S-wave $N\pi$ scattering lengths from lattice QCD

C. Alexandrou[1,2], Y. Li[1], G. Koutsou[1], S. Paul[2], F. Pittler[1]

[1]The Cyprus Institute [2]University of Cyprus

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Motivation

The near-threshold $N\pi$ interaction is characterized at leading order by the two isospin scattering lengths $a_{1/2}$ and $a_{3/2}$. It is also tied to the pion-nucleon sigma term,

$$\sigma_{\pi N} = \widehat{m}\,\langle N|\bar{u}u + \bar{d}d|N\rangle, \tag{1}$$

which measures explicit chiral-symmetry breaking in the nucleon mass. Phenomenological extractions use the Cheng-Dashen low-energy theorem and a dispersive continuation of the πN amplitude away from threshold. The small isoscalar combination $a^+ = (a_{1/2} + 2a_{3/2})/3$ results from a strong cancellation, making the sigma-term analysis sensitive to small amplitude shifts, isospin breaking and threshold systematics.

- Determine finite-volume $N\pi$ energy shifts close to threshold.
- Extract $m_\pi a_{1/2}$ and $m_\pi a_{3/2}$ using Lüscher's method.
- Provide first-principles threshold constraints relevant to dispersive determinations of $\sigma_{\pi N}$.
- Quantify the effect of charged–neutral pion mass splitting in the twisted-mass calculation.

Ensembles, interpolating fields and methods

Ensemble	Action	$L^3 \times T$	m_π	Operators
MILC	HISQ/valence setup	$64^3 \times 96$	140.6(1)MeV	p, $N\pi_{I=1/2}$,$N\pi_{I=3/2}$
cA2.09.48	twisted mass	$48^3 \times 96$	130.62(6)MeV	p, $p\pi^0$, $n\pi^+$

For the twisted-mass ensemble, the charged and neutral pion thresholds are distinct. The correlation matrix is

$$C_{ij}(t) = \langle 0|\mathcal{O}_i(t)\mathcal{O}_j^\dagger(0)|0\rangle, \qquad \mathcal{O}_i \in \{p, p\pi^0, n\pi^+\}.$$

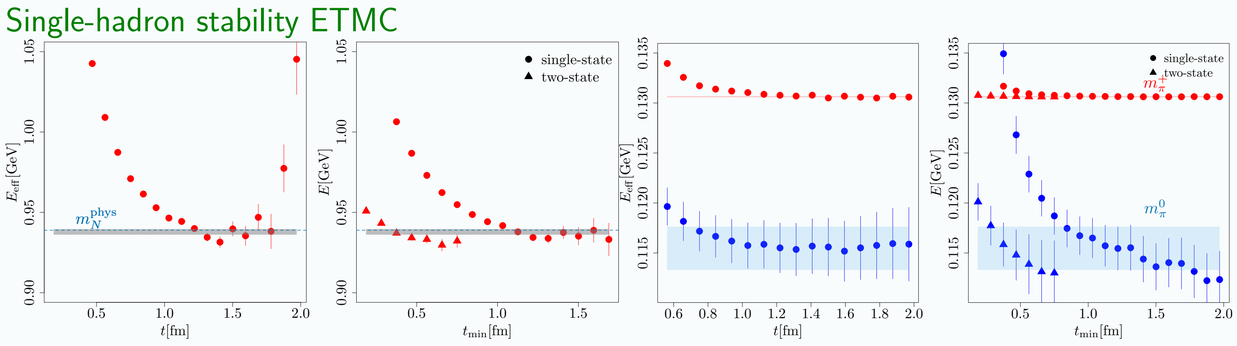
The spectrum is obtained from

$$C(t)v_n(t,t_0) = \lambda_n(t,t_0)C(t_0)v_n(t,t_0).$$

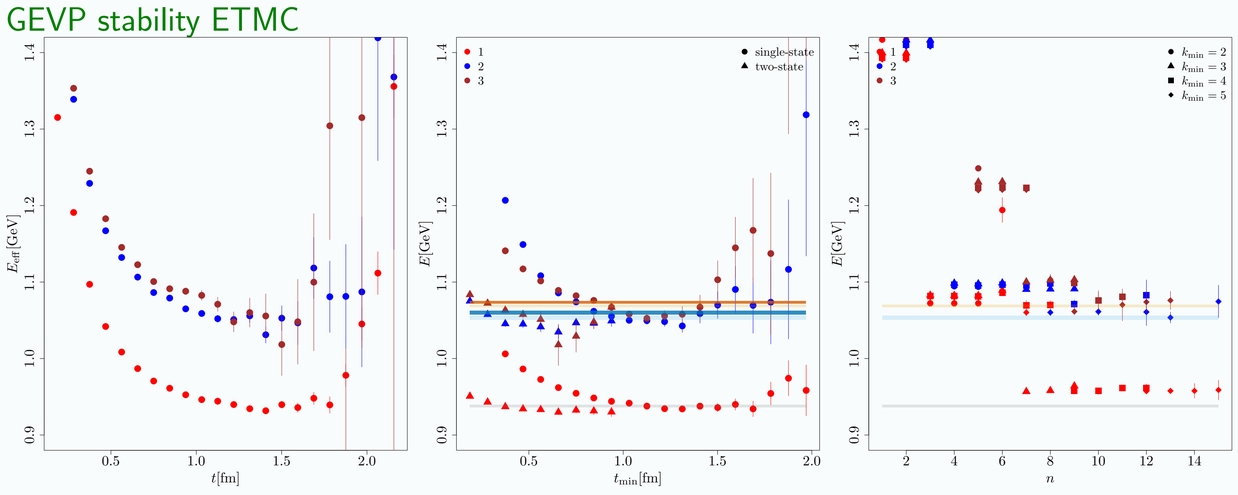
Truncated Hankel correlator method

The THC method enlarges the variational basis using time-shifted copies of the correlation functions. A truncated singular-value decomposition removes poorly constrained modes, and the finite-volume energies are extracted from the resulting principal correlators. Stability is tested by varying the truncation parameter $k_{\min}$.

Single-hadron stability ETMC



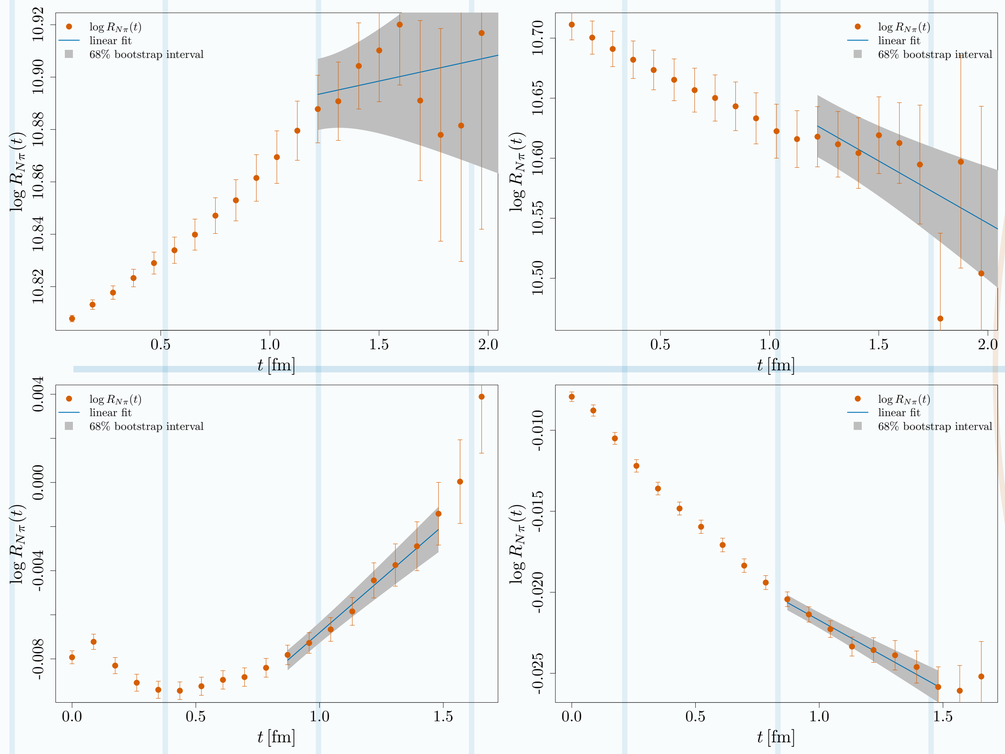
GEVP stability ETMC



Correlated ratio method

For each principal correlator, ratios to the corresponding non-interacting correlators isolate the energy shift:

$$R_n^{(c)}(t) = \frac{\lambda_n(t,t_0)}{C_N(t)C_{\pi_c}(t)} \longrightarrow A_n^{(c)} e^{-\Delta E_n^{(c)}(t-t_0)}, \Delta E_n^{(c)} = E_n - m_N - m_{\pi_c}, \qquad c \in \{\pi^0, \pi^+\}.$$



Finite-volume quantization conditions

Single-channel case. For an S-wave level with energy E, the center-of-mass momentum is $k(E) = \frac{\sqrt{\lambda(E^2,m_N^2,m_\pi^2)}}{2E}$, $q = \frac{kL}{2\pi}$, where λ is the Källén function. Lüscher's condition reads $\tan\delta_0(k) = \frac{\pi^{3/2}q}{Z_{00}(1;q^2)}$. Using the leading effective-range expansion in our sign convention, $\tan\delta_0(k) = a_0 k + \mathcal{O}(k^3)$, the scattering length follows from

$$\boxed{a_0 = \frac{\tan\delta_0(k)}{k} = \frac{\tan\delta_L(E,L)}{k(E)}}.$$

Equivalently, defining $B(E,L) = \frac{k(E)}{\tan\delta_L(E,L)}$, the quantization condition is $\boxed{a_0^{-1} - B(E,L) = 0}$. Close to threshold, the equivalent large-volume expansion is

$$\Delta E = -\frac{2\pi a_0}{\mu L^3}\left[1 + c_1\frac{a_0}{L} + c_2\left(\frac{a_0}{L}\right)^2\right] + \mathcal{O}(L^{-6}),$$

with

$$\mu = \frac{m_N m_\pi}{m_N + m_\pi}, \qquad c_1 = -2.837297, \qquad c_2 = 6.375183.$$

Coupled charge-channel case. For twisted-mass fermions, $m_{\pi^0} \neq m_{\pi^+}$ and the $p\pi^0$ and $n\pi^+$ thresholds are distinct. We define

$$B_0(E,L) = \frac{k_0(E)}{\tan\delta_{L,0}(E,L)}, \qquad B_+(E,L) = \frac{k_+(E)}{\tan\delta_{L,+}(E,L)}.$$

The interaction is assumed diagonal in the isospin basis,

$$K^{-1} = \mathrm{diag}\left(a_{1/2}^{-1}, a_{3/2}^{-1}\right),$$

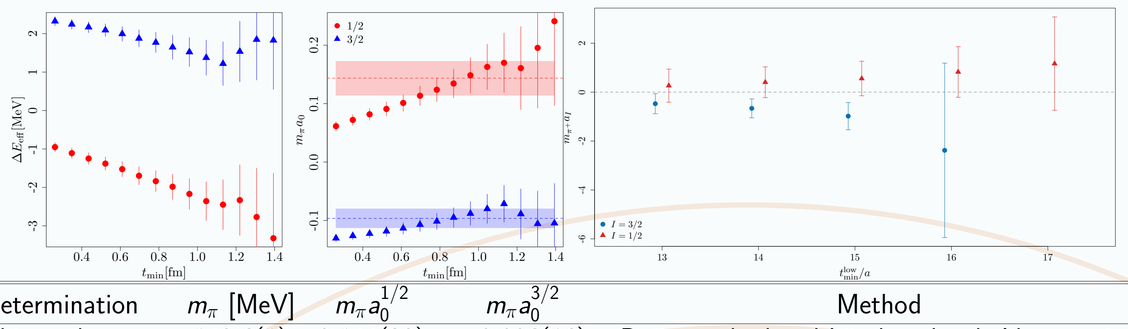
whereas the finite-volume functions are diagonal in the charge basis. After rotating to the isospin basis, each energy level satisfies

$$\boxed{\left[a_{1/2}^{-1} - \frac{B_0 + 2B_+}{3}\right]\left[a_{3/2}^{-1} - \frac{2B_0 + B_+}{3}\right] - \frac{2}{9}(B_0 - B_+)^2 = 0}.$$

The two measured finite-volume energies E_1 and E_2 provide two nonlinear equations, which are solved simultaneously for $a_{1/2}$ and $a_{3/2}$. All masses and energies are propagated sample by sample through the jackknife analysis.

Results: MILC, ETMC ensemble

Channel	$m_\pi a_I$(MILC)	$m_\pi a_I$(ETMC)	Interpretation
$I = \frac{1}{2}$	0.144(29)	0.40(62)	attractive
$I = \frac{3}{2}$	−0.096(16)	−0.66(39)	repulsive



Determination	m_π [MeV]	$m_\pi a_0^{1/2}$	$m_\pi a_0^{3/2}$	Method
This work	140.6(1)	0.144(29)	−0.096(16)	Ratio method and Lüscher threshold expansion
Hoferichter et al.	139.57	0.1698(20)	−0.0863(18)	Pionic atoms and Roy–Steiner analysis
Wang et al.	physical	0.157(31)	−0.104(18)	Lattice QCD, GEVP and Lüscher method
Alexandrou et al.	139	—	−0.13(4)	Lattice QCD, ratio method and Lüscher analysis
Bulava et al.	200	0.142(22)	−0.2735(81)	Lattice QCD, finite-volume spectrum and ERE
Silvi et al.	255.4(1.6)	—	0.057(67)(45)	Lattice QCD, global S- and P-wave analysis

Conclusions and outlook

- ETMC: Both $N\pi^0$ and $N\pi^+$ reference thresholds are propagated with matched jackknife samples.
- ETMC: The $I = \frac{3}{2}$ interaction is predominantly negative and is better constrained by the upper level.
- ETMC: The spectrum gives $a_{1/2}^{-1}$ compatible with zero over several fit windows; inversion therefore produces an unbounded or strongly non-Gaussian $a_{1/2}$ distribution.
- The MILC spectrum yields the expected signs: attractive $I = \frac{1}{2}$ and repulsive $I = \frac{3}{2}$.
- Charged–neutral pion splitting requires the two-threshold coupled quantization condition on the twisted-mass ensemble.
- The ratio method improves correlated energy shifts, but does not add an independent isospin constraint.
- Using the MILC scattering lengths as input to the linearized Roy–Steiner relation, we obtain the exploratory estimate

  $$\sigma_{\pi N} = 43.8(16.3)\,\mathrm{MeV},$$

  where the uncertainty includes the statistical scattering-length errors and the intrinsic Roy–Steiner uncertainty, but not yet the full lattice systematics.
- A maximal-isospin $p\pi^+$ calculation would directly constrain $a_{3/2}$ and stabilize $a_{1/2}$.
- Future work: additional volumes/momenta, effective-range terms and a controlled treatment of lattice isospin breaking.

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Contact: f.pittler@cyi.ac.cy

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