Carleman routes to the quantum simulation of fluid flows

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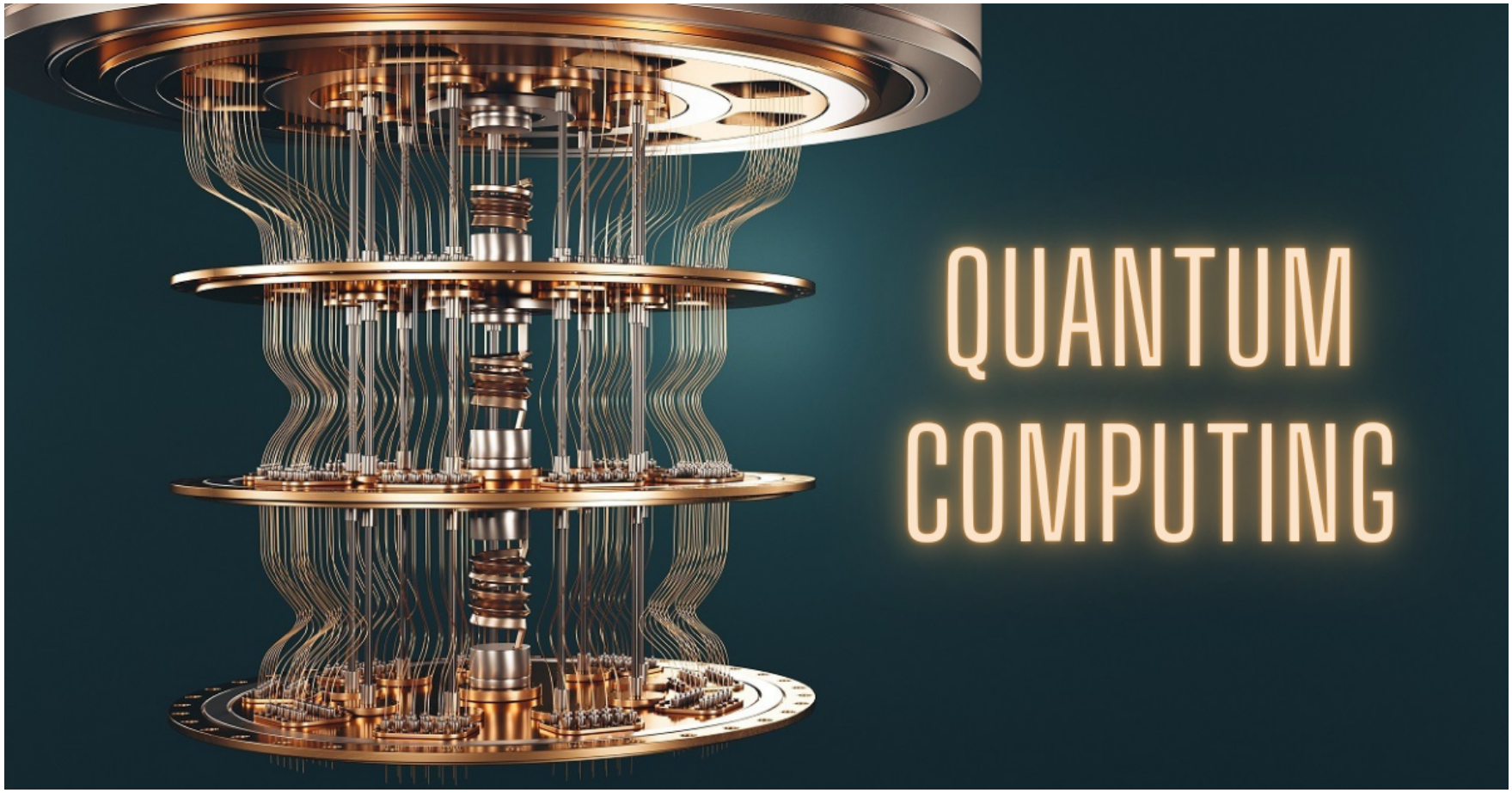
AQTIVATE

Cyprus, Oct 1, 2026

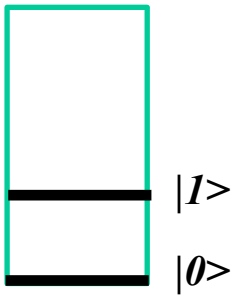
Basics of Quantum Computing

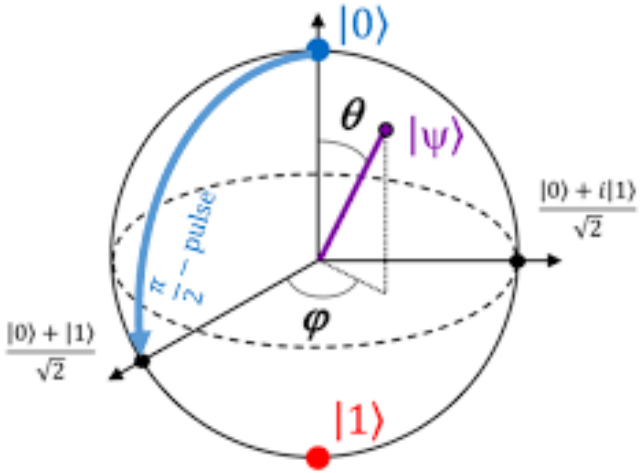


Qubits: the Bloch sphere

The qubit is parametrized by two angles: quantum superposition of two basic states: **|0> (ground), |1> (excited)**.
The famous Schroedinger cat is both dead and alive, (until you find out)

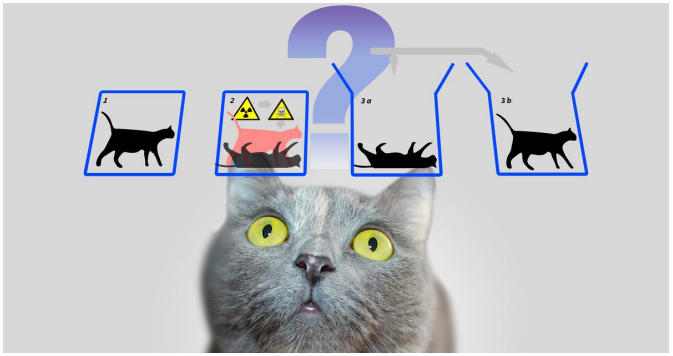
$$|\psi> = c_0|0> + c_1|1>$$





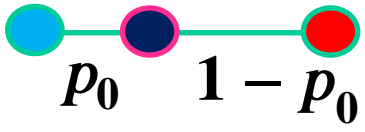
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$





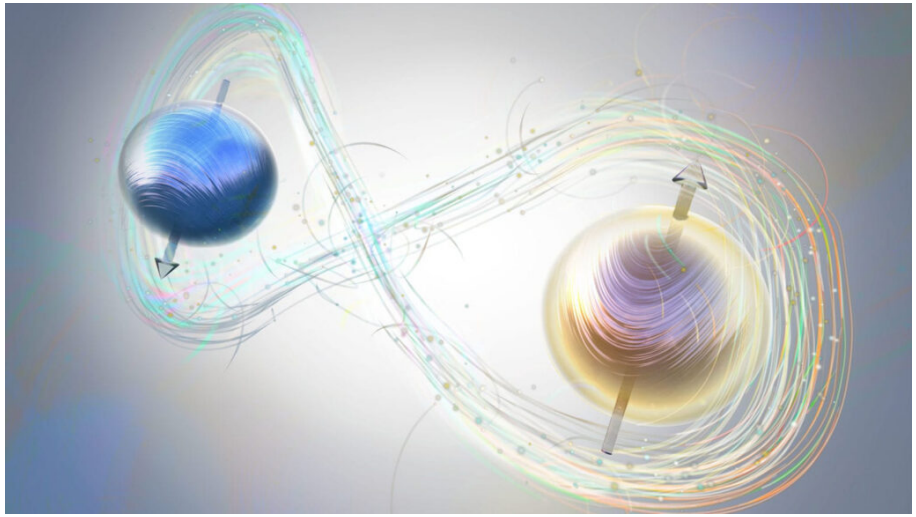
The qubit can be in any superposition of |0> and |1> (2d Bloch surface)

Measure: qubit to bit {0 or 1}



Classical state: (4^q) measurements

Quantum Entanglement



If A and B are entangled
any measure on A/B affects B/A, no matter how far apart.
Nonlocality,Unseparability,

Space is erased $(1/r^0)$

(No superluminar propagation!)

Purely quantum extra-information

Alice

$$A = a_0|0> + a_1|1>$$

Bob

$$B = b_0|0> + b_1|1>$$

$$AB = a_0b_0|00> + a_0b_1|01> + a_1b_0|10> + a_1b_1|11>$$

$$\widetilde{AB} = c_{00}|00> + c_{01}|01> + c_{10}|10> + c_{11}|11>$$

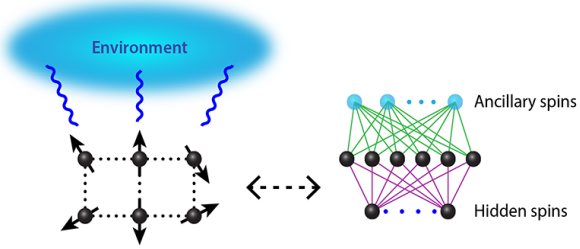
$$c_{ij} = a_i b_j + e_{ij}$$ ***Entanglement***

Entanglement is VERY fragile

Open quantum systems are strongly coupled to the environment («noise» NISQ):

Decoherence

The entanglement dies out very quickly
tdec ~ 10-1000 microseconds



Quantum Noise

The quantum circuit fails the unitary update (Fidelity)
In and Out the Hilbert sphere
10-100-1000 Error Correction qubits per logical qubit!

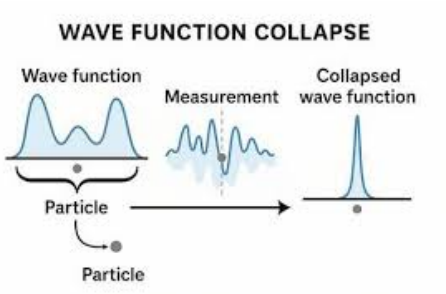
HUUUGE technological challenge

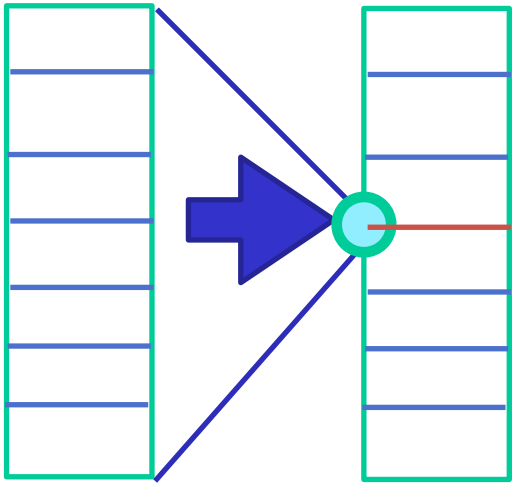
(pre-transistor stage, but fast moving…)

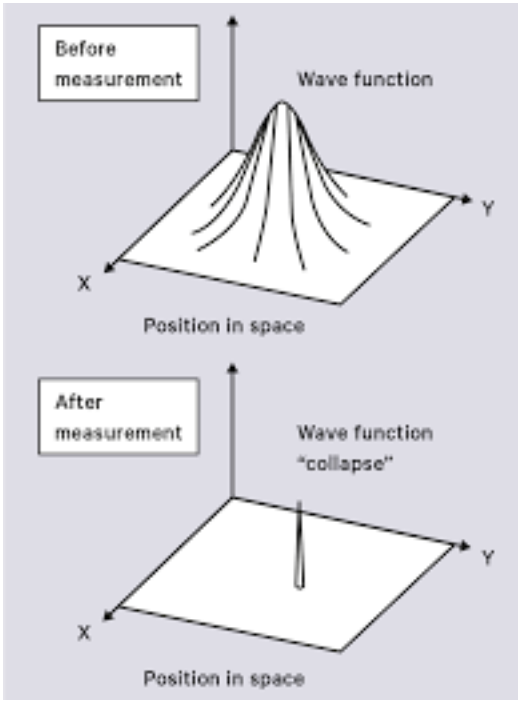
Collapse of the wavefunction

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n(t)\varphi_n(x) = \sum_{n=0}^{\infty} c_n(t)\,|n>$$

$$\hat{M}\,|\psi> = |m>$$







Projective Measurement: ***Emergence of Classical from Quantum***
The interaction with the macroscopic apparatus
forces the wavefunction into a single state (collapse)
This is NOT part of the Schroedinger equation!!!

Unitary Gates

Pauli

X Gate
Bit–flip, Not

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta|0\rangle + \alpha|1\rangle$$

Z Gate
Phase–flip

$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle - \beta|1\rangle$$

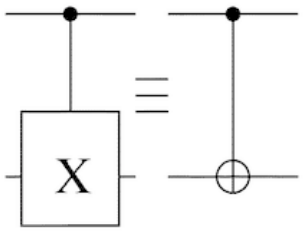
H Gate
Hadamard

$$H \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\alpha+\beta|0\rangle + \alpha-\beta|1\rangle}{\sqrt{2}}$$

T Gate

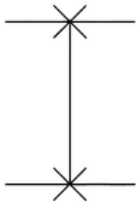
$$T \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + e^{i\pi/4}\beta|1\rangle$$

Controlled Not
Controlled X
CNot



$$\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{matrix} a|00\rangle + b|01\rangle + \\ d|10\rangle + c|11\rangle \end{matrix}$$

Swap

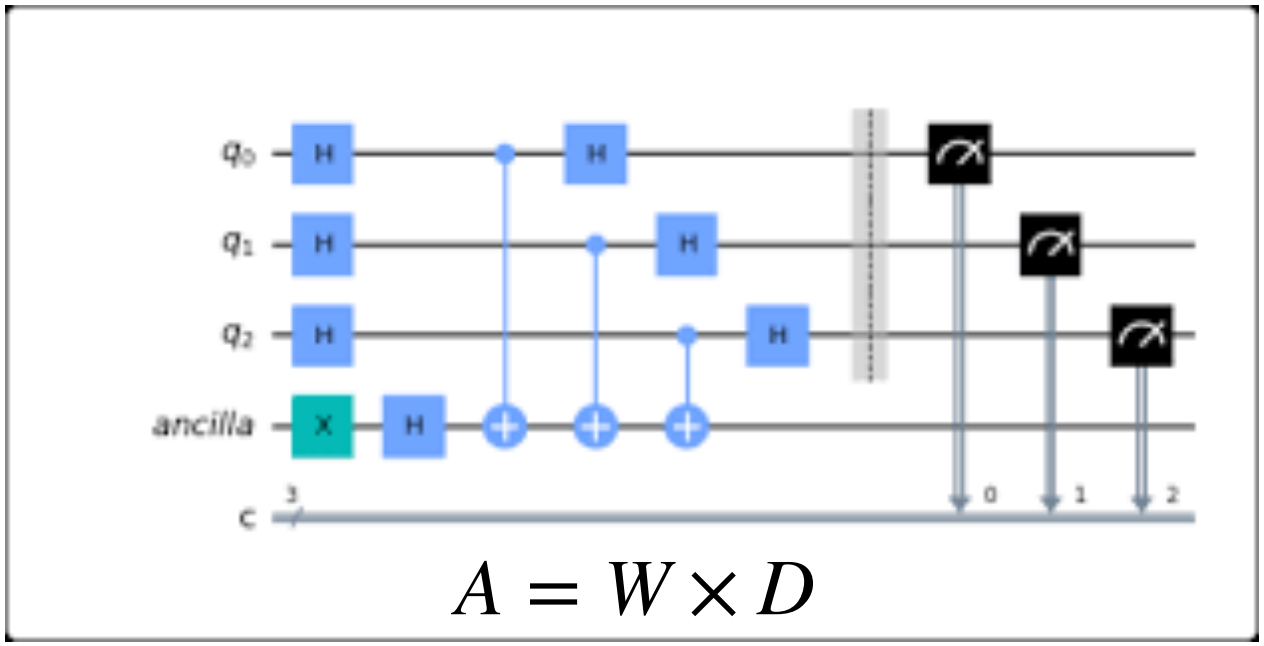


$$\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{matrix} a|00\rangle + c|01\rangle + \\ b|10\rangle + d|11\rangle \end{matrix}$$

The circuit "executes" the unitary evolution

Quantum Circuit

Depth



$$A = W \times D$$

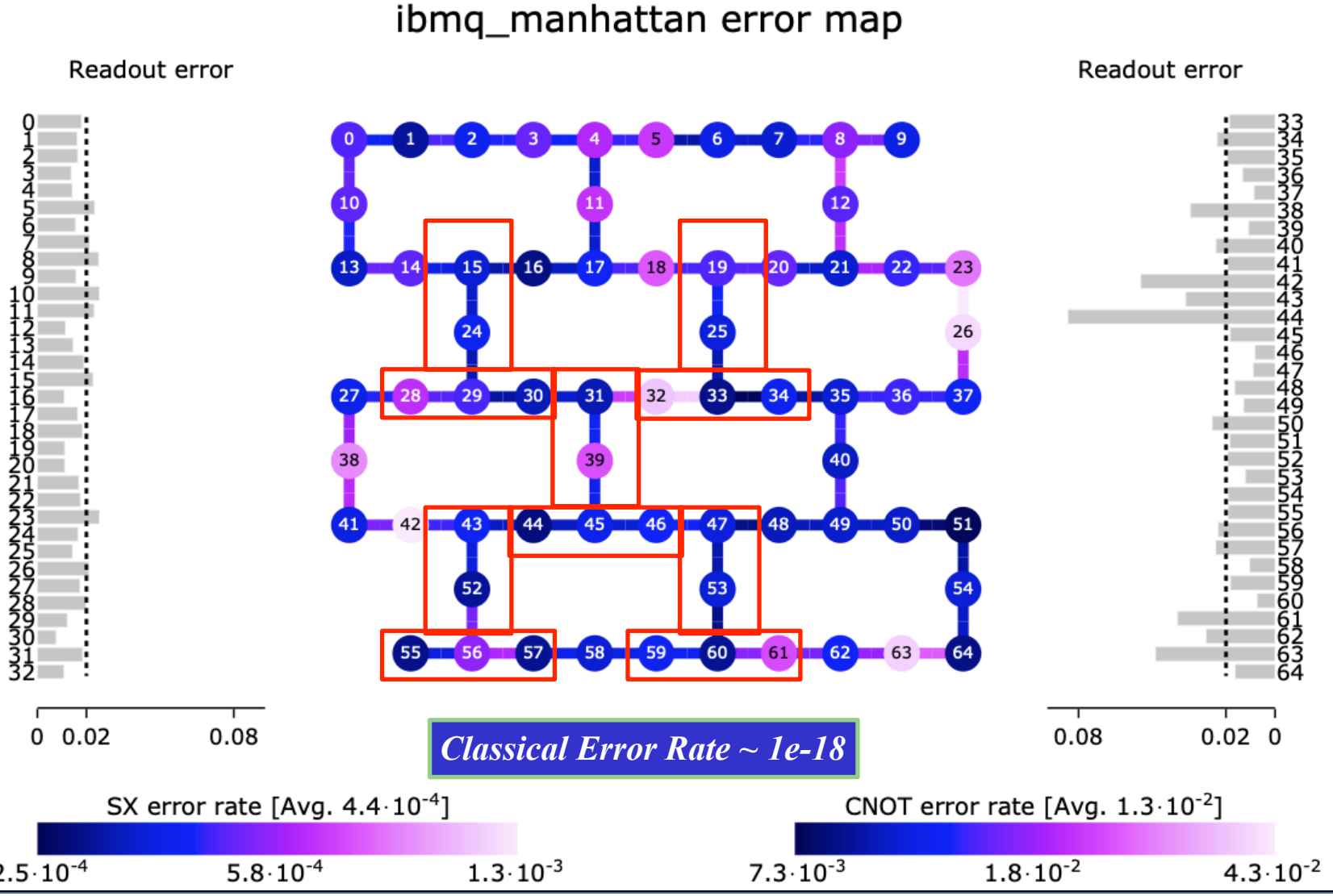
Width

Depth is exposed to Noise:

$$p_D = p_1^D$$

Bipartite 65-entangled state

https://www.ibm.com/quantum/blog/whole-device-entanglement



Quantum Computers today

Superconducting: ***(IBM, Google,Rigetti…)***

Google : Superconducting qubits, Willow chip enhanced EC

Microsoft : Topological qubits (?)

Trapped ions: ***IonQ, Quantinuum***

Photonic: ***PsiQuantum, Xanadu***

Neutral Atoms: ***QuEra, Pasqal,Eniquantica***

Silicon spin: ***Diraq,Intel***

Dwave: ***Quantum annealer (Q ~ 10000's)***

QC market projected to 20G$ in 2030

QC for fluids: why?



"Beyond" Feynman…

	bits	qubits
bits	CC	CQ
qubits	QC	QQ

Extend the portfolio of quantum computing app's

Two-way algorithmic transfer Q2C/C2Q

New angles to the foundations of Quantum Mechanics

N-body Schroedinger in the limit N >> 1 ? (W. Kohn)

New algorithms for CLASSICAL CFD?

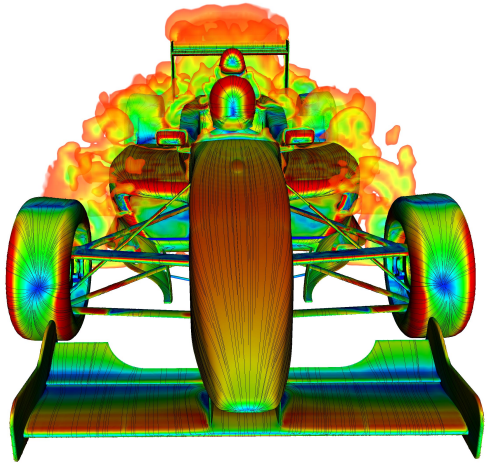
Computational Fluid Dynamics

$$Re = UL/\nu$$

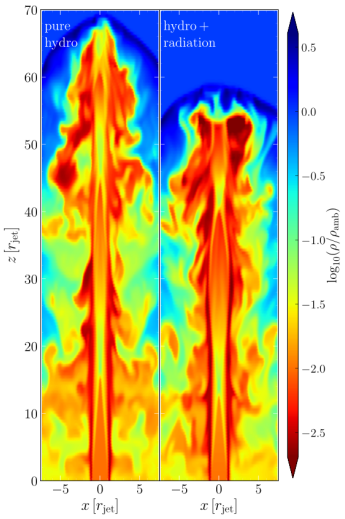
(Mpsc)

$$Dof \sim Re^{9/4}$$

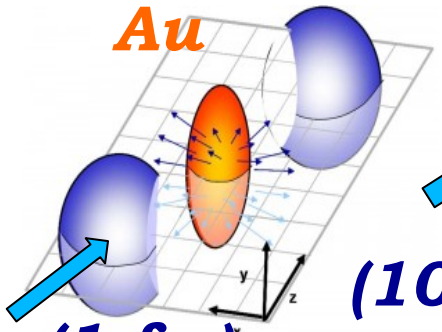
$$CC \sim Re^{3}$$



(10^7)



(10^35)



(1 fm)

(10^2)

$$\frac{L}{Ld} \sim Re^{3/4}$$

$$N = \frac{L}{dx} \sim Re^{3/4}$$

$$G = N^3 \sim Re^{9/4}$$

$$CC = N^4 \sim Re^{3}$$

Hydrodynamics is universal across scales

NATO OTAN

QCFD potential

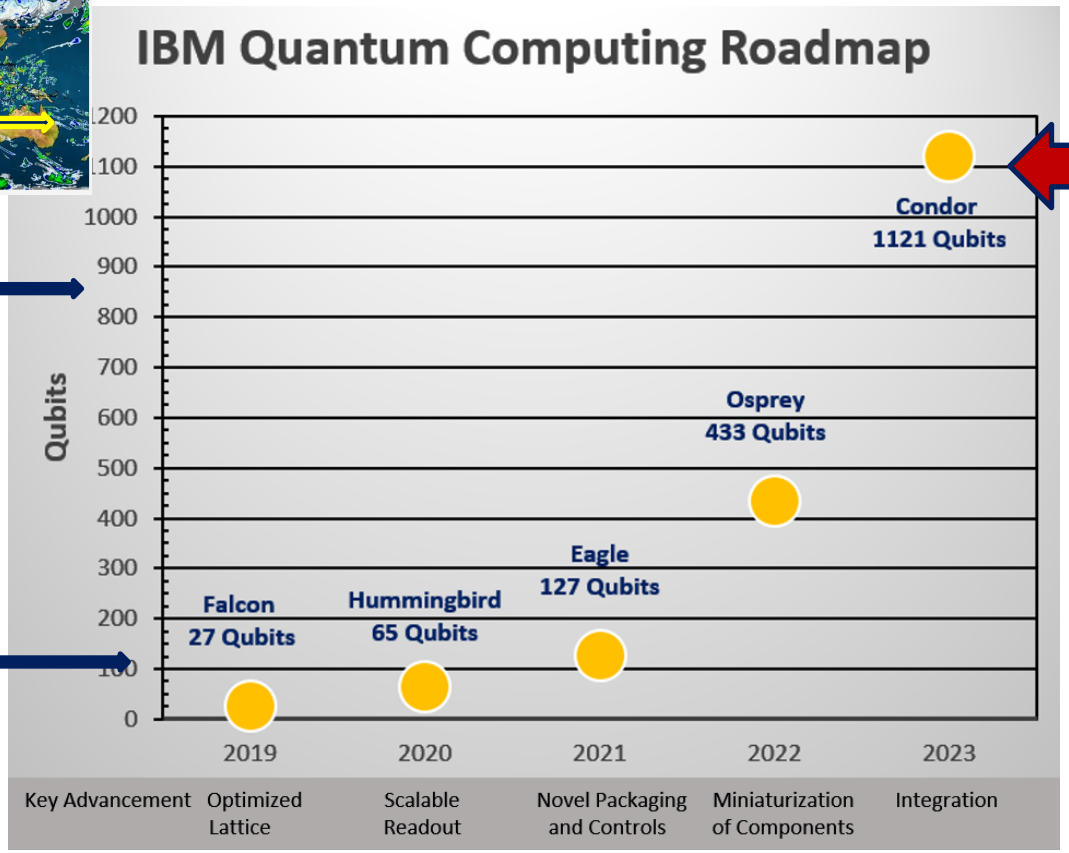
$2^{Q} \sim Re^{3}$: Q~3Log_2Re~10Log_10Re

1:10 Phys/Logical

$$q = \frac{10}{f} Log_{10} Re$$

f ; Fidelity





Exascale





You are here

Current QCFD

Blue Sky scenario: iff everything goes fine..

NATO OTAN

End-to-End Simulation

1. Preparation of Quantum state from Classical

$$\psi_0 \rightarrow |\psi_0> = \sum_{p=(0,1)^q} c_p |p>$$

$(N^2 = 2^{2q})$

$N \rightarrow q = Log_2 N$

Complexity

2. Quantum Evolution

$$|\psi_0> \rightarrow |\psi_t>$$

*Pow(T)*Polylog(N,T,1/precision)*

Complexity
Circuit Width, Depth
Noise, NISQ

3. Measurement: Classical from Quantum

$$|\psi_t> \rightarrow \psi_t \qquad (N^2 = 2^{2n})$$

Complexity

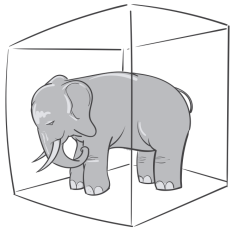
*Every measurement returns 1 classical state out of N and destroys the quantum state: N*N replicas needed*

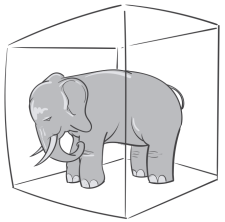
Two elephants in the QCFD room

Quantum mechanics is Linear and Unitary.
The physics of fluids is typically neither...

Non-linearity (Inertia)

Non-unitarity (Dissipation)





Nonlinearity:

Loss of Orthogonality, No-cloning theorem

Dissipation:

Restoring unitarity implies low-success probabilistic updates

QCFD Strategies

1. QODE's	*Nonlinear quantum ODE's solvers*
2. Variational	*Nonlinear quantum eigensolvers*
3. Linearized	*Carleman embedding*
4. Linear functional	*Koopman Von Neumann*
5. Fluid wavefunctions	*Inverse Madelung Transform*
6. Homotopy methods	*Nonlinear deformations*
7. Quantum Lattice Gas	*Second Quantization*
8. Quantum spectral	*Quantum Fourier Transform*
9. Replicas	*?*
10. Schroedingerization	

General Strategy

UPLIFT

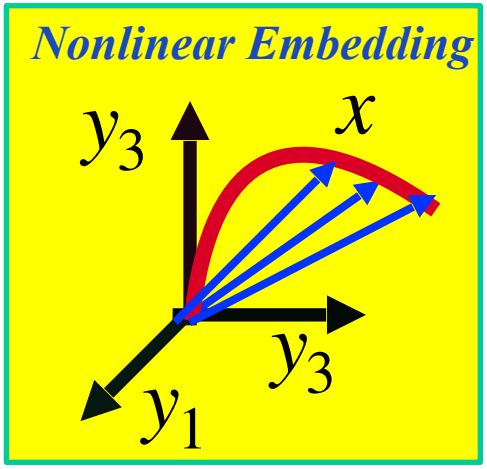
Non linear and Dissipative in finite dimension = Linear and Unitary in infinite dimensions

$$\dot{x} = f(x) \leftrightarrow \dot{y} = Ay$$

$x \in R^d$

$y \in R^D$

$D \to \infty, \quad D \gg d$



Carleman Logistic Equation

Homogeneous quadratic kinetic equation:

$$\partial_t f = -f(1 - Rf); \quad f(0) = f_0$$

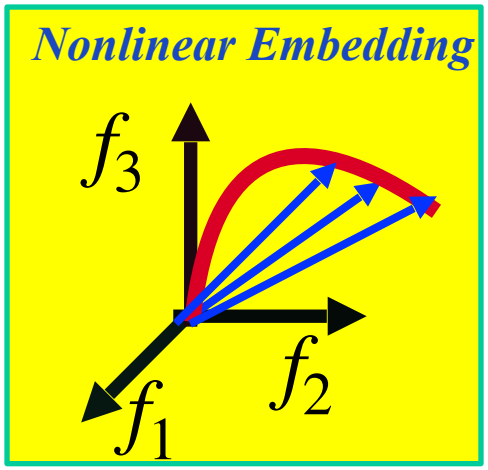
$$f(t) = f_0 \frac{e^{-t}}{r + (1-r)e^{-t}} = f_0 e^{-t}(1 + c_1 e^{-t} + c_2 e^{-2t} + \ldots)$$

$$r \equiv R f_0 \equiv f_0 / f_u^*$$

How fast does it converge?

$$\partial_t f_1 = -1(f_1 - R f_2)$$
$$\partial_t f_2 = -2(f_2 - R f_3)$$
$$\partial_t f_k = -k(f_k - R f_{k+1})$$

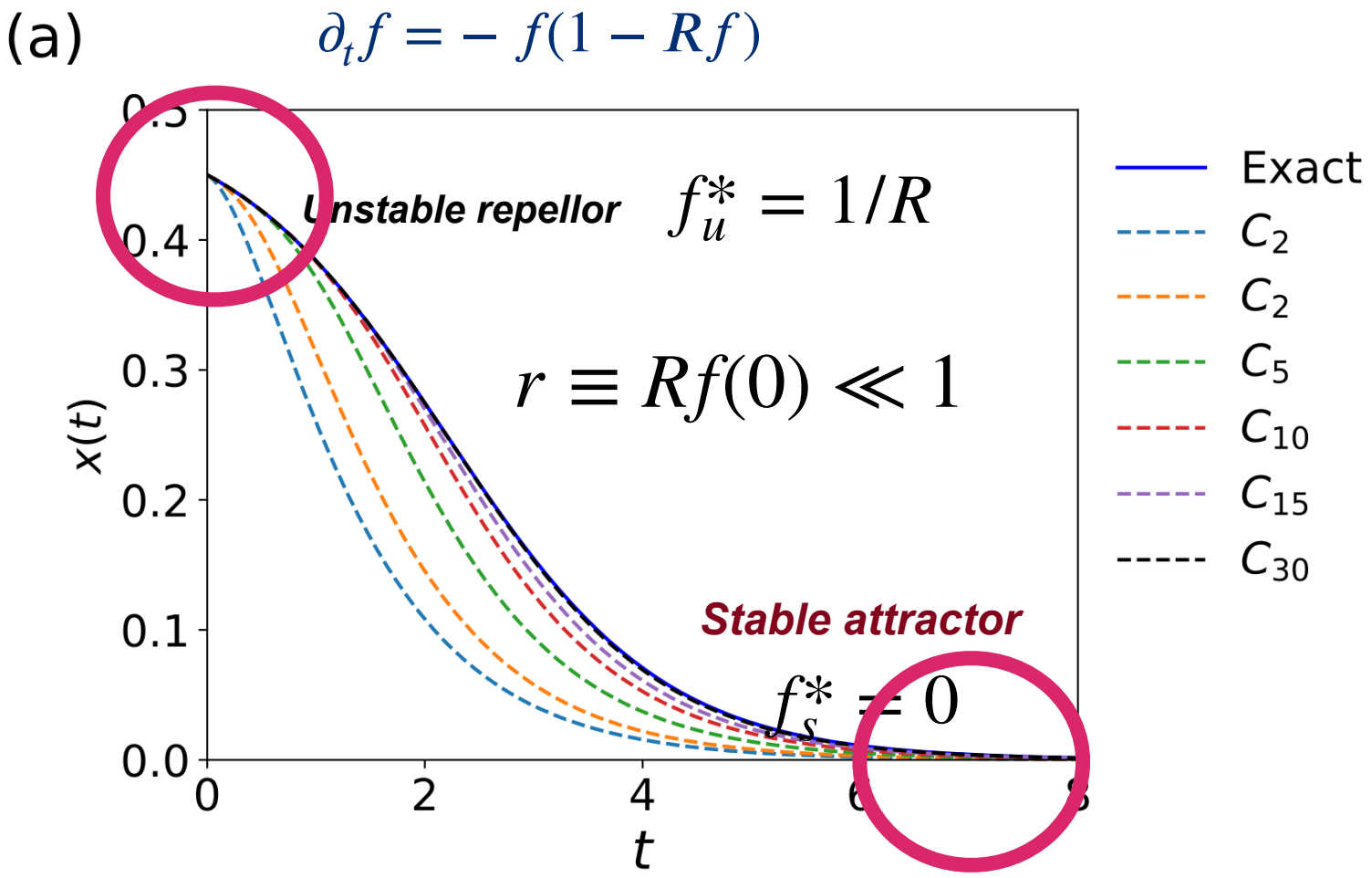
$$f_1 \equiv f$$
$$f_2 \equiv f^2$$
$$f_3 \equiv f^3$$



Truncation at level K: $f_{K+1} = 0$

$$f \to \{f_l = (f)^l\}$$

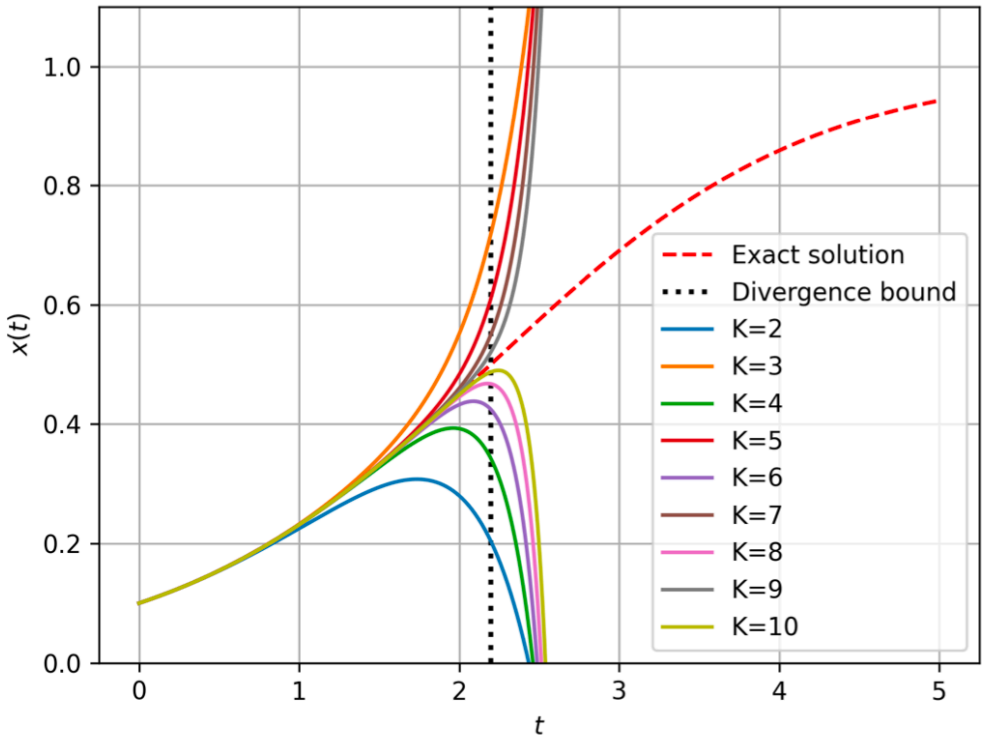
Carleman-Logistic Solutions



Nonlinear relaxation towards the stable attractor f=0
In the limit R to infinity the two attractors merge

Growing Logistic

$$\partial_t f = f - Rf^2 \qquad f(t) = f_0 \frac{e^t}{re^t + (1-r)}$$



Positive eigenvalues:

$$\lambda = 1,2\ldots\ k$$

Diverges above t* because e^t is unbounded

REGULARIZATION:

$$\tilde{f} = 1/R - f$$

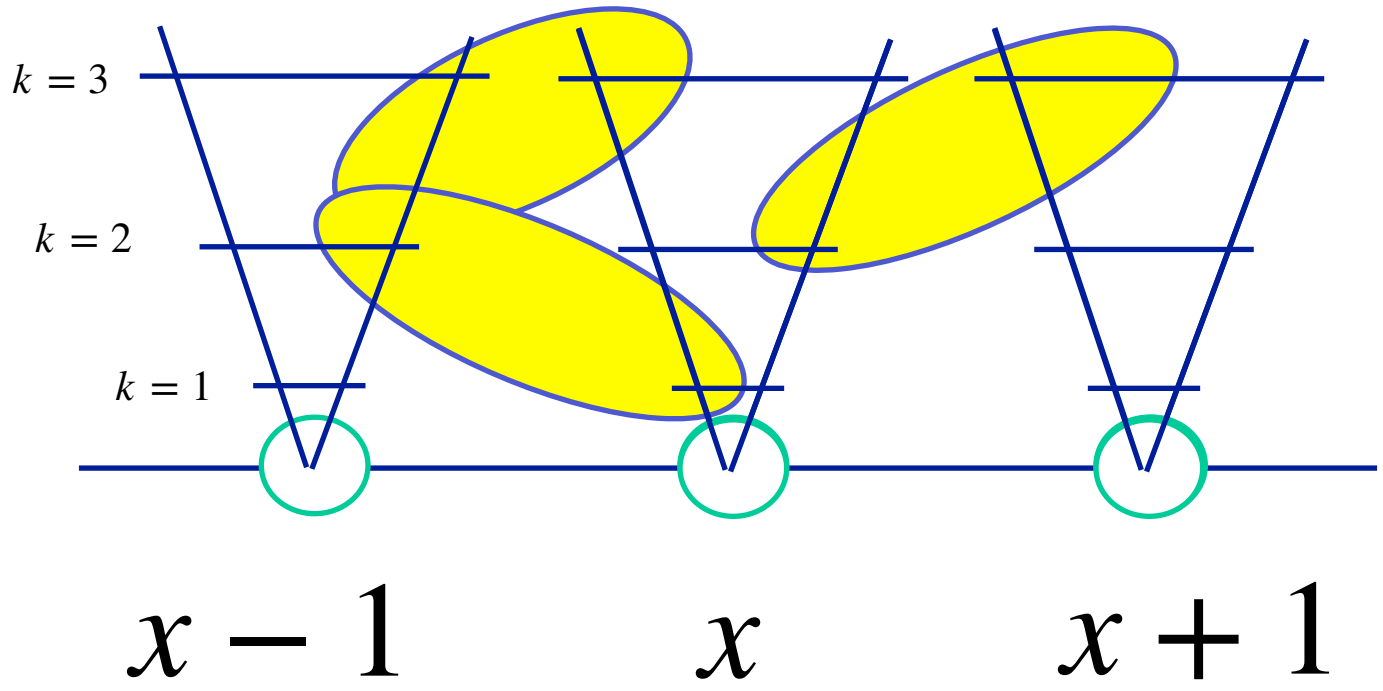
$$f = e^t g(t)$$

Turns it back to a decaying logistic.

Carleman Extended Systems

Nonlinear &Nonlocal correlations

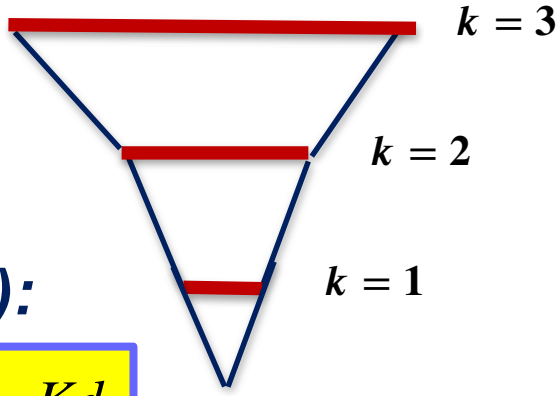
$$f_{kl}(x_1, x_2) \equiv f_k(x_1) f_l(x_2)$$



Carleman fluids

The inflationary Carleman ladder

$$\partial_t V_k + C_k V_k = 0 \qquad k = 1, K$$

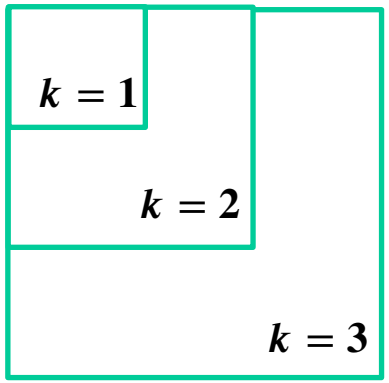


$$V_k(t) = e^{-C_k t} V_k(0)$$

The Carleman matrix C is HUGE (tensorial):

$$\{G, z, K\} \qquad N_c \sim (Gz)^K \qquad G = L^d$$

$$N_C(K) \sim L^{Kd}$$



G gps, z variables/gp, K Carleman levels:

$$N = z + \frac{z(z+1)}{2} + \frac{z(z+1)(z+2)}{6} + \ldots$$

$$N_c \sim G z^K$$

Locality: $V_1(x), V_2(x,x), V_3(x,x,x)\ldots$

Nonlocality: $V_1(x1), V_2(x1,x2), V_3(x1,x2,x3)\ldots$

Childs group, PNAS 2021: Burgers, d=1,G=16, K=4

Carleman-Burgers

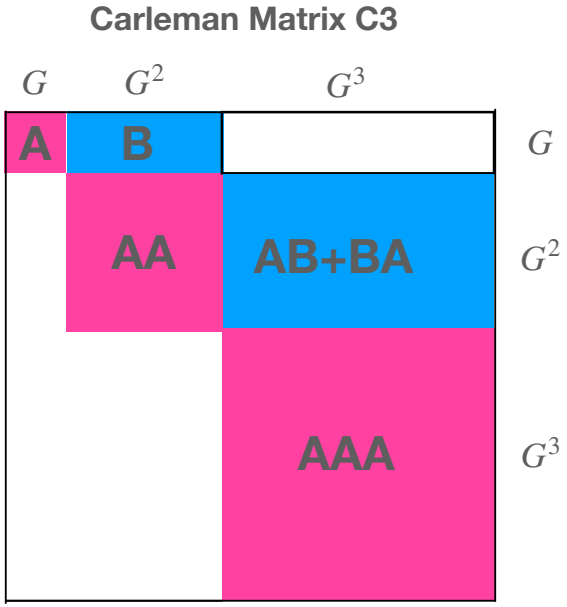


Burgers: $u_i^+ = A_{i,j}u_j + B_{i,jk}u_j u_k$

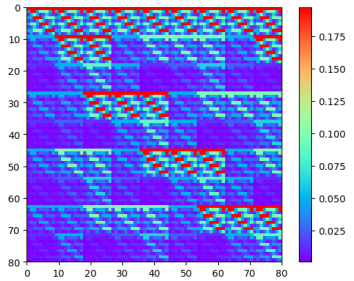
u_i

$u_i u_j$

$u_i u_j u_k$



B matrix



Sparsity s: # of non-zero off-diag elements

Block structured: Important to bypass NO-GO's

Carleman-Burgers



G=16, 4000 steps, Carleman=4

$$C \sim sqT^2 Polylog(logT, logN, log(1/\epsilon))$$

STABILITY:

$$Re < Ma/2$$

Suff, not necessary!

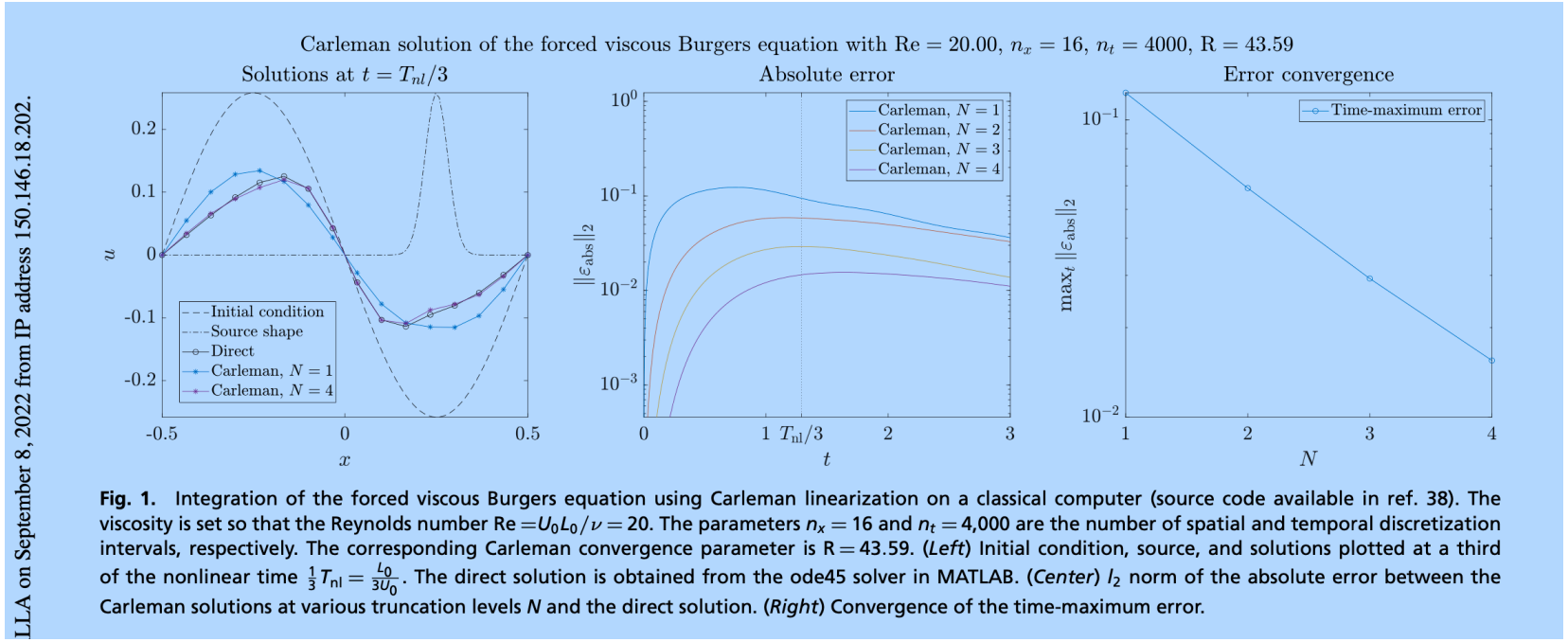


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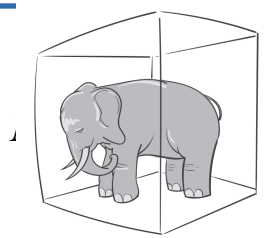
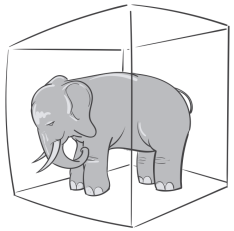
Fig. 1. Integration of the forced viscous Burgers equation using Carleman linearization on a classical computer (source code available in ref. 38). The viscosity is set so that the Reynolds number Re $= U_0 L_0/\nu = 20$. The parameters $n_x = 16$ and $n_t = 4{,}000$ are the number of spatial and temporal discretization intervals, respectively. The corresponding Carleman convergence parameter is R = 43.59. (*Left*) Initial condition, source, and solutions plotted at a third of the nonlinear time $\frac{1}{3}T_{nl} = \frac{L_0}{3U_0}$. The direct solution is obtained from the ode45 solver in MATLAB. (*Center*) l_2 norm of the absolute error between the Carleman solutions at various truncation levels N and the direct solution. (*Right*) Convergence of the time-maximum error.

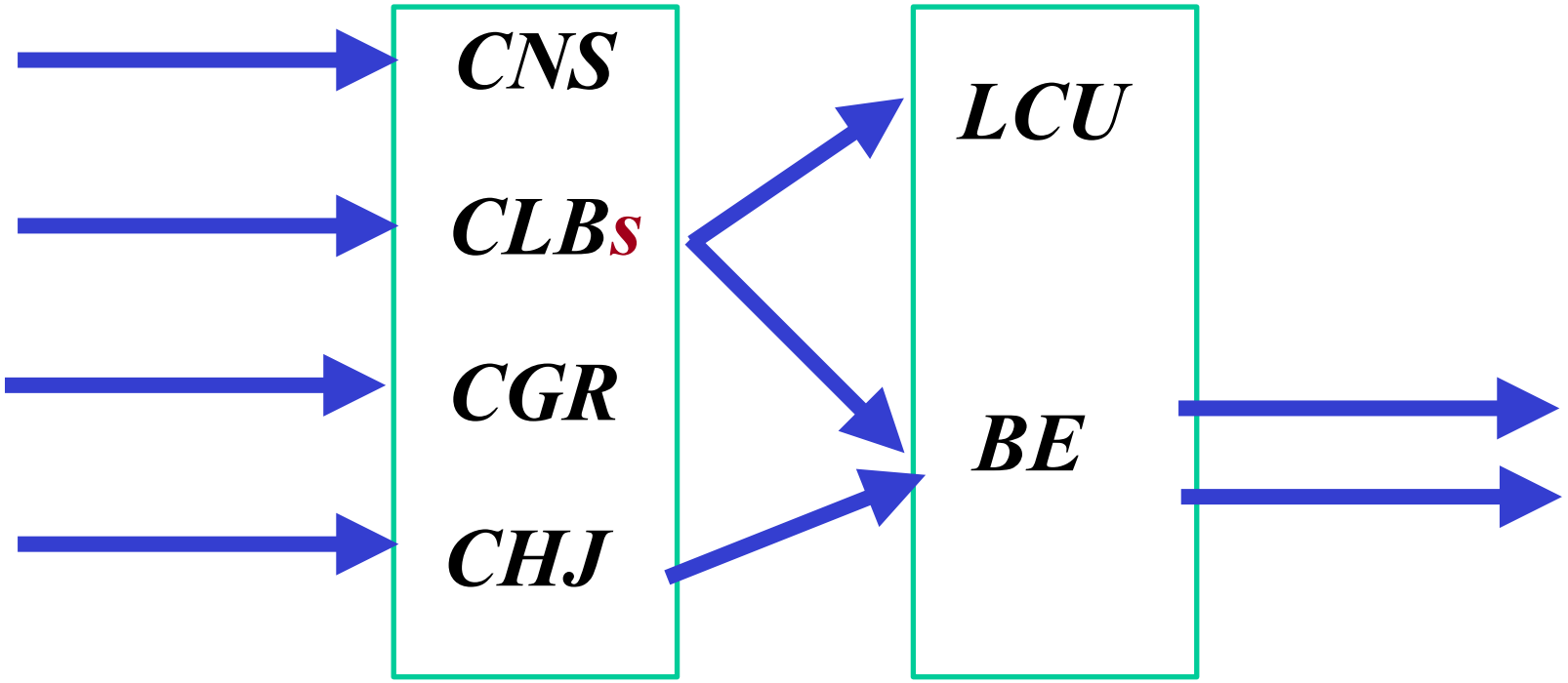
Child's group: PNAS 2021

Nice, but real fluids are MUCH more complicated

Carleman routes to NS







De-classicalization: *Put Nonlinearity and Dissipation back into a Linear and Unitary mathematical box.* ***EXTRA-DIMENSIONS***

Carleman NS hierarchy

$$\partial_t(\rho u_a) + \partial_b[\rho u_a u_b + p\delta_{ab} + 2\mu(\partial_a u_b + \partial_b u_a)] = 0$$

$$u_a(x) -> u_a(x)u_b(y) -> u_a(x)u_b(y)u_c(z) \to \ldots$$

$$u_a -> u_c\partial_b\partial_a u_b \to \ldots \qquad u_a -> u_a\partial_b p \to \ldots$$

uu, up, pp non-local correlators uuu,upu,…

Cumbersome/excruciating

Numerical errors: Truncation,Discretization, Unitarity

||CNS-NS|| = Error(K,Re;dx,dt)

Theory: $||e^{C_K t}|| > 1,\ Re > Re_{crit}(K)$ ***But structured C's may tunnel the NO-GO's***

Carleman Navier-Stokes

Euler time-stepping + Finite Differences:

$$\rho^+ = \rho + h\hat{D}_a J_a$$

$$J_a^+ = J_a + h[\hat{D}_b(\rho^{-1} J_a J_b + p(\rho)\delta_{ab} + \mu(\hat{D}_a J_b + \hat{D}_b J_a)]$$

$$K_{ab} \equiv \rho^{-1} J_a J_b \sim 2J_a J_b - \rho J_a J_b$$

Non-local quadratic and cubic

$$p(\rho) = \rho c_s^2$$

Ideal weakly compressible

$$K_{ab}^+ \quad \textit{Involves} \quad \overset{(2,0)}{(JJ}, \overset{(2,2)}{JDDJ}); (\overset{(3,1)}{JDJJ}, \overset{(4,1)}{JD\rho JJ}, \overset{(2,1)}{JDp})$$

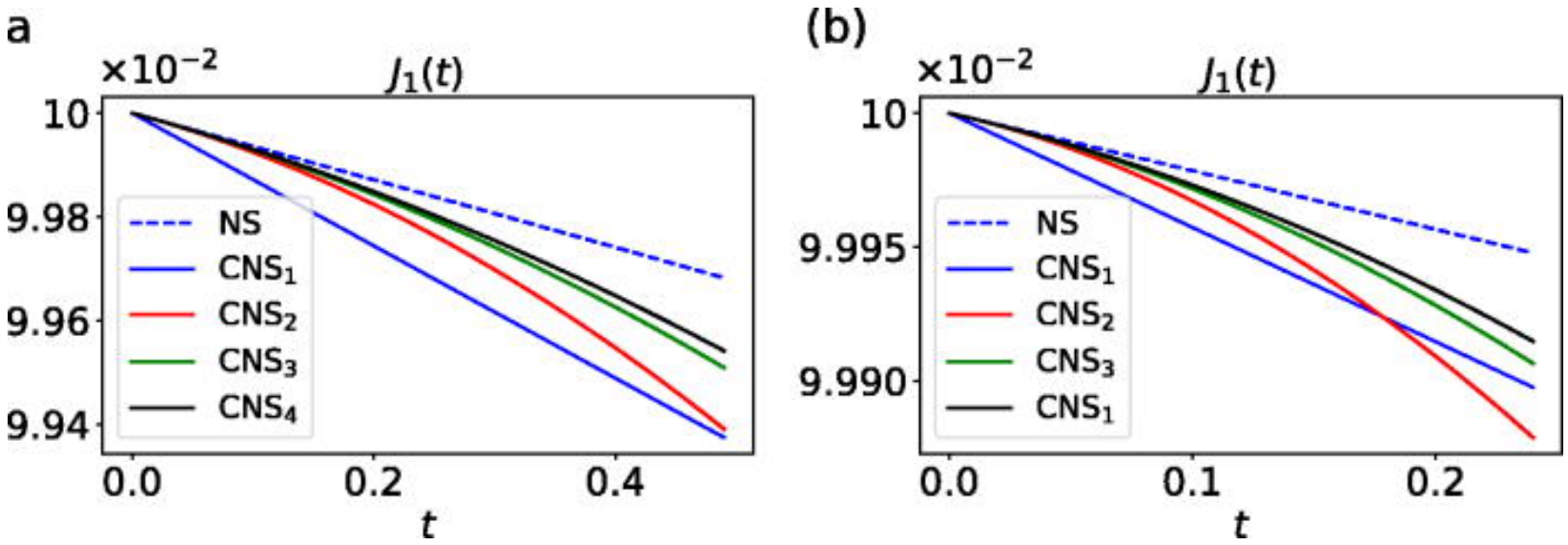
A tangle of nonlinear and non-local terms

(Carleman-Feynman diagrams)

Carleman NS

From: **Three Carleman routes to the quantum simulation of classical fluids**

Physics of Fluids 36, 057143 (2024)



$$dt = 0.01 \qquad T_A = L/U \sim 300$$

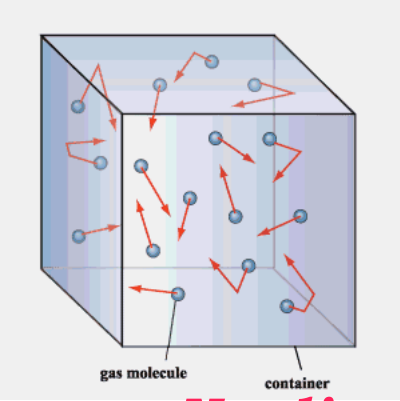
$$T_C/T_A \sim 0.001$$

Lattice Boltzmann

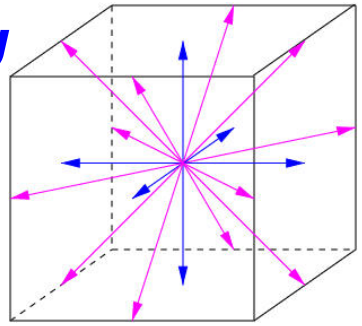
$$f(\vec{x},\vec{v};t)=\sum_{i=0}^{b} f_i(\vec{x},t)\,\delta(\vec{v}-\vec{c}_i)$$

$$\left\{\begin{array}{l}\rho=\sum_{i=0}^{b} f_i \\ \rho\vec{u}=\sum_{i=0}^{b} f_i\vec{c}_i\end{array}\right.$$

Exact sampling (low-order moments)



Universality



Syms& Cons_Laws

Nonlinearity goes with Mach^2

No U grad U

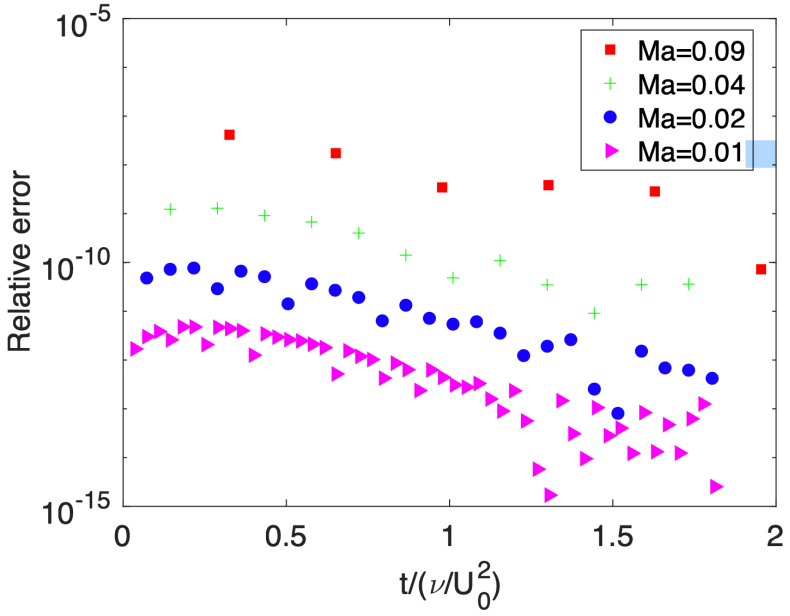
$$f_i\left(\vec{r}+\vec{c}_i,t+1\right)-f_i\left(\vec{r},t\right)=-\Omega_{ij}\left(f_j-f_j^{eq}\right)+S_i$$

Exact **Free-Stream** **Colli-Relax** **Drive**

Mc Namara, Zanetti, Higuera, Jimenez, Benzi,SS 1988-89

Carleman-LB-QLSA

Margaret Cheung et al (2023):
Two-dimensional Taylor Green Vortex, 32x32
Fully classical, excellent agreement (10 digits) with K=3
But still emulated …..



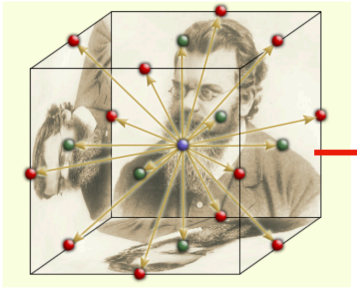
The Mach argument seems to apply!

FIG. 2. Relative error of the maximum u_x between Carleman-linearized LBM and LBM simulations for different initial Ma (Ma = 0.01, 0.02, 0.04, 0.09 are represented by magenta triangles, blue dots, green pluses, and red squares, respectively) at fixed initial Re = 100. The x-axis represents the simulation time t normalised by the kinetic diffusion time scale ν/U_0^2, i.e., kinetic viscosity divided by square of the initial flow velocity.

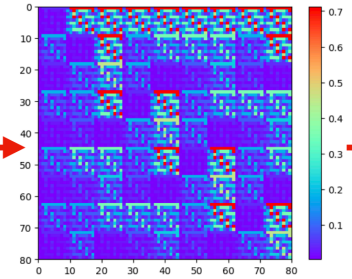
Carleman LB setup

$$\dot{f}_K = C_K f_K$$

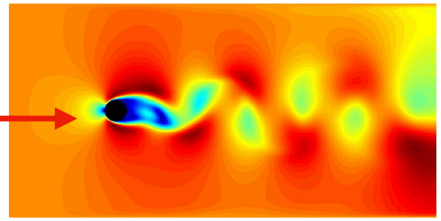
LB method



Carleman-2 LB matrix

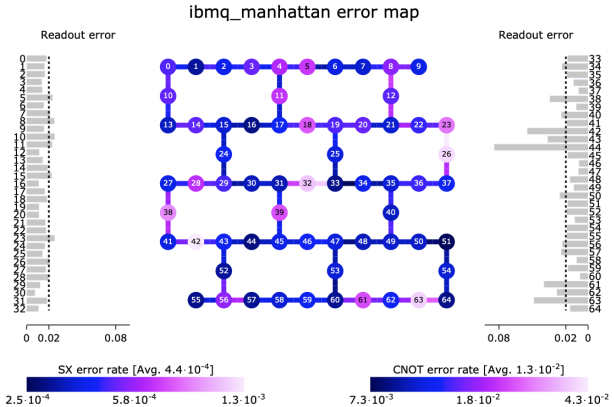


Pre-Turbulent flows



But the CLB matrix is "Pauli unfriendly"

Classical CLB shows excellent convergence !



$$C = \sum_n c_n P_n$$

$$|c_n| \sim 1/n$$

Depth scales like

$$\mathscr{D} \sim 2^{2q}$$

(Itani-SS, 2022, Sanavio-Succi, 2024-2025)

Carleman-LB

Post-collision: $f_i^*(x,t) = (1-\omega) f_i(x,t) + \omega f_i^{eq}$

Mode coupling:

$$\rho^{-1} \sim 2 - \rho \quad \Rightarrow \quad f_i^{eq} = L_{ij} f_j + Q_{ijk} f_j f_k$$

$$f_i^+ \equiv f_i(x + c_i, t+1) = A_{ij} f_j + B_{ijk} f_j f_k$$

CLB generators

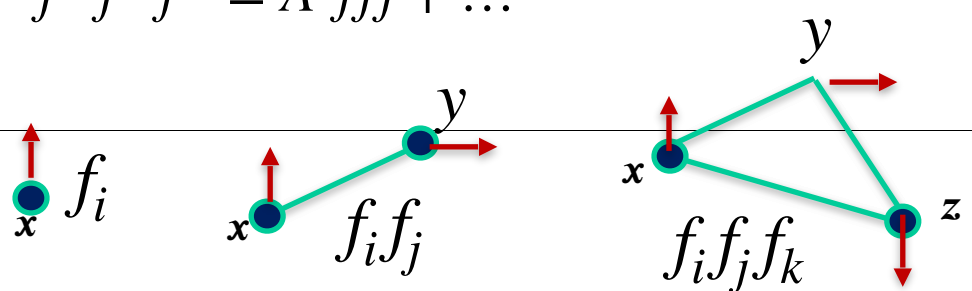
$$A_{ij} = \delta_{ij} - \omega L_{ij}$$
$$B_{ijk} = \omega Q_{ijk}$$

CLB

$$f^+ = Af + Bff$$

$$f^+ f^+ = A^2 ff + (AB + BA) fff + B^2 ffff$$

$$f^+ f^+ f^+ = A^3 fff + \ldots$$



$$\omega \to 0: \; A \to I, B \to 0$$

Multi-streaming is still EXACT

$$Re \to 0: \; B \to 0$$

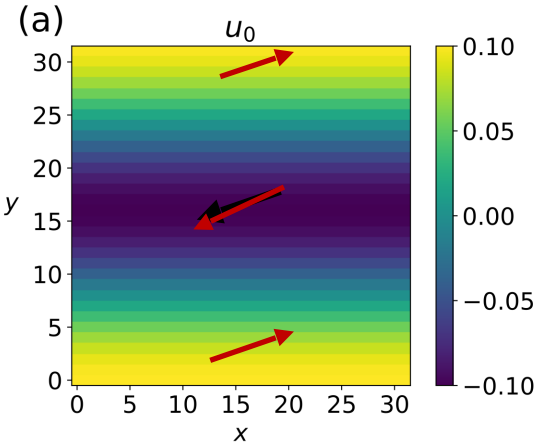
Carleman = Block Diagonal

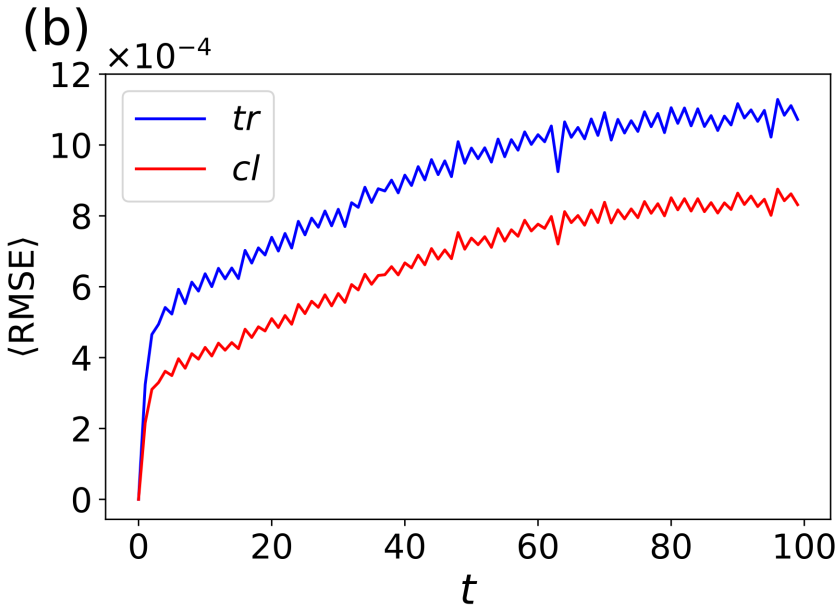
$$\omega \to 2; \; Re \to \infty: \; A \to I - 2L, B \to 2Q$$

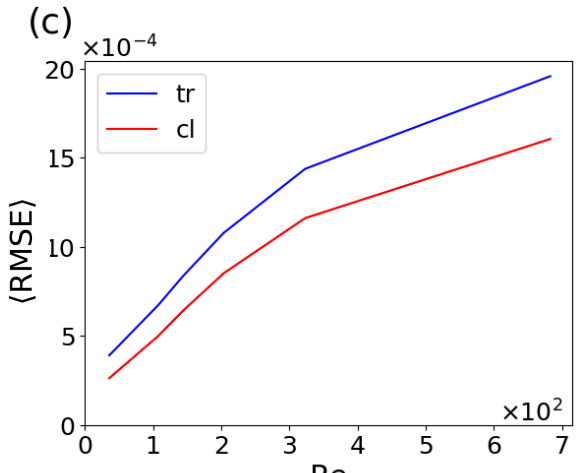
Singular

CLB vs LB: Kolmogorov-like flow

$$\left\langle \text{RMSE} \right\rangle = \sqrt{\sum_{i=1}^{N} \frac{1}{N} \left(\frac{f_i - f_i^{CL}}{f_i} \right)^2}$$





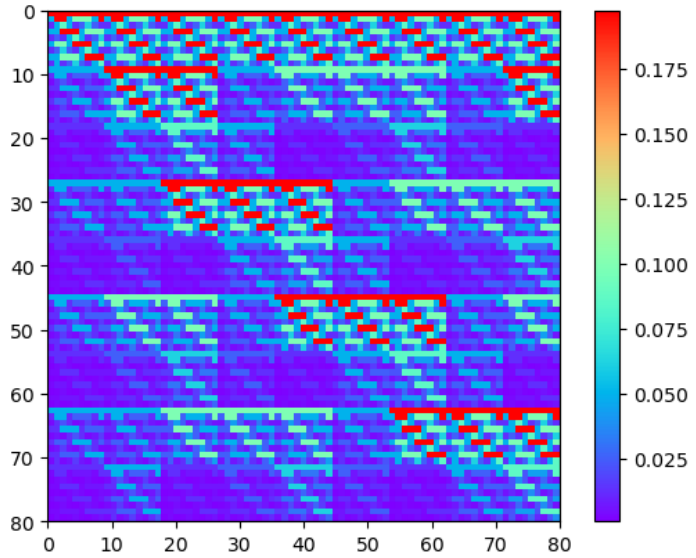


CLB excellent convergence at Re ~ O(10)

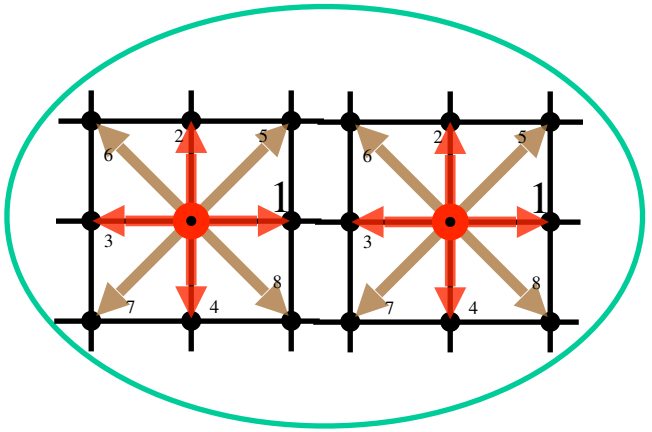
Caveat: Other flows should be investigated

PsiQuantum: Loss of convergence at high Re ?

The quantum CLB algo

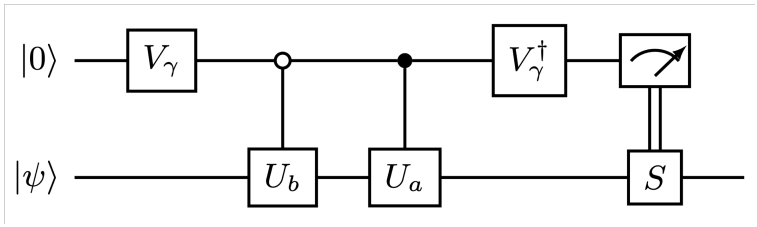


Single-site K=2, 81 states f_ij



The Collision Matrix projects over ALL Pauli gates !!!

$$C = \sum_n c_n P_n \quad |c_n| \sim 1/n$$



Carleman level *Velocities* *Ancilla*

$$q = 1 + 2\left\lceil \log_2 9 \right\rceil + 1 = 10$$

Circuit Depth: $4^9 = 256K$

Linear Combination of Unitaries

CLB_2.0

Quantum CLB-LCU: Exponential circuit depth

Block-Encoding oracles for sparse matrices

Depth is (log_2N)^2 but p_success <0.001

Modified Oblivious Amplitude Amplification

ps <0.01 for Advection-Diffusion-Reaction

Telescopic Hamiltonian simulation (QSVT)

Zamora et al report ps ~ 0.01 (PRE 2026)

Single-step success probability

$$p \sim \frac{e^{-\gamma \Delta t}}{s^2}$$

Nonlocality + Nonlinearity: Sparsity

Dissipation: Non-Unitarity

$$s \sim b + b(b+1)/2 = 54$$

$$\gamma \Delta t \ll 1$$

$$p \sim 10^{-4}$$



Nonlocality + Nonlinearity seems to hurt the most, but not sure…

Technical hurdle or fundamental barrier?

Second generation CLB

LB Single-Time Relaxation: $P_{success} \ll 1$

LB Multi-Time Relaxation: $P_{success} \to 1$

Regularised LB : $P_{success} \to 1$?

Entropic LB : $P_{success} \to 1$?

MTR-CLB

Historical context:

LB was born in matrix form (1988-89), then made diagonal (1992)
Then matrix form again, but optimised (Multi-Time Relaxation)
D'Humieres, Lallemand, Y. Qian, L. Luo, (1992-1996)

Classical LB: Very popular to improve stability

Quantum LB: From Strong to Weak coupling, KEY!

$$p_s : \epsilon \rightarrow 1 - \epsilon$$

Huadong Yao and SS, arXiv, Sep 2026

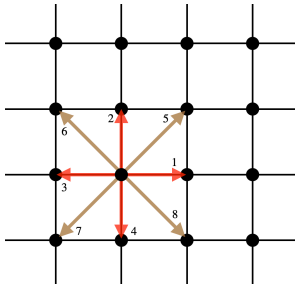
MTR-LB algorithm

1) f2m: $m_k = \sum_i T_{ki} f_i$ $\quad m_{0,1,2} = [\rho, J_x, J_y]$ **Conserved**

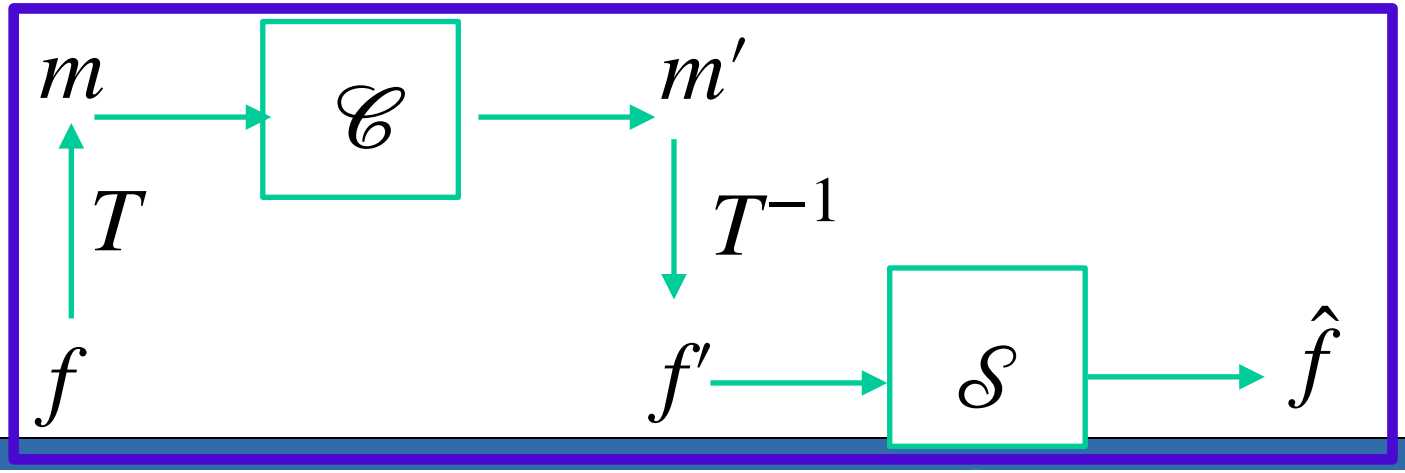
2) MRT collisions: $m'_k = (1 - \omega_k) m_k + \omega_k m_k^{eq}$

3) m2f: $f'_i = \sum_k T_{ik} m'_k$

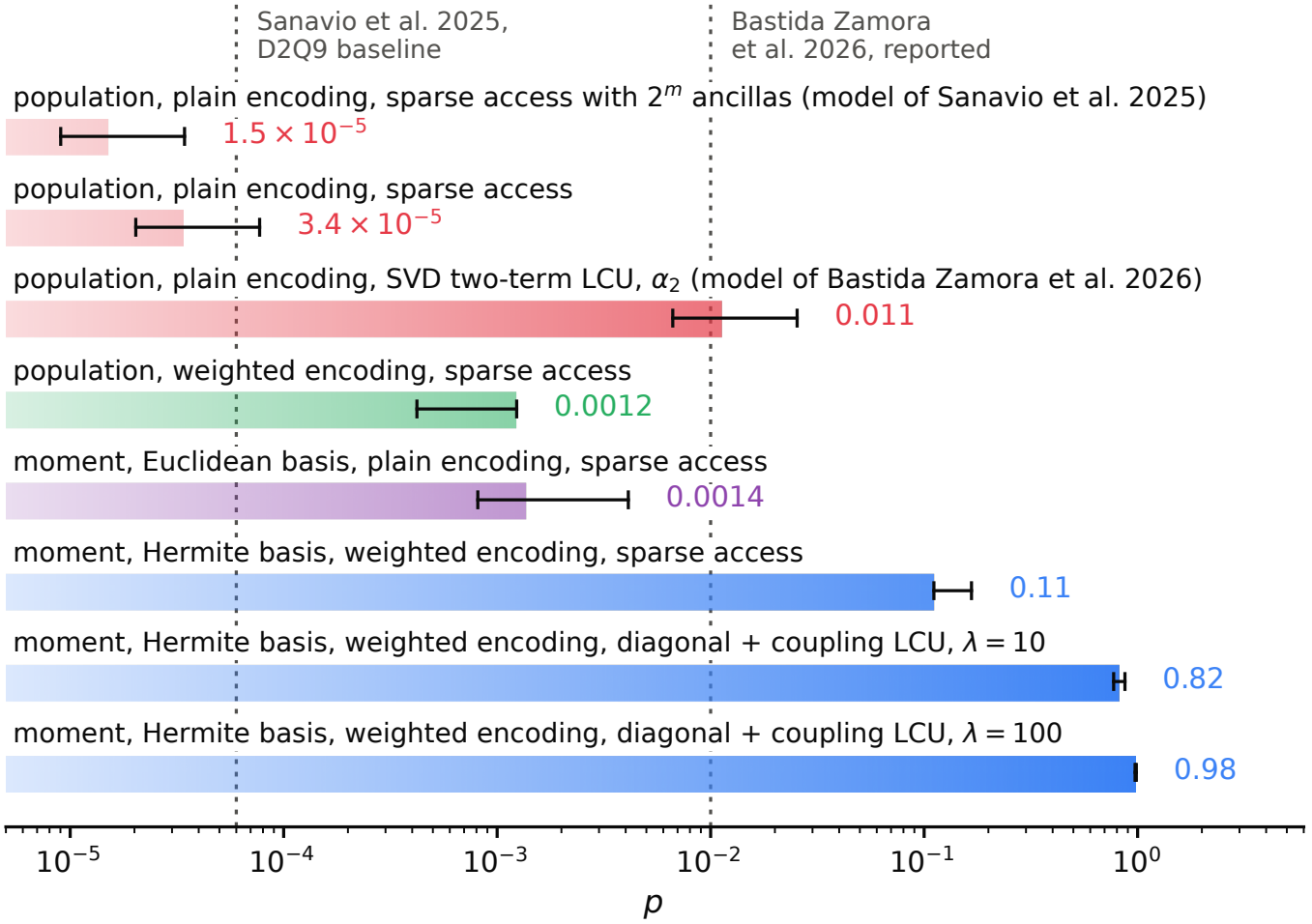
4) MRT streaming: $f_i(x + c_i, t + 1) = f'_i$



The LB-MTR algorithm:



Single-step success prob



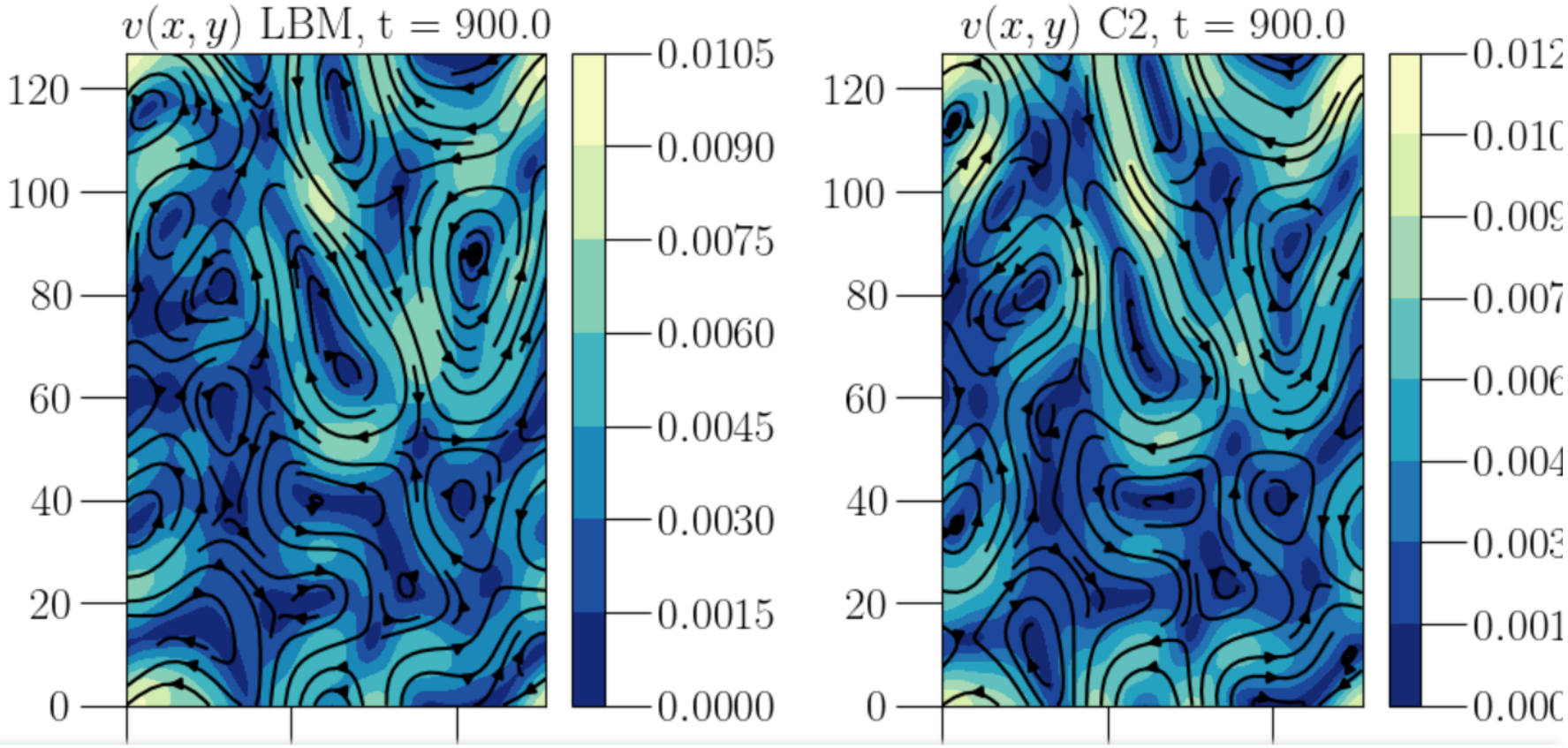
CLB2: moderate Reynolds

Freely decaying turbulence (f=0)

$$||e^{Ct}|| < 1$$

128x128: Re ~ 100~ 2000* Ma/2

(8*90*128^4 = 180 GB)



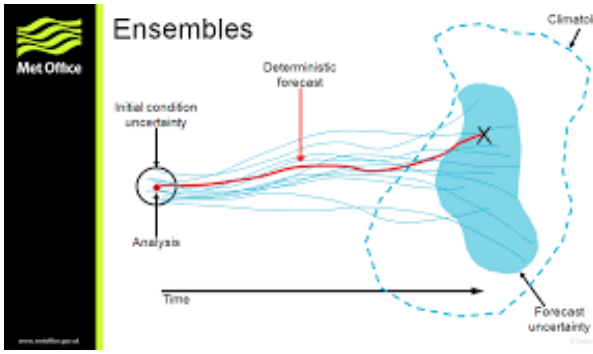
Functional Approach

(Liouville-Kolmogorov-KvN)

Ensemble quantum sim's

Classical Stochastic dynamics (Ensemble Simulations)

$$\dot{u} = f[u] + \xi$$



Classical Statistical Dynamics (Functional Fokker-Planck)

$$\partial_t P[u,t] + \partial_u \{ f(u)P + D\partial_u P \} = 0$$

Schroedingerization: $P(u,t) = g(u)\phi(u,t)$

$$f(u)g + 2D\partial_u g = 0$$

$$g = Z^{-1} e^{-W/2D}$$

$$\partial_u W = f(u)$$

$$\boxed{\partial_t \psi = (\Delta_u - V[g])\,\psi}$$

$$V[u] = W_{uu} - W_u^2$$

G-body Schroedinger in imaginary time: G = # grid points

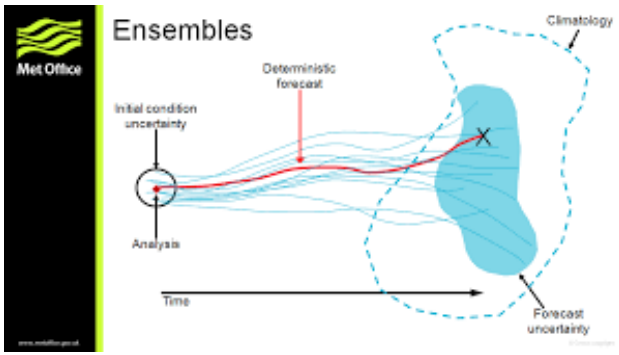
Ensemble quantum sim's

Koopman-von Neumann:

$$\Psi(u,t) = P^{1/2} e^{i\theta}$$

$$\partial_t \theta + f(u)\partial_u \theta = -W/\hbar$$

$$H_{KvN} = \frac{i\hbar}{2}(f\nabla + \nabla f) + W$$



$$i\hbar\partial_t \Psi = i\hbar L_{KvN}\Psi \equiv H_{KvN}\Psi$$

Looks quantum but it is NOT: no interference, no tunnel effect, no entanglement
Dirac-like looking (first order); import Hilbert space techniques.
It is a classical G-body problem, G ~ 10^9

Monster ODE problem

$$\dot{p} = \mathscr{L} p$$

On a grid with G grid points, this lives in G dimensions, G ~ 10^9

With n discretization levels/dimension: Monster Problem

$$M \sim n^G$$

Closures / Tensor Networks Reduction

$$p_G(x_1 \ldots x_G) \rightarrow p_g(x_1, x_2, x_g; g \ll G)$$

$$m \sim n^g \ll M$$

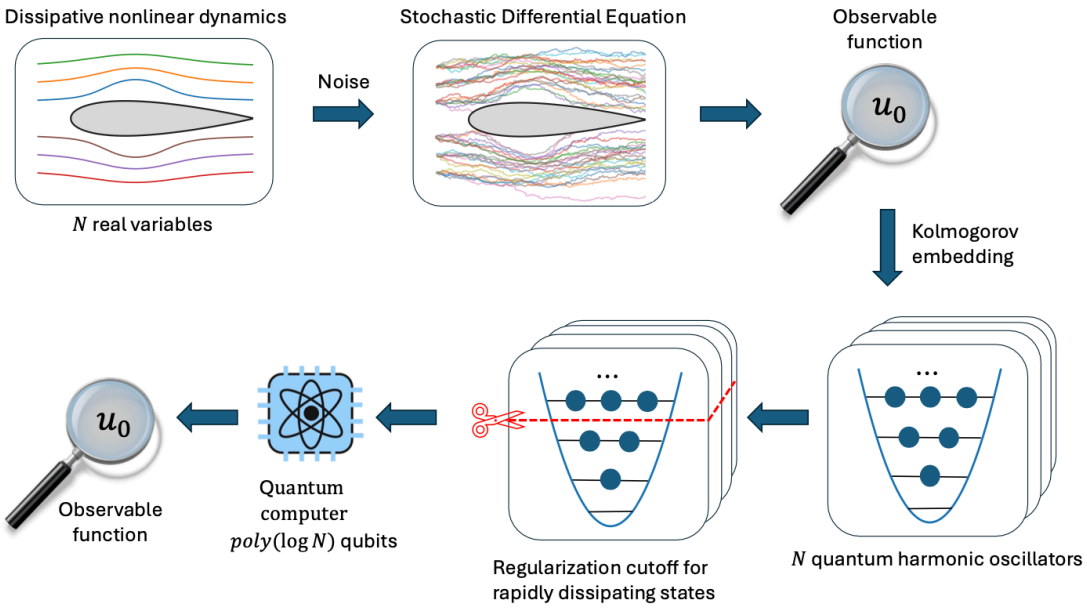
Now doable (?)

Kolmogorov

$$\dot{x}_i = f_i(x) + \xi_i \qquad \partial_t \phi(x,t) = \sum_{i=1}^{G} [f_i(x)\partial_i \phi + D\partial_{ii}\phi]$$

$$\phi(x,t) = \mathcal{F}[x_1(t)\ldots x_G(t)]$$

Qalgo : poly(log G)



S. Bravyi et al (IBM)

~100 pages, 200+ eqs

Figure 1: *Quantum simulation workflow.* A quadratic ODE for a state vector $X(t) \in \mathbb{R}^N$ is converted into a stochastic differential equation (SDE) by adding random noise. Rather than tracking the full state vector, the quantum algorithm estimates the noise-averaged value of a scalar observable $u_0 : \mathbb{R}^N \to \mathbb{R}$ evaluated at the solution $X(t)$ for a given target initial condition $x \in \mathbb{R}^N$. As a function of t and x, this expected value satisfies the Kolmogorov equation. Embedding this equation into the Hilbert space of N quantum harmonic oscillators and applying a regularization procedure then enables efficient quantum simulation.

Summary

Classical Storage: G

N-body SE:	G^{N_A}	**Kohn's point**
Carleman-K:	G^{K}	$K(Re) \to \infty ?$ **Regularization ?**
Liouville-KvN:	$n^{G} \sim \epsilon^{-G}$	**Clever closures?**

Turbulence is a very long shot: maybe not impossible

Credit and Thanks

W. Itani (AUB), *K. Sreenivasan (NYU)*

C. Sanavio (ENIquantica)

A. Zecchi, S. Perotto (Polimi) *A. Roggero (Uni TN)*

M. Moeller, M. Lacatus (Delft)

H. Yao (Chalmers) *L. Cappelli (IIT)*

THANKS!

