Entanglement Asymmetry for Higher and Noninvertible Symmetries

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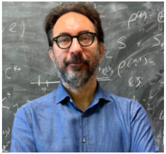
Developments on Generalized Symmetries of Quantum Systems
12 May 2026, Universidad de Murcia (Spain)

NP-QFT

The collaboration



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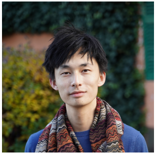

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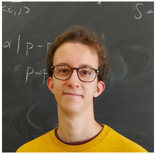

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Based on:	FB, V. Godet, A. Singh:	2407.07969
	FB, P. Calabrese, M. Fossati, A. Singh, M. Venuti	2509.16311
	FB, E. García-Valdecasas, S. Vitouladitis	2512.15898

Symmetries and their Breaking

⋆ Standard approach: use local order parameter $\mathcal{O}(x)$

Landau paradigm

$\langle \mathcal{O}(x) \rangle \neq 0 \qquad \Rightarrow \qquad$ symmetry breaking

⋆ Quantum-information-based approach [Ares, Murciano, Calabrese 22]

[see also: Vaccaro, Anselmi, Wiseman, Jacobs 08; Gour, Marvian, Spekkens 09; Chitambar, Gour 19]

[Casini, Huerta, Magan, Pontello 19 & 20]

Entanglement asymmetry $\Delta S[A]$

- $\Delta S[A] > 0$ if and only if the symmetry is broken
- Depends on a subsystem A (probe the RG flow)
- Applicable to ground states $|0\rangle$
  as well as to excited (or out-of-equilibrium) states

Standard setup used for entanglement entropy:

Subsystem A (and complement B) of a full system.

$\mathcal{H}_{\text{full}} = \mathcal{H}_A \otimes \mathcal{H}_B$ factorized Hilbert space (*e.g.*: spin chain)

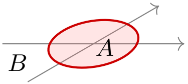
$\rho_{\text{full}} = |\psi\rangle\langle\psi|$ pure state (or mixed state)

$\rho_A = \mathrm{Tr}_B\ \rho_{\text{full}}$ reduced density matrix

$$S_A = -\,\mathrm{Tr}\big(\rho_A \log \rho_A\big) \qquad \text{(entanglement entropy)}$$

(In the following simply $\rho \equiv \rho_A$)

⋆ Can be extended to QFTs after suitable regularization.
Typically A is a spatial region.



Asymmetry

[Ares, Murciano, Calabrese 22]

Suppose the system has a symmetry.

- *E.g.*: $U(1)$ symmetry with charge Q s.t. $Q = Q_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes Q_B$

⋆ The symmetry is broken in Subsystem A if and only if $[Q_A, \rho] \neq 0$.

Symmetrized density matrix:

$$\boxed{\begin{aligned} \rho_Q &= \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi}\, e^{-i\alpha Q_A}\, \rho\, e^{i\alpha Q_A} \\ &= \sum\nolimits_{\text{charges } q} \Pi_q\, \rho\, \Pi_q \end{aligned}}$$

ρ_Q is a density matrix, and retains only the diagonal terms

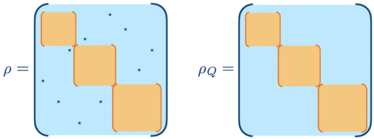


image from Ares, Murciano, Calabrese, Nature Commun. 14 (2023) 2036

- Entanglement asymmetry ΔS: measures "difference" between density matrices

$$\Delta S = S(\rho \,\|\, \rho_Q) = \mathrm{Tr}\Big[\rho\left(\log\rho - \log\rho_Q\right)\Big] = S[\rho_Q] - S[\rho]$$

⋆ Entanglement asymmetry is a relative entropy:

- $\Delta S \geq 0$, and $\Delta S = 0$ if and only if $[Q_A, \rho] = 0$.
- Only depends on the state, not on the Hamiltonian
  (Useful for both spontaneous and explicit breaking)
- If Q is broken spontaneously, or explicitly but softly, no divergences in QFT.

- Rényi asymmetries $\Delta S_n = \frac{1}{1-n}\log\frac{\mathrm{Tr}(\rho_Q^n)}{\mathrm{Tr}(\rho^n)}$ then $\Delta S = \lim_{n\to 1}\Delta S_n$

Asymmetry has been computed in a variety of quantum systems:

spin chains, 2d free fermions, 2d integrable systems,
matrix product states, random quantum circuits,
topological theories (TQFTs), conformal field theories (CFTs), free QFTs, . . .

Mainly in 1d and 2d systems, often numerically.

• Highlights of a few selected results •

⋆ Spontaneous Symmetry Breaking (SSB) in the vacuum: [Capizzi, Mazzoni 23] [Capizzi, Vitale 23] [FB, Calabrese, Fossati, Singh, Venuti 25]

$$\Delta S_A = \log\Big(|G|\,/\,|H|\Big)$$

for finite symmetry breaking $G \to H$.

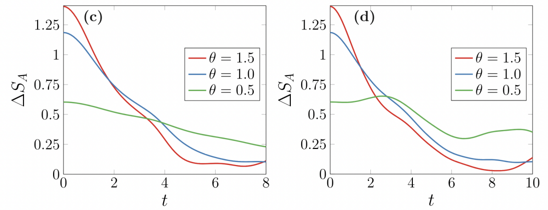
See *Eduardo García-Valdecasas' talk* for the case of a $U(1)$ symmetry and entropic proof of Coleman–Mermin–Wagner theorem

[FB, García-Valdecasas, Vitouladitis 25]

⋆ Quantum Mpemba effect in spin chains

(in closed quantum systems)

[Mpemba, Osborne 69]
[Ares, Murciano, Calabrese 22]



periodic and open boundary conditions

images from Ares, Murciano, Calabrese, Nature Commun. 14 (2023) 2036

This effect has been *experimentally measured*:

[Joshi *et al.* 24]

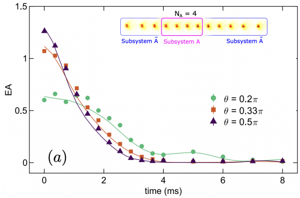


image from Joshi *et al.* Phys. Rev. Lett. 133 (2024) 010402

Quantum Mpemba Effect in CFTs

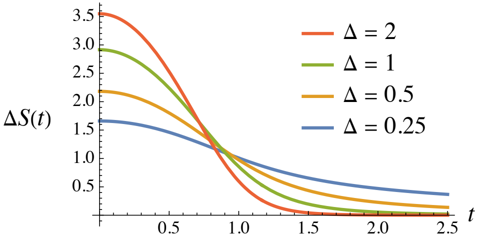
[FB, Godet, Singh 24]

⋆ The CFT vacuum $|0\rangle$ does not break internal symmetries

Study instead a simple class of excited states $\quad |\psi\rangle \simeq \big(1 + i\lambda V(x_-)\big)|0\rangle$

Use integral rep. for quadratic expansion of relative entropy (Fisher information metric)

$$\Delta S_{\text{Rindler}} = \frac{\lambda^2}{r^{2\Delta}} \frac{\sqrt{\pi}\,\Gamma(\Delta+1)}{4^{\Delta}\Gamma(\Delta+\frac{1}{2})} \left\{ 1 - \frac{\left(1-\tan^2\left(\frac{\tau_0}{2}\right)\right)\Delta}{\Delta+\frac{1}{2}}\, {}_2F_1\left[1, \tfrac{1}{2}-\Delta, \tfrac{3}{2}+\Delta, -\tan^2\left(\tfrac{\tau_0}{2}\right)\right] \right\} + \mathcal{O}(\lambda^4)$$



We observe a quantum Mpemba effect in any CFT, in any dimension.

⋆ “Page curve” of entanglement asymmetry [Ares, Murciano, Piroli, Calabrese 23] [Russotto, Ares, Calabrese 24]

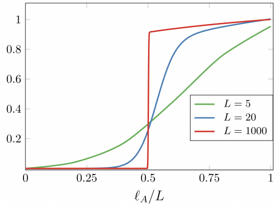


image from Ares, Murciano, Piroli, Calabrese, Phys. Rev. D 110 (2024) L061901

Here second Rényi asymmetry w.r.t. a $U(1)$ symmetry

Generalized symmetries

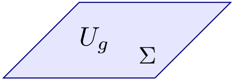
New paradigm:	symmetry operations, or elements, in a continuum QFT	=	topological defect operators, of any dimension

[Gaiotto, Kapustin, Seiberg, Willett 14]

⋆ This leads to a substantial widening of the concept of symmetry (at least in continuum QFT)

as well as of all <u>related constructions</u> and <u>consequences</u> (representations, anomalies, spontaneous breaking, . . .)

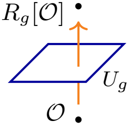
⋆ Standard 0-form symmetry G: codimension-1 defects $U_g[\Sigma]$, $g \in G$
along submanifolds Σ



that fuse according to group structure of G:

$$U_g \times U_h = U_{gh}$$

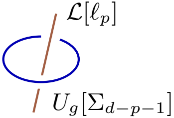
and act on *local* operators through representations:



⋆ Higher p-form symmetries: (necessarily Abelian)

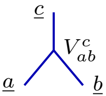
Topological operators of higher co-dimension

Charges carried by p-dimensional extended operators



⋆ Noninvertible or categorical symmetries: fusion algebras instead of group laws

$$U_a \times U_b = \sum_c N_{ab}^c \ U_c$$



Familiar from Verlinde lines in 2d RCFT's

"Noninvertible symmetries": $\quad U_g \times U_g^\dagger = \mathbb{1} + \ldots$

⋆ Peculiarities of noninvertible symmetries:

- reshuffle untwisted with twisted sectors
- physically inequivalent vacua upon spontaneous breaking

[Huse 84; Aguilera-Damia, Argurio, Benini, Benvenuti, Copetti, Tizzano 23]

⋆ Symmetries no longer characterized by groups

$\longrightarrow$ "Categorical" or "Noninvertible" symmetries

Mathematically: described by (higher) categories

⋆ Question: how can we define asymmetry for categorical symmetries?

ρ_{tot}	$\to$	ρ	$\to$	ρ_{S}
(pure or mixed) state		reduced density matrix on A		symmetrized density matrix

- Need to define "symmetrization" $\mathsf{S} : \rho \to \rho_{\mathsf{S}}$

- Construction of reduced density matrix is subtle [Ohmori, Tachikawa 14]
  Care must be paid to *boundary conditions*

Similarities with symmetry-resolved entanglement entropy

[Saura-Bastida, Das, Sierra, Molina-Vilaplana 24; Benedetti, Casini, Kawahigashi, Longo, Magán 24]
[Choi, Rayhaun, Zheng 24; Heymann, Quella 24]

One should distinguish:

- symmetry of the theory — local property, described by a (higher) fusion category
- implementation on a physical setup — depends on spatial manifold and subregion A, described by standard algebra $\mathcal{A}$

⋆ Physical setup
(spacetime dimension, spatial manifold, subsystem A, inclusion or not of twisted sectors, . . .)

$\Rightarrow$ Hilbert space $\mathcal{H}$ on A

Density matrices $\rho \in \mathrm{End}(\mathcal{H})$

- Space-like topological operators act on $\mathcal{H}$ forming an algebra $\mathcal{A}$

Here a finite-dimensional, semisimple C^*-algebra (from unitarity) $\mathcal{A}$:

$$\mathcal{X}_a \times \mathcal{X}_b = \sum_c T_{ab}^c \, \mathcal{X}_c$$

$\mathcal{X}_a$: generators, T_{ab}^c: structure constants

Symmetrizer

[FB, Calabrese, Fossati, Singh, Venuti 25]

★ Goal: construct a symmetrizer $\mathsf{S} : \rho \to \rho_{\mathsf{S}}$

- Linear projector
- Commuting: $\mathcal{X}_a \, \rho_{\mathsf{S}} = \rho_{\mathsf{S}} \, \mathcal{X}_a \qquad \forall \, \mathcal{X}_a \in \mathcal{A}$
- If ρ is already commuting, then $\rho_{\mathsf{S}} = \rho$
- Trace preserving: $\mathrm{Tr}\, \rho_{\mathsf{S}} = \mathrm{Tr}\, \rho$
- If ρ is density matrix, ρ_{S} is density matrix: $\rho_{\mathsf{S}}^\dagger = \rho_{\mathsf{S}}, \quad \rho_{\mathsf{S}} \geq 0$

- These requirements are enough to fix the *unique* solution:

$$\boxed{\rho_{\mathsf{S}} = \sum\nolimits_{ab} (K^{-1})^{ab} \, \mathcal{X}_a \, \rho \, \mathcal{X}_b}$$

$K_{ab} = \sum_{mn} T^m_{an} \, T^m_{bn}$ non-degenerate bilinear pairing

symmetric
separability
idempotent

★ Quantum channel $\Rightarrow$ monotonicity of entangl. asymmetry along RG flows

[FB, García-Valdecasas, Vitouladitis 25]

Examples:

Noninvertible 0-form symmetries in (1+1)d QFTs

See *Eduardo García-Valdecasas' talk* for the case of higher-form symmetries

[FB, García-Valdecasas, Vitouladitis 25]

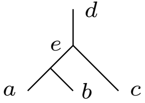
Noninvertible Symmetries in 2d

In 2 dimensions, (internal, finite) symmetries are described by fusion categories.

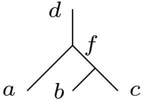
$\rightarrow$ Symmetry Fusion category $\mathscr{C}$

- Fusion algebra: $\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \, \mathcal{L}_c \qquad N_{ab}^c \in \mathbb{N}$

- F-symbols:



$= [F_d^{abc}]_{ef}$



They are constrained to satisfy the pentagon equation

Language familiar from 2d diagonal RCFTs [Moore, Seiberg 89]

Includes standard 0-form symmetry G with 't Hooft anomaly: $F \in H^3\big(BG, U(1)\big)$

Asymmetry of the full system

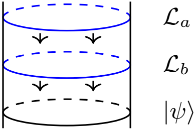
1. Consider asymmetry of full system A. Compact space: $A = S^1$

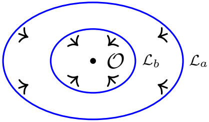
Fusion algebra is algebra of line operators acting on $\mathcal{H}[S^1]$ (untwisted Hilbert space)

$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \, \mathcal{L}_c$$

Here $\mathcal{A}$ is the fusion algebra itself.

$$\rho_{\mathsf{S}} = \sum_{ab} (K^{-1})^{ab} \, \mathcal{L}_a \, \rho \, \mathcal{L}_b \qquad K_{ab} = \sum_{mn} N_{an}^m N_{bm}^n$$





[see also: Molina-Vilaplana, Saura-Bastida, Sierra 24]

Example: the Ising Symmetry

The "Ising fusion category $\mathscr{C}$" is the symmetry of the Ising CFT

Symmetry elements: $\mathcal{L}_1, \ \mathcal{L}_\eta, \ \mathcal{L}_\mathcal{N}$

Fusion algebra:

$$\mathcal{L}_\eta \times \mathcal{L}_\eta = \mathcal{L}_1 \qquad \mathcal{L}_\eta \times \mathcal{L}_\mathcal{N} = \mathcal{L}_N \qquad \mathcal{L}_\mathcal{N} \times \mathcal{L}_\mathcal{N} = \mathcal{L}_1 + \mathcal{L}_\eta$$

The symmetrizer turns out to be:

$$\rho_{\mathsf{S}} = \frac{1}{8}\Big[3\,\rho + 3\,\mathcal{L}_\eta\,\rho\,\mathcal{L}_\eta - \mathcal{L}_\eta\,\rho - \rho\,\mathcal{L}_\eta + 2\,\mathcal{L}_\mathcal{N}\,\rho\,\mathcal{L}_\mathcal{N}\Big]\,.$$

Example: Spontaneous Symmetry Breaking

Assume that the system is in a gapped phase (in 1+1 d)

$\Rightarrow$ 2d TQFT with local operators and action of Fusion Category $\mathcal{C}$

[Huang, Lin, Seifnashri 21; Bhardwaj, Bottini, Pajer, Schäfer-Nameki 23]

It describes Hilbert space of zero-energy states $\mathcal{H} = \mathbb{C}^N$

Preferred basis of "vacua" $|v_i\rangle$: (1-1 with simple objects in a module category)

- Satisfy cluster decomposition: $\langle v_i|\mathcal{O}_\mu \mathcal{O}_\nu|v_i\rangle = \langle v_i|\mathcal{O}_\mu|v_i\rangle \langle v_i|\mathcal{O}_\nu|v_i\rangle$
- Transform as non-negative integer-valued matrix (NIM) representation:

$$\mathcal{L}_a|v_i\rangle = \sum_j n_{ij}^{(a)}|v_j\rangle$$

Regular rep. of fusion ring $n_{ij}^{(a)} = N_{ij}^a$ describes complete symmetry breaking.

⋆ *E.g.*: Tricritical Ising deformed by $\varepsilon'_{(\frac{3}{5},\frac{3}{5})}$ [Huse 84; Chang, Lin, Shao, Wang, Yin 18]

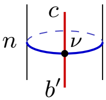
Spontaneous breaking of Ising symmetry $\rightarrow$ 3 vacua $|+\rangle,\ |-\rangle,\ |f\rangle$

$\Delta S[|f\rangle] = \log 2 \qquad \Delta S[|\pm\rangle] = \frac{3}{2}\log 2$ ($\Delta S = \log|G|$ for groups)

Tube Algebra

❷ Non-invertible symmetries not only act on the untwisted sector, they also have an action connecting different (un)twisted sectors.

Total Hilbert space: $\mathcal{H}_{\text{Tube}} = \sum_{\text{twist by } a} \mathcal{H}_a$



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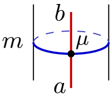
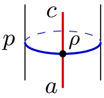


$= \delta_{bb'} \sum_{p,\rho} M^{[p,\rho]}_{a,[m,\mu],b,[n,\nu],c}$



Tube algebra. Fusion coefficients $M \sim F^* \, F \, F^*$ in terms of F-symbols

[Ocneanu 94]

Tube algebra is non-Abelian: representations can be higher dimensional

⋆ Construct a symmetrizer S under the full tube algebra.
More powerful than fusion ring.

Example: Excited States in Ising Model

Compare how different symmetrizers detect symmetry breaking in excited states.

Primaries: $\mathbb{1},\ \sigma(x),\ \varepsilon(x)$ Symmetries: $\mathcal{L}_1,\ \mathcal{L}_\eta,\ \mathcal{L}_\mathcal{N}$

⋆ $\mathbb{Z}_2$ spin-flip symmetry. [M. Chen, H.-H. Chen 23]

$\mathbb{1}$ and $\varepsilon_{(\frac{1}{2},\frac{1}{2})}(x)$ are neutral, $\big(\mathbb{1}+\sigma(x)\big)|0\rangle$ breaks the symmetry

$\sigma_{(\frac{1}{16},\frac{1}{16})}(x)$ is eigenstate

⋆ Ising fusion algebra ($\mathbb{Z}_2$ Tambara-Yamagami).

$\varepsilon(x)$ is eigenstate, $\big(\mathbb{1}+\varepsilon(x)\big)|0\rangle$ breaks the symmetry

⋆ Ising tube algebra.

$\sigma(x)$ and $\mu_{(\frac{1}{16},\frac{1}{16})}(x)$ form a 2d rep $\Rightarrow$ $\sigma(x)\,|0\rangle$ breaks the symmetry

Asymmetry for Subsystems

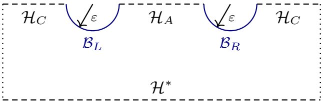
3. Consider a subsystem A: single interval

- Need to *regularize* the entangling surface: in QFT $\mathcal{H}_{\text{tot}} \neq \mathcal{H}_A \otimes \mathcal{H}_C$

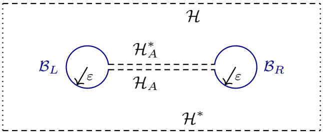
★ Define a map $\psi_{\mathcal{B}} : \mathcal{H}_{\text{tot}} \to \mathcal{H}_A \otimes \mathcal{H}_C$ [Ohmori, Tachikawa 14]

Path integral realization, which requires boundary conditions:

$\psi_{\mathcal{B}_L,\mathcal{B}_R}$:



$\Psi_{\mathcal{B}_L,\mathcal{B}_R}$:



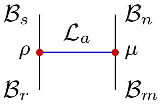
Extends to a map $\Psi_{\mathcal{B}} : \mathrm{End}(\mathcal{H}_{\text{tot}}) \to \mathrm{End}(\mathcal{H}_A)$ to reduced density matrices

In path integral: remove small disks, impose boundary conditions, then $\varepsilon \to 0$

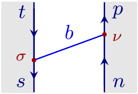
Strip Algebra

Now symmetry acts on a strip $\rightarrow$ strip algebra

[Kitaev, Kong; Inamura, Ohyama; C.Cordova, García-Sepulveda, Holfester; C.Cordova, Holfester, Ohmori; Choi, Rayhaun, Zheng; Copetti, L.Cordova, Komatsu; ...]



Strip algebra is finite-dimensional C^*-algebra ($\rightarrow$ semisimple), also weak Hopf



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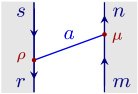
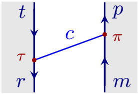


$$= \sum_{c,\tau,\pi} Q^{(c;\tau\pi)}_{m,r,(a;\rho\mu),n,s,(b;\sigma\nu),p,t}$$



$\Rightarrow$ symmetrizer can be computed as before.

[Slightly different proposal in: Ali Ahmad, Klinger, Y.Wang 25]

Role of Boundary Conditions

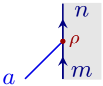
Ideally, use symmetric boundary conditions
(in order not to contamitate the (a)symmetry of the state)

(Invertible sym.-breaking boundaries in [Fossati, Rylands, Calabrese 24; Kusuki, Murciano, Ooguri, Pal 24])

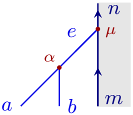
★ 't Hooft anomalies prevent the existence of symmetric boundary conditions!

[Jensen, Shaverin, Yarom 17; Thorngren, Wang 20]

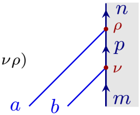
- Boundary conditions form "multiplets" $\{\mathcal{B}_m\}$ similar to representations.
  Structures called (indecomposable) module categories $\mathscr{M}$



$$\mathcal{L}_a \times \mathcal{B}_m = \sum\nolimits_n \tilde{N}_{am}^n \, \mathcal{B}_n$$



$$= \sum_{p,\nu,\rho} \left[\tilde{F}^n_{abm}\right]_{(e;\alpha\mu)(p;\nu\rho)}$$



⋆ Strongly symmetric boundary condition [Choi, Rayhaun, Sanghavi, Shao 23]

Module category with a single (simple) boundary condition $\mathcal{B}$

$\mathcal{B}$ is symmetry-invariant under fusion

⋆ Weakly symmetric boundary condition

Module category has multiple boundary conditions $\mathcal{B}_a$

however one of them, $\hat{\mathcal{B}} \in \{\mathcal{B}_a\}$, can be joined by all bulk lines:

$$\mathcal{L}_a \times \hat{\mathcal{B}} = \hat{\mathcal{B}} + \dots \qquad \forall\, \mathcal{L}_a$$

Restrict to $\hat{\mathcal{B}}$: reduced strip algebra

⋆ No (simple) symmetric boundary condition

Construct an "average" over boundary conditions:

$$\rho = \frac{1}{D} \sum\nolimits_{m,n} \tilde{d}_m\, \tilde{d}_n\, \rho_{mn} \qquad D = \left(\sum\nolimits_m \tilde{d}_m\right)^2 \qquad \tilde{d}_m \text{ quantum dimension}$$

Symmetric states of $\mathcal{H}_{\text{full}}$ give vanishing asymmetry on A

[Similar to, but different from, "cloaking boundary state" of [Brehm, Runkel 24]]

Example: Ising ⊠ Ising model

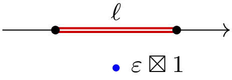
Take two copies of the Ising model, [Choi, Rayhaun, Sanghavi, Shao 23]

consider symmetry subcategory $\mathsf{TY}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ with simple lines $\{1,\, \eta,\, \eta',\, \eta\eta',\, \mathcal{V}\}$.

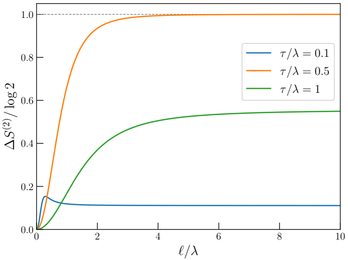
There $\exists$ a strongly-symmetric boundary condition $|\mathcal{N}\rangle$

⋆ Strip algebra H_8 has 8 generators: $\{H_1,\, H_\eta,\, H_{\eta'},\, H_{\eta\eta'},\, H_{\mathcal{V}}^{\alpha\beta}\}$ with $\alpha, \beta \in \{0,1\}$

Excited state: $|\psi\rangle = 1 \boxtimes 1 + \lambda\, \varepsilon \boxtimes 1\, (\tau, x)$



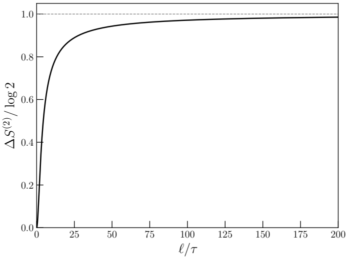
Second Rényi asymmetry:



- Excited state: $|\psi\rangle = \sigma \boxtimes 1$

Second Rényi asymmetry:

$$\Delta S^{(2)}[\rho] = \log 2 - \log\left[1 + \frac{\left\langle \sigma(z_1)\,\sigma(\bar{z}_1)\,\mu(z_2)\,\mu(\bar{z}_2)\right\rangle_{\mathcal{M}_2}}{\left\langle \sigma(z_1)\,\sigma(\bar{z}_1)\,\sigma(z_2)\,\sigma(\bar{z}_2)\right\rangle_{\mathcal{M}_2}}\right]$$



★ Excited states in the Ising CFT can be studied, using the averaging protocol. Gives similar results.

- ⋆ For (invertible) group symmetries,

  the averaging protocol gives the same asymmetry
  as the symmetric boundary condition.

$\Rightarrow$ boundary conditions can be "neglected",

and anomalies do not affect the asymmetry.

(but can affect vanishing terms in $\varepsilon \to 0$) [Ali Ahmad, Klinger, Wang 25]

Conclusions

Entanglement asymmetry is a quantum-information-based observable well suited to detect symmetry breaking in QFT, in localized subsystems and in out-of-equilibrium states (but also in ground states).

Some questions under investigation:

- Unbroken / broken 1-form symmetries
  related to confinement / deconfinement (*e.g.* in 4d)
  or topological order

- Detect anomalies using asymmetry [WIP with E.García-Valdecasas, S.Vitouladitis]

- Entanglement asymmetry in gravitational theories
  and relation to black-hole physics
  and holography [WIP with O.Mamroud, N.Talwar, M.Tortora, P.Veltman]