The nucleons generalized form factors and Mellin moments up to fourth order

Christian Kummer

with: C. Alexandrou, S. Bacchio, M. Constantinou, Y. Li, G. Spanoudes

C. Alexandrou *et al.*, (2026), 2607.03458

Funded by the European Union

Πανεπιστήμιο Κύπρου
University of Cyprus

TECHNISCHE UNIVERSITÄT BERLIN

AQTIVATE

The *AQTIVATE* project receives funding from the European Union's HORIZON MSCA Doctoral Networks programme, under Grant Agreement No. 101072344.

Recap – Lattice QCD

“Through lattice chromodynamics, we can arrange time and space in a grid, which allows us to simulate the quarks binding into the protons.”

Recap – Lattice QCD

"Through lattice chromodynamics, we can arrange time and space in a grid, which allows us to simulate the quarks binding into the protons."

Bruce Banner,

in *Spider-Man: Brand New Day*



Recap – Lattice QCD

- Formulate QCD path integral on discretized space-time

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\left[\psi, \overline{\psi}\right] \mathcal{D}[U] \, \mathrm{e}^{-S_F[\psi,\overline{\psi},U] - S_G[U]} \, O[\psi, \overline{\psi}, U]$$

Recap – Lattice QCD

- Formulate QCD path integral on discretized space-time

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\left[\psi, \overline{\psi}\right] \mathcal{D}[U] \, \mathrm{e}^{-S_F[\psi,\overline{\psi},U] - S_G[U]} \, O[\psi, \overline{\psi}, U]$$

- Calculate correlators

$$C(t) = \langle O(t) O^\dagger(0) \rangle$$

Recap – Lattice QCD

- Formulate QCD path integral on discretized space-time

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\left[\psi, \overline{\psi}\right] \mathcal{D}[U] \, \mathrm{e}^{-S_F[\psi,\overline{\psi},U] - S_G[U]} \, O[\psi, \overline{\psi}, U]$$

- Calculate correlators

$$C(t) = \langle O(t) O^{\dagger}(0) \rangle$$

- Extract hadron observables

Recap – Lattice QCD

- Formulate QCD path integral on discretized space-time

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\left[\psi, \overline{\psi}\right] \mathcal{D}[U] \, \mathrm{e}^{-S_F[\psi,\overline{\psi},U] - S_G[U]} \, O[\psi, \overline{\psi}, U]$$

- Calculate correlators

$$C(t) = \langle O(t) O^{\dagger}(0) \rangle$$

- Extract hadron observables
- Here: higher moments of GPDs

Motivation

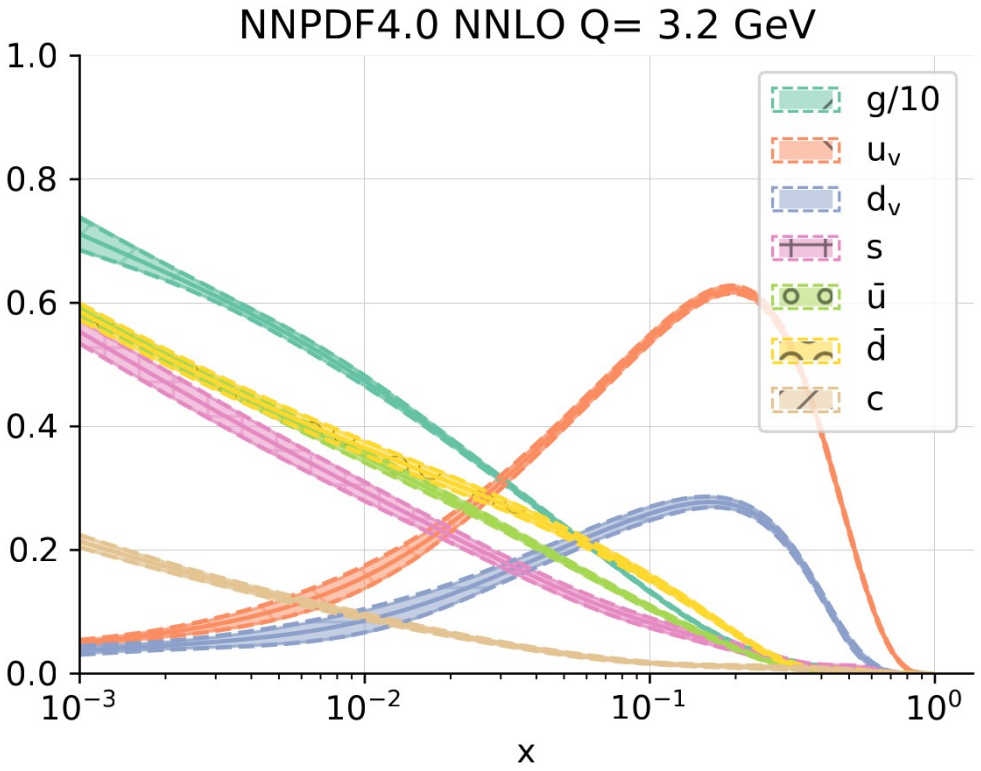
- Proton is an excellent probe for strong processes

Motivation

- Proton is an excellent probe for strong processes
- Internal structure highly non-perturbative ➡ Lattice QCD

Motivation

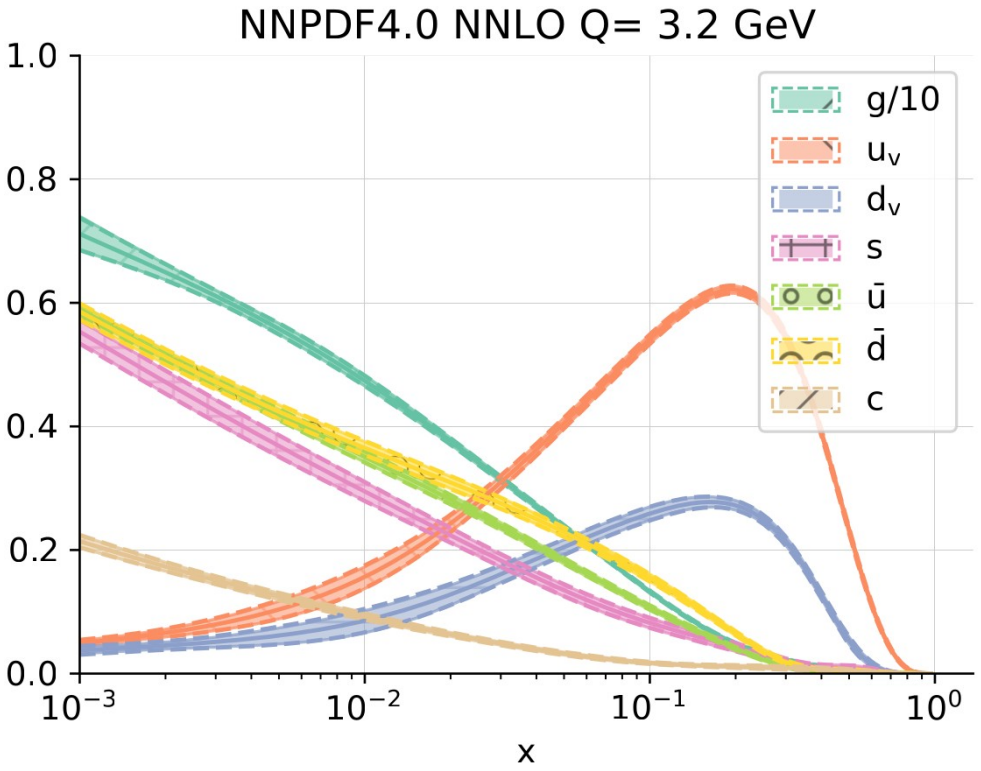
- Proton is an excellent probe for strong processes
- Internal structure highly non-perturbative ➡ Lattice QCD
- Interest in generalized parton distributions (GPDs)



NNPDF, EPJC 82, 428 (2022)

Motivation

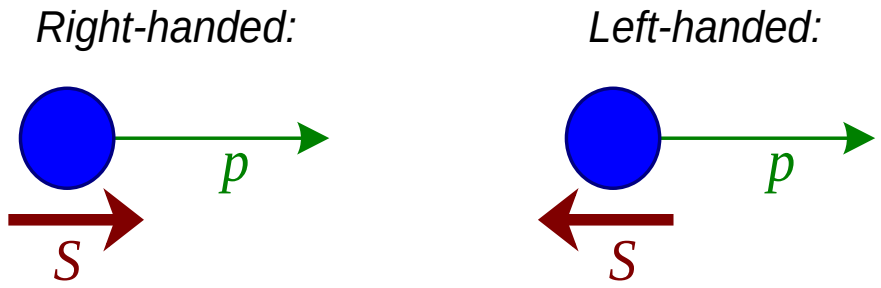
- Proton is an excellent probe for strong processes
- Internal structure highly non-perturbative ➡ Lattice QCD
- Interest in generalized parton distributions (GPDs)
- Direct calculation of Mellin moments



NNPDF, EPJC 82, 428 (2022)

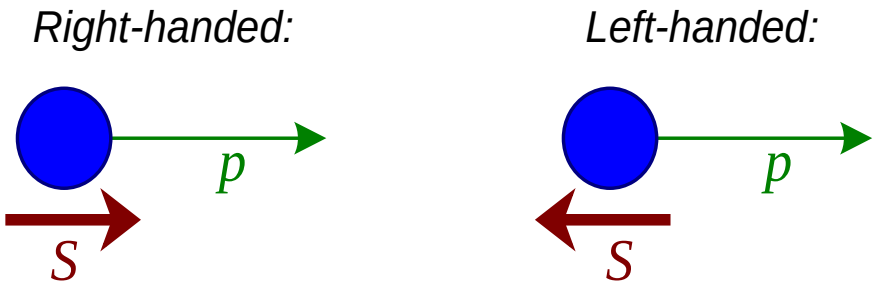
Polarization

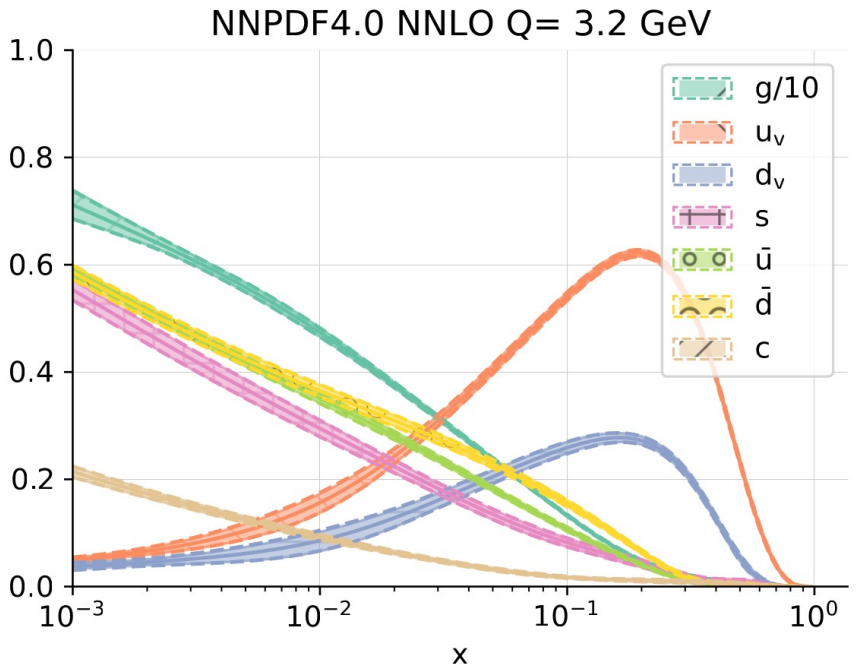
- Quantum number: Helicity



Polarization

- Quantum number: Helicity
- Polarized: RH + LH
  - Positive definite in forward limit
  - Experimentally very precisely determined
  - Many sum rules known

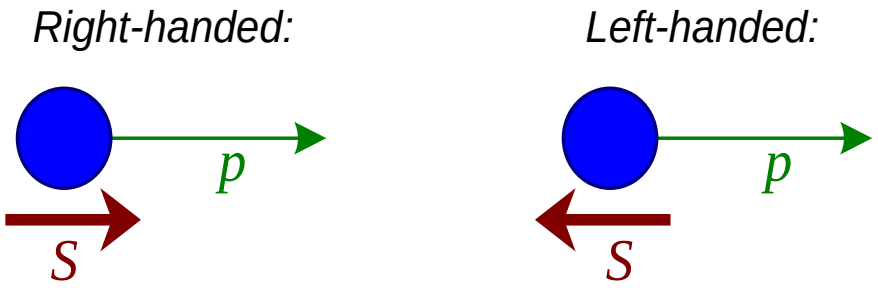


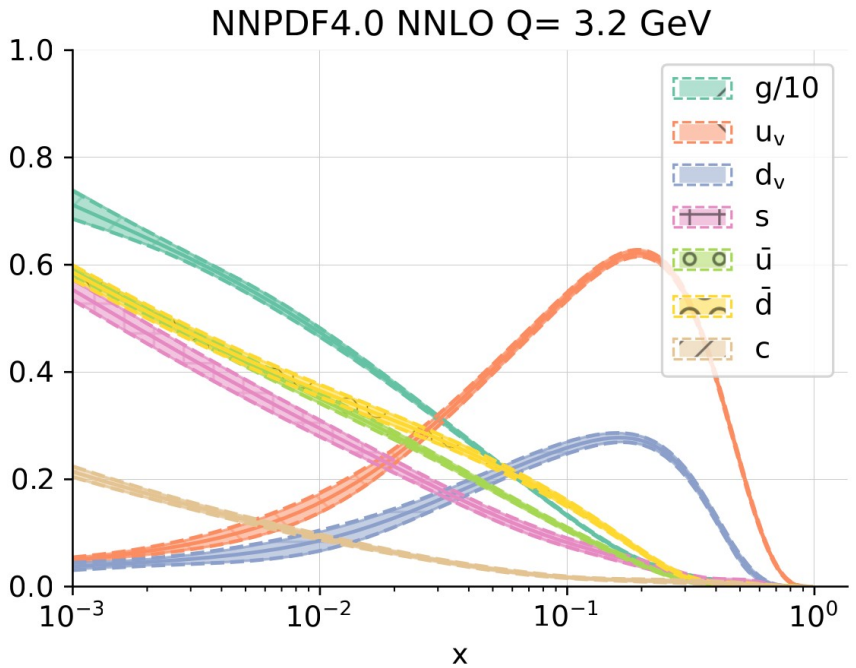


NNPDF, EPJC 82, 428 (2022)

Polarization

- Quantum number: Helicity
- Polarized: RH + LH
  - Positive definite in forward limit
  - Experimentally very precisely determined
  - Many sum rules known
- Unpolarized: RH - LH
  - Can be negative
  - Much less precise experimentally
  - Few sum rules known

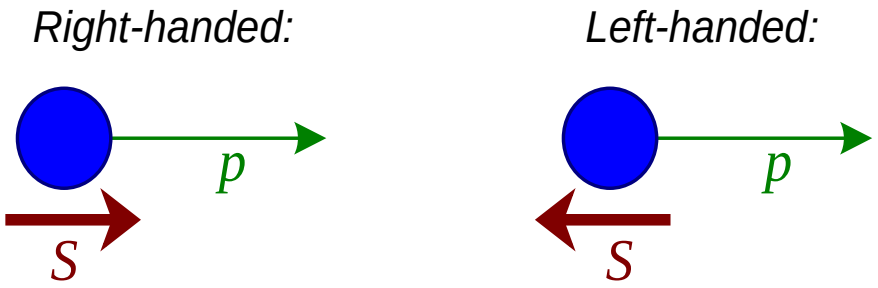


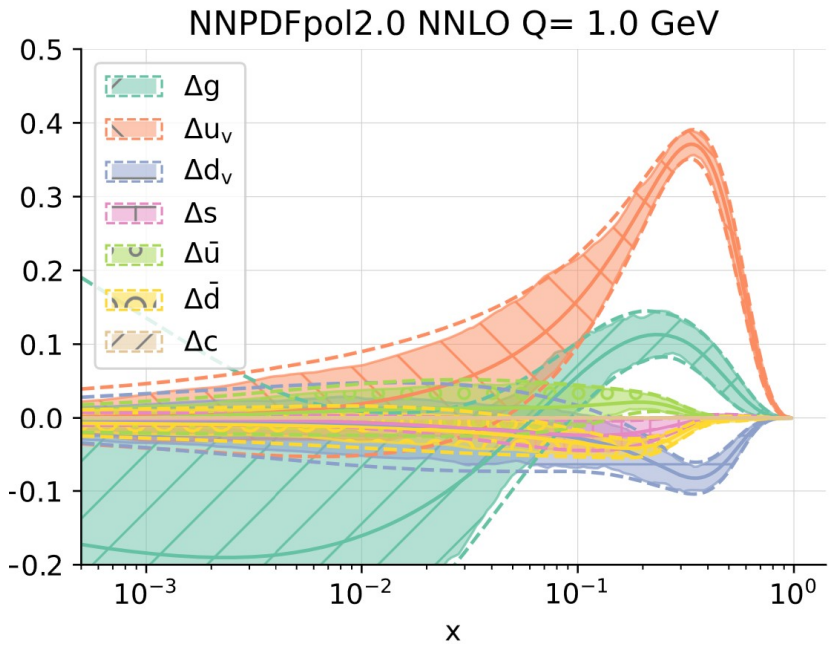


NNPDF, EPJC 82, 428 (2022)

Polarization

- Quantum number: Helicity
- Polarized: RH + LH
  - Positive definite in forward limit
  - Experimentally very precisely determined
  - Many sum rules known
- Unpolarized: RH - LH
  - Can be negative
  - Much less precise experimentally
  - Few sum rules known





NNPDFpol, JHEP 7, 168 (2025)

Definitions

- Leading twist operators

$$\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}} = \overline{\psi}\,\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\,\psi$$

- Matrix elements

Definitions

- Leading twist operators

$$\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}} = \overline{\psi}\,\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\,\psi$$

- Matrix elements

$$\langle N(p')|\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}|N(p)\rangle =$$

$$\overline{u}(p')\Bigg[\sum_{\substack{i=0\\ \text{even}}}^{n-1}\Big\{q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\gamma^{\mu_n\}}A_{n,i}(q^2)$$

$$+\,q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\frac{i\sigma^{\mu_n\}\alpha}q_\alpha}{2m_N}B_{n,i}(q^2)\Big\}$$

$$+\,\frac{q^{\{\mu_1}q^{\mu_2}\ldots q^{\mu_n\}}}{m_N}C_{n,0}(q^2)\Big|_{n\text{ even}}\Bigg]u(p)$$

X.-D. Ji, JPG 24, 1181 (1998)

Definitions

- Leading twist operators

$$\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}} = \overline{\psi}\,\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\,\psi$$

- Matrix elements

$$\langle N(p')|\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}|N(p)\rangle =$$

$$\overline{u}(p')\Bigg[\sum_{\substack{i=0\\ \text{even}}}^{n-1}\Big\{q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\gamma^{\mu_n\}}A_{n,i}(q^2)$$

$$+\,q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\frac{i\sigma^{\mu_n\}\alpha}q_\alpha}{2m_N}B_{n,i}(q^2)\Big\}$$

$$+\,\frac{q^{\{\mu_1}q^{\mu_2}\ldots q^{\mu_n\}}}{m_N}C_{n,0}(q^2)\Big|_{n\text{ even}}\Bigg]u(p)$$

GFFs

X.-D. Ji, JPG 24, 1181 (1998)

Definitions

- Similarly for helicity operators

$$\mathcal{O}_A^{\{\mu_1\mu_2\ldots\mu_n\}} = \overline{\psi}\,\gamma^5\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\,\psi$$

- Matrix elements

$$\langle N(p')|\mathcal{O}_A^{\{\mu_1\mu_2\ldots\mu_n\}}|N(p)\rangle =$$

$$\overline{u}(p')\left[\sum_{\substack{i=0\\ \text{even}}}^{n-1}\left\{q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\gamma^5\gamma^{\mu_n\}}\tilde{A}_{n,i}(q^2)\right.\right.$$

$$\left.\left.+\,\gamma^5\frac{q^{\{\mu_1}}{2m_N}\ldots q^{\mu_i+1}P^{\mu_{i+2}}\ldots P^{\mu_n}\tilde{B}_{n,i}(q^2)\right\}\right]u(p)$$

GFFs

P. Hägler, PLB 594, 164 (2004)

Definitions

- Leading twist operators

$$\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}} = \overline{\psi}\,\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\,\psi$$

- Connection to GPDs

$$\int_{-1}^{1}\mathrm{d}x\,x^{n-1}H(x,\xi,q^2) = \sum_{\substack{i=0\\ \text{even}}}^{n-1} A_{n,i}(q^2)(-2\xi)^i + C_{n0}(q^2)(-2\xi)^n\Big|_{n\text{ even}}$$

$$\int_{-1}^{1}\mathrm{d}x\,x^{n-1}E(x,\xi,q^2) = \sum_{\substack{i=0\\ \text{even}}}^{n-1} B_{n,i}(q^2)(-2\xi)^i - C_{n0}(q^2)(-2\xi)^n\Big|_{n\text{ even}}$$

Operator Mixing

- Continuum: Lorentz group $SO(1,3)$
- Lattice: Hypercubic group $H(4)$

G. Beccarini *et al.*, NPB 456, 271 (1995)
M. Göckeler *et al.*, PRD 54, 5705 (1996)

Operator Mixing

- Continuum: Lorentz group $SO(1,3)$
- Lattice: Hypercubic group $H(4)$
- Mixing!
- In general: Avoid repeated indices
- Two derivatives $\mathcal{O}^{\{\mu\nu\rho\}}$ with $\mu \neq \nu \neq \rho$
- Three derivatives

$$\mathcal{O}^{\{1234\}} \qquad \mathcal{O}^{\{44ij\}} \qquad \mathcal{O}^{\{44ij\}} - \mathcal{O}^{\{kkij\}}$$

G. Beccarini *et al.*, NPB 456, 271 (1995)
M. Göckeler *et al.*, PRD 54, 5705 (1996)

Operator Mixing

- Continuum: Lorentz group $SO(1,3)$
- Lattice: Hypercubic group $H(4)$
- Mixing!
- In general: Avoid repeated indices
- Two derivatives $\mathcal{O}^{\{\mu\nu\rho\}}$ with $\mu \neq \nu \neq \rho$
- Three derivatives $\mathcal{O}^{\{1234\}}$ $\mathcal{O}^{\{44ij\}}$ $\mathcal{O}^{\{44ij\}} - \mathcal{O}^{\{kkij\}}$
- Special case: Forward limit
- Leads to boosted frame

G. Beccarini *et al.*, NPB 456, 271 (1995)
M. Göckeler *et al.*, PRD 54, 5705 (1996)

$$\langle N(p')|\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}|N(p)\rangle =$$
$$\overline{u}(p')\Bigg[\sum_{\substack{i=0\\ \text{even}}}^{n-1}\Big\{q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\gamma^{\mu_n\}}A_{n,i}(q^2)$$
$$+ q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\frac{i\sigma^{\mu_n\}\alpha}q_\alpha}{2m_N}B_{n,i}(q^2)\Big\}$$
$$+ \frac{q^{\{\mu_1}q^{\mu_2}\ldots q^{\mu_n\}}}{m_N}C_{n,0}(q^2)\Big|_{n\text{ even}}\Bigg]u(p)$$

Gradient flow?

- New approach using gradient flow

A. Francis *et al.*, PRL 136, 17 (2026)

A. Francis *et al.*, PRD 113, 7 (2026)

Gradient flow?

- New approach using gradient flow
- Flowed fields eliminate operator mixing
- Allow calculation of even higher moments

A. Francis *et al.*, PRL 136, 17 (2026)

A. Francis *et al.*, PRD 113, 7 (2026)

Gradient flow?

- New approach using gradient flow
- Flowed fields eliminate operator mixing
- Allow calculation of even higher moments
- Not yet proven to work for nucleon at physical point

A. Francis *et al.*, PRL 136, 17 (2026)

A. Francis *et al.*, PRD 113, 7 (2026)

Gradient flow?

- New approach using gradient flow
- Flowed fields eliminate operator mixing
- Allow calculation of even higher moments
- Not yet proven to work for nucleon at physical point
- Gets very expensive for non-zero momentum transfer

A. Francis *et al.*, PRL 136, 17 (2026)

A. Francis *et al.*, PRD 113, 7 (2026)

Gradient flow?

- New approach using gradient flow
- Flowed fields eliminate operator mixing
- Allow calculation of even higher moments
- Not yet proven to work for nucleon at physical point
- Gets very expensive for non-zero momentum transfer

A. Francis *et al.*, PRL 136, 17 (2026)

A. Francis *et al.*, PRD 113, 7 (2026)

For gradient flow see Poster by B. Hu

Methods

- Ensemble at physical point

Ensemble	a [fm]	$(L/a)^3 \times T/a$	m_π [GeV]	L [fm]
cB211.072.64	0.0801(4)	$64^3 \times 128$	0.1393(7)	5.12(3)

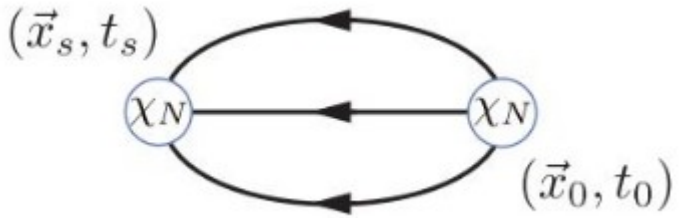
C. Alexandrou *et al.*, PRD 98, 054518 (2018)



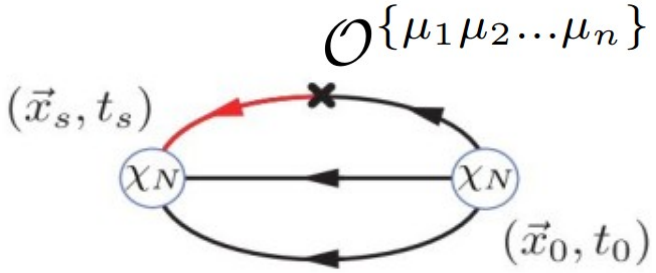
Methods

- Two- and three-point-functions
- Neglect disconnected

$$C(\Gamma_0, \vec{p}, t_s, t_0) = \sum_{\vec{x}_s} e^{-\mathrm{i}(\vec{x}_s - \vec{x}_0)\cdot\vec{p}} \times$$
$$\mathrm{tr}[\Gamma_0 \langle J_N(t_s, \vec{x}_s) \bar{J}_N(t_0, \vec{x}_0) \rangle]$$



$$C^{\{\mu_1\mu_2\ldots\mu_n\}}(\Gamma, \vec{q}, \vec{p}', t_s, t_{\mathrm{ins}}, t_0) =$$
$$\sum_{\vec{x}_{\mathrm{ins}}, \vec{x}_s} e^{-\mathrm{i}(\vec{x}_{\mathrm{ins}} - \vec{x}_0)\cdot\vec{q}} e^{-\mathrm{i}(\vec{x}_s - \vec{x}_0)\cdot\vec{p}'} \times$$
$$\mathrm{tr}[\Gamma \langle J_N(t_s, \vec{x}_s) \mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}(t_{\mathrm{ins}}, \vec{x}_{\mathrm{ins}}) \bar{J}_N(t_0, \vec{x}_0) \rangle]$$



Methods

- Statistics

$\vec{n}'$	N_{mom}	t_s/a	N_{conf}	N_{srcs}	N_{meas}
two derivatives					
$(1,1,0)$	12	8	735	1	8820
$(1,1,0)$	12	10	735	3	26460
$(1,1,0)$	12	12	735	9	79380
$(1,1,0)$	12	14	735	18	158760
three derivatives					
$(1,1,1)$	8	8	735	1	5880
$(1,1,1)$	8	10	735	3	17640
$(1,1,0)$	12	14	735	18	158760

Methods

- Construct Ratio to extract matrix element

$$R^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\rm ins}) = \frac{C^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\rm ins})}{C(\Gamma_0;\vec{p}\,';t_s)} \times$$

$$\sqrt{\frac{C(\Gamma_0,\vec{p};t_s-t_{\rm ins})C(\Gamma_0,\vec{p}\,';t_{\rm ins})C(\Gamma_0,\vec{p}\,';t_s)}{C(\Gamma_0,\vec{p}\,';t_s-t_{\rm ins})C(\Gamma_0,\vec{p};t_{\rm ins})C(\Gamma_0,\vec{p};t_s)}}$$

$$R^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\rm ins}) \xrightarrow[t_{\rm ins}\gg a]{t_s-t_{\rm ins}\gg a} \Pi^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p})$$

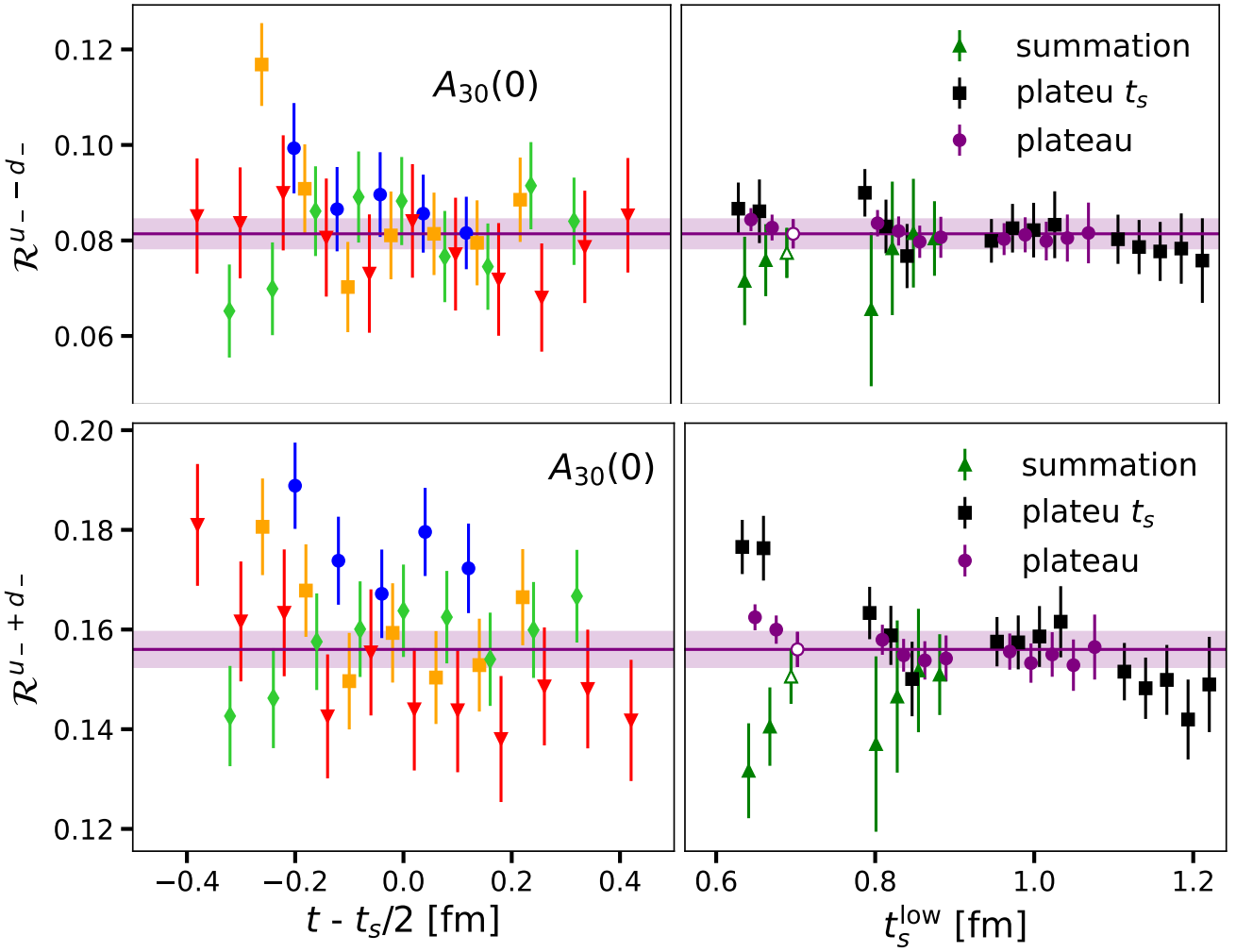
Methods

- Multiplicative renormalization

non-perturbative renormalization: $\mathrm{RI}'\text{-}\mathrm{MOM}$
perturbative renormalization: $\overline{\mathrm{MS}}$
matching at $\bar{\mu} = 2\ \mathrm{GeV}$

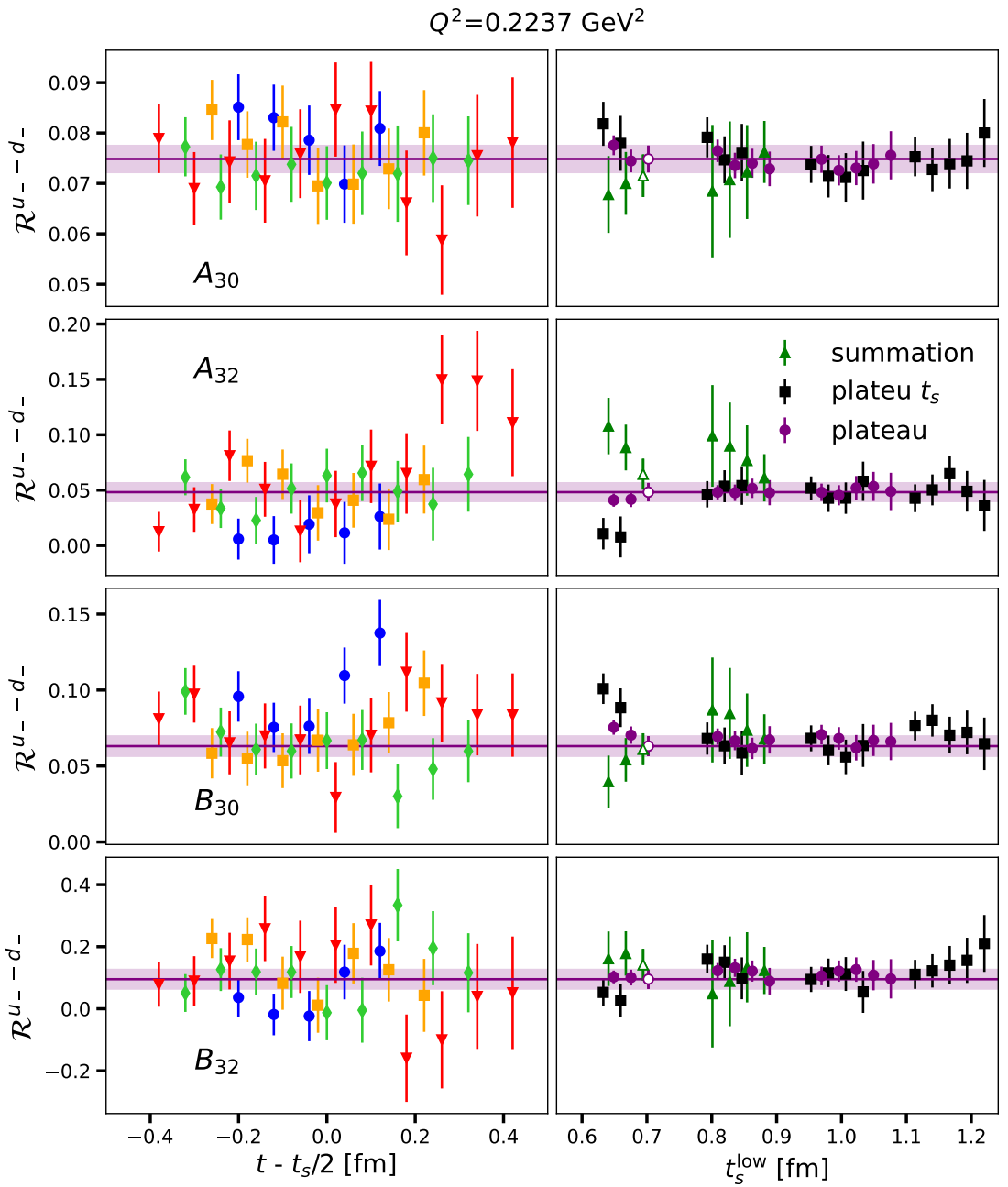
Results – Two-derivative

- Unpolarized
- Forward limit



Results – Two-derivative

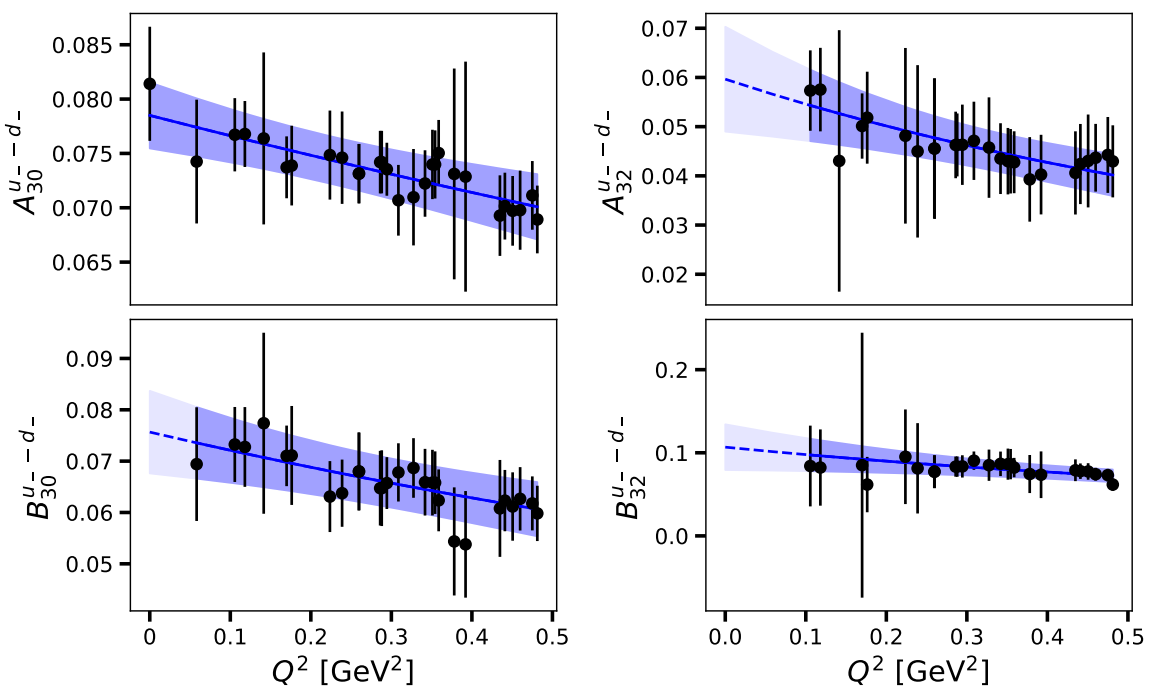
- Unpolarized
- Momentum transfer

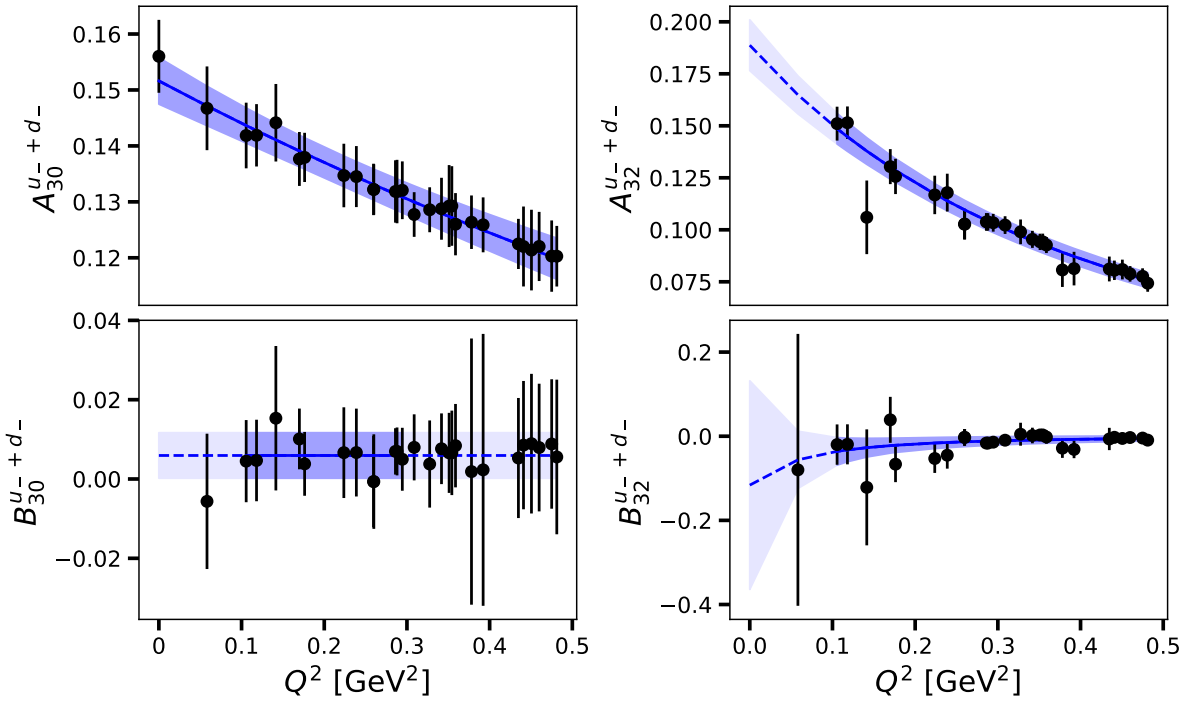


Results – Two-derivative

- Unpolarized
- Dipole fit

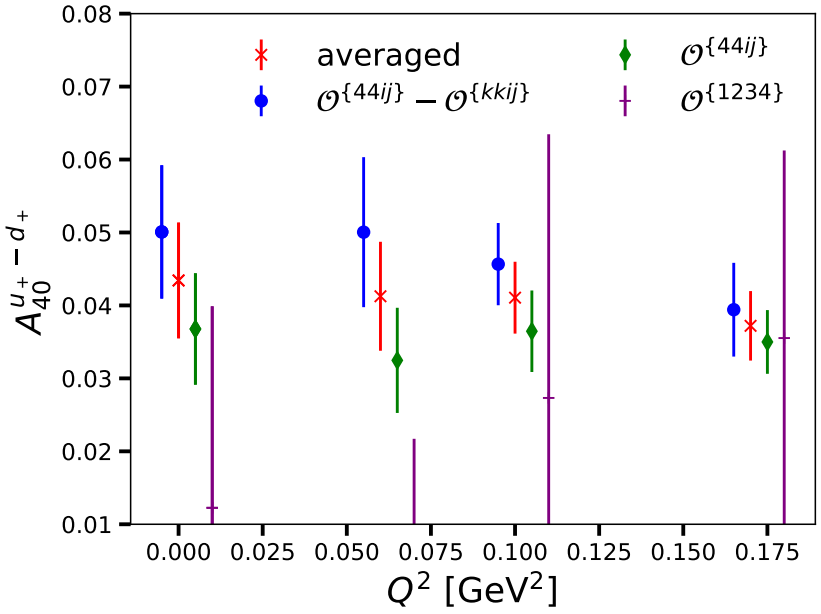
$$G(Q^2) = \frac{G(0)}{(1 + \frac{Q^2}{M^2})^2}$$

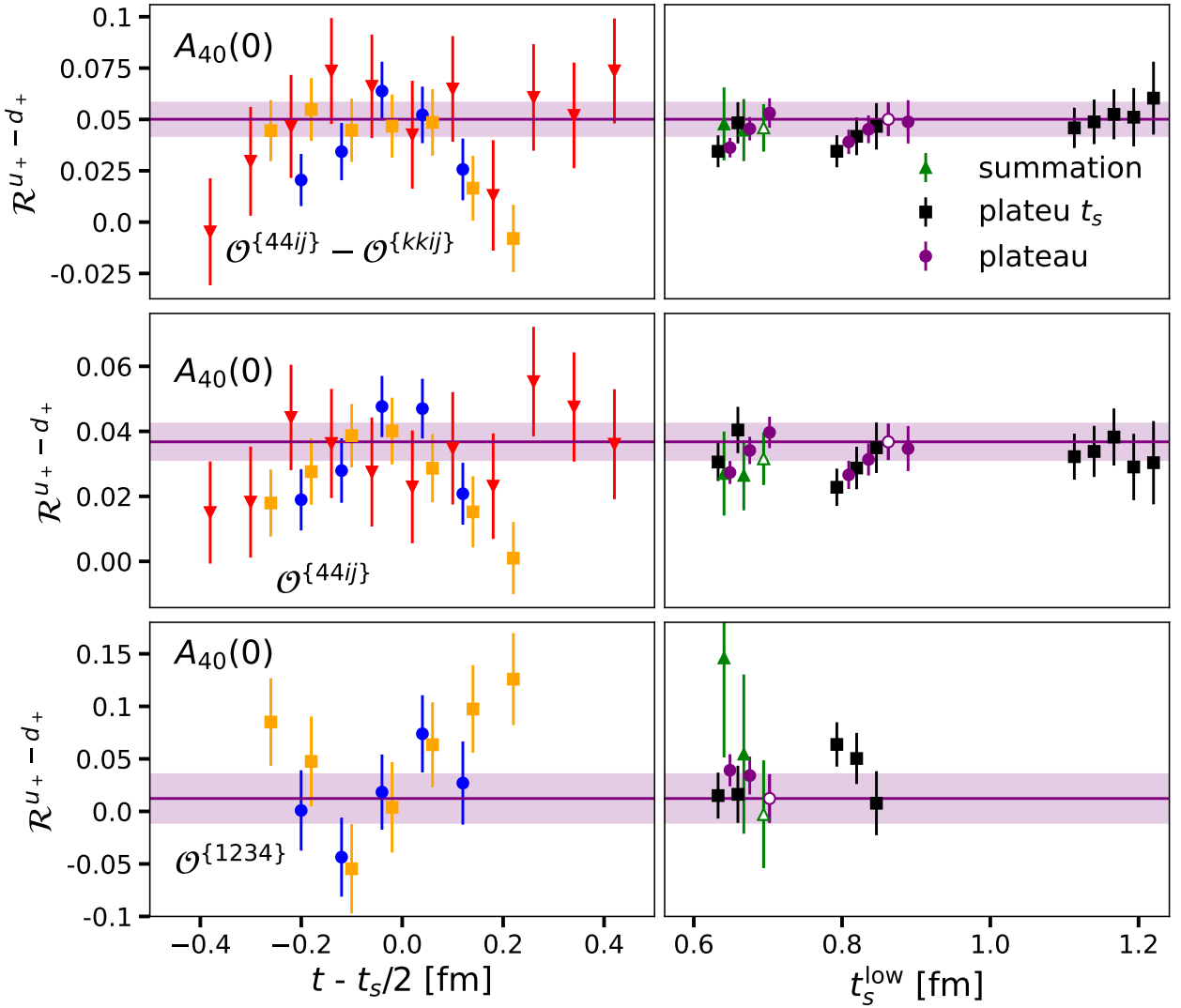




Results – Three-derivative

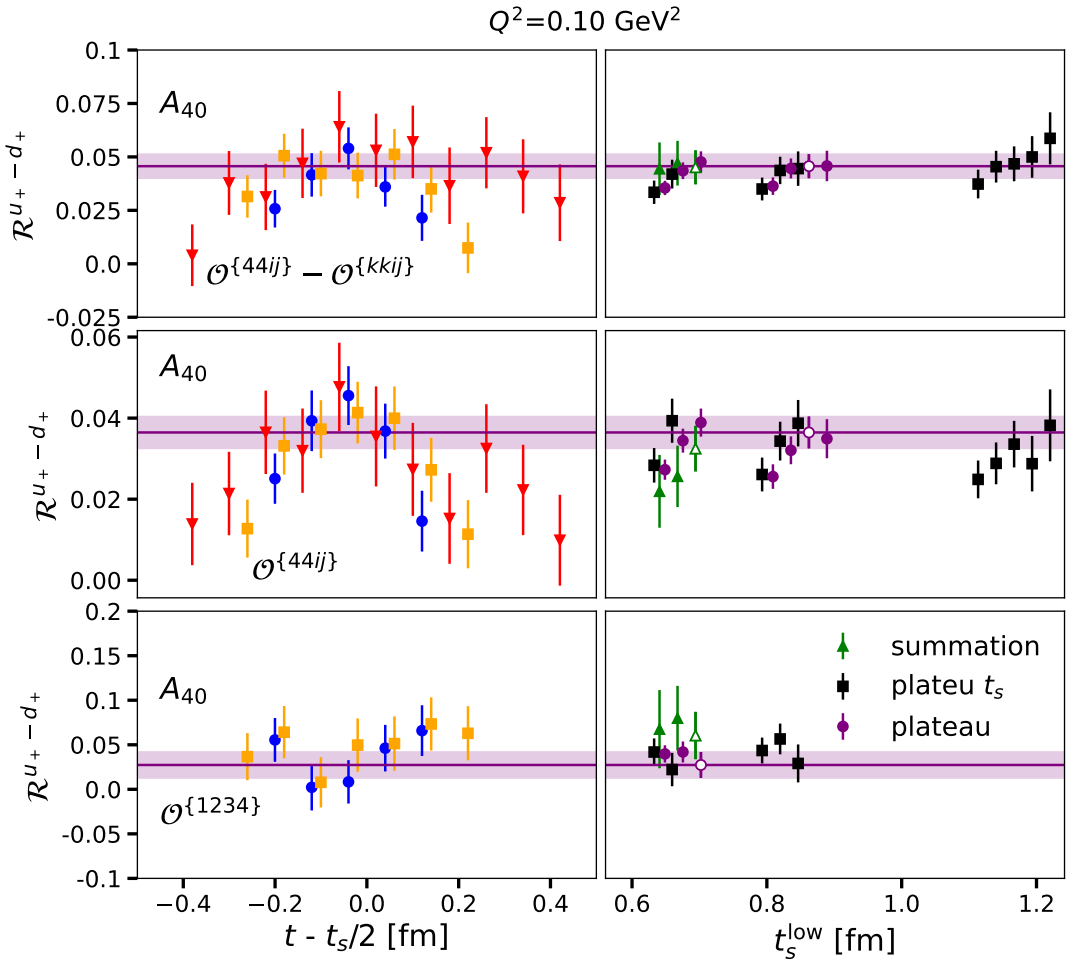
- Unpolarized, forward limit
- Comparison of different operators
- Note: *y*-axis scale

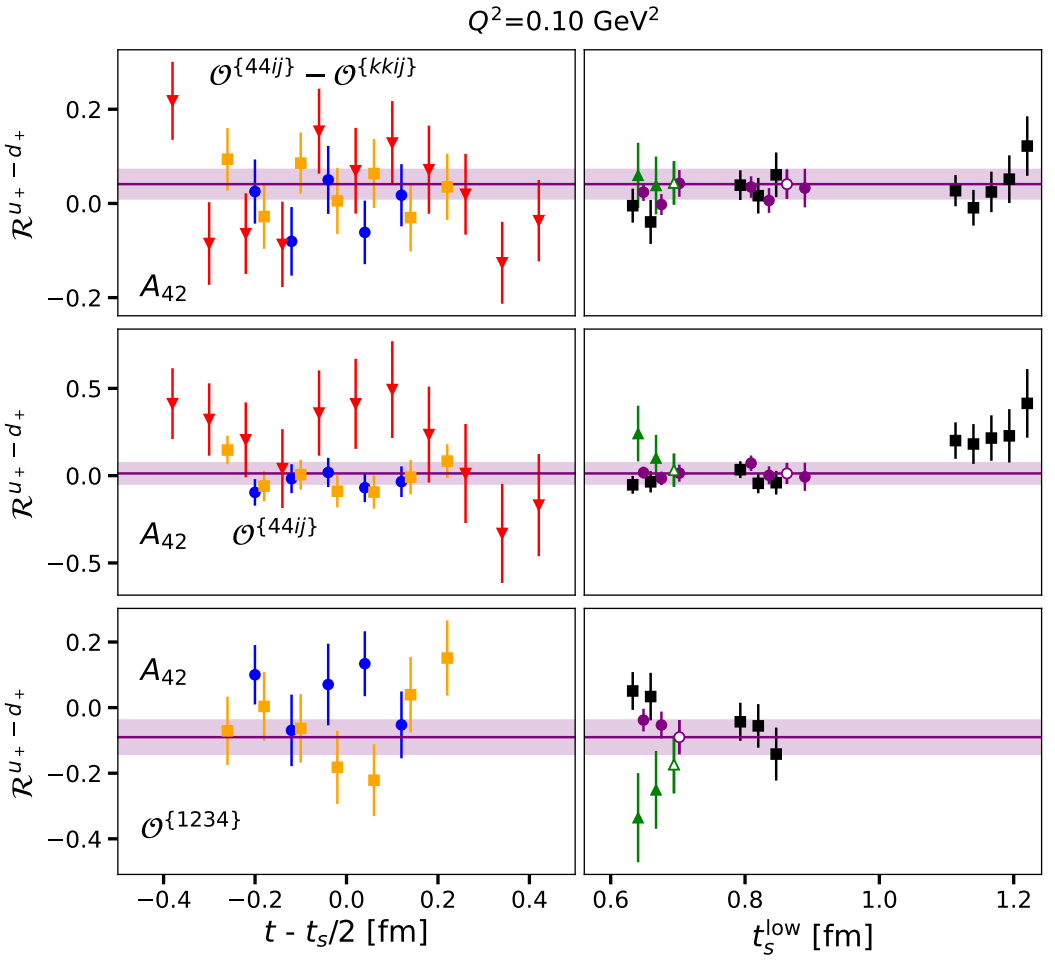




Results – Three-derivative

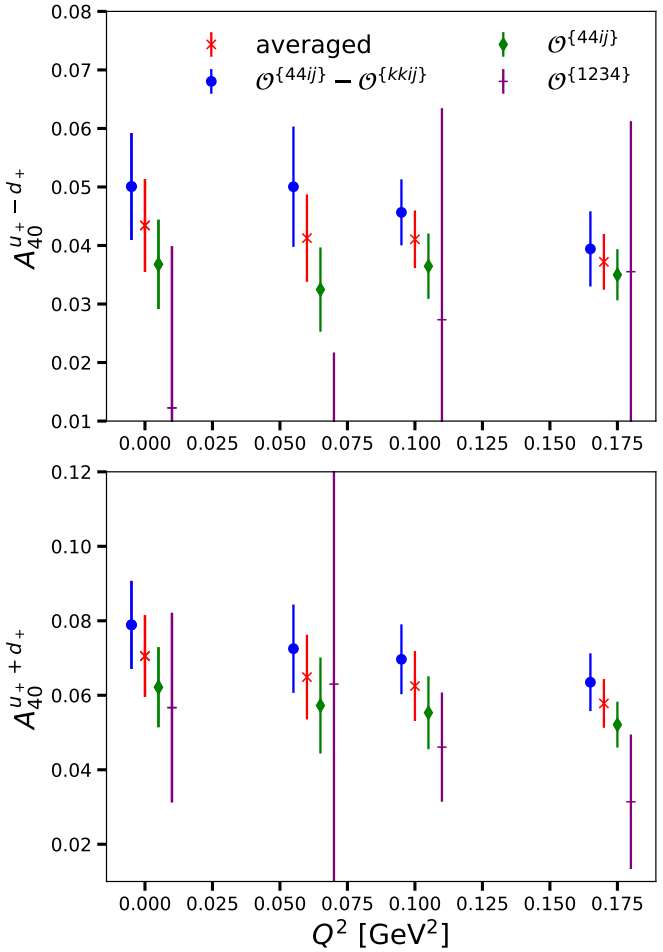
- Unpolarized, momentum transfer

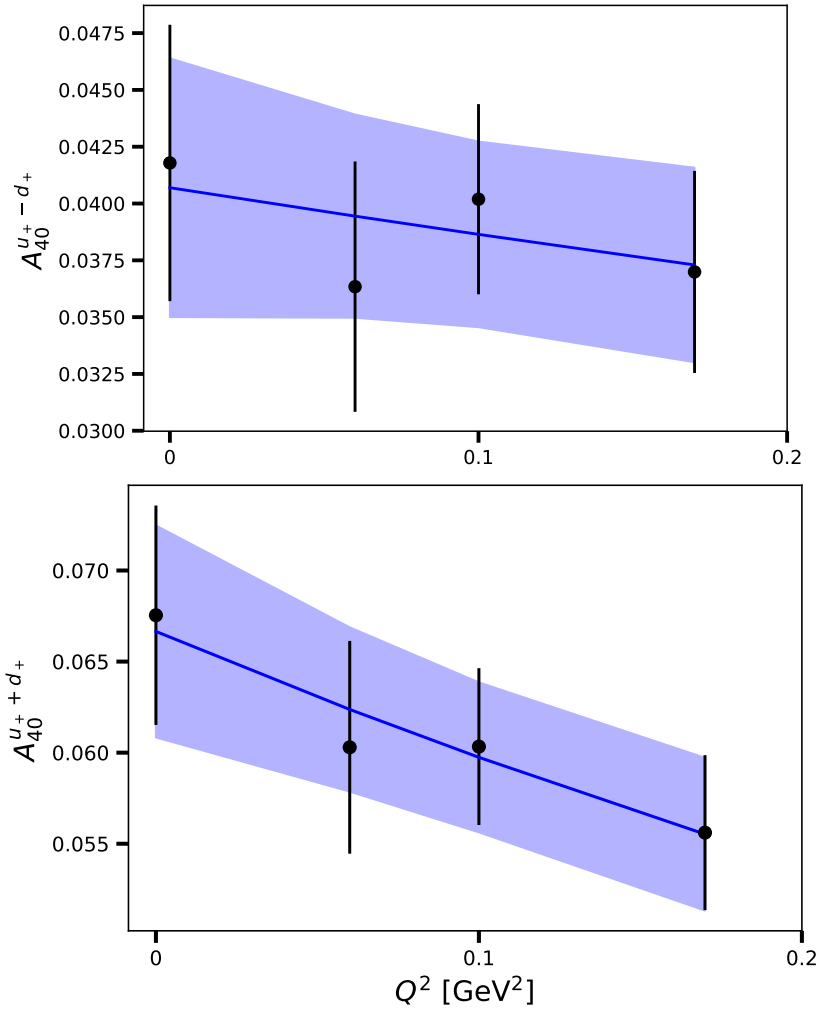




Results – Three-derivative

- Only A_{40} nonzero





Comparison with literature

- JAM and NNPDF are phenomenological studies

JAM, PRD 106, L051502 (2022)

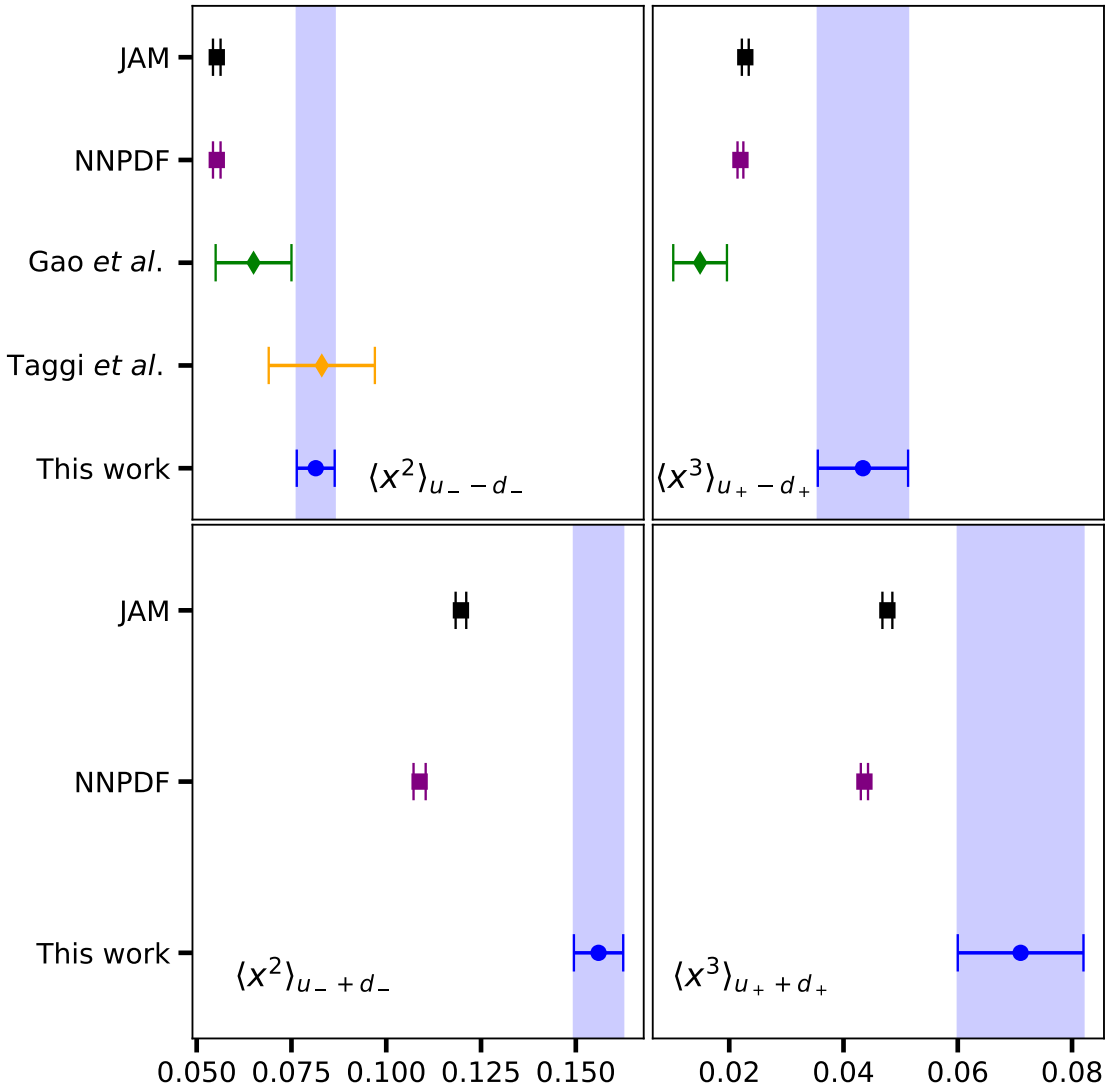
NNPDF, EPJC 82, 428 (2022)

- Gao uses non-local operators

X. Gao *et al.*, PRD 107, 074509 (2023)

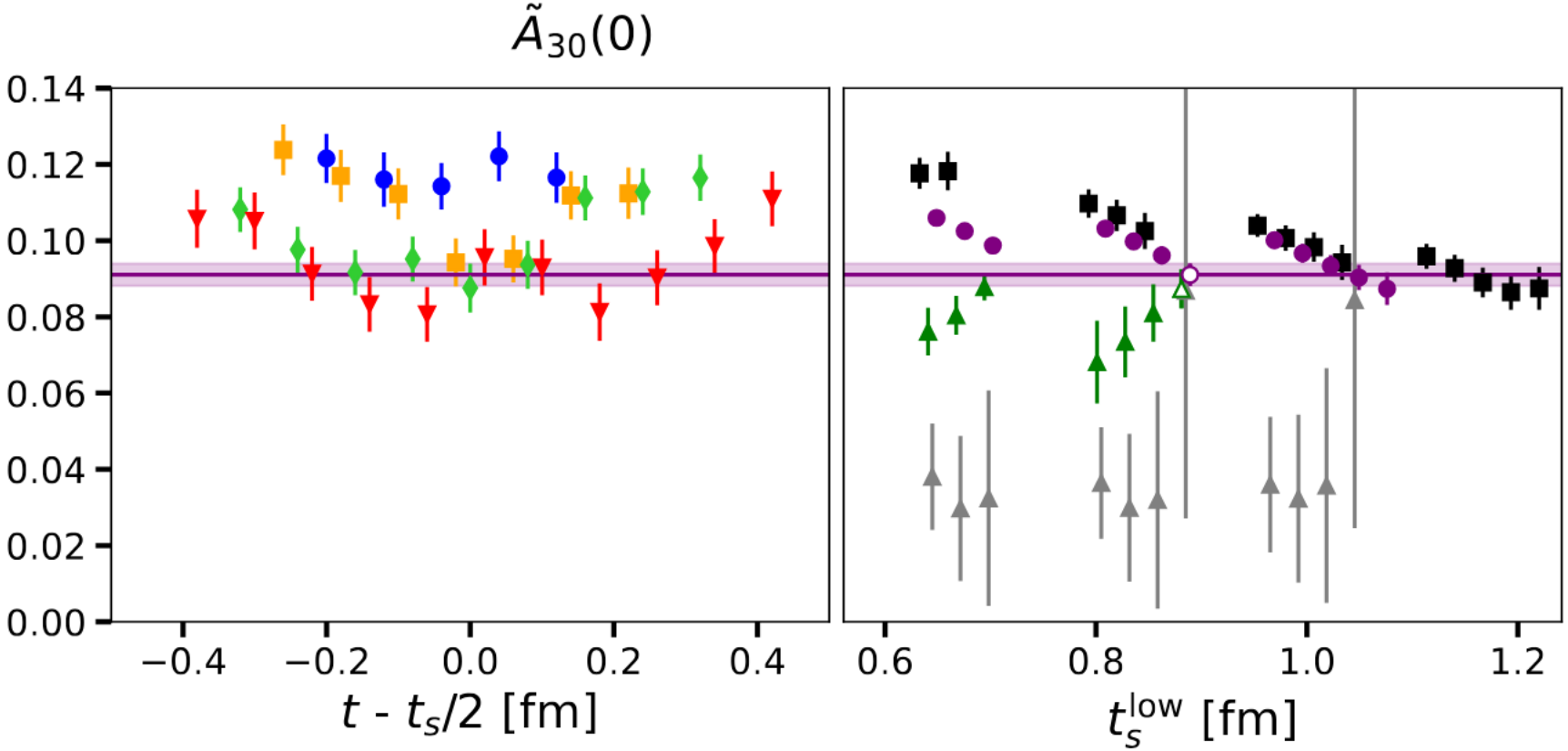
- Taggi is a direct computation of the third moment

E. Taggi *et al.*, (2026), 2605.02808



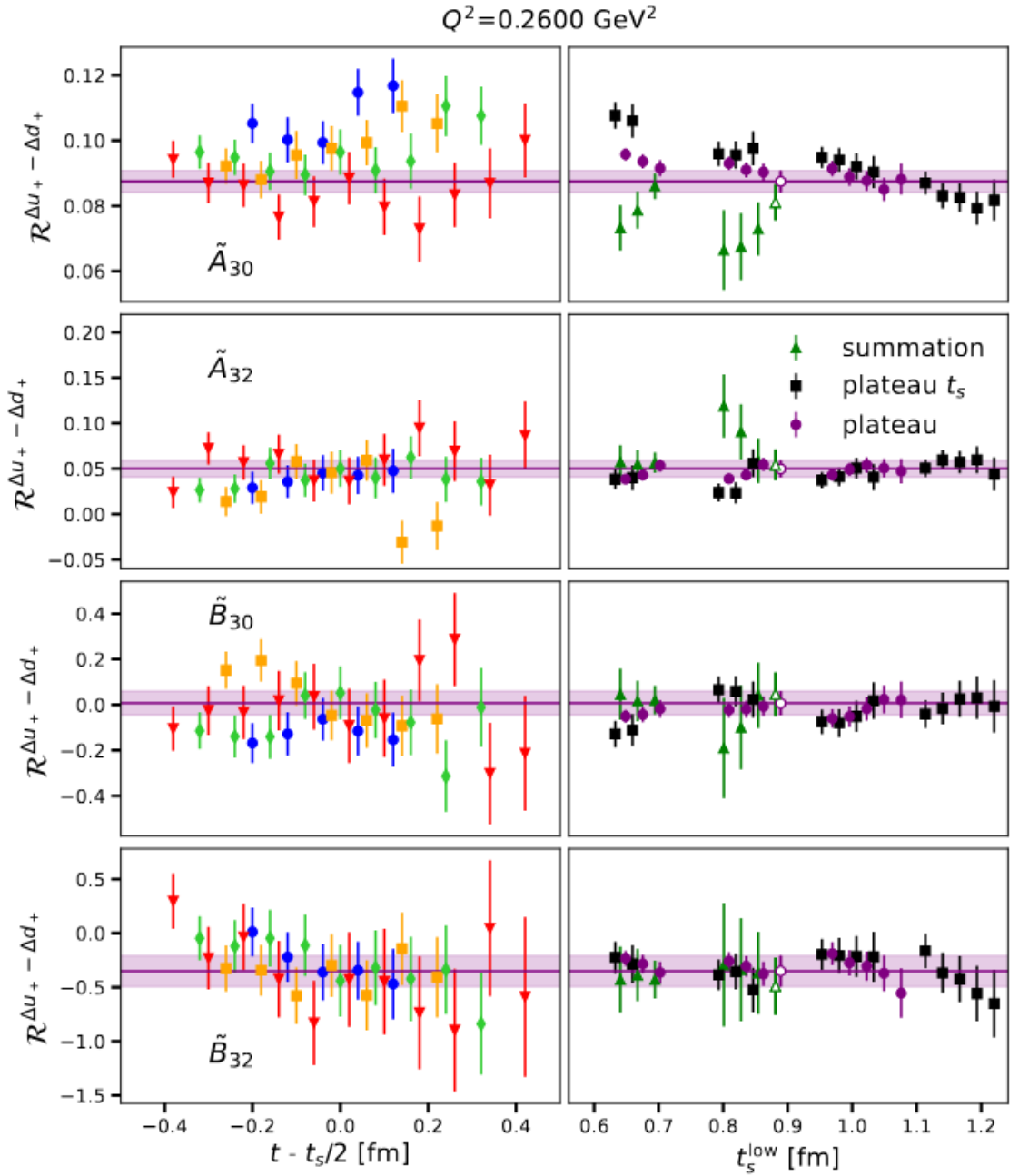
Results – Two-derivative

- Helicity
- Isovector
- Forward limit



Results – Two-derivative

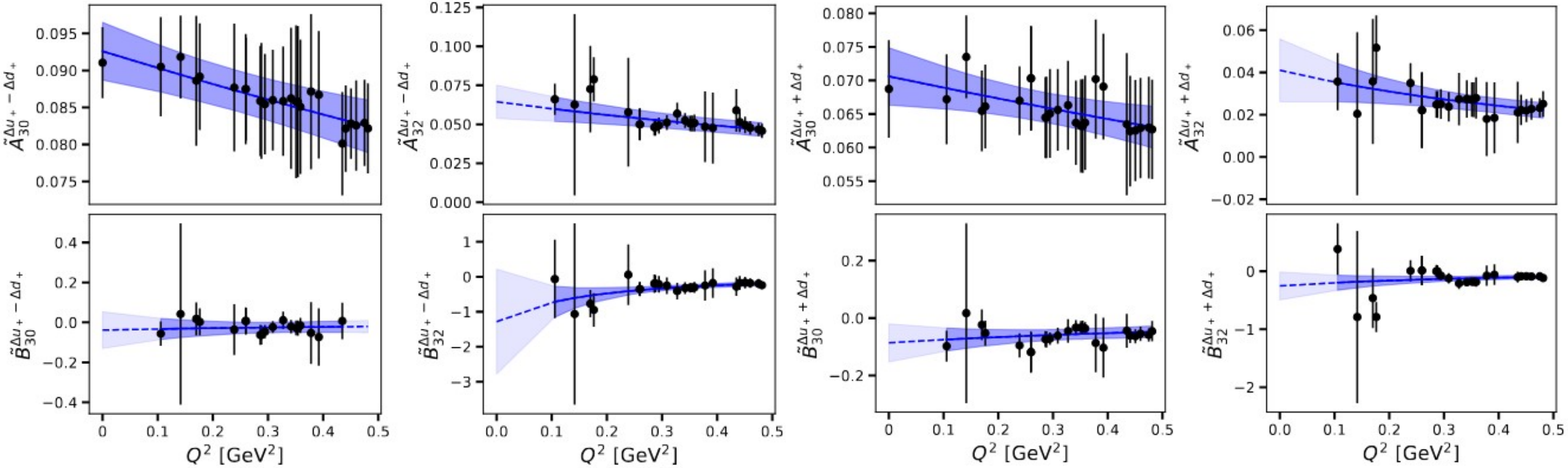
- Helicity
- Momentum transfer



Results – Two-derivative

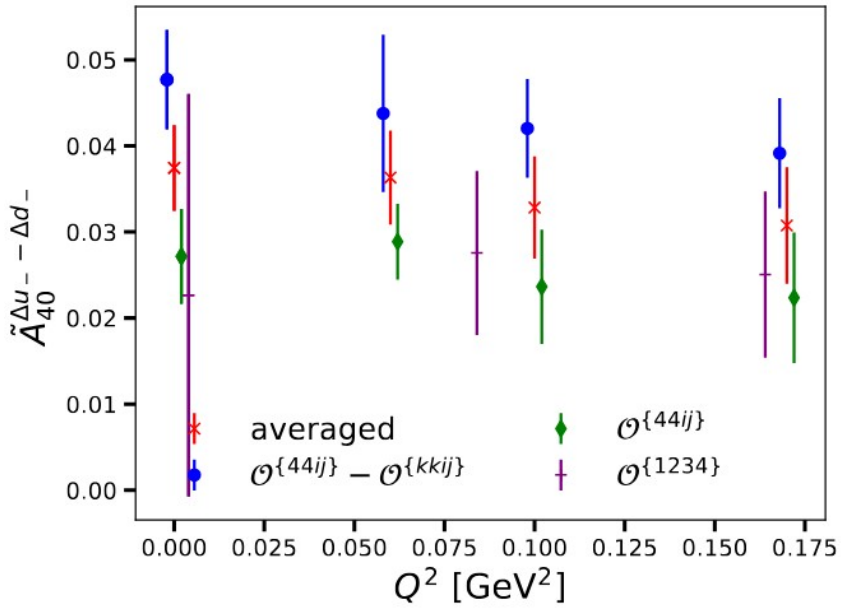
- Helicity
- Dipole fit

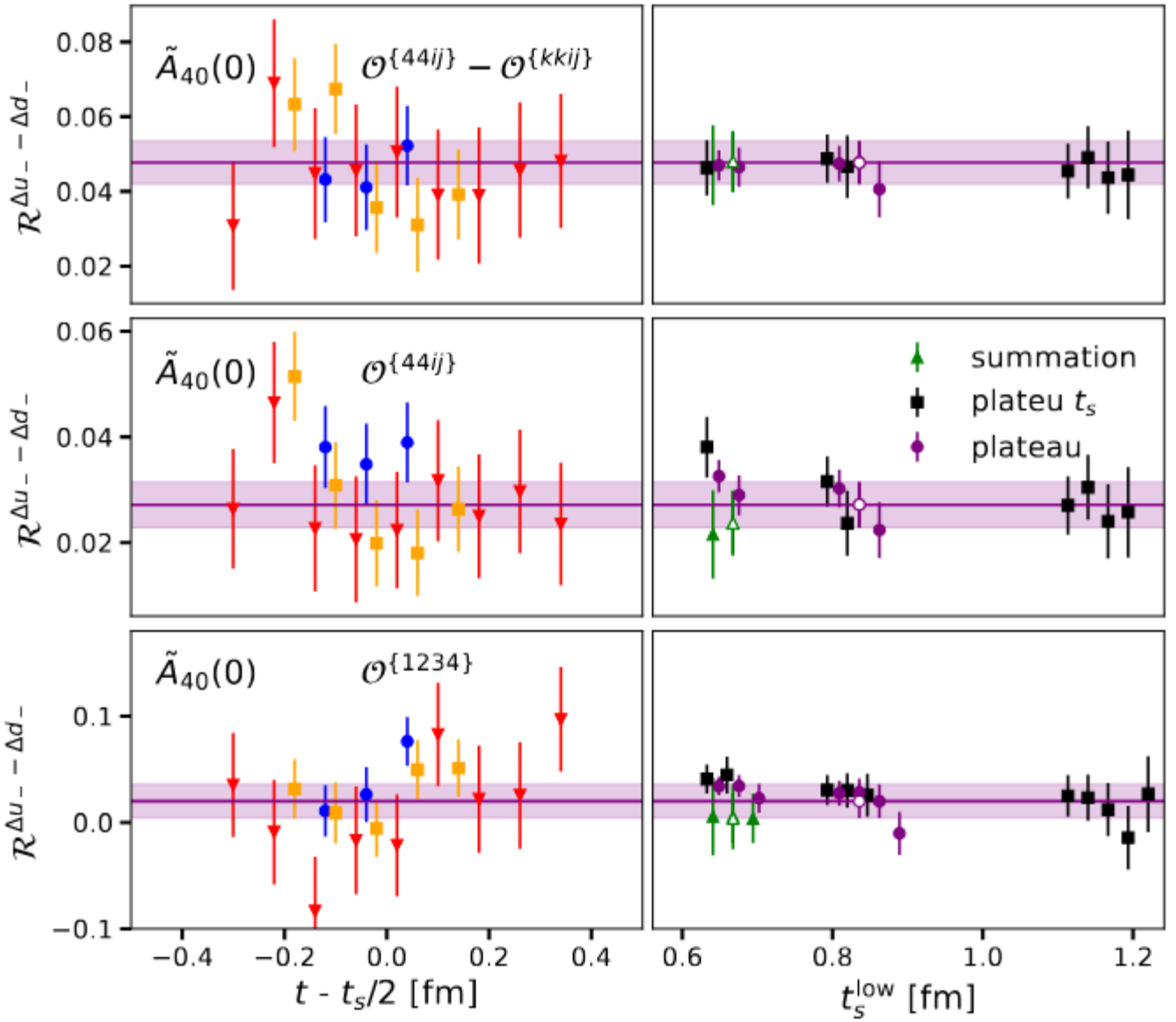
$$G(Q^2) = \frac{G(0)}{(1 + \frac{Q^2}{M^2})^2}$$



Results – Three-derivative

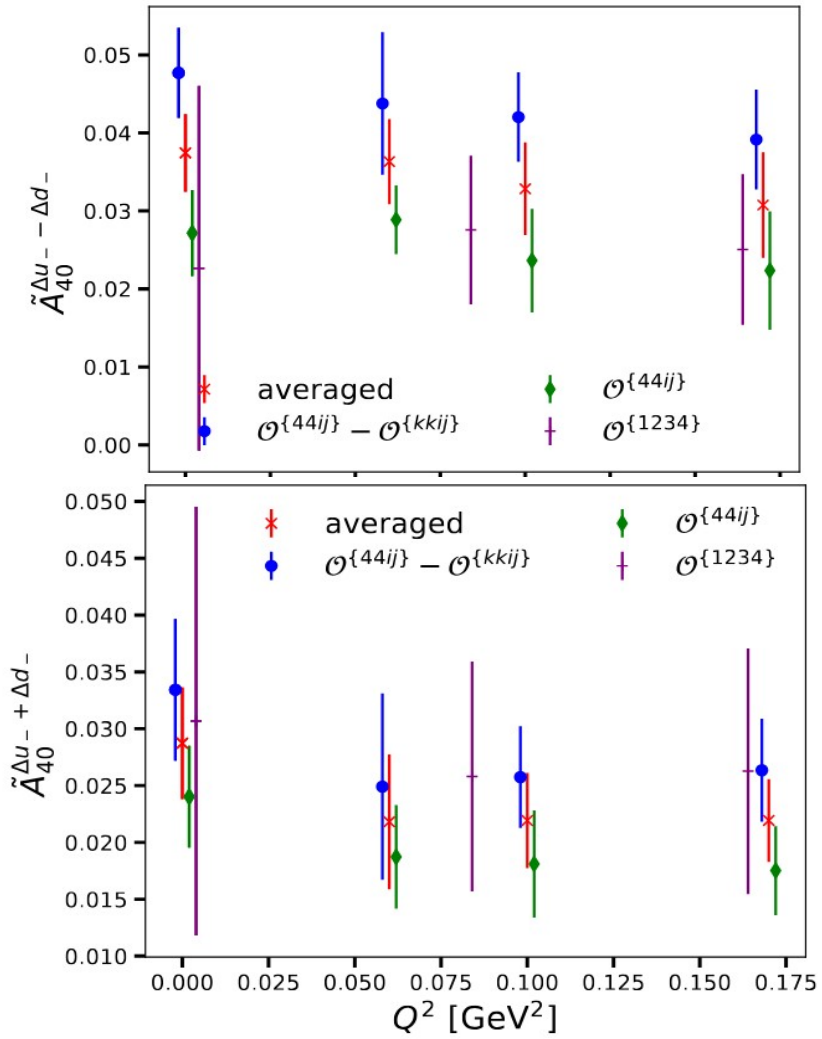
- Helicity, forward limit
- Comparison of different operators
- Note: *y*-axis scale

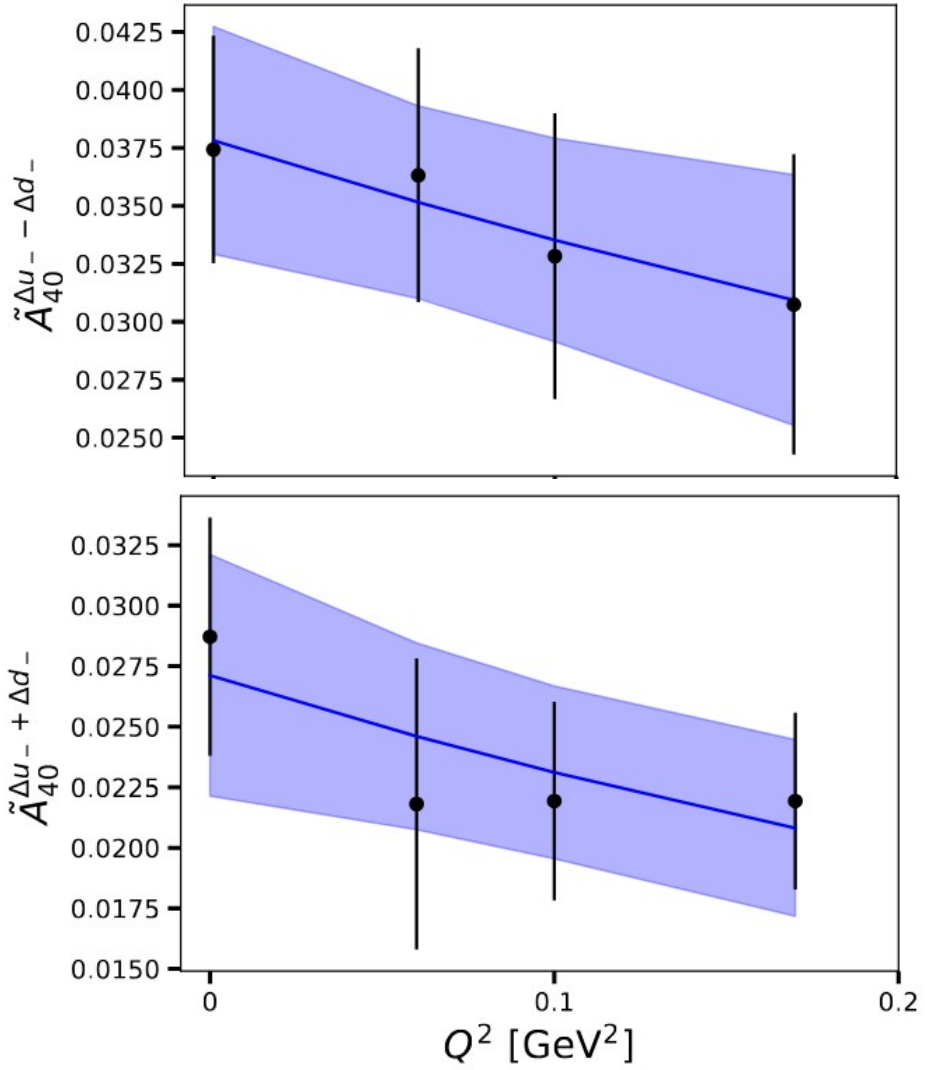




Results – Three-derivative

- Helicity





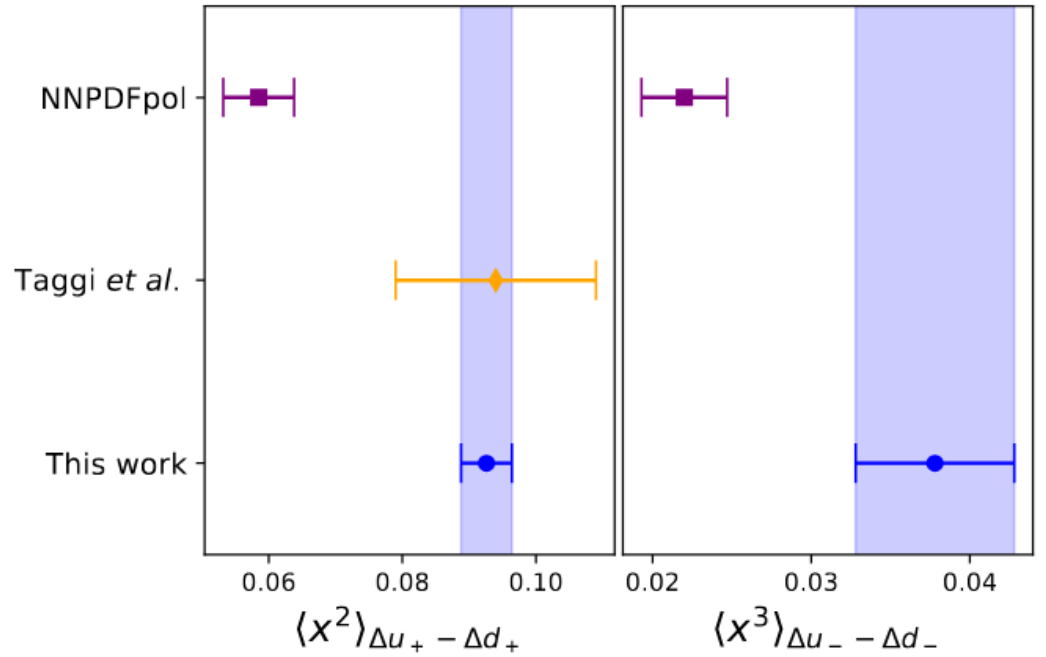
Comparison with literature

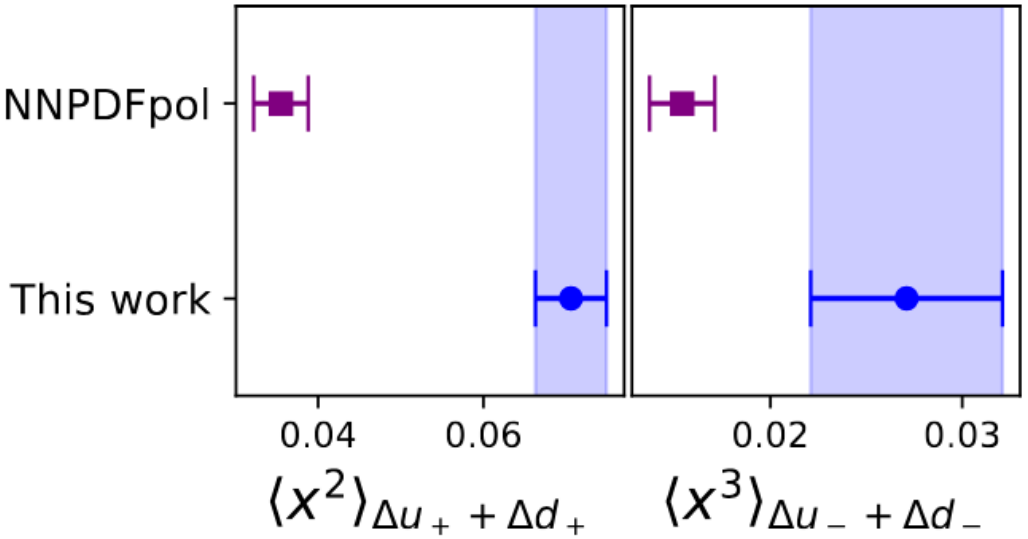
- NNPDFpol is a phenomenological study

NNPDFpol, JHEP 7, 168 (2025)

- Taggi is a direct computation of the third moment

E. Taggi *et al.*, (2026), 2605.02808





PDF reconstruction

- Ansatz

$$q(x) = N x^{\alpha}(1-x)^{\beta}$$

PDF reconstruction

- Ansatz

$$q(x) = N x^{\alpha}(1-x)^{\beta}$$

- Problem with antiquarks

$$\langle x^{n-1} \rangle = \int_0^1 \mathrm{d}x\, x^{n-1}[q(x) - (-1)^{n-1}\bar{q}(x)]$$

PDF reconstruction

- Ansatz

$$q(x) = N x^{\alpha}(1-x)^{\beta}$$

- Problem with antiquarks

$$\langle x^{n-1} \rangle = \int_0^1 \mathrm{d}x\, x^{n-1}[q(x) - (-1)^{n-1}\bar{q}(x)]$$

- Even powers give valence distribution

$$\langle 1 \rangle_{q_-} = \langle 1 \rangle_{q_-}$$
$$\langle x^2 \rangle_{q_-} = \langle x^2 \rangle_{q_-}$$

PDF reconstruction

- Ansatz

$$q(x) = N x^{\alpha}(1-x)^{\beta}$$

- Problem with antiquarks

$$\langle x^{n-1} \rangle = \int_0^1 \mathrm{d}x\, x^{n-1}[q(x) - (-1)^{n-1}\bar{q}(x)]$$

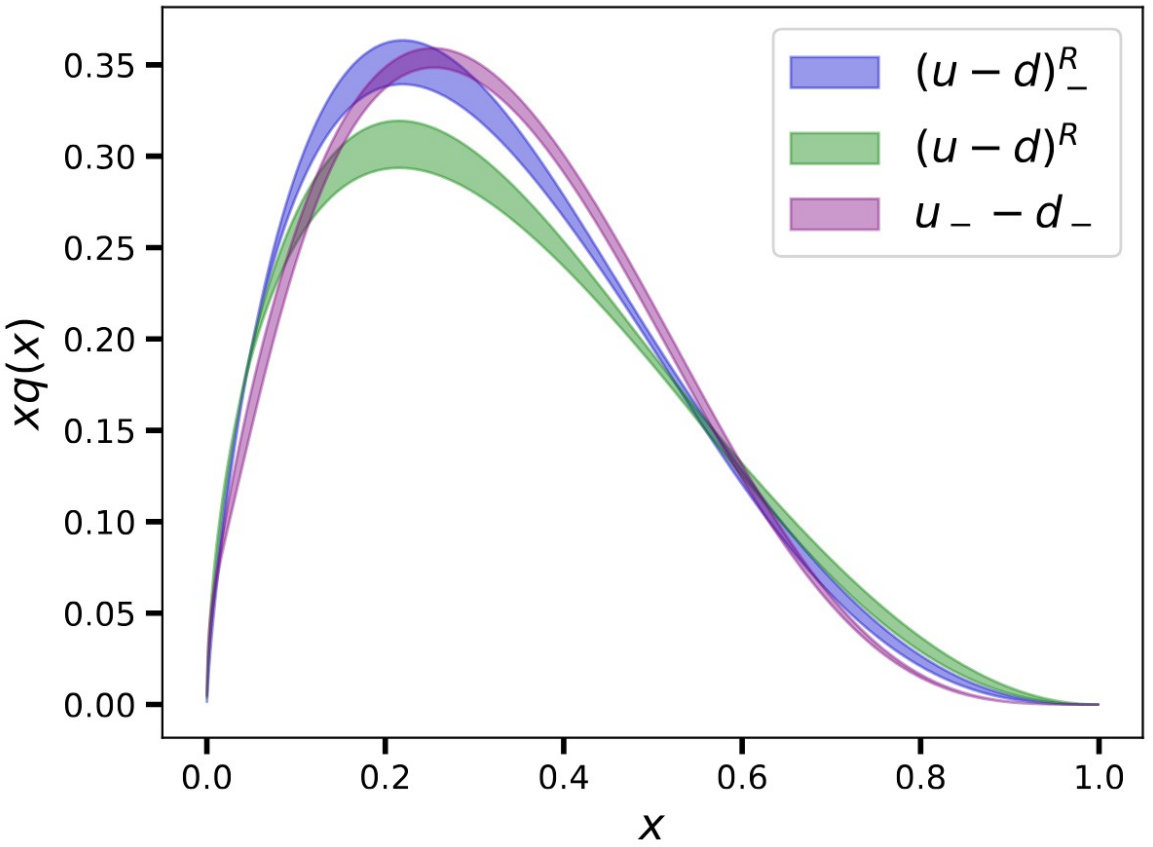
- Even powers give valence distribution
- Odd powers do not

$$\langle 1 \rangle_{q_-} = \langle 1 \rangle_{q_-}$$
$$\langle x^2 \rangle_{q_-} = \langle x^2 \rangle_{q_-}$$

$$\langle x \rangle_{q_-} = \langle x \rangle_{q_+} - 2\langle x \rangle_{\bar{q}}$$
$$\langle x^3 \rangle_{q_-} = \langle x^3 \rangle_{q_+} - 2\langle x^3 \rangle_{\bar{q}}$$

PDF reconstruction

- Comparison between approaches with JAM data
- JAM data: $u_- - d_-$
- Calculate moments
  $\langle x \rangle_{q_-}$, $\langle x^2 \rangle_{q_-}$, $\langle x^3 \rangle_{q_-}$
  and reconstruct PDF: $(u-d)^R_-$
- Calculate moments
  $\langle x \rangle_{q_+}$, $\langle x^2 \rangle_{q_-}$, $\langle x^3 \rangle_{q_+}$
  and reconstruct PDF: $(u-d)^R$



JAM, PRD 106, L051502 (2022)

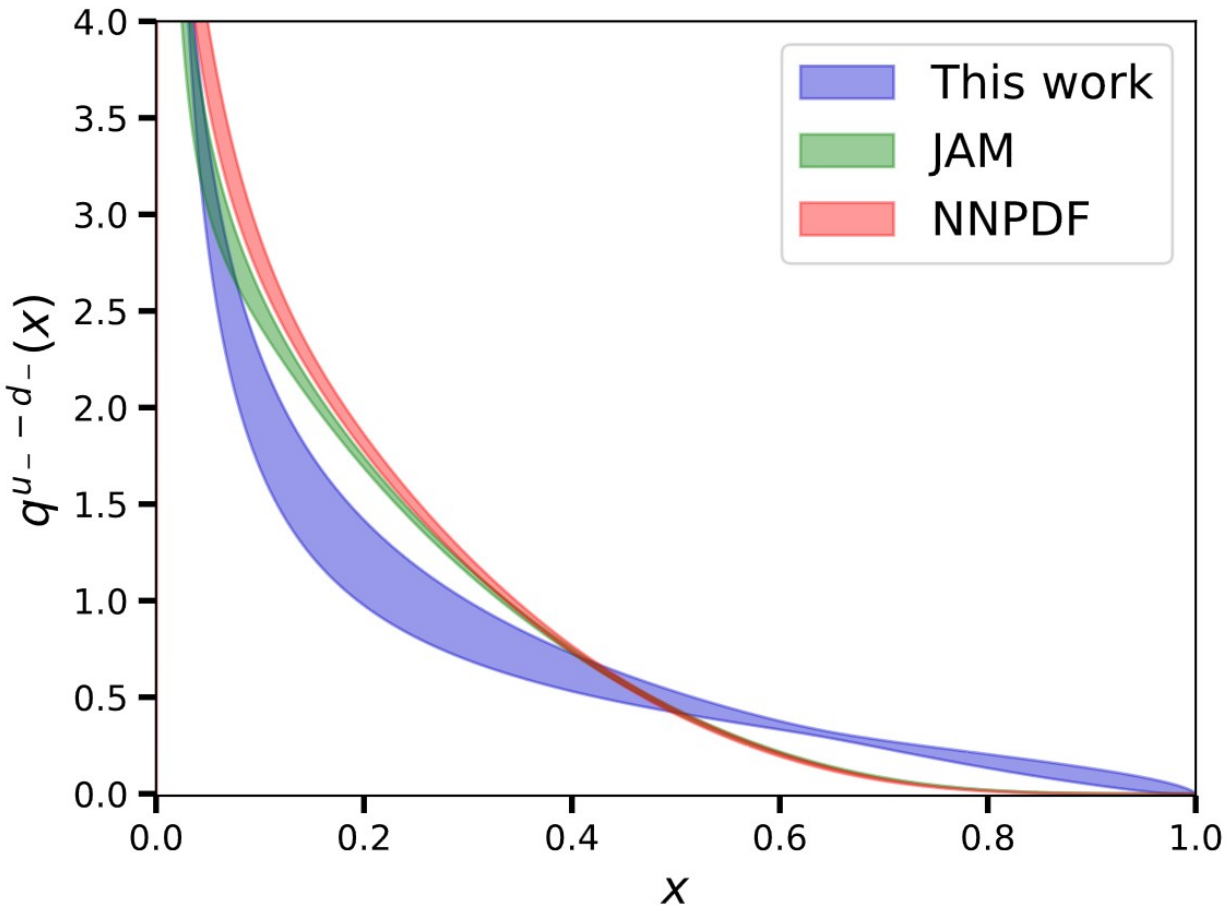
PDF reconstruction

- Ansatz

$$q(x) = N x^{\alpha}(1-x)^{\beta}$$

- Include JAM data for $\langle x^n \rangle_{\bar{q}}$ in reconstruction

For second moment see Poster by Y. Li



JAM, PRD 106, L051502 (2022)

NNPDF, EPJC 82, 428 (2022)

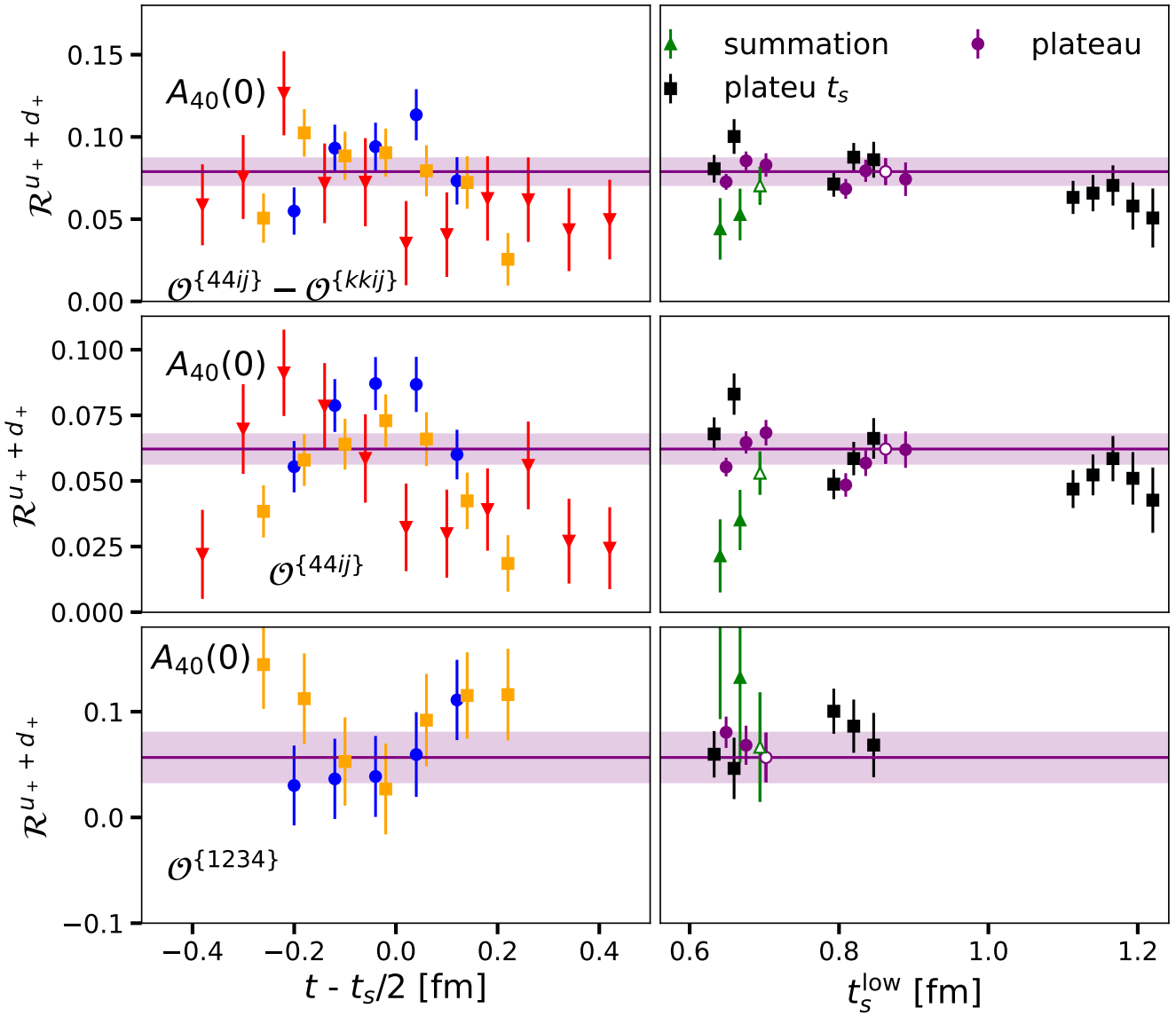
Conclusions

- Direct calculation of third nucleon moment
- Direct calculation of fourth moment
- Determination of GFFs
- Reconstruction of unpolarized isovector PDF

Backup

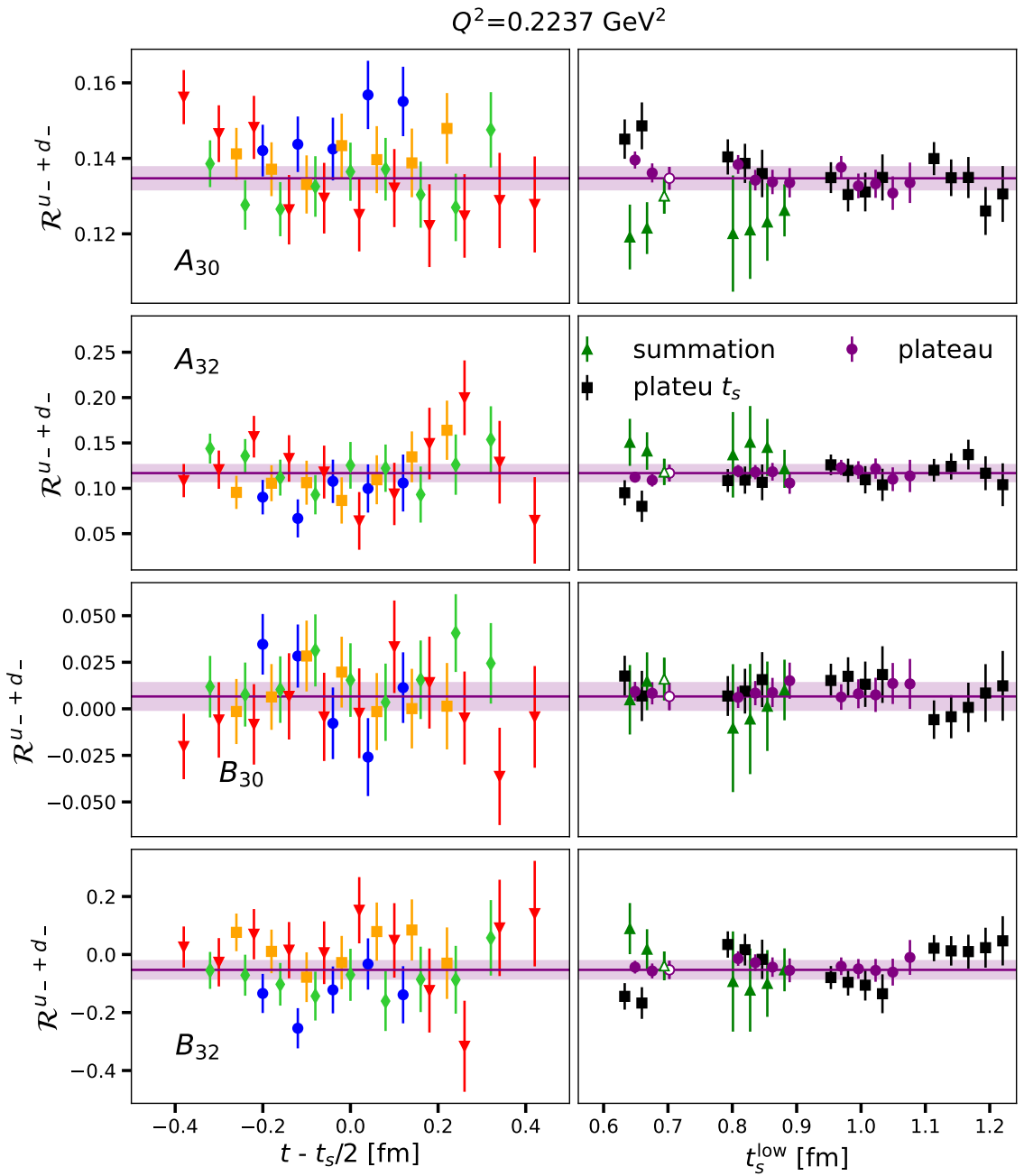
Results

- Forward limit
- Three derivative operators
- isoscalar



Results

- Momentum transfer
- Two-derivative operators
- isoscalar



AQTIVATE

Results

- Three-derivative operators, isoscalar

