Data Assimilation in Turbulence

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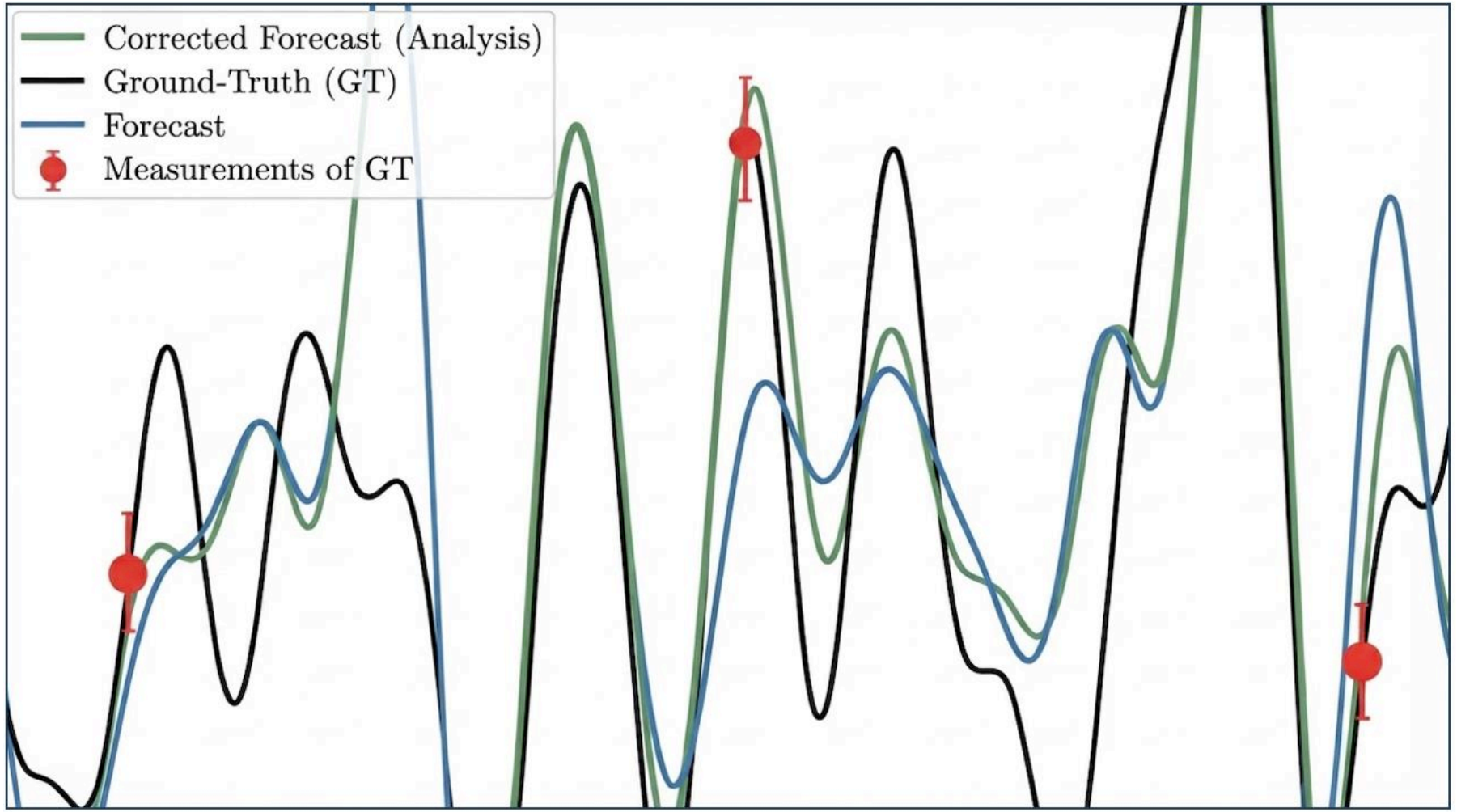
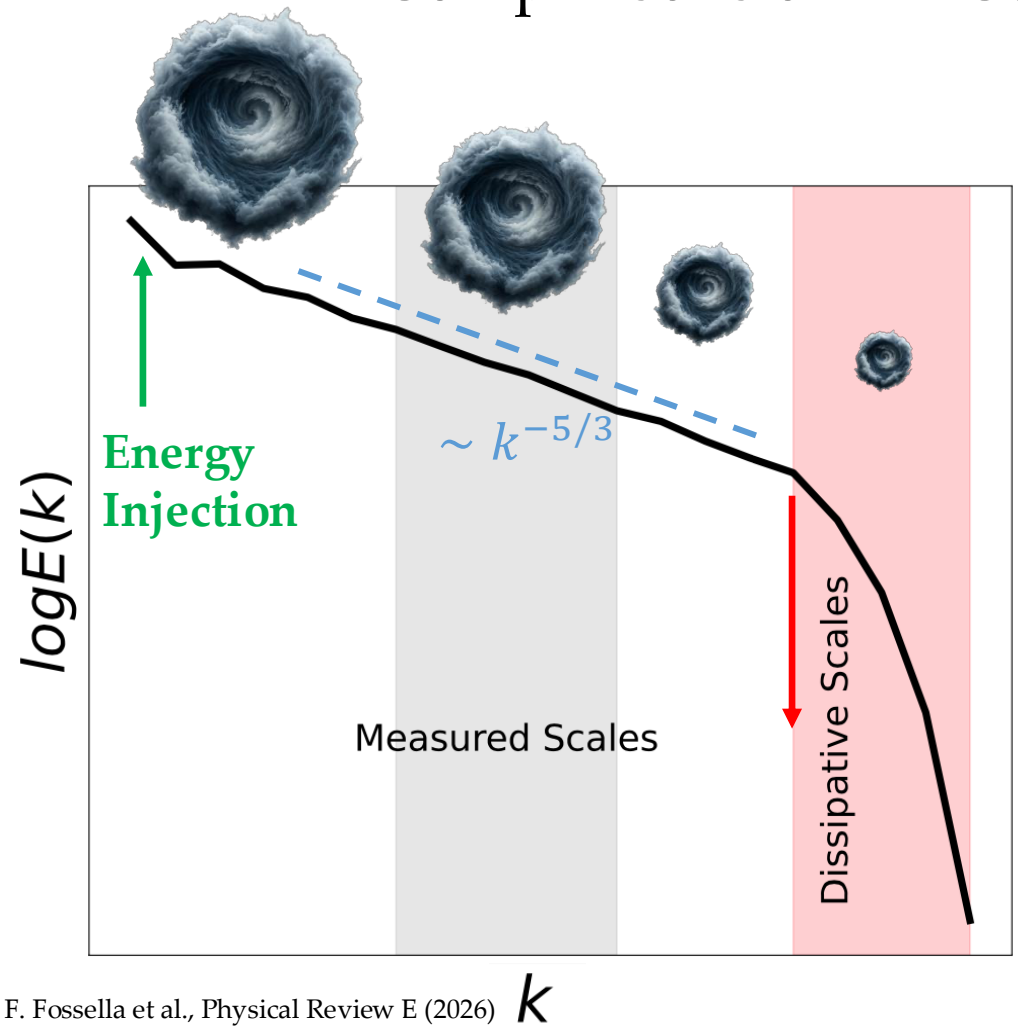


Multi-Scale Data Assimilation in Turbulent Models:

- Nudging & Ensemble Kalman Filter (EnKF)
- Comparisons on Shell Model (ODE)
- Comparisons on 2D Convection (PDE)



[1] F. Fossella et al., Physical Review E (2026)

➢ Nudging

$$\frac{d\tilde{u}_n}{dt} = \Phi_n(\tilde{\boldsymbol{u}}) + \gamma_n \delta_{nm}(z_m - \tilde{u}_m)$$

➢ Ensemble Kalman Filter

$$\frac{d\tilde{u}_n}{dt} = \Phi_n(\tilde{\boldsymbol{u}}) + \gamma_{nm}(t)\,(z_m - \tilde{u}_m)$$

$$\boldsymbol{\gamma}(t) : \frac{\partial}{\partial \boldsymbol{\gamma}} tr\langle(\boldsymbol{u} - \tilde{\boldsymbol{u}})(\boldsymbol{u} - \tilde{\boldsymbol{u}})^T\rangle = 0$$

A Model For Atmospheric Convection

$$\begin{cases} \dfrac{dx(t)}{dt} = \sigma(y(t) - x(t)) \\ \dfrac{dy(t)}{dt} = x(t)\big(\rho - z(t)\big) - y(t) \\ \dfrac{dz(t)}{dt} = x(t)y(t) - \beta z(t) \end{cases}$$

$$\sigma = 10\ ,\rho = 1\ ,\beta = 8/3$$

Lyapunov Spectrum

$$\lambda_x = 0.90\ ,\lambda_y = 0.001\ ,\lambda_z = -14.6$$

Fractal Dimension

$$D_{KY} = 2.06$$

[2] E. N. Lorenz, J. Atmos. Sci. (1963). [3] M. Cencini et al., "Chaos: from simple models to complex systems", World Scientific, Singapore, September 2009.

Nudging on Atmospheric Convection

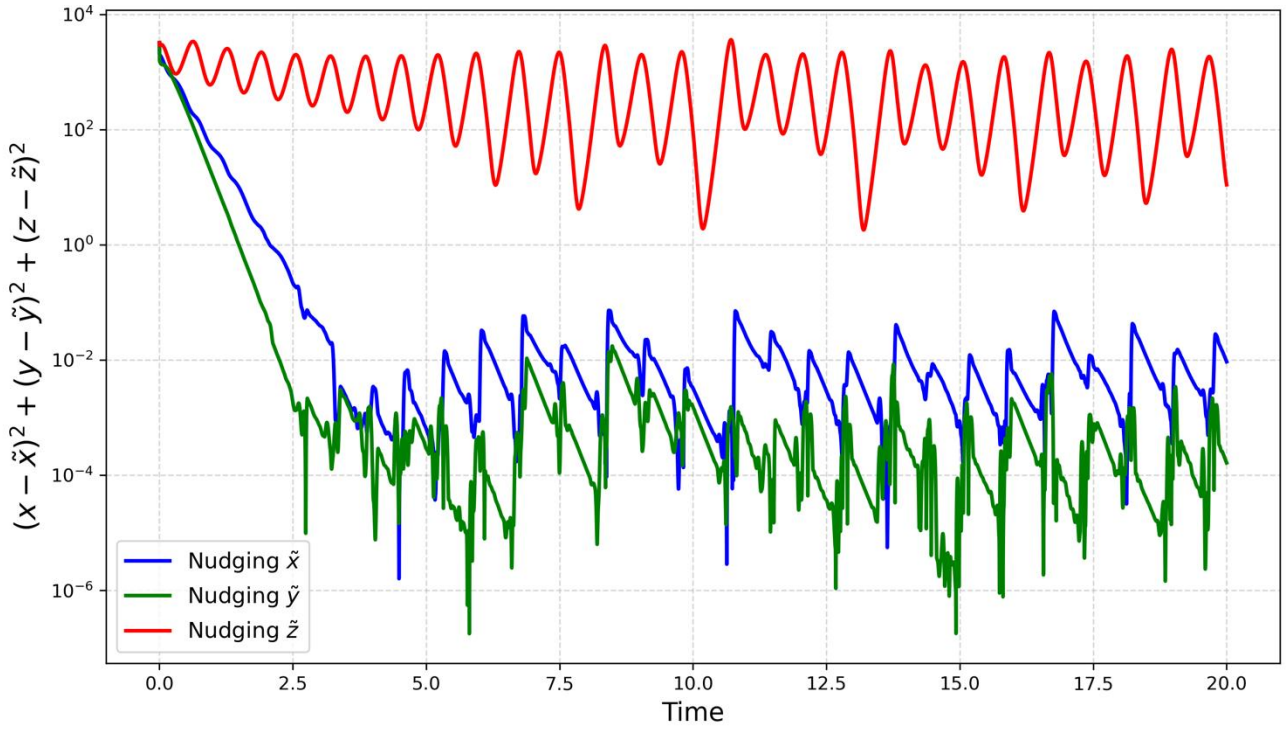
Ground-Truth

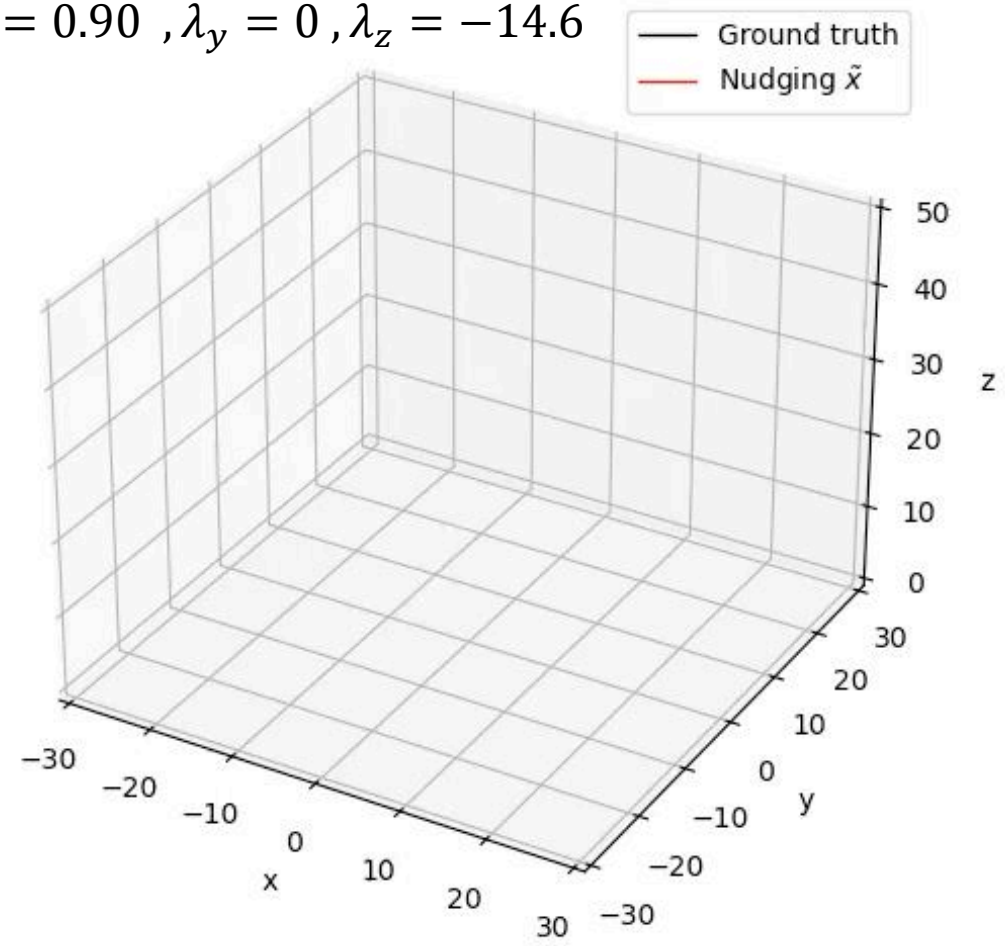
$$\begin{cases} x_{k+1} = \sigma(y_k - x_k) \\ y_{k+1} = x_k(\rho - z_k) - y_k \\ z_{k+1} = x_k y_k - \beta z_k \end{cases}$$

Forecast

$$\begin{cases} \tilde{x}_k = \boldsymbol{x_k} \quad (\boldsymbol{\gamma = 1000}) \\ \tilde{y}_{k+1} = \boldsymbol{x_k}(\rho - \tilde{z}_k) - \tilde{y}_k. \\ \tilde{z}_{k+1} = \boldsymbol{x_k}\tilde{y}_k - \beta\tilde{z}_k \end{cases}$$

$\lambda_x = 0.90$, $\lambda_y = 0$, $\lambda_z = -14.6$





We have to know the variables to Nudge

[4] L.M. Pecora & T.L. Carroll, *PRL* (1990). [5] G.S. Duane, & J.J. Tribbia, *JAS* (2004).

A Multi-Scale Challenge

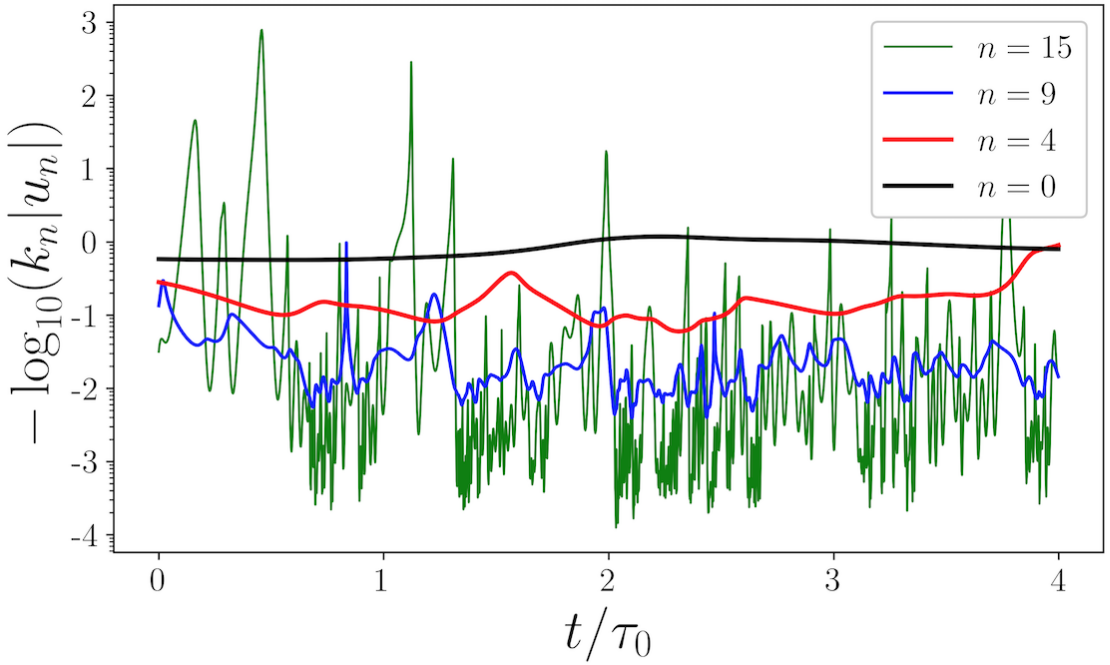
$$\{u_n \in \mathbb{C}, n = 1,2,..,N\} \qquad k_n = k_0 2^n$$

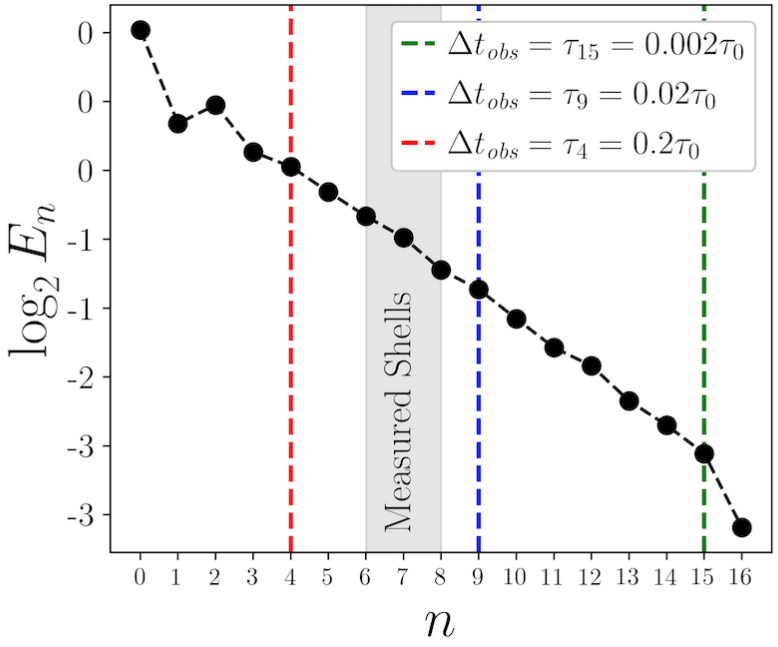
Dissipation Forcing

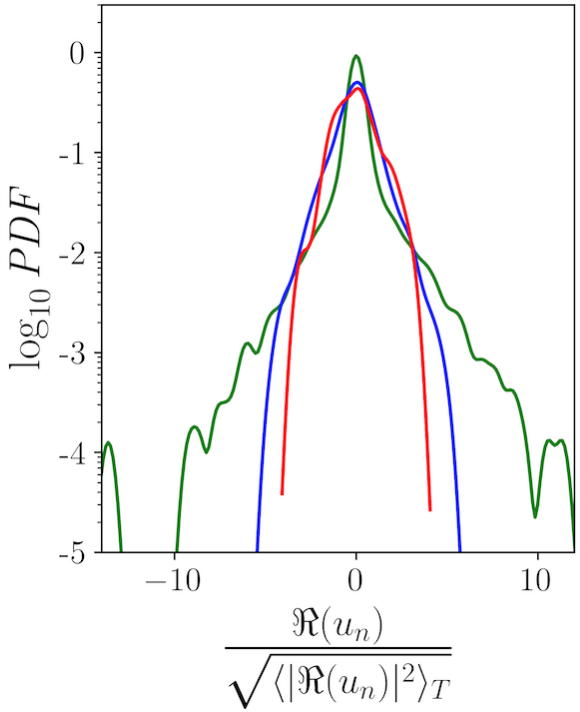
$$\frac{du_n}{dt} = i(ak_{n+1}u^*_{n+1}u_{n+2} + bk_n u^*_{n-1}u_{n+1} - ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n$$

Energy Flux across the scales

$$\Pi_n = \Im(a\, k_{n+1}u^*_n u^*_{n+1}u_{n+2} - bk_n u^*_{n-1}u^*_n u_{n+1})$$





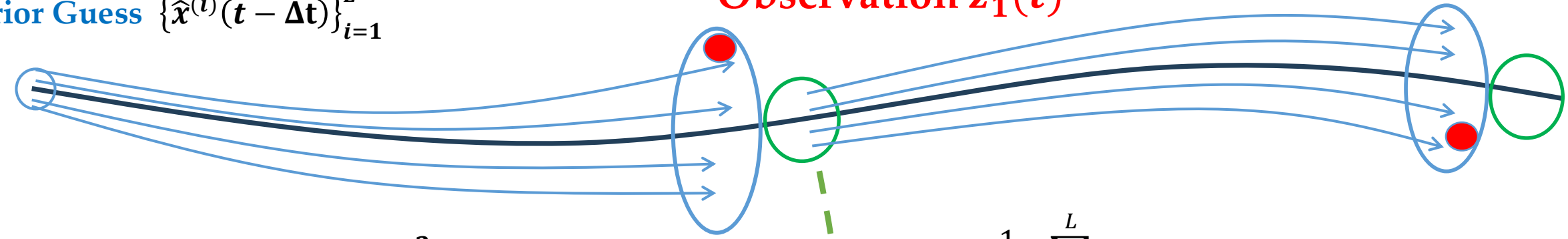


[6] L. Biferale, Annual Review of Fluid Mechanics (2003).
[7] V. S. L'vov et al., Physical Review E (1998).

Ensemble Kalman Filter

Prior Guess $\{\hat{x}^{(i)}(t-\Delta t)\}_{i=1}^{L}$

Observation $z_1(t)$

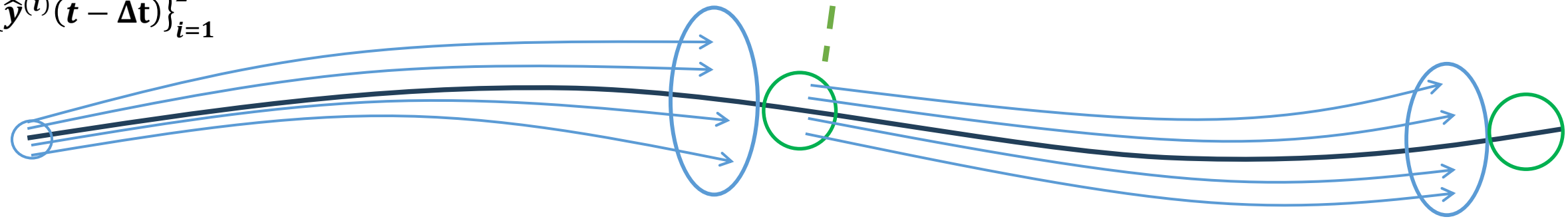


$$\tilde{x}^{(i)}(t) = \hat{x}^{(i)}(t) + \frac{\hat{\sigma}_x^2}{\hat{\sigma}_x^2 + \sigma^2}\{z_1(t) - \hat{x}^{(i)}(t)\}$$

$$\hat{\sigma}_x^2 = \frac{1}{L-1}\sum_{i=1}^{L}\left(\hat{x}^{(i)}(t) - \langle \hat{x}^{(i)}(t) \rangle\right)^2$$

$$\hat{\sigma}_{xy}(t) = \frac{1}{L-1}\sum_{i=1}^{L}\left(\hat{x}^{(i)}(t) - \langle \hat{x}^{(i)}(t) \rangle\right)\left(\hat{y}^{(i)}(t) - \langle \hat{y}^{(i)}(t) \rangle\right)$$

$\{\hat{y}^{(i)}(t-\Delta t)\}_{i=1}^{L}$

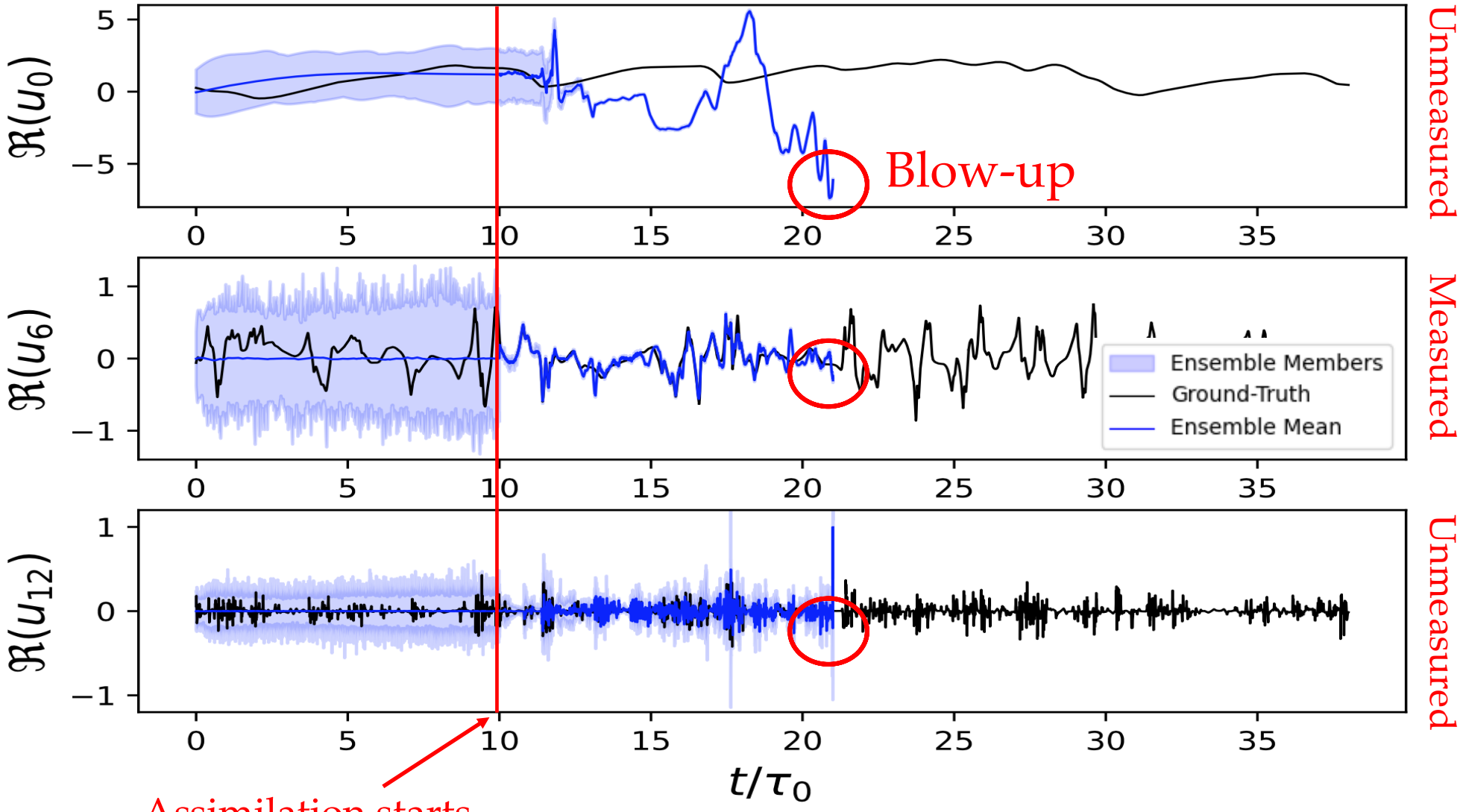


$$\tilde{y}^{(i)}(t) = \hat{y}^{(i)}(t) + \frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x^2 + \sigma^2}\{z_1(t) - \hat{x}^{(i)}(t)\}$$

[8] G. Evensen, Journal of Geophysical Research: Oceans (1994).
[9] G. Burgers et al., Monthly Weather Review (1998).
[10] P. J. van Leeuwen, Quarterly Journal of the Royal Meteorological Society (2020).

Catastrophic Divergence

Measuring 6,7, 8 $\Delta t_{obs} = \tau_{15} = 0.002\,\tau_0$ L=1000 ensemble members



[11] G. A. Gottwald et al., Nonlinear Processes in Geophysics (2013).

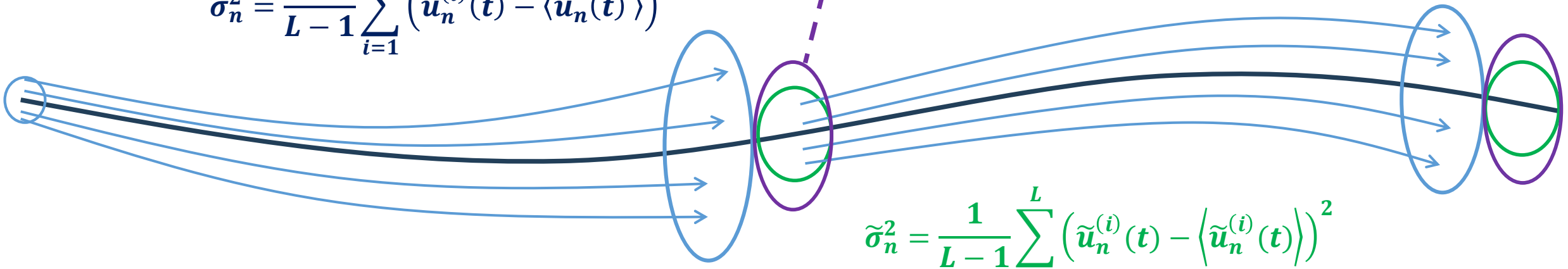
Scale Dependent Adaptive Inflation

After the filter correction, we Inflate ALL the variables $\tilde{u}_n^{(i)}$

$$g_n = 1 + \lambda * \frac{\hat{\sigma}_n - \tilde{\sigma}_n}{\hat{\sigma}_n}$$

$$\tilde{u}_n^{(i)} \leftarrow \tilde{u}_n^{(i)} * g_n + (1 - g_n) * \langle \tilde{u}_n \rangle$$

$$\hat{\sigma}_n^2 = \frac{1}{L-1} \sum_{i=1}^{L} \left(\hat{u}_n^{(i)}(t) - \langle \hat{u}_n(t) \rangle \right)^2$$

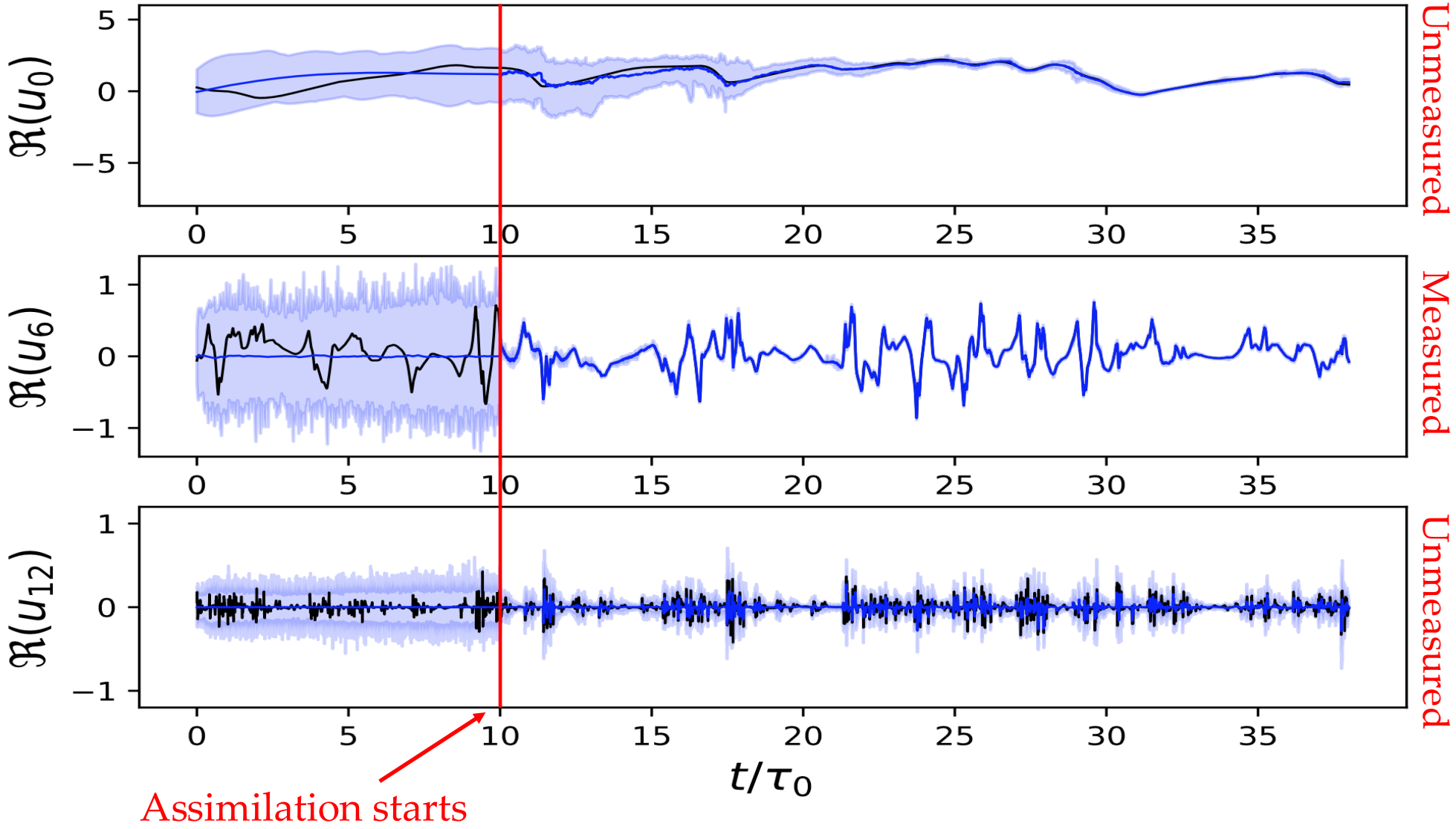


$$\tilde{\sigma}_n^2 = \frac{1}{L-1} \sum_{i=1}^{L} \left(\tilde{u}_n^{(i)}(t) - \left\langle \tilde{u}_n^{(i)}(t) \right\rangle \right)^2$$

[12] F. Zhang et al., Monthly Weather Review (2004).

Measuring meso scales

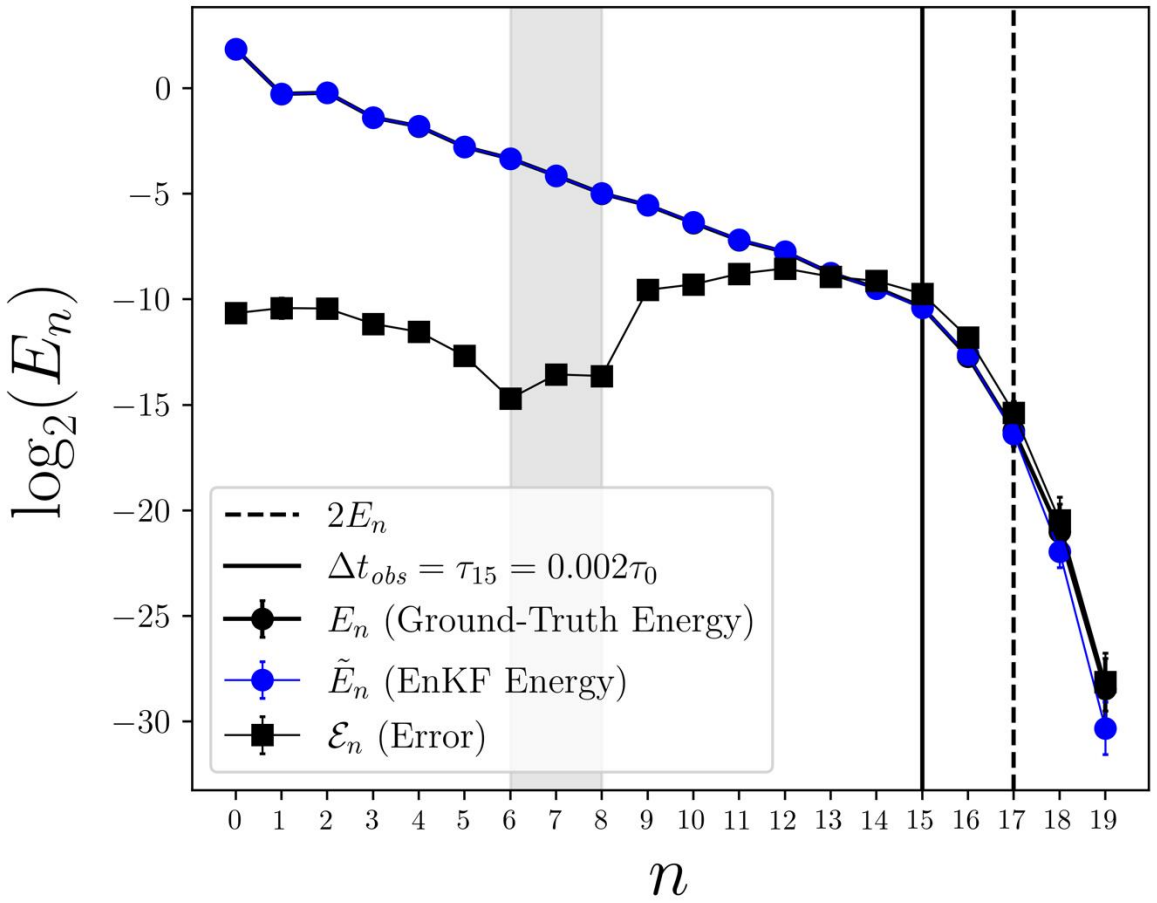
Measuring 6,7, 8 $\Delta t_{obs} = \tau_{15} = 0.002\,\tau_0$ L=1000 ensemble members

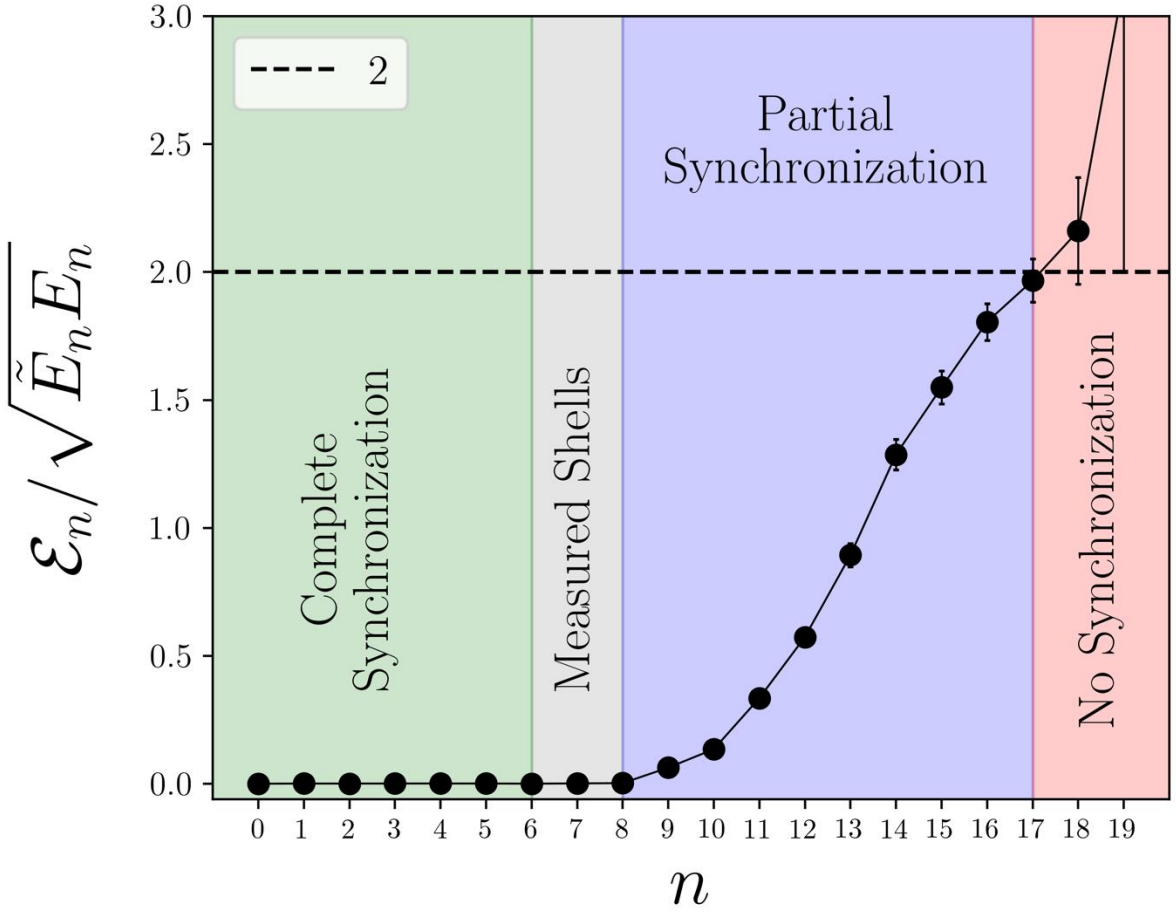


Measuring meso scales 6, 7, 8 with $\Delta t_{obs} = \tau_{15} = 0.002\tau_0$

$$\frac{\mathcal{E}_n}{\sqrt{E_n \tilde{E}_n}} = \frac{\langle |u_n - \tilde{u}_n|^2 \rangle_{T,L}}{\sqrt{\langle |u_n|^2 \rangle_T \langle |\tilde{u}_n|^2 \rangle_{T,L}}} = \frac{E_n + \tilde{E}_n - \langle u_n \tilde{u}_n^* + u_n^* \tilde{u}_n \rangle_{T,L}}{\sqrt{E_n \tilde{E}_n}} = 2$$

$E_n = \tilde{E}_n$ $\quad \langle u_n \tilde{u}_n^* + u_n^* \tilde{u}_n \rangle_{T,L} = 0$





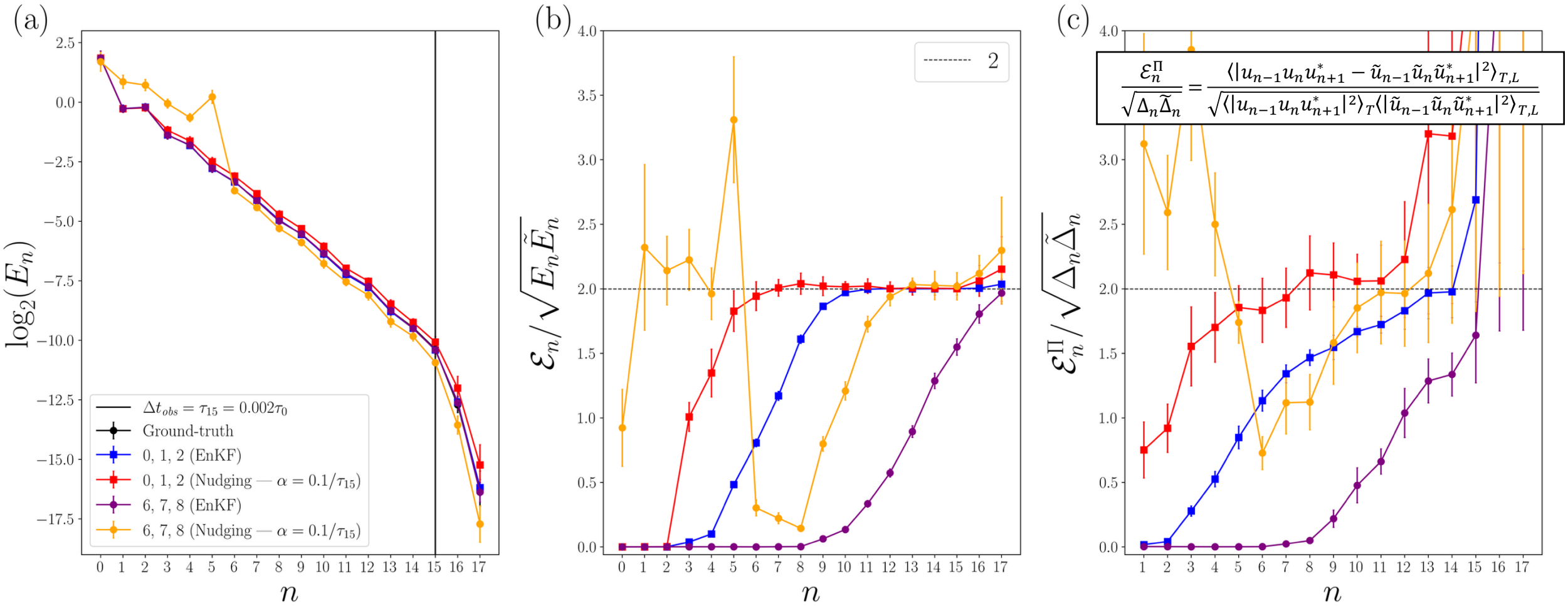
Comparisons EnKF and Nudging

$$\Pi_5 = \Im(a\, k_6 u_6^* u_7^* u_8 - b k_5 u_5^* u_6^* u_7) \qquad \text{6, 7, 8 Nudging}$$

$$\Pi_5 = \Im(a\, k_6 u_6^* u_7^* u_8 - b k_5 u_5^* u_6^* u_7) \qquad \text{6, 7, 8 EnKF}$$

Corrected

Weakly Corrected

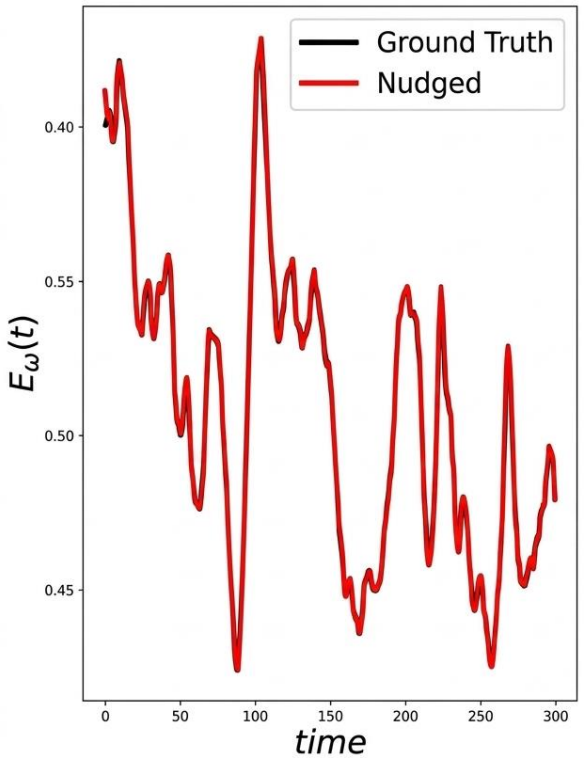


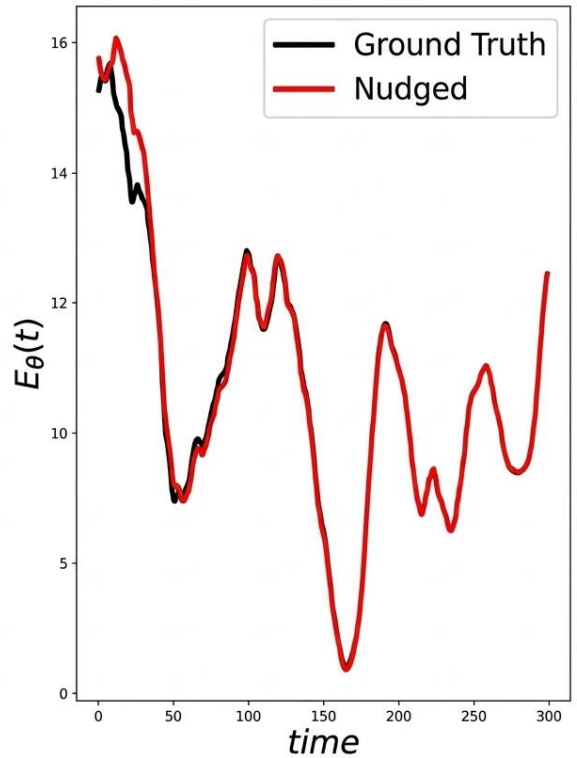
Nudging 2D Rayleigh-Benard

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \Delta^{\xi} \omega + \alpha \partial_x \theta + \mu \Delta^{-1} \omega \\ \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta^{\beta} \theta + \gamma u_z \end{cases}$$

➢ Nudging Vorticity

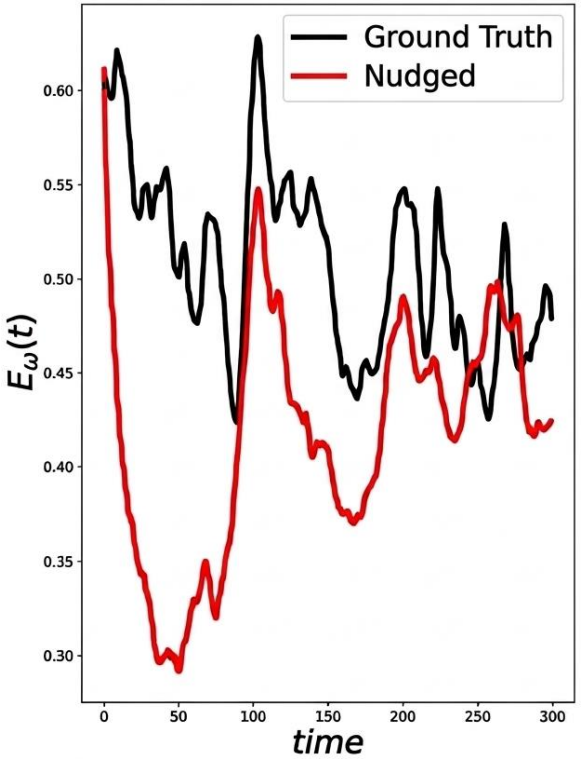
$$\begin{cases} \omega = \tilde{\omega} \qquad \delta\theta = \theta - \tilde{\theta} \\ \partial_t \delta\theta + u \cdot \nabla \delta\theta = \kappa \Delta^{\beta} \delta\theta \end{cases}$$

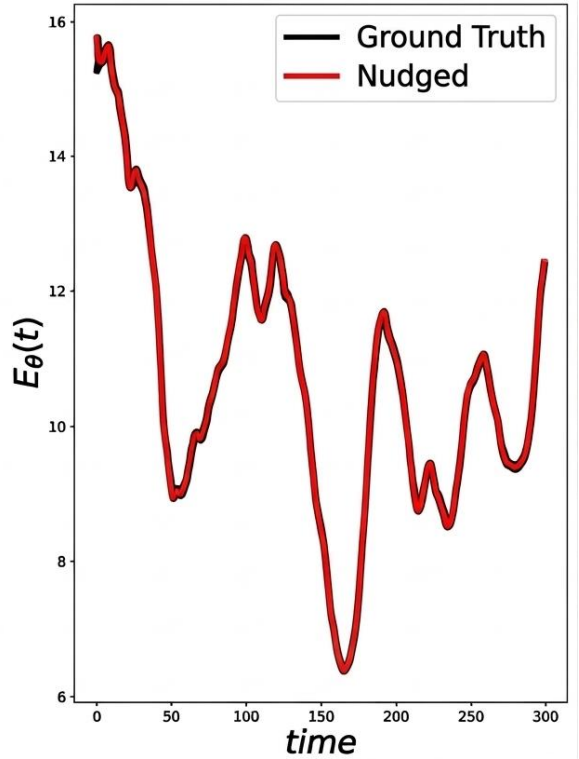




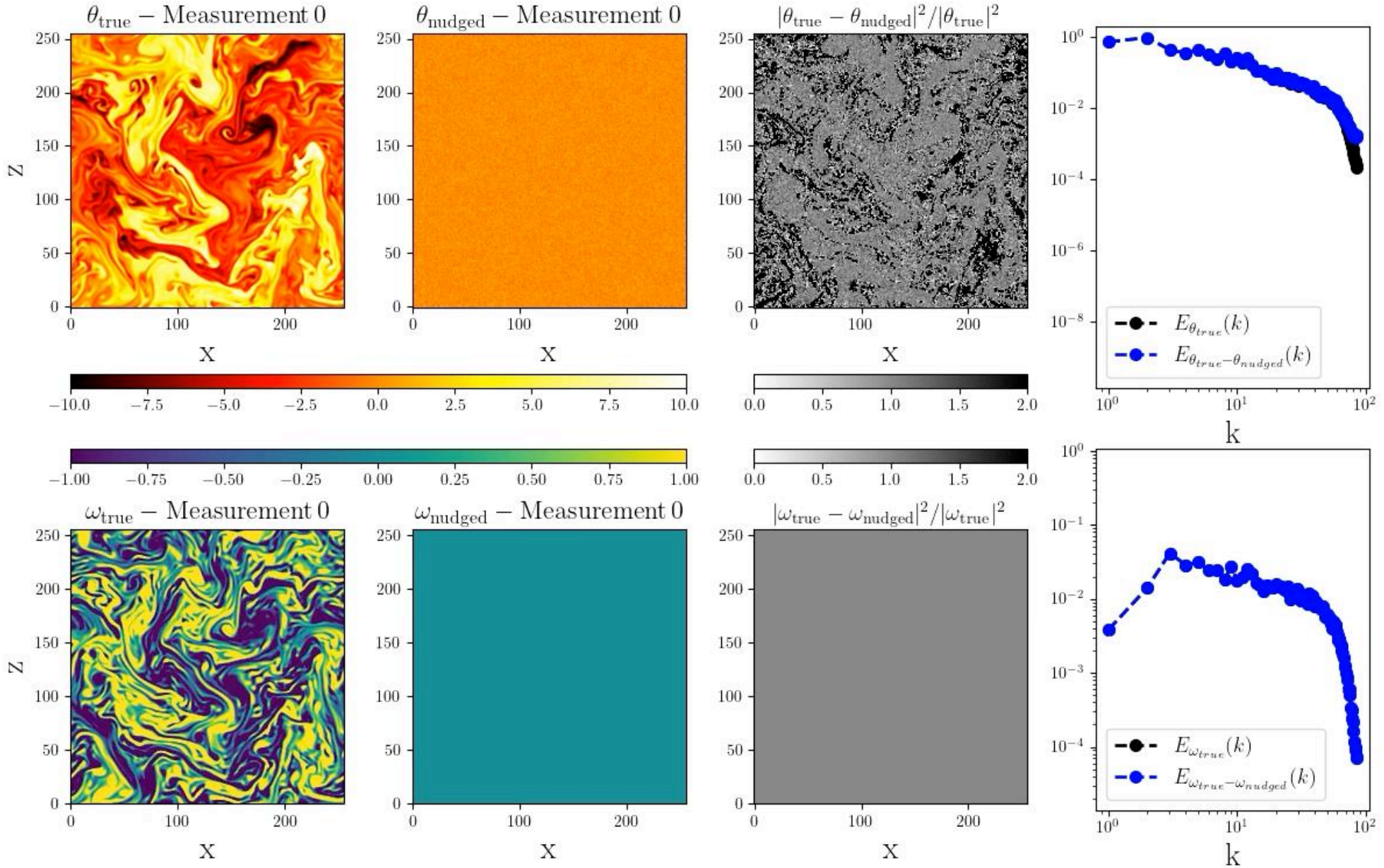
➢ Nudging Temperature

$$\begin{cases} \theta = \tilde{\theta} \qquad \delta\omega = \omega - \tilde{\omega} \\ \partial_t \delta\omega + u \cdot \nabla \omega - \tilde{u} \cdot \nabla \tilde{\omega} = \nu \Delta^{\xi} \delta\omega + \mu \Delta^{-1} \delta\omega \end{cases}$$

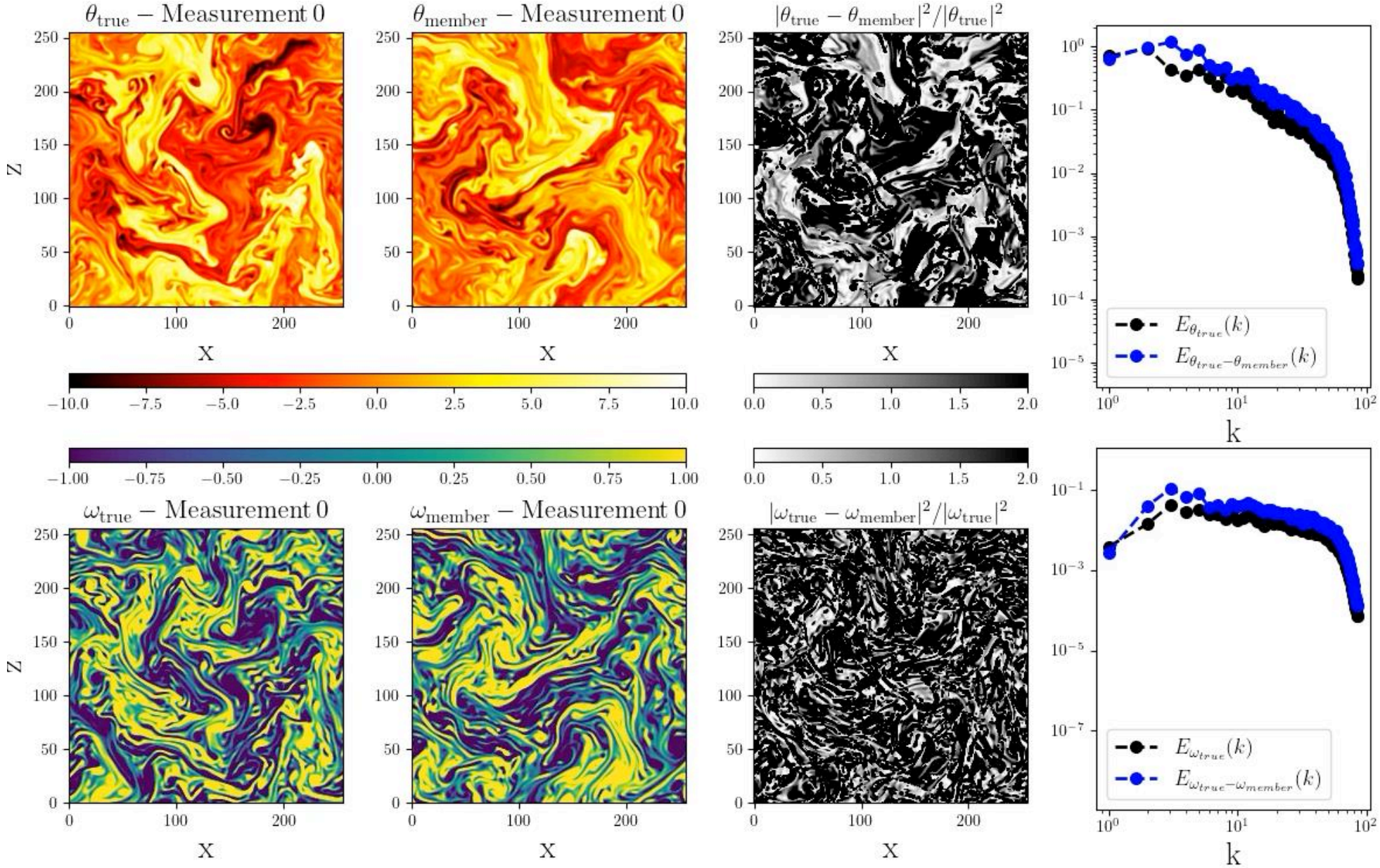




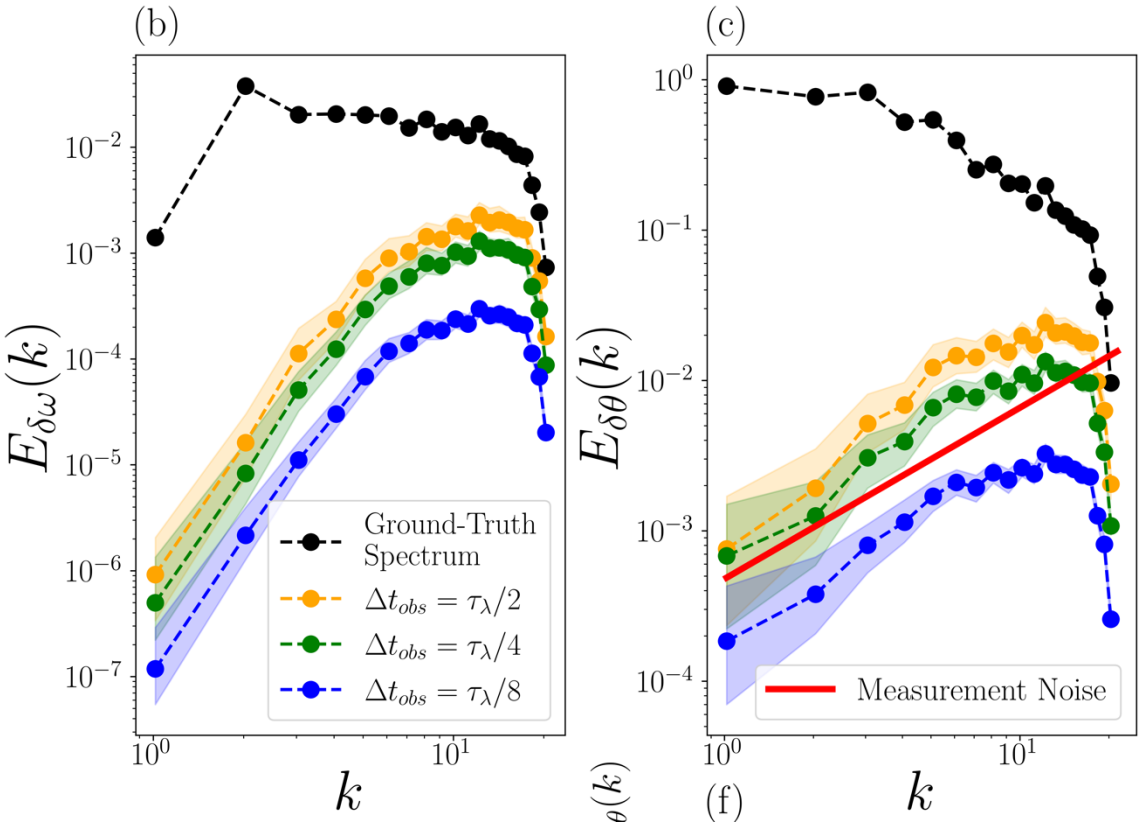
Nudging 2D Rayleigh-Benard – Measuring Entire Domain



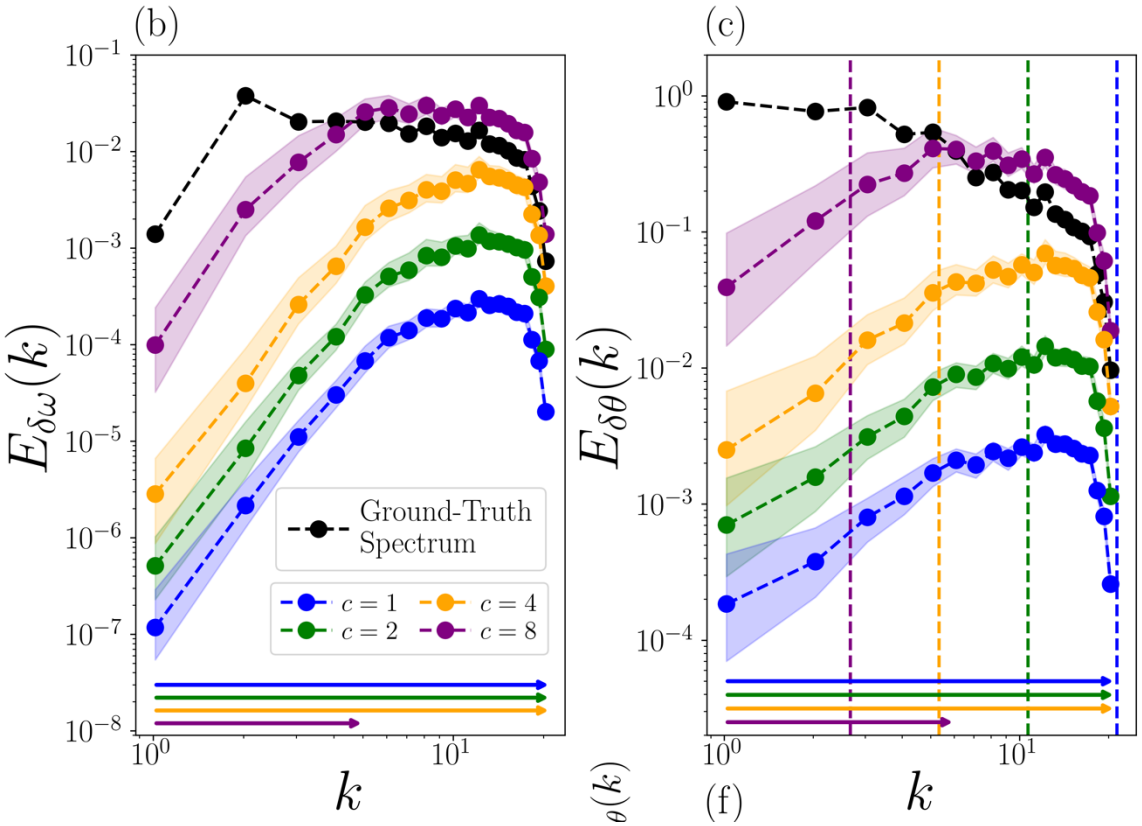
EnKF 2D Rayleigh-Benard – Measuring Entire Domain



No Need to be Dense in Time



No Need to be Dense in Space



Conclusion

[13] L. Agasthya et al., Physics of Fluids (2022).
[14] P. Clark Di Leoni et al., European Physical Journal E (2023).
[15] K. Um et al., Advances in Neural Information Processing Systems. (2020).

- Nudging fails to modify Lyapunov Spectrum with sparse information
- EnKF does, making Synchronization

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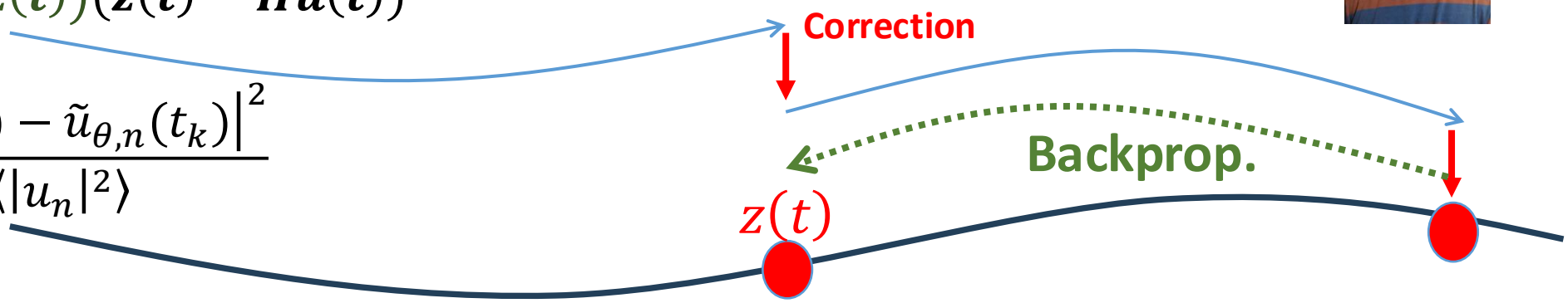
($L = 10^4$ + Inflation) → essentially an HPC problem

A. Freitas



$$\tilde{\boldsymbol{u}}_{\boldsymbol{\theta}} = \hat{\boldsymbol{u}}(\boldsymbol{t}) + \boldsymbol{K}_{\boldsymbol{\theta}}\big(\hat{\boldsymbol{u}}(\boldsymbol{t}), \boldsymbol{z}(\boldsymbol{t})\big)(\boldsymbol{z}(\boldsymbol{t}) - \boldsymbol{H}\hat{\boldsymbol{u}}(\boldsymbol{t}))$$

$$Loss = \sum_k \sum_n \frac{\left|u_n(t_k) - \tilde{u}_{\theta,n}(t_k)\right|^2}{\langle |u_n|^2 \rangle}$$





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[8] G. Evensen (1994). **Sequential Data Assimilation with a Nonlinear Quasi-Geostrophic Model Using Monte Carlo Methods to Forecast Error Statistics**. Journal of Geophysical Research: Oceans.
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