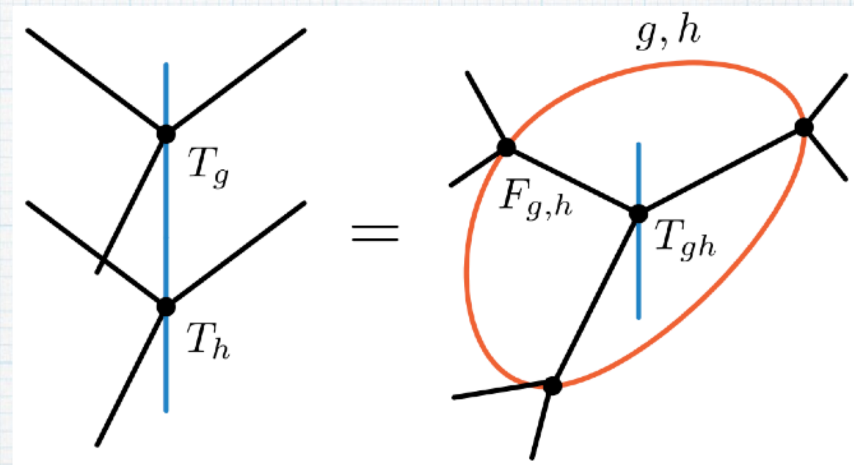
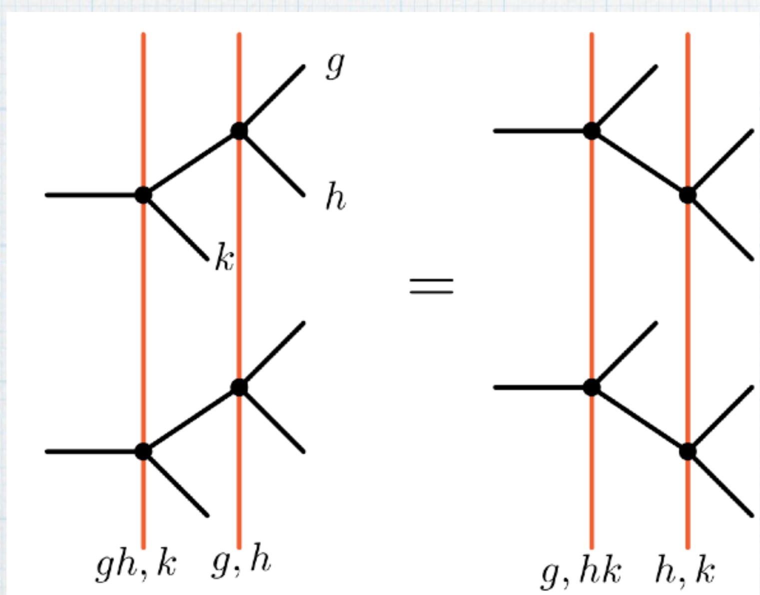
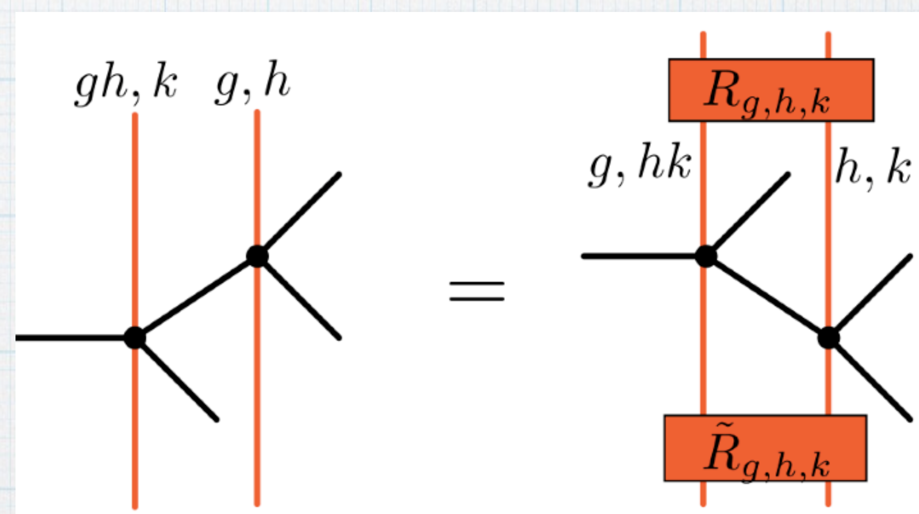
THANKS

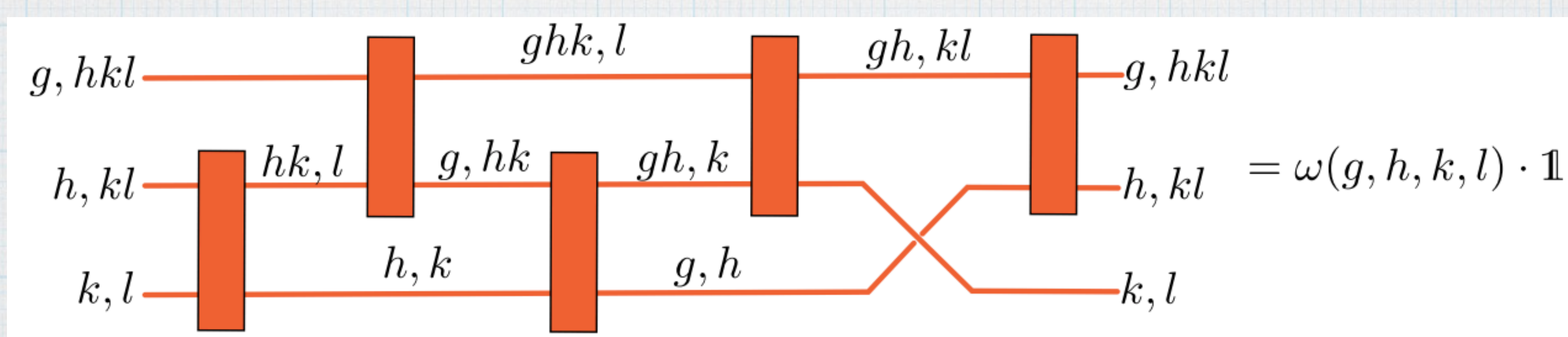
Anomalous 2D symmetries: TN approach











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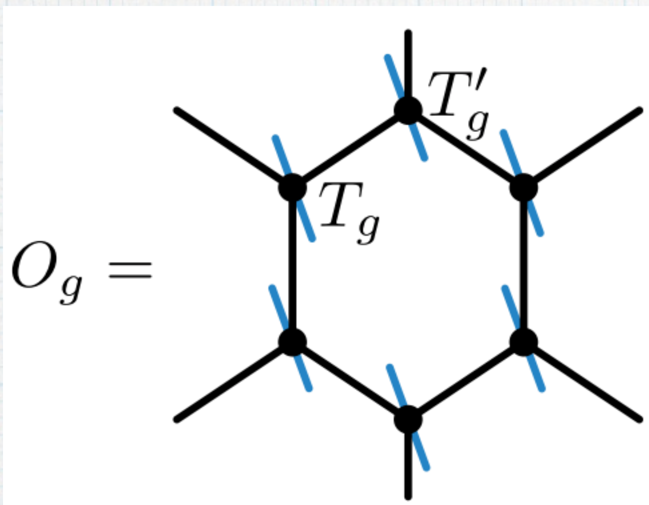

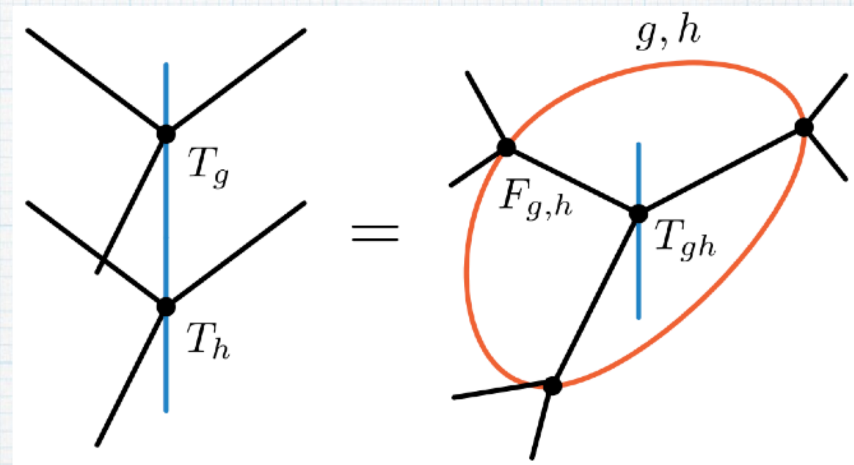

$$O_g^{\Omega} O_h^{\Omega} = O_{gh}^{\Omega} \mathscr{F}_{g,h}^{\partial\Omega}$$