Gapless Exotic/Foliated dual models

Based on 2504.11449 w/ F. Apruzzi and F. Bedogna; 2602.03926 w/ F. Bedogna

Developments on Generalized Symmetries of Quantum Systems - Murcia ’26

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Introduction

Subsystem symmetries

Theories with non-conventional symmetries require further extension of the symmetry properties.

We are interested in conservation equations of the form

$$\partial_0 J_0 = \partial_i \partial_j \ldots \partial_k J^{ij\ldots k}$$

with the time and spatial components of the current in some representation of the rotation group.

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with the time and spatial components of the current in some representation of the rotation group.

Examples are *Subsystem Symmetries*, whose conserved charge is integrated over a closed subspace

$$Q = \int_\Sigma J_0$$

In [Seiberg, Shao ’18, ’20 - Gorantla, Lam, Seiberg, Shao ’20, ’21, ’22 - Rudelius, Seiberg, Shao ‘20], models with such $U(1)$ global symmetry has discontinuous field configuration; and Lorentz symmetry is broken. Such theories are not yet systematically understood.

Introduction

Example: XY-Plaquette

In (2+1)d the XY-plaquette model with Lagrangian $\mathscr{L} = \frac{\mu_0}{2}\left(\partial_0\phi\right)^2 + \frac{1}{2\mu}\left(\partial_x\partial_y\phi\right)^2$

$\phi \sim \phi + 2\pi w^x(x) + 2\pi w^y(y)$, $w^i(x^i) \in \mathbb{Z}$ discontinuous functions.

[Seiberg, Shao ’18 - Gorantla, Lam, Seiberg, Shao ’21]

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- Subgroup $\mathbb{Z}_4$ of the space rotation group;

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Symmetries:

- Subgroup $\mathbb{Z}_4$ of the space rotation group;
- $U(1)$ momentum dipole symmetry, $J_0 = \mu_0\partial_0\phi \in \mathbf{1}_0$, $J^{xy} = \frac{1}{\mu}\partial^x\partial^y\phi \in \mathbf{1}_2$,

  shifting the field as $\phi(t,x,y) \to \phi(t,x,y) + c^x(x) + c^y(y)$;

$$Q_{x^i}(x^i) = \oint dx^j J_0$$

$$\int_{x_1^i}^{x_2^i} Q_{x^i}(x^i) \in \mathbb{Z}$$

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- $U(1)$ winding dipole symmetry, $J_0^{xy} = \partial^x\partial^y\phi$, $J = \partial_0\phi$, $Q^{xy}_{x^i}(x^i) = \oint dx^j J_0^{xy}$

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Symmetries: $\mathbb{Z}_4$ rotations, $U(1) \times U(1)$ dipole symmetry with mixed anomaly [Burnell, Devakul, Gorantla, Lam, Shao ‘ 21 - Gorantla, Lam, Seiberg, Shao ’21]

The model has a duality exchanging - $\phi \leftrightarrow \phi^{xy}$,

- $\left(\mu_0\,, \mu\right) \leftrightarrow \left(\mu/4\pi^2\,, 4\pi^2\mu_0\right)$,

- momentum modes ↔ winding modes,

reminiscent of T-duality of a compact boson in (1+1)d.

Introduction

Motivation

Can we capture theories with subsystem symmetries within the framework of SymTFT?

- Problem: Lorentz-invariance is broken, we cannot define a topological theory in the bulk.

- Solution: use a partially topological theory, i.e. topological only in the bulk and time directions.

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Can we capture theories with subsystem symmetries within the framework of SymTFT?

- Problem: Lorentz-invariance is broken, we cannot define a topological theory in the bulk.

- Solution: use a partially topological theory, i.e. topological only in the bulk and time directions.
  [Cao, Jia ‘23]

A natural model realising that is a *Foliated BF theory*. Foliated along (x^i) means that fields have components on a plane orthogonal to (x^i):

for a field $B^{(x)}$ foliated along (x), the component $B^{(x)}_x = 0$;

it can be realised by writing terms as $\left(B^{(x)} \wedge dx\right)$ in the action.

Introduction

Motivation - Foliated theory

Foliated BF theory: for a (3+1)d Sym(Partially)TFT, such a theory has action [Spieler '23]

$$S_{\text{fol}} = \frac{i}{2\pi} \int_{M \times I} c \wedge db + C^{(x)} \wedge dB^{(x)} \wedge dx + C^{(x)} \wedge b \wedge dx$$
$$+ C^{(y)} \wedge dB^{(y)} \wedge dy + C^{(y)} \wedge b \wedge dy$$

- $(t, x, y) \in M =$ three-torus, and $r \in I = [0\,, L]$
- b is a 2-form; c , $B^{(x^i)}$, $C^{(x^i)}$ are 1-forms. All fields in $\mathbb{R}$.

Introduction

Motivation - Foliated theory

Foliated BF theory:

$$S_{\mathrm{fol}} = \frac{i}{2\pi}\int_{M\times I} c \wedge db + \sum_{x^i=x,y} C^{(x^i)} \wedge dB^{(x^i)} \wedge dx^i$$

Introduction

Motivation - Exotic/Foliated duality

Foliated BF theory:

$$S_{\text{fol}} = \frac{i}{2\pi}\int_{M\times I} c \wedge db + \sum_{x^i = x,y} C^{(x^i)} \wedge dB^{(x^i)} \wedge dx^i$$

By integrating out components $C_r^{(x^i)}$, b_{rx^i} , b_{rt} , and field redefinitions as $A_r = c_r$, $\tilde{A}_r = B_r^{(x)} - B_r^{(y)}$

"Dual" to the exotic field theory (exotic = manifest $\mathbb{Z}_4$ space rotation) [Spieler '23]

$$S = \frac{i}{2\pi}\int_{M\times I} d^4x \, A_t\left(\partial_r \tilde{A}_{xy} - \partial_x\partial_y \tilde{A}_r\right) + A_r\left(\partial_x\partial_y \tilde{A}_t - \partial_t \tilde{A}_{xy}\right) + A_{xy}\left(\partial_r \tilde{A}_t - \partial_t \tilde{A}_r\right)$$

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Using this gapped exotic/foliated duality, we construct the Sym(Partially)TFT for continuous subsystem symmetries.

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Using this gapped exotic/foliated duality, we construct the Sym(Partially)TFT for continuous subsystem symmetries.

We add a topological boundary and a scale invariant boundary. After interval compactification, we realise the first instances of gapless exotic/foliated theories [Apruzzi, Bedogna, SM ‘25], together with the work [Ohmori, Shimamura ‘25].

Within the framework, we construct [Bedogna, SM ’26] non-invertible symmetries and generalise the exotic model.

Recap

- Subsystem symmetry: $Q = \int_\Sigma J_0$, spatial Σ subspace;

- XY-plaquette model: $S = \int d^3x \, \frac{\mu_0}{2} \left(\partial_0 \phi\right)^2 + \frac{1}{2\mu} \left(\partial_x \partial_y \phi\right)^2$, $\phi \sim \phi + 2\pi w^x(x) + 2\pi w^y(y)$;

- Foliated theory in (3+1)d: $S_{\text{fol}} = \frac{i}{2\pi} \int_{M \times I} c \wedge db + \sum_{x^i = x,y} C^{(x^i)} \wedge dB^{(x^i)} \wedge dx^i$;

- Exotic theory in (3+1)d:

dual

$$S = \frac{i}{2\pi} \int_{M \times I} d^4x \, A_t \left(\partial_r \tilde{A}_{xy} - \partial_x \partial_y \tilde{A}_r\right) + A_r \left(\partial_x \partial_y \tilde{A}_t - \partial_t \tilde{A}_{xy}\right) + A_{xy} \left(\partial_r \tilde{A}_t - \partial_t \tilde{A}_r\right)$$

SymTFT for Subsystem Sym

Foliated

[Argurio, Collinucci, Galati, Hulik, Paznokas ’24 - Apruzzi, Bedogna, SM ‘25]

$$S_L = \frac{1}{4\pi}\int_{M_L} (b + dB^{(x)} + dB^{(y)}) \wedge \star_2 (b + dB^{(x)} + dB^{(y)}) \wedge dt$$
$$+ g\, (B^{(x)} - B^{(y)}) \wedge \star_1 (B^{(x)} - B^{(y)}) \wedge dx \wedge dy$$

$$S_{\text{fol}} = \frac{i}{2\pi}\int_{M\times I} c \wedge db$$
$$+ \sum_{x^i = x,y} C^{(x^i)} \wedge dB^{(x^i)} \wedge dx^i$$

$$S_0 = \frac{iR}{2\pi}\int_{M_0} \sum_{x^i=x,y} \Phi^{(x^i)} \left(dB^{(x^i)} + b\right) \wedge dx^i$$
$$+\phi\, db$$

$r = 0$

Top

$r = L$

Scale Inv

r

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$$S_{\text{fol}} = \frac{i}{2\pi}\int_{M\times I} c \wedge db + \sum_{x^i=x,y} C^{(x^i)} \wedge dB^{(x^i)} \wedge dx^i$$

$$S_0 = \frac{iR}{2\pi}\int_{M_0} \sum_{x^i=x,y} \Phi^{(x^i)} \left(dB^{(x^i)} + b\right) \wedge dx^i + \phi\ db$$

b.c. :

$$c = R\left(d\phi - \sum_{x^i} \Phi^{(x^i)} dx^i\right)\ ,$$
$$C^{(x^i)} \wedge dx^i = -\,R d\Phi^{(x^i)} \wedge dx^i$$

$r = 0$

Top

$r = L$

Scale Inv

r

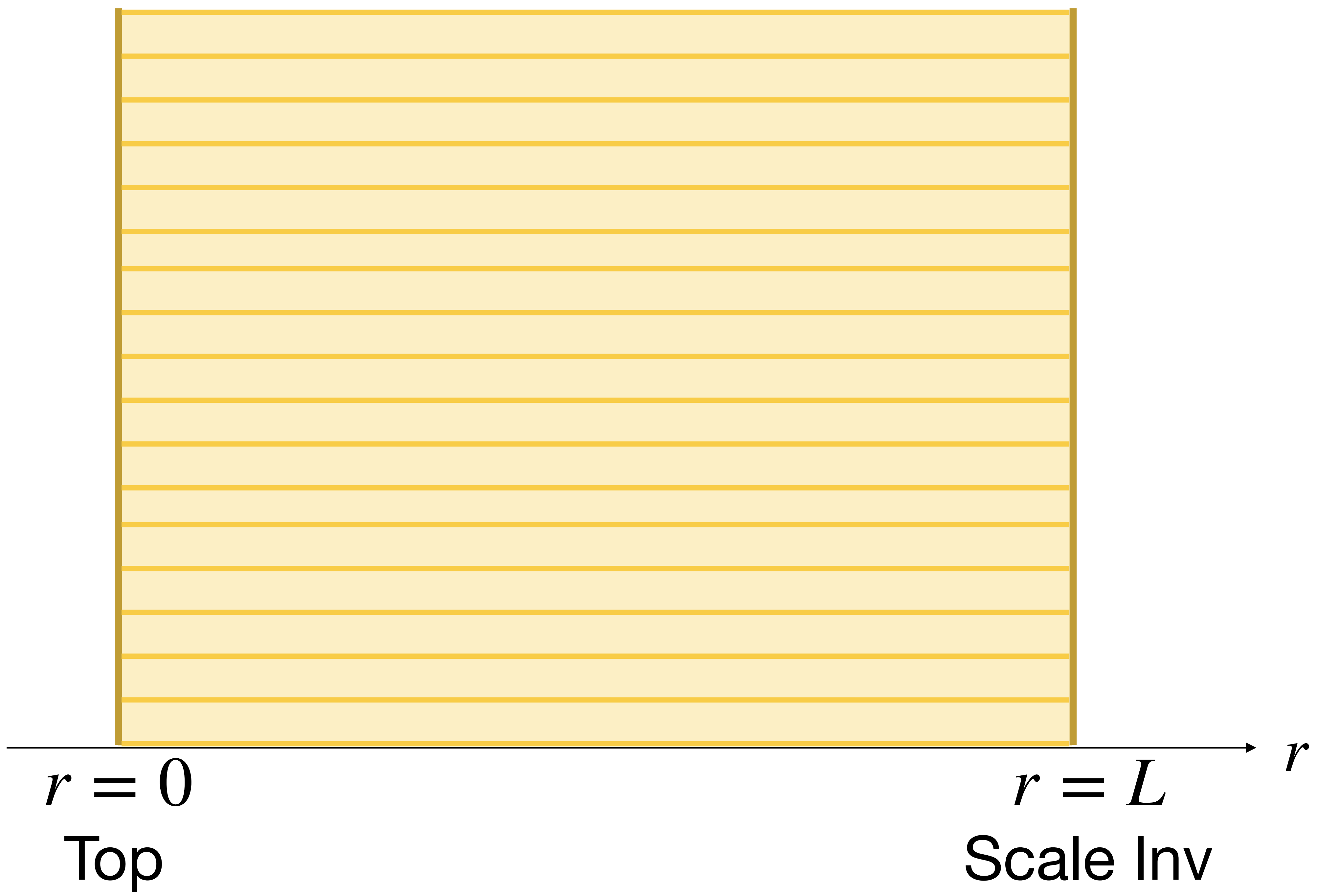
b.c. :

$$c = i \star_2 \left(b + \sum_{x^i} dB^{(x^i)} \wedge dt\right)\ ,$$
$$C^{(x^i)} \wedge dx^i = \dots$$

SymTFT for Subsystem Sym

Foliated

$$S_{\text{fol}} = \frac{i}{2\pi}\int_{M\times I} c \wedge db + \sum_{x^i = x,y} C^{(x^i)} \wedge dB^{(x^i)} \wedge dx^i$$



Mille-feuille

SymTFT for Subsystem Sym

Exotic

$$S_L = \frac{1}{4\pi}\int_{M_L} d^3x \left[g\left(\tilde{A}_t\right)^2 - \left(\tilde{A}_x y\right)^2\right]$$

$$S = \frac{i}{2\pi}\int d^4x\, A_t\left(\partial_r\tilde{A}_{xy} - \partial_x\partial_y\tilde{A}_r\right) + A_r\left(\partial_x\partial_y\tilde{A}_t - \partial_t\tilde{A}_{xy}\right) + A_{xy}\left(\partial_r\tilde{A}_t - \partial_t\tilde{A}_r\right)$$

$$S_0 = \frac{iR}{2\pi}\int_{M_0} d^3x\, \phi\left(\partial_t\tilde{A}_{xy} - \partial_x\partial_y\tilde{A}_t\right)$$

b.c. :

$$\tilde{A}_t = \frac{i}{g}A_{xy}\,,$$
$$\tilde{A}_{xy} = iA_t$$

b.c. :

$$A_t = -R\partial_t\phi\,,$$
$$A_{xy} = -\partial_x\partial_y\phi$$

r

$r=0$ Top

$r=L$ Scale Inv

SymTFT for Subsystem Sym

Gapless Exotic/Foliated duality

After interval compactification:

Foliated frame

$$S = \frac{1}{2\pi}\int iR^2 \left[\phi\, db + \sum_{x^i} \Phi^{(x^i)} \left(dB^{(x^i)} + b\right) \wedge dx^i\right]$$
$$+\frac{1}{2}\left(b + dB^{(x)} + dB^{(y)}\right) \wedge \star_2\left(b + dB^{(x)} + dB^{(y)}\right) \wedge dt$$
$$+\frac{1}{2} g\left(B^{(x)} - B^{(y)}\right) \wedge \star_1\left(B^{(x)} - B^{(y)}\right) \wedge dx \wedge dy$$

Exotic frame

XY-plaquette model

$$S = \frac{1}{4\pi}\int d^3x\, R^2 \left(\partial_0\phi\right)^2 + \frac{R^2}{g}\left(\partial_x\partial_y\phi\right)^2$$

$$R^2 = \mu_0\ , \quad \frac{R^2}{g} = \frac{1}{\mu}$$

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By integrating out $\Phi^{(x)}$, $\Phi^{(y)}$ and using b.c.

SymTFT for Subsystem Sym

Gapless Exotic/Foliated duality

After interval compactification:

Foliated frame

By integrating out ϕ: $db = 0 \Rightarrow b = da$

$$S = \frac{1}{2\pi}\int iR^2 \left[\sum_{x^i} \Phi^{(x^i)} d\left(B^{(x^i)} + a\right) \wedge dx^i \right] + \frac{1}{2} d\left(a + B^{(x)} + B^{(y)}\right) \wedge \star_2 d\left(a + B^{(x)} + B^{(y)}\right) \wedge dt + \frac{1}{2} g\left(B^{(x)} - B^{(y)}\right) \wedge \star_1 \left(B^{(x)} - B^{(y)}\right) \wedge dx \wedge dy$$

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$$R^2 = \mu_0\,, \quad \frac{R^2}{g} = \frac{1}{\mu}$$

SymTFT for Subsystem Sym

Spectrum of gauge-invariant operators

Gauge transformations:

$$\delta b = d\chi_1 \,, \quad \delta c = d\lambda_0 - \sum_{x^i} \lambda^{(x^i)} dx^i \,,$$

$$\delta B^{(x^i)} = d\chi_0^{(x^i)} - \chi_1 \,, \quad \delta C^{(x^i)} = d\lambda^{(x^i)} \,,$$

$$\delta A_t = \partial_t \lambda \,, \quad \delta A_r = \partial_r \lambda \,, \quad \delta A_{xy} = \partial_x \partial_y \lambda \,,$$

$$\delta \tilde{A}_t = \partial_t \tilde{\lambda} \,, \quad \delta \tilde{A}_r = \partial_r \tilde{\lambda} \,, \quad \delta \tilde{A}_{xy} = \partial_x \partial_y \tilde{\lambda} \,,$$

Eom:

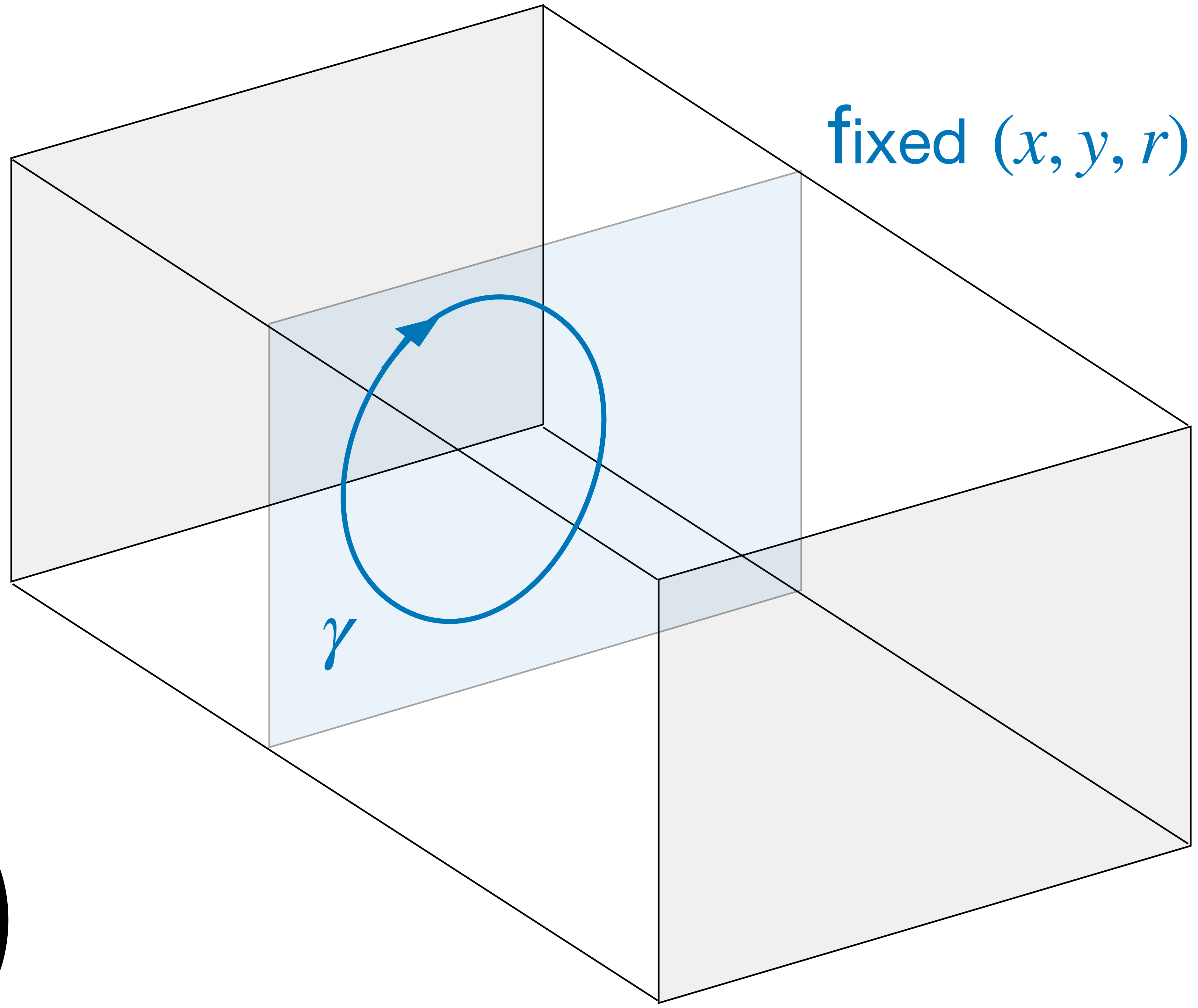
$$\left(dB^{(x)} + b\right) \wedge dx^i = 0 \,, \quad db = 0 \,,$$

$$dC^{(x^i)} \wedge dx^i = 0 \,, \quad \sum_i dC^{(x^i)} \wedge dx^i + dc = 0$$

(Partially) topological lines, at fixed (x, y)

$$V_\alpha(x,y)[\gamma] = \exp\left(i\alpha \oint_\gamma c \right) \longrightarrow \exp\left(i\alpha \oint_\gamma A_t dt + A_r dr \right)$$

$$U_\beta(x,y)[\gamma] = \exp\left(i\beta \oint_\gamma B^{(x)} - B^{(y)} \right) \longrightarrow \exp\left(i\beta \oint_\gamma \tilde{A}_t dt + \tilde{A}_r dr \right)$$



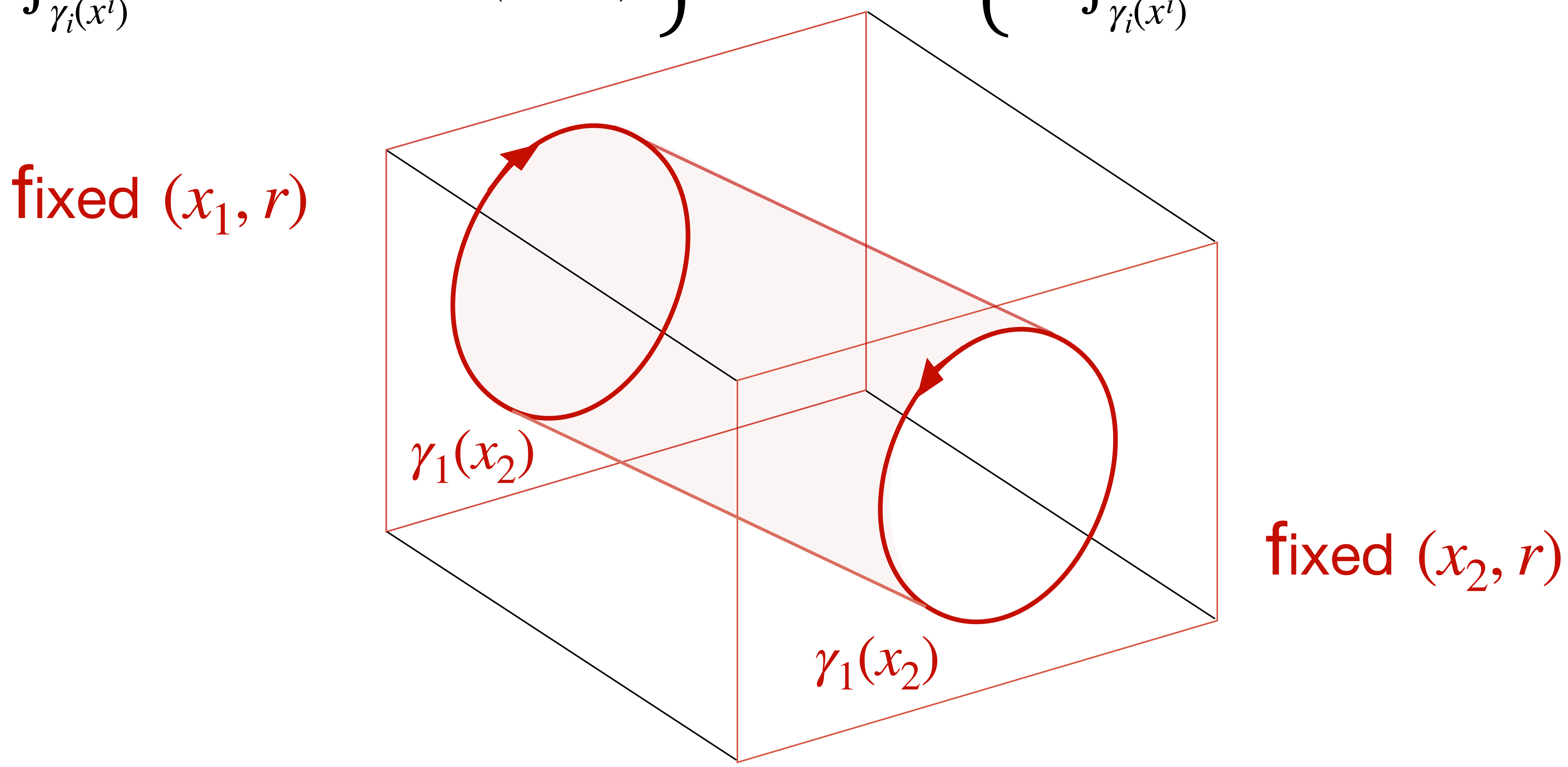
SymTFT for Subsystem Sym

Spectrum of gauge-invariant operators

- (Partially) topological strips, on surface swiped by a $\gamma_i(x^i)$ from x_1^i to x_2^i

$$\tilde{U}_{\tilde{\beta}}\left(x_1^i, x_2^i\right)[\sigma^i] = \exp\left\{ i\tilde{\beta}\int_{x_1^i}^{x_2^i}\left(\oint_{\gamma_i(x^i)} b + dB^{(x^i)}\right)dx^i\right\} \longrightarrow \exp\left\{ i\tilde{\beta}\int_{x_1^i}^{x_2^i}\left(\oint_{\gamma_i(x^i)} \tilde{A}_{xy}dx^j + \partial_i\tilde{A}_r dr + \partial_i\tilde{A}_t dt\right)dx^i\right\}$$

$$\tilde{V}_{\tilde{\alpha}}(x_1^i, x_2^i)[\sigma^i] = \exp\left\{ i\tilde{\alpha}\oint_{\gamma_i(x^i)} C^{(x^j)} \wedge dx^j + d\left(c_j dx^j\right)\right\} \longrightarrow \exp\left\{ i\tilde{\alpha}\oint_{\gamma_i(x^i)} A_{xy}dx^j + \partial_i A_r dr + \partial_i A_t dt\right\}$$



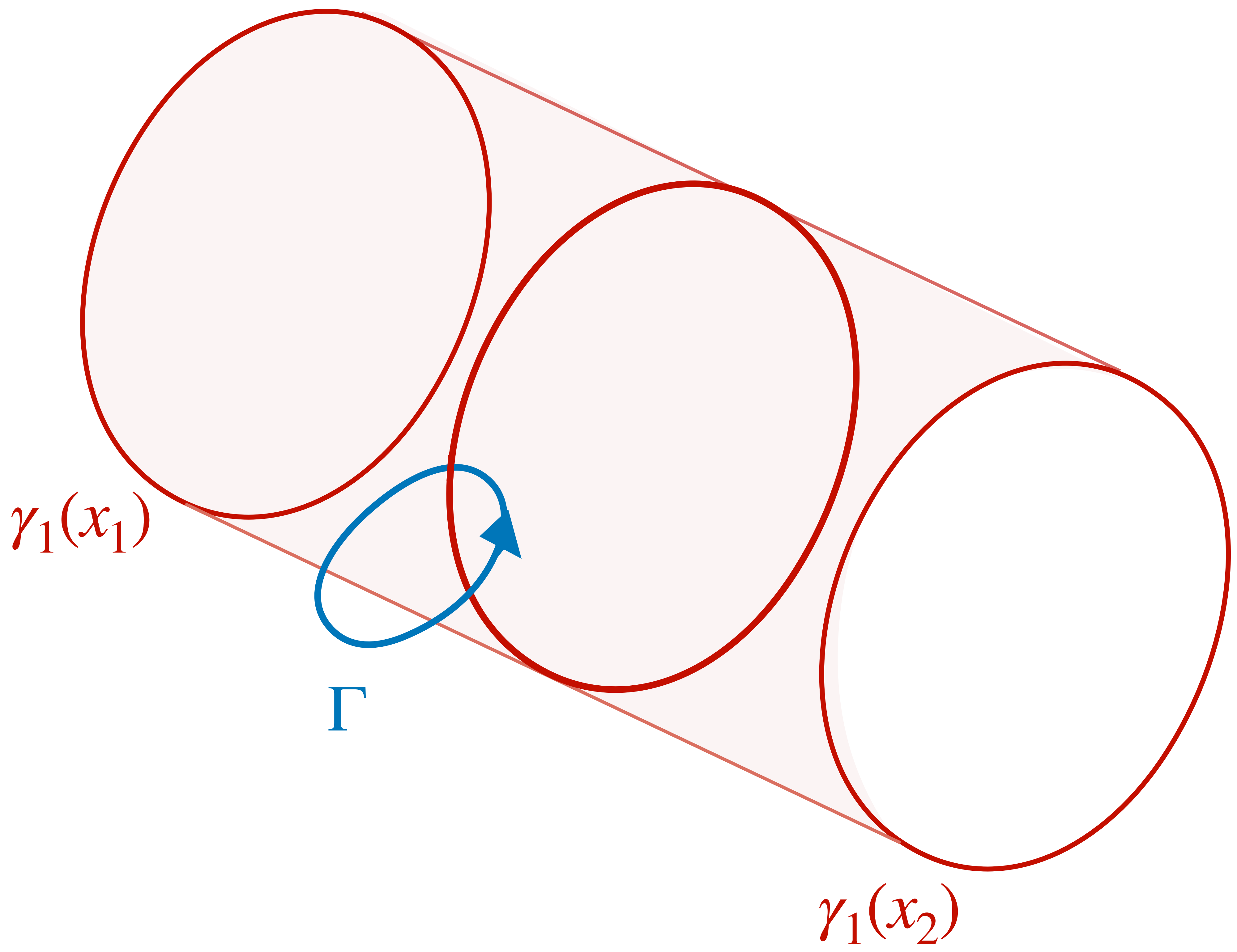
SymTFT for Subsystem Sym

Spectrum of gauge-invariant operators

Lines and surfaces link

$$\langle V_\alpha[\Gamma], \tilde{U}_{\tilde{\beta}}[\sigma^i]\rangle = \exp\left(2\pi i\, \alpha\tilde{\beta}\, \mathrm{Link}\left(\Gamma, \gamma_i\right)\right)$$

$$\langle U_\beta[\Gamma], \tilde{V}_{\tilde{\alpha}}[\sigma^i]\rangle = \exp\left(2\pi i\, \beta\tilde{\alpha}\, \mathrm{Link}\left(\Gamma, \gamma_i\right)\right)$$



SymTFT for Subsystem Sym

Boundary operators

Generators and charges of the subsystem symmetry.

On the gapped boundary: $c = R\left(d\phi - \sum_{x^i} \Phi^{(x^i)} dx^i\right)$, $\int_{\Sigma \subset M_0} b \in \frac{2\pi}{R}\mathbb{Z}$, $\int_{\gamma \subset M_0} B^{(x^i)} \in \frac{2\pi}{R}\mathbb{Z}$

$$C^{(x^i)} \wedge dx^i = -Rd\Phi^{(x^i)} \wedge dx^i$$

$$V_\alpha(x,y)[\gamma] = \exp\left(i\alpha \oint_\gamma c\right) , \qquad U_\beta(x,y)[\gamma] = \exp\left(i\beta \oint_\gamma B^{(x)} - B^{(y)}\right)$$

- Lines trivialises if $\alpha = p/R$ and $\beta = qR$, $p, q \in \mathbb{Z}$.
- Non-trivial lines V_α , U_β , $\alpha \sim \alpha + 1/R$, $\beta \sim \beta + R$ generate the $U(1) \times U(1)$ dipole symmetry

  [Antinucci, Benini ‘ 24 - Argurio, Collinucci, Galati, Hulik, Paznokas ’24].

SymTFT for Subsystem Sym

Boundary operators

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$$C^{(x^i)} \wedge dx^i = -R d\Phi^{(x^i)} \wedge dx^i$$

$$\tilde{U}_{\tilde{\beta}}\left(x_1^i, x_2^i\right)[\sigma^i] = \exp\left\{ i\tilde{\beta} \int_{x_1^i}^{x_2^i} \left(\oint_{\gamma_i(x^i)} b + dB^{(x^i)} \right) dx^i \right\} = \exp\left\{ \int_{x_1^i}^{x_2^i} Q_m^{x^i}(x^i) \right\} ,$$

$$\tilde{V}_{\tilde{\alpha}}(x_1^i, x_2^i)[\sigma^i] = \exp\left\{ i\tilde{\alpha} \oint_{\gamma_i(x^i)} C^{(x^j)} \wedge dx^j + d\left(c_j dx^j \right) \right\} = \exp\left\{ \int_{x_1^i}^{x_2^i} Q_w^{x^i}(x^i) \right\} .$$

Exotic Theta Term

Add a theta term on the gapped boundary of the SymTFT [Pretko ’17 - Bedogna, SM ’26 - Furukawa ‘26]

$$S = \int_{M_0} d^3x \, \frac{iR}{2\pi} \left(\partial_t \tilde{A}_{xy} - \partial_x \partial_y \tilde{A}_t \right) - \frac{i\theta^{xy}}{2\pi^2 R} \phi \left(\partial_t A_{xy} - \partial_x \partial_y A_t \right) + \frac{i\theta^{xy}}{4\pi^2 R^2} A_t A_{xy} \quad , \qquad \theta^{xy} \in \mathbf{1}_2 \, .$$

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By interval compactification we obtain

$$S = \int d^3x \, \frac{R^2}{4\pi} \left(\partial_0 \phi \right)^2 + \frac{R^2}{4\pi} \left(\partial_x \partial_y \phi \right)^2 + \frac{i\theta^{xy}}{4\pi^2} \partial_0 \phi \, \partial_x \partial_y \phi$$

We can rewrite it in a compact notation as

$$D^{\pm}\phi = \partial_0 \phi \pm \partial_x \partial_y \phi \in \mathbf{1}_0 \pm \mathbf{1}_2 \, , \quad \tau = \frac{\theta^{xy}}{2\pi} + iR^2$$

$$S = \frac{i}{8\pi} \int d^3x \, \bar{\tau} \left(D^+ \phi \right)^2 - \tau \left(D^- \phi \right)^2$$

$$\left(D^+ \phi \, , D^- \phi \right) \, , \quad (\bar{\tau} \, , \, -\tau)$$

Exotic Theta Term and Duality

$$S = \frac{i}{8\pi}\int d^3x\ \bar{\tau}\left(D^+\phi\right)^2 - \tau\left(D^-\phi\right)^2$$

Introduce a gauge field $\left(\chi^+, \chi^-\right)$ for the shift symmetry and couple it to $\left(D^+\phi^{xy},\ -D^-\phi^{xy}\right)$.

By integrating out χ

$$S = \frac{i}{8\pi}\int d^3x\ -\frac{1}{\bar{\tau}}\left(D^+\phi^{xy}\right)^2 + \frac{1}{\tau}\left(D^-\phi^{xy}\right)^2$$

$$(\bar{\tau},\ -\tau)\ \rightarrow\ \left(-\frac{1}{\bar{\tau}}, \frac{1}{\tau}\right)$$

Duality Symmetry of the XY plaquette

Only exotic frame

Following [Paznokas '25]

$$S = \frac{i}{2\pi}\int_{M\times I} d^4x\; A_t\left(\partial_r\tilde{A}_{xy} - \partial_x\partial_y\tilde{A}_r\right) + A_r\left(\partial_x\partial_y\tilde{A}_t - \partial_t\tilde{A}_{xy}\right) + A_{xy}\left(\partial_r\tilde{A}_t - \partial_t\tilde{A}_r\right)$$

It has a $SL(2,\mathbb{R})$ symmetry rotating the fields as: $\begin{pmatrix} a & b\,1^{xy} \\ c\,1^{xy} & d \end{pmatrix}\begin{pmatrix} A \\ \tilde{A} \end{pmatrix}\,, \quad ad - bc = 1$

By the Iwasawa decomposition, the generators are

$$\mathscr{K}_\omega = \begin{pmatrix} \cos\omega & -1^{xy}\sin\omega \\ 1^{xy}\sin\omega & \cos\omega \end{pmatrix}\,, \quad \omega \in \mathbb{R}$$

$$\mathscr{T}_l = \begin{pmatrix} 1 & 0 \\ l1^{xy} & 1 \end{pmatrix}\,, \quad l \in \mathbb{R}/\{0\} \qquad\qquad \mathscr{A}_s = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}\,, \quad s \in \mathbb{R}$$

Duality Symmetry of the XY plaquette

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Only this preserves the b. c., if $g = 1$

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Duality Symmetry of the XY plaquette

Condensation Defect

Constructing the condensation defect for the $SO(2)$ generator $\mathscr{K}_\omega$, supported on a codim-1 surface Σ at fixed r. For gauge-invariance and the correct action on the bulk operators

$$
\begin{aligned}
S^{\mathscr{K}}_\omega[\Sigma] = \frac{i}{2\pi}\int_\Sigma & \left(v_t A_{xy} + v_{xy}A_t + v_t\partial_x\partial_y\varphi + v_{xy}\partial_0\varphi\right) + \left(\tilde{v}_t\tilde{A}_{xy} + \tilde{v}_{xy}\tilde{A}_t + \tilde{v}_t\partial_x\partial_y\tilde{\varphi} + \tilde{v}_{xy}\partial_0\tilde{\varphi}\right) \\
& + \frac{1^{xy}\sin\omega}{2(\cos\omega - 1)}\left(v_{xy}v_t - \tilde{v}_{xy}\tilde{v}_t\right) \\
+ \frac{i}{2\pi}\int_{\partial\Sigma} & \left(\sigma A_{xy} + \sigma\partial_x\partial_y\varphi\right) + \left(\tilde{\sigma}\tilde{A}_{xy} + \tilde{\sigma}\partial_x\partial_y\tilde{\varphi}\right) \\
& + \frac{1^{xy}\sin\omega}{4(\cos\omega - 1)}\left(\sigma\partial_x\partial_y\sigma + 2\sigma v_{xy} + \tilde{\sigma}\partial_x\partial_y\tilde{\sigma} + 2\tilde{\sigma}\tilde{v}_{xy}\right)
\end{aligned}
$$

Duality Symmetry of the XY plaquette

Condensation Defect

The fusion of two such defects give

$$S^{\mathscr{K}_1\mathscr{K}_2^{-1}} = \frac{i}{2\pi}\int_{\partial\Sigma} \eta A_{xy} + \tilde{\eta}\tilde{A}_{xy} + \eta\partial_x\partial_y\varphi^+ + \tilde{\eta}\partial_x\partial_y\tilde{\varphi}^+$$

with $\varphi^+ = \varphi_1 + \varphi_2$, $v^+ = v_1 + v_2$, and $\eta = \partial_0 v^+$.

Duality Symmetry of the XY plaquette

Condensation Defect

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$$S^{\mathscr{K}_1\mathscr{K}_2^{-1}} = \frac{i}{2\pi}\int_{\partial\Sigma} \eta A_{xy} + \tilde{\eta}\tilde{A}_{xy} + \eta\partial_x\partial_y\varphi^+ + \tilde{\eta}\partial_x\partial_y\tilde{\varphi}^+$$

with $\varphi^+ = \varphi_1 + \varphi_2$, $v^+ = v_1 + v_2$, and $\eta = \partial_0 v^+$.

The defect generates a continuous non-invertible $SO(2)$ symmetry, at any value of the couplings R and θ, or at any τ.

The same symmetry arises with a similar mechanism for Maxwell in (3+1)d.

[Paznokas ’25] and [Niro, Roumpedakis, Sela ’22 - Hasan, Meynet, Migliorati ’24 - Arbalestreier, Argurio, Tizzano ’24 - Argurio, Collinucci, Galati, Hulik, Paznikas - ‘24]

Conclusion

We showed the construction of

- The first instance of gapless exotic/foliated duality via SymTFT;
- Subsystem symmetries with the Mille-feuille framework;
- The (2+1)d continuous theory proposed for the XY-plaquette model, allows for an exotic theta-term;
- The XY-plaquette has a continuous non-invertible $SO(2)$ symmetry. This model stands in between a compact boson in (1+1)d and Maxwell theory in (3+1)d.

Thank you

SymTFT for Subsystem Sym

Foliated

Bulk: $S_{\text{fol}} = \frac{i}{2\pi} \int_{M \times I} c \wedge db + \sum_{x^i = x,y} C^{(x^i)} \wedge dB^{(x^i)} \wedge dx^i$

Topological boundary: $S_0 = \frac{iR}{2\pi} \int_{M_0} \sum_{x^i = x,y} \Phi^{(x^i)} \left(dB^{(x^i)} + b \right) \wedge dx^i + \phi \, db$

b.c. :

$$c = R\left(d\phi - \sum_{x^i} \Phi^{(x^i)} dx^i \right) \, ,$$

$$C^{(x^i)} \wedge dx^i = -R d\Phi^{(x^i)} \wedge dx^i$$

Scale-invariant boundary:

$$S_L = \frac{1}{4\pi} \int_{M_L} (b + dB^{(x)} + dB^{(y)}) \wedge \star_2 (b + dB^{(x)} + dB^{(y)}) \wedge dt$$

$$+ g \, (B^{(x)} - B^{(y)}) \wedge \star_1 (B^{(x)} - B^{(y)}) \wedge dx \wedge dy$$

b.c. :

$$c = i \star_2 \left(b + \sum_{x^i} dB^{(x^i)} \wedge dt \right) \, ,$$

$$C^{(x)} \wedge dx = + ig(B_t^{(x)} - B_t^{(y)}) \, dx \wedge dy + id \star_2 (b + dB^{(x)} + dB^{(y)}) \wedge dt \, ,$$

$$C^{(y)} \wedge dy = + ig(B_t^{(x)} - B_t^{(y)}) \, dx \wedge dy + id \star_2 (b + dB^{(x)} + dB^{(y)}) \wedge dt$$

Duality Symmetry

Gauge the $U(1)$ winding symmetry

$$S = \frac{i}{8\pi}\int_M d^3x\bar{\tau}\left(D^+\phi\right)^2 - \tau\left(D^-\phi\right)^2 + 2\left(D^+\phi\right)\eta^+ - 2\left(D^-\phi\right)\eta^- - \frac{2}{R^2}\left(D^+b\right)\eta^+ + \frac{2}{R^2}\left(D^-b\right)\eta^-$$

$$b \in \mathbb{R}\,, \quad D^+\eta^- = 0 = D^-\eta^+$$

Integrating out η: $$S = \frac{1}{4\pi}\int_M d^3x(\partial_0 b)^2 + (\partial_x\partial_y b)^2$$

Gauge a $\mathbb{Z} \subset U(1)$ momentum:

$$S = \frac{1}{8\pi}\int_M d^3x\left(D^+b - c^+\right)^2 - \left(D^-b - c^-\right)^2 - 2\tilde{R}^2\left(D^+\tilde{\phi}\right)c^+ - 2\tilde{R}^2\left(D^-\tilde{\phi}\right)c^- + \frac{i\tilde{\theta}^{xy}\tilde{R}^2}{2}\left(D^+\tilde{\phi}\right)^2 - \frac{2}{R^2}\left(D^-\tilde{\phi}\right)^2$$

$$\Rightarrow S = \frac{i}{8\pi}\int d^3x\,\bar{\tilde{\tau}}\left(D^+\tilde{\phi}\right)^2 - \tilde{\tau}\left(D^-\tilde{\phi}\right)^2$$

Foliated condensation defect

Condensing an arbitrary number N of lines requires summing over insertion points and 1-cycles

$$\sum_{N=1}^{\infty} \prod_{i=1}^{N} \int dx_i dy_i \int_{\mathbb{R}} d\alpha \; W(x_i, y_i)_\alpha [S_1^t]$$

Condensing an arbitrary number of strips requires defining a measure in the space of all partition of an interval