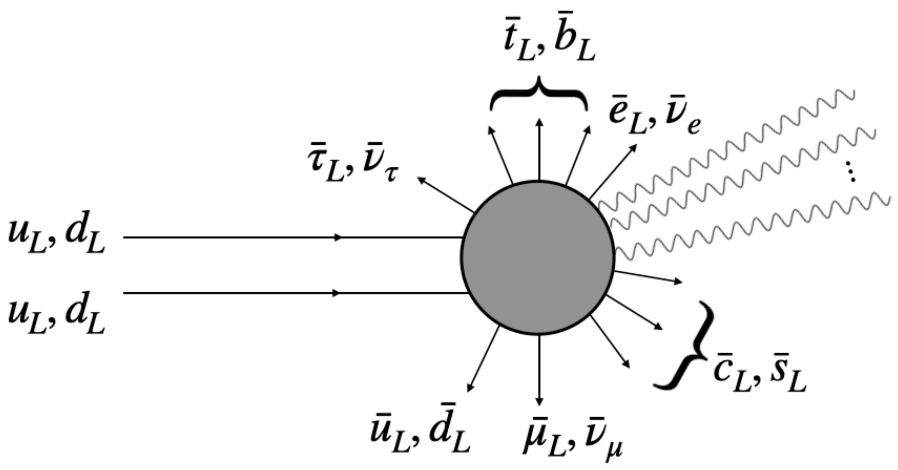

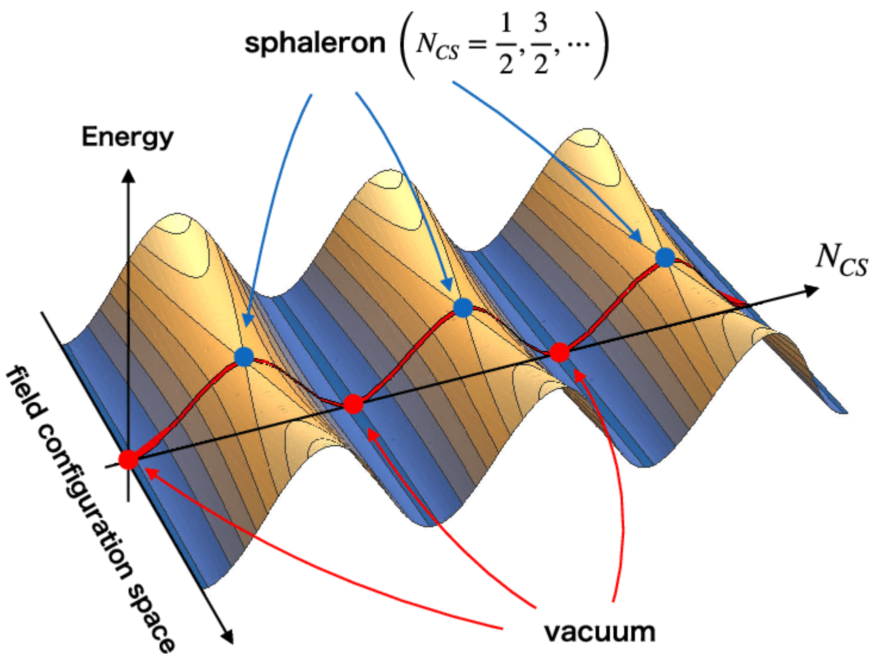

Sphalerons and their phenomenology

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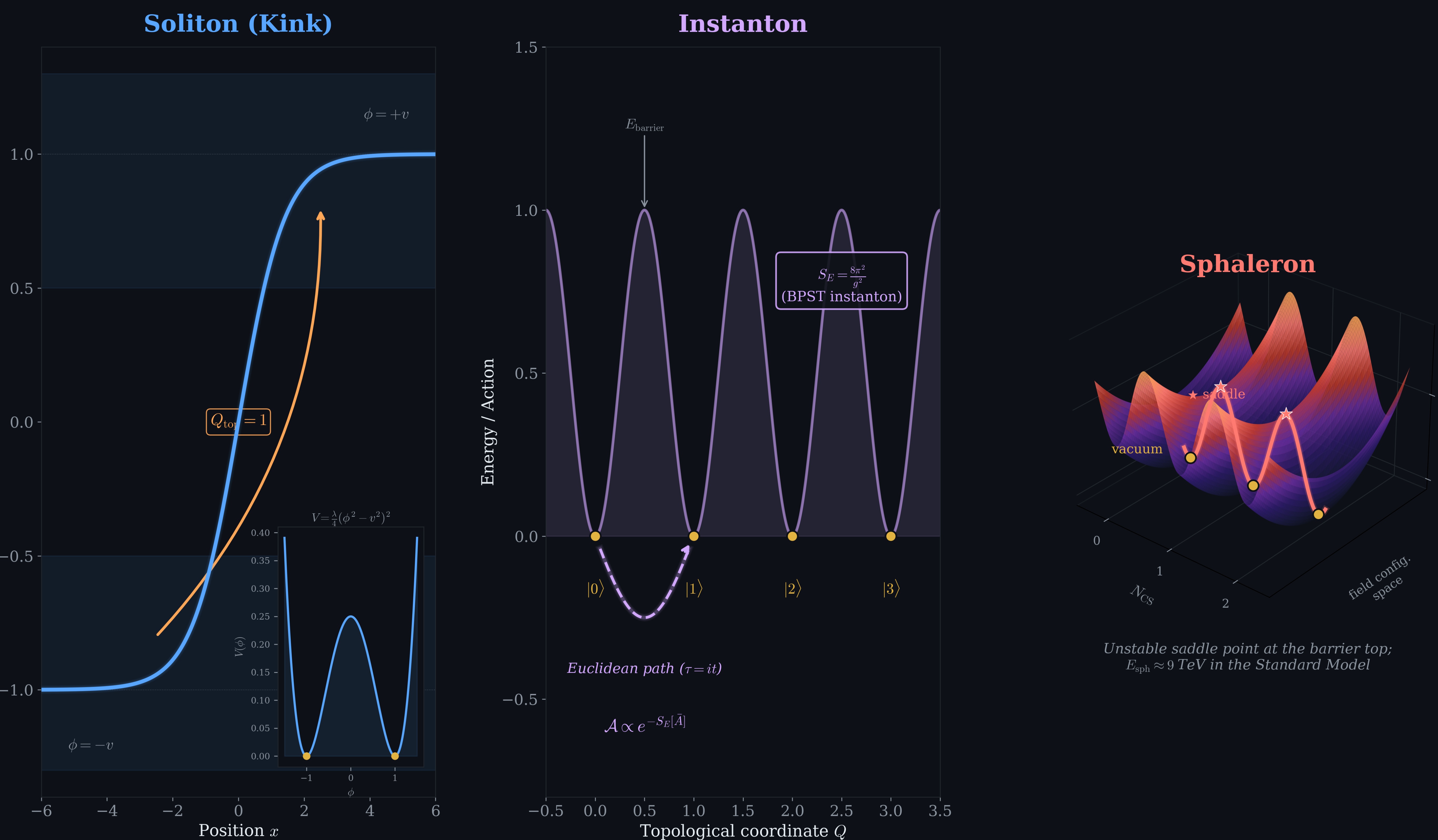
Topological Objects in Gauge Field Theory

Three ways the vacuum topology manifests as physical configurations



Stable, localised, real-time solution interpolating between degenerate vacua

Euclidean saddle point encoding tunneling between topological vacua

Soliton $\longleftrightarrow$ stable topological object | Instanton $\longleftrightarrow$ tunneling amplitude | Sphaleron $\longleftrightarrow$ barrier saddle point $\Rightarrow$ $B+L$ violation

What are solitons, instantons and sphalerons?

Type	Soliton	Instanton	Sphaleron
What it is	A localized, finite-energy classical solution of nonlinear field equations	A finite-action solution of the **Euclidean** field equations	A static, finite-energy **saddle-point** solution of the energy functional
Time signature	Usually real time / static in Minkowski space	Euclidean time	Real time / static in Minkowski space
Stability	typically stable or long-lived, often because of topology	Not a real-time stable object	Unstable, with at least one negative mode
Physical meaning	Particle-like nonperturbative object	Describes tunneling between vacua	Sits at the top of the barrier between vacua
Typical example	Kink, vortex, monopole, skyrmion, etc	BPST instanton in Yang-Mills	Electroweak sphaleron

- **Solitons** are nonperturbative classical configurations that behave like localized objects in real time.
- **Instantons** are Euclidean saddle points that encode tunneling amplitudes.
- **Sphalerons** are static unstable saddles of the energy functional, representing the top of a barrier between topologically distinct vacua. It means "ready to fall".

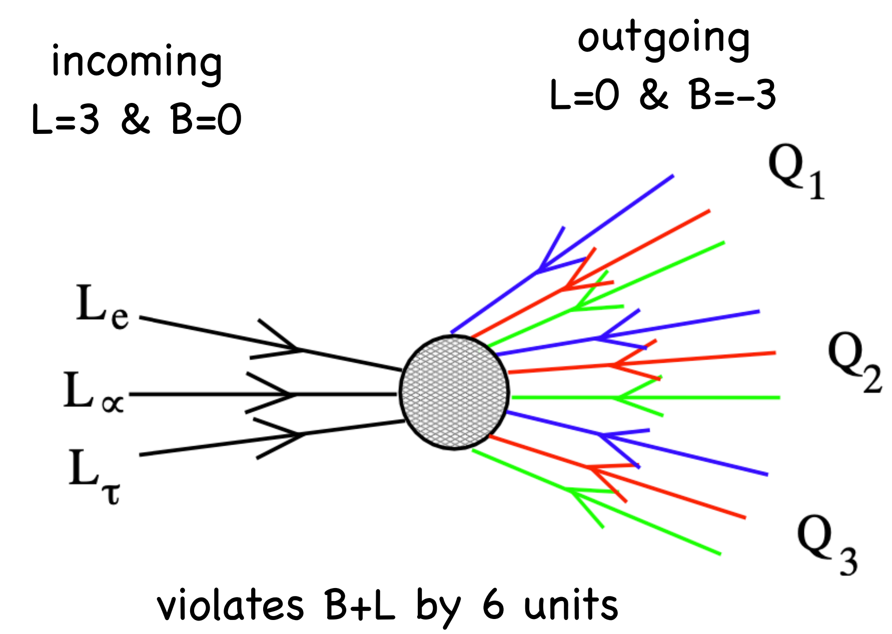
Why care about sphalerons?

- They are the lowest-energy saddle points separating inequivalent electroweak vacua.
  -> They show the elw. vacuum is not unique, but has topological sectors labelled by N_{CS}

- They make baryon plus lepton number violation possible in the Standard Model:
  $\Delta(B+L) \neq 0$ and $\Delta(B-L) = 0$
  and connect topology to real quantum numbers

- They are a genuinely nonperturbative Standard Model effect, not just "another rare process" and set a scale at 10 TeV in the SM

- Phenomenologically, they matter in at least four ways:
  - ➡ baryogenesis and washout in the early Universe
  - ➡ sensitive to the global structure of the Higgs potential
  - ➡ possible ultra-rare high-energy scattering at colliders
  - ➡ ultra-high-energy cosmic-ray or neutrino interactions.

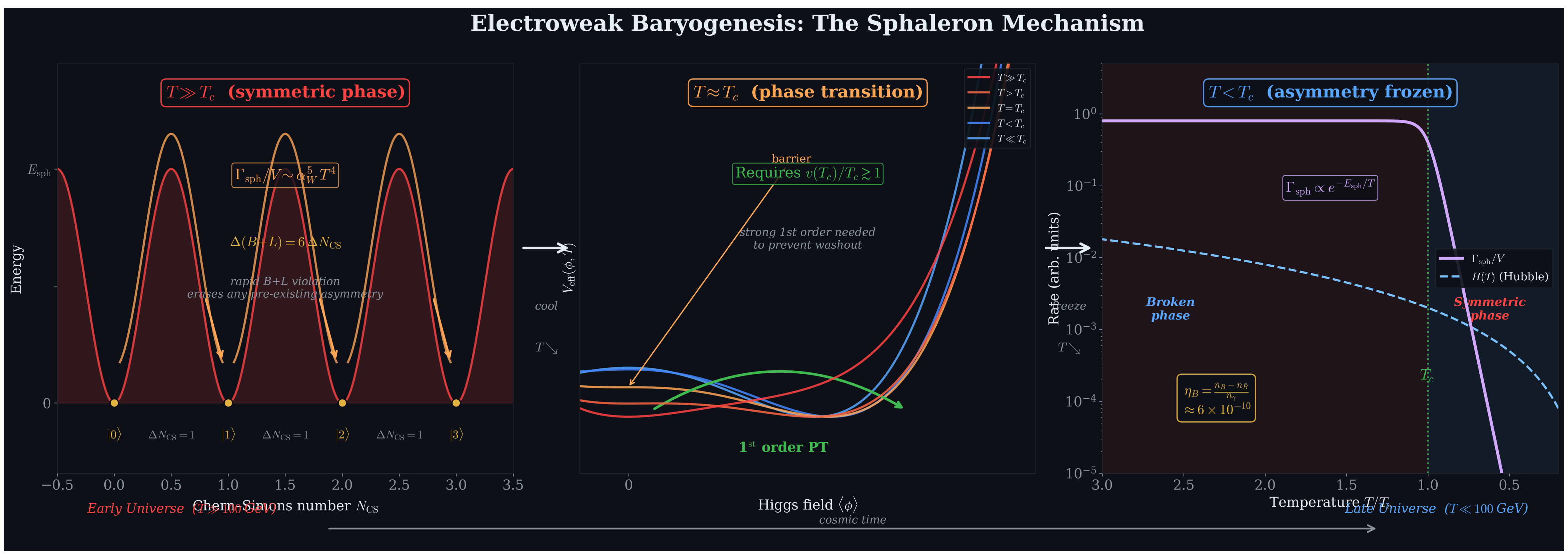


- They are a clean entry point to topology, anomalies, and semiclassical methods in gauge theory.

Sphalerons are SM ingredient to Baryogenesis

Sakharov conditions: (for dynamical generation of Baryon asymmetry)

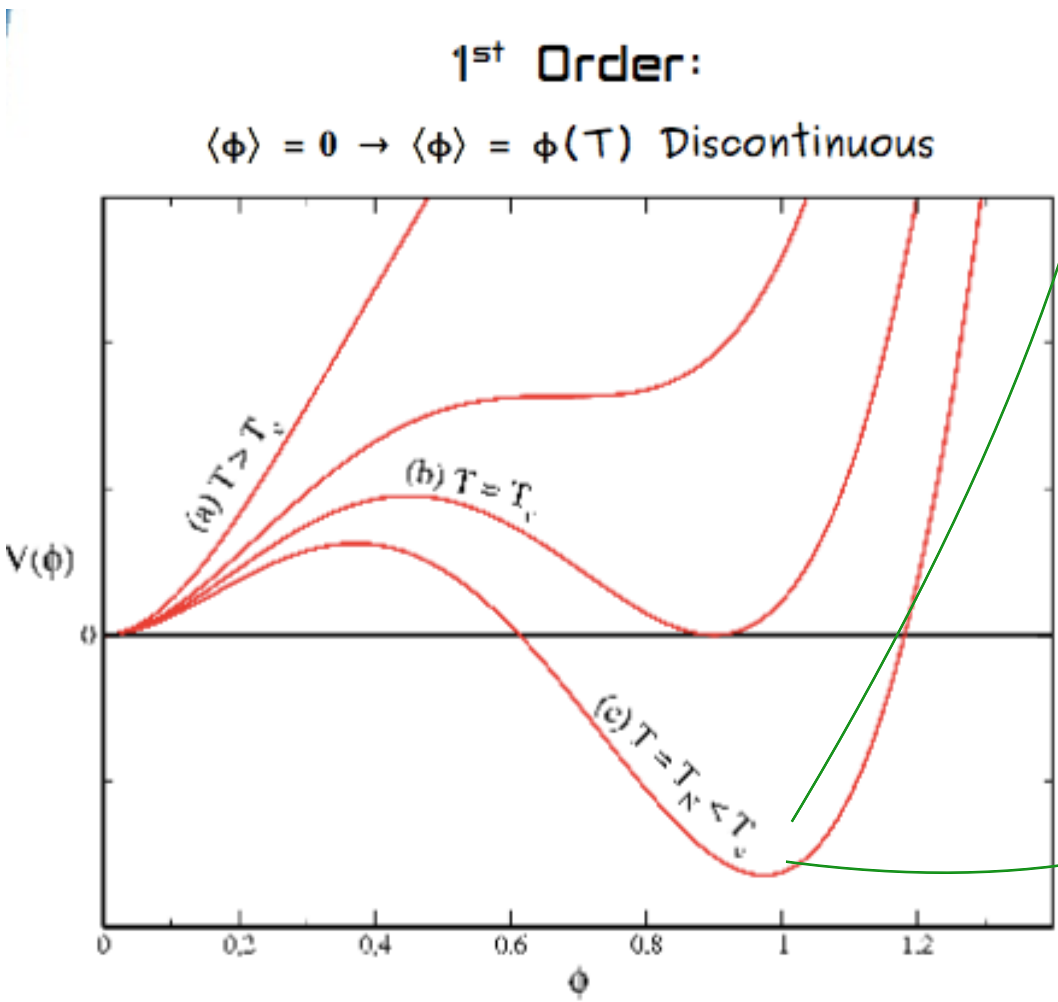
- B violation ✓ **Sphaleron**
- C & CP violation ⚡ not enough
- Departure from thermal equilibrium ⚡ not enough

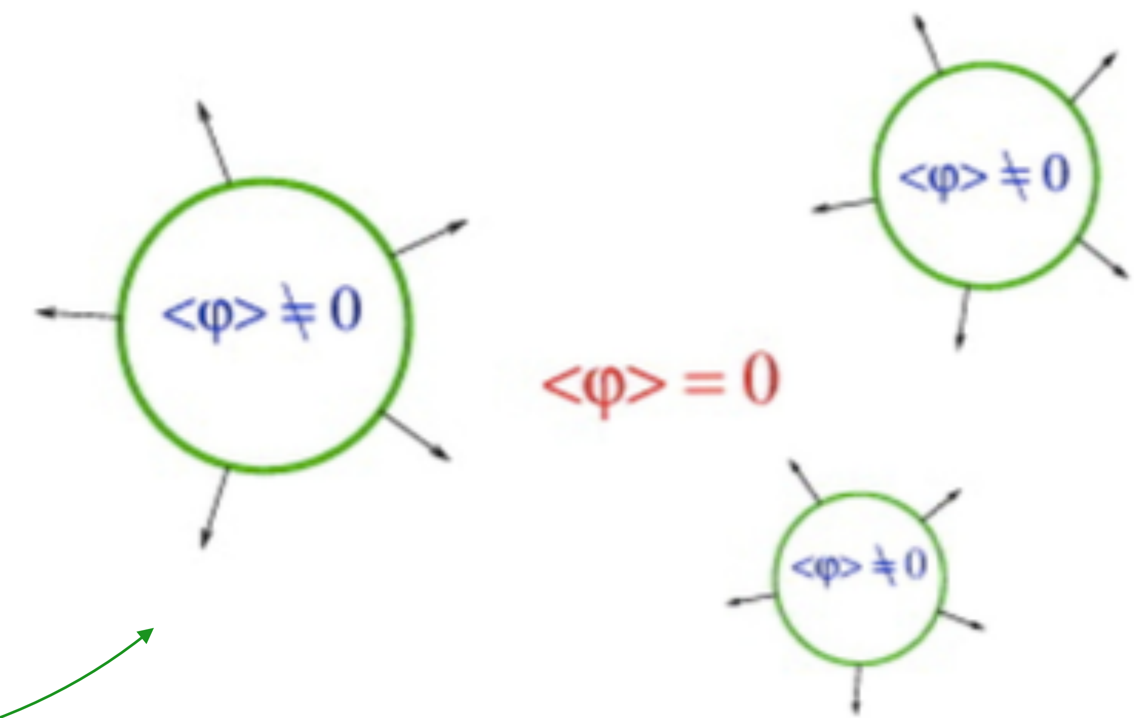


Baryogenesis in pictures

Nucleation of true vacuum bubbles

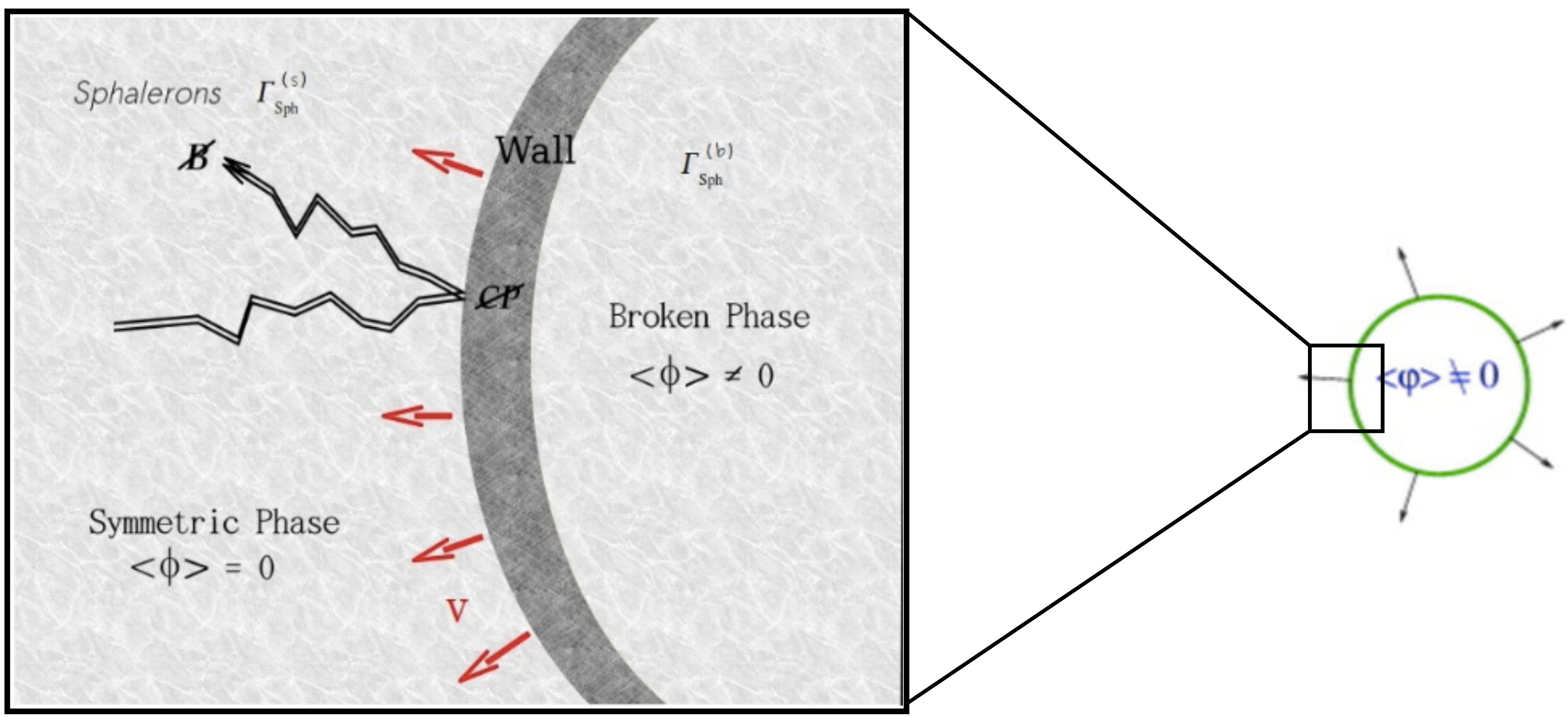
(in false vacuum sea)





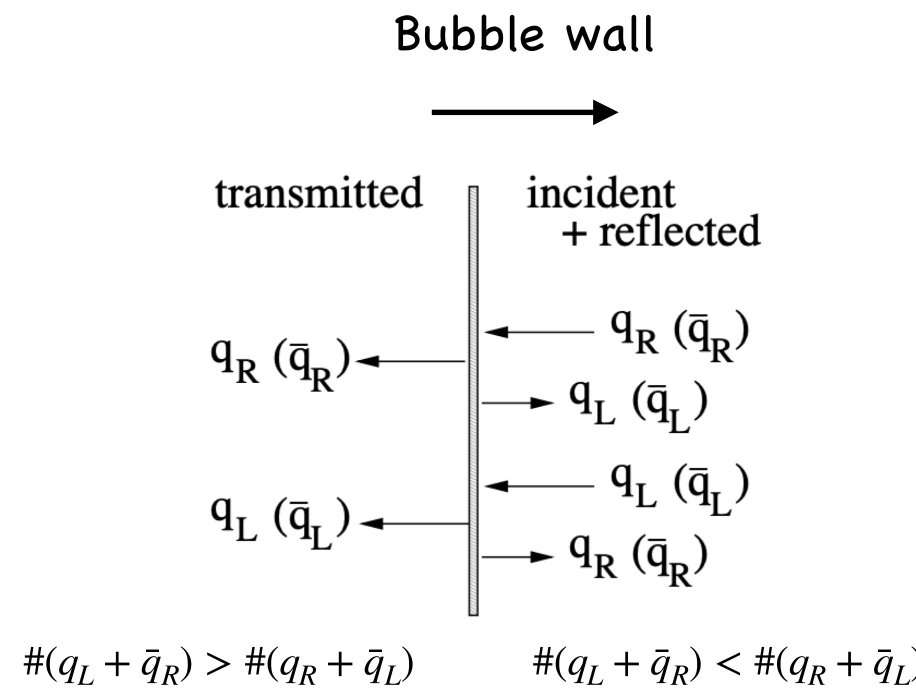
Sudden change in Higgs vev

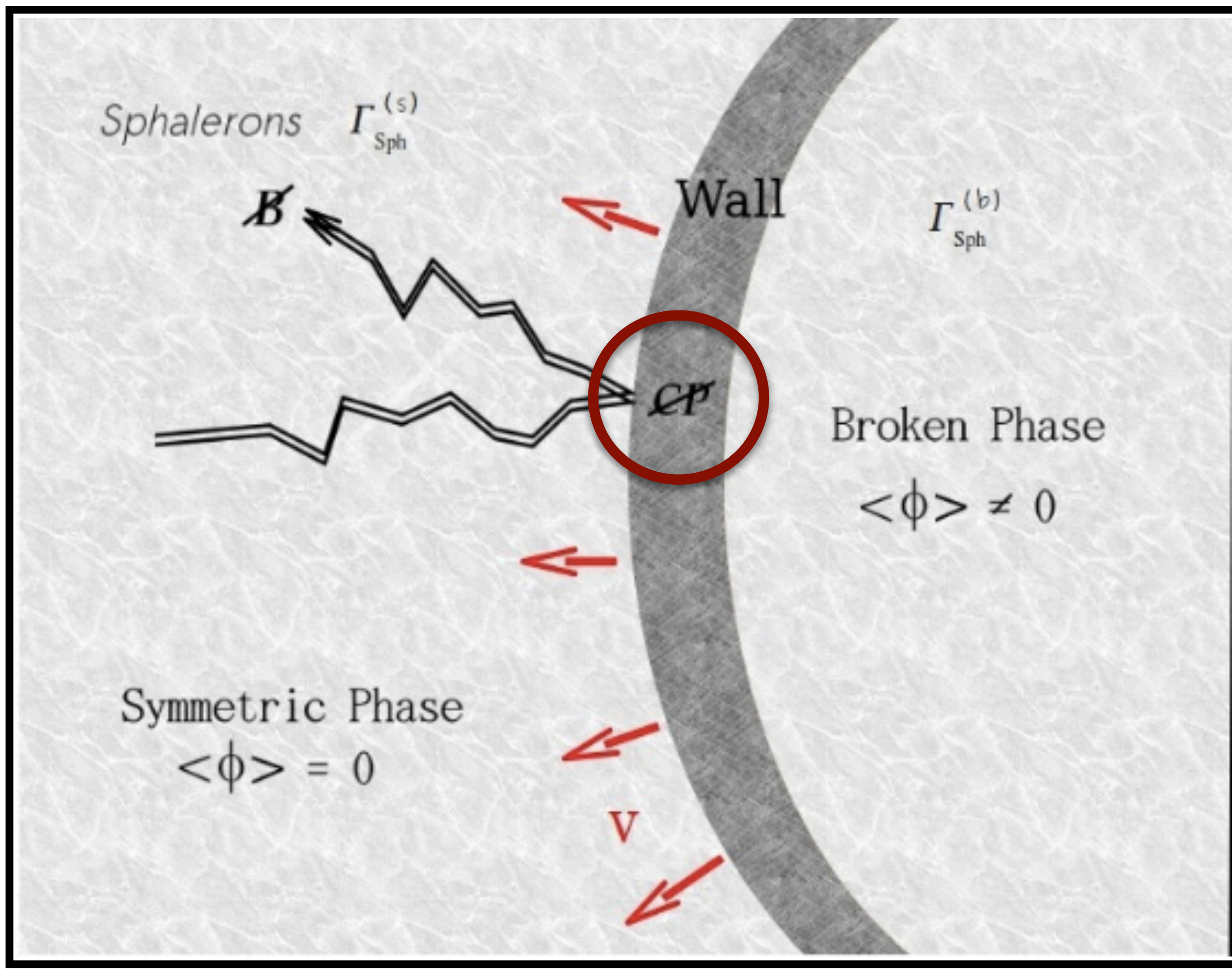
Baryogenesis



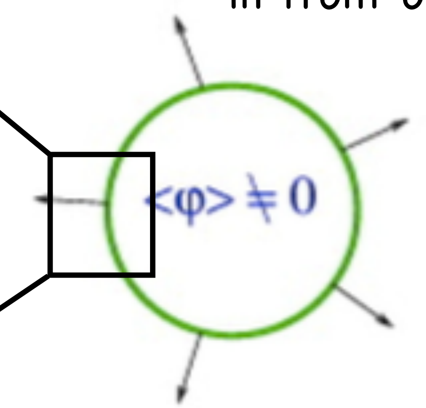
Baryogenesis

- CP Violation + Transport (diffusion)





more left-handed antiparticles (than particles) in front of wall

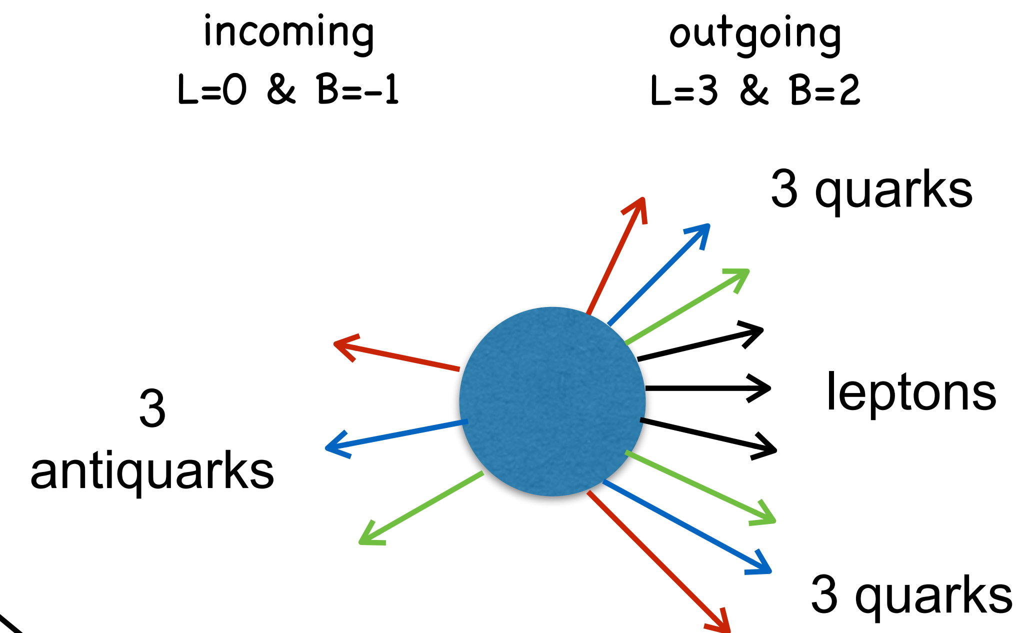


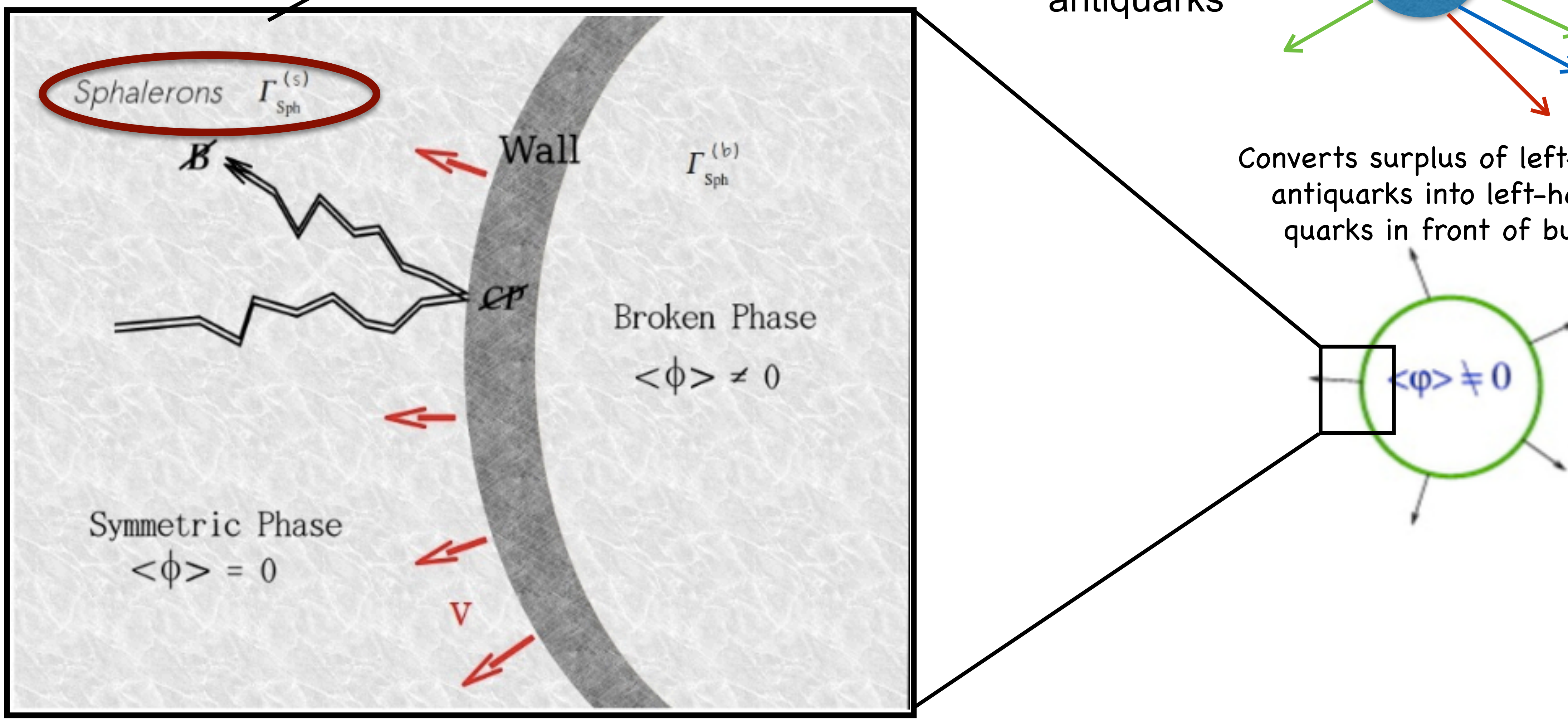
Particles scattering off bubble wall with C and CP violation generate net chiral asymmetry, i.e. more left-handed quarks than right-handed quarks

Baryogenesis

- Baryon number violation

$\Gamma^{S}_{Sph} \sim \alpha_W^5 T_N^4 \sim$ UNSUPPRESSED





Surplus of left-handed antiquarks fuel sphaleron processes, generating more baryons than antibaryons $\Delta(B+L) = 6$

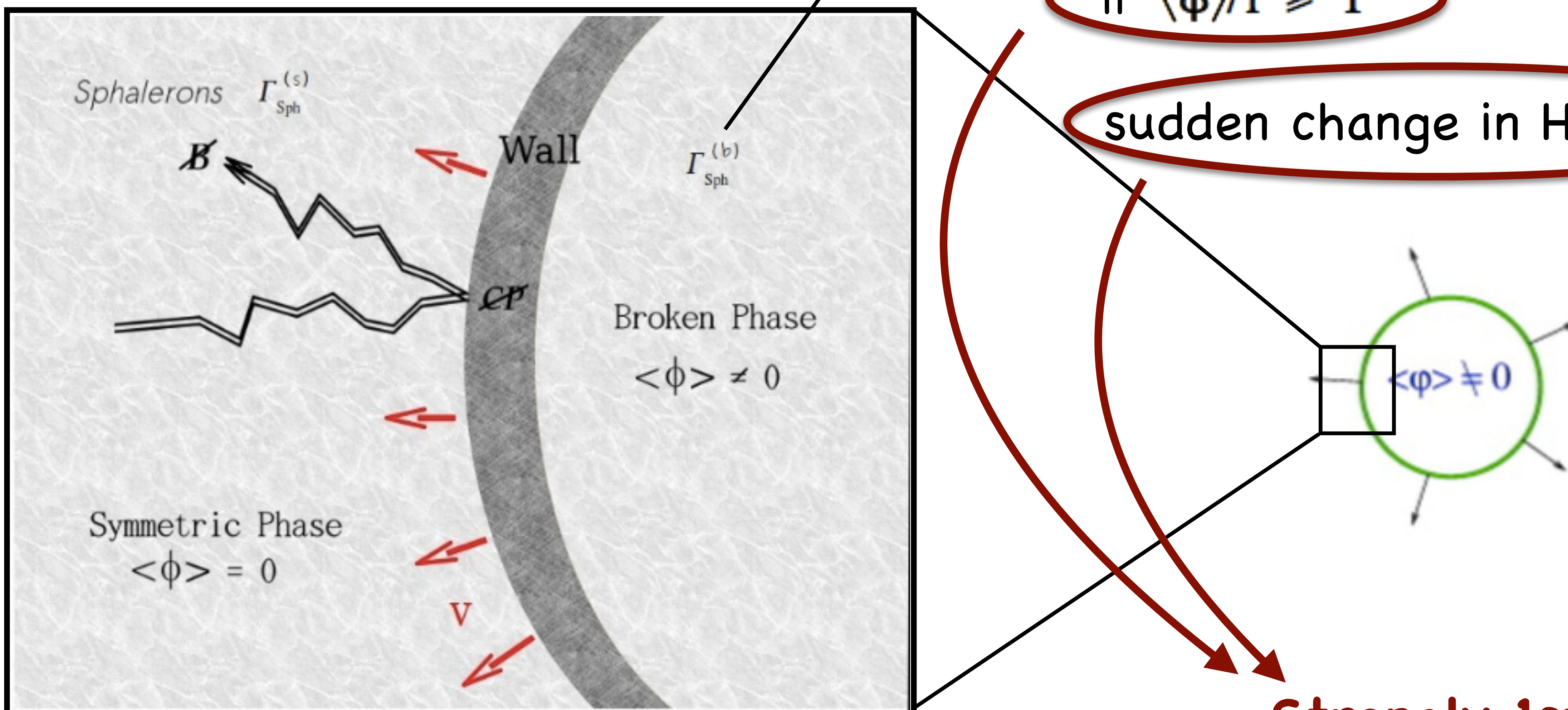
Baryogenesis

- Out of equilibrium

$\Gamma^{b}_{Sph} \sim Exp(-E_{Sph}/T_N) \sim$ SUPPRESSED (at nucleation temperature)

if $\langle\phi\rangle/T \geqslant 1$

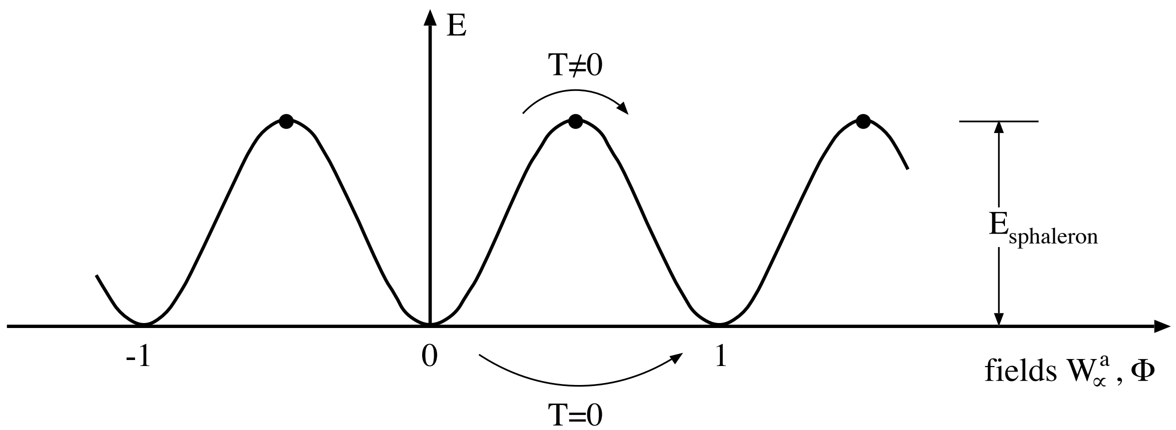
sudden change in Higgs vev

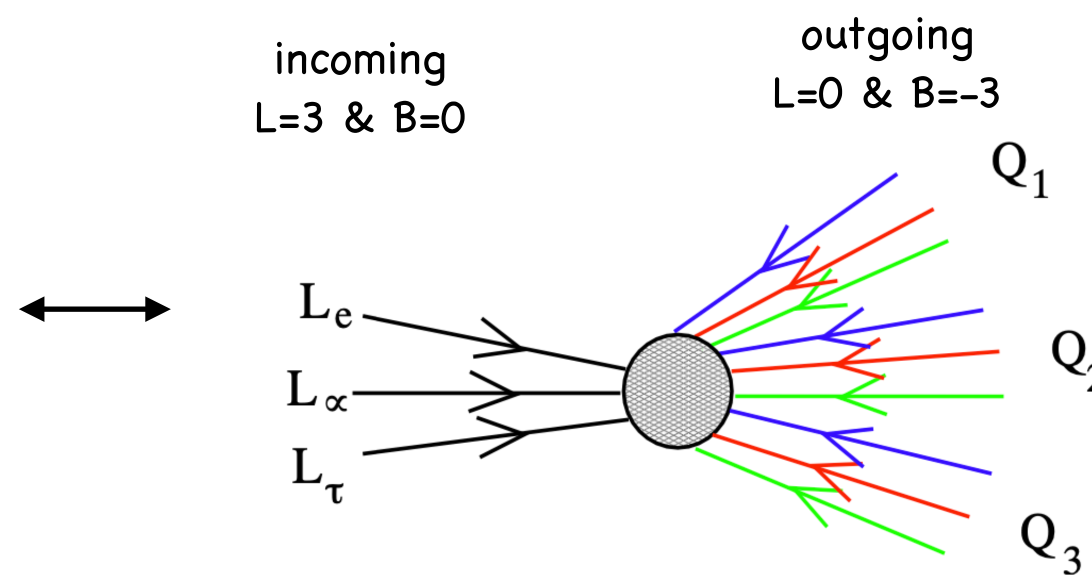


Strongly 1st order EW phase transition

Once the baryons are generated, they are quickly swept up into the interior of the expanding bubble (broken phase), where sphalerons transitions are exponentially suppressed -> asymmetry frozen

Find stationary solution for $E[field] = \int d^3x \left\{ \frac{1}{4g^2} W^a_{ij} W^a_{ij} + D_i H^\dagger D_i H + V(H) \right\} + \xi N_{\rm CS}$





Instanton rate at zero temperature (T=0):

$$\Gamma_{inst} \propto e^{-16\pi^2/g_2^2} \simeq 10^{-320}$$

['t Hooft '76]

At finite temperature T:

$$\Gamma_{sp} \sim \begin{cases} T^4\, e^{-4\pi\langle H\rangle/gT} & \langle H\rangle \neq 0 \\ \kappa\, \alpha_w^5\, T^4 & \langle H\rangle = 0 \end{cases}$$

[Klinkhammer, Manton '84]

[Arnold, McLerran '87]

[Bodeker, Moore, Rummukainen '99]

In broken phase called sphalerons, transition exponentially suppressed

In unbroken phase unsuppressed B+L violation transitions within SM

The word sphaleron refer to a **static**, **finite-energy solution** of the **classical gauge-Higgs field equations**, equivalently a saddle point of the corresponding energy functional

In EW theory, sphaleron sits on top of the energy barrier that separates neighbouring topological vacuum sectors, and it has exactly one unstable fluctuation mode.

Importance comes from the fact that those topological sectors are tied, through chiral anomaly, to baryon and lepton number violation: $\Delta B = \Delta L = n_g \Delta N_{CS}$
where $n_g = 3$, and N_{CS} is the Chern-Simons number of the $SU(2)_L$ gauge fields

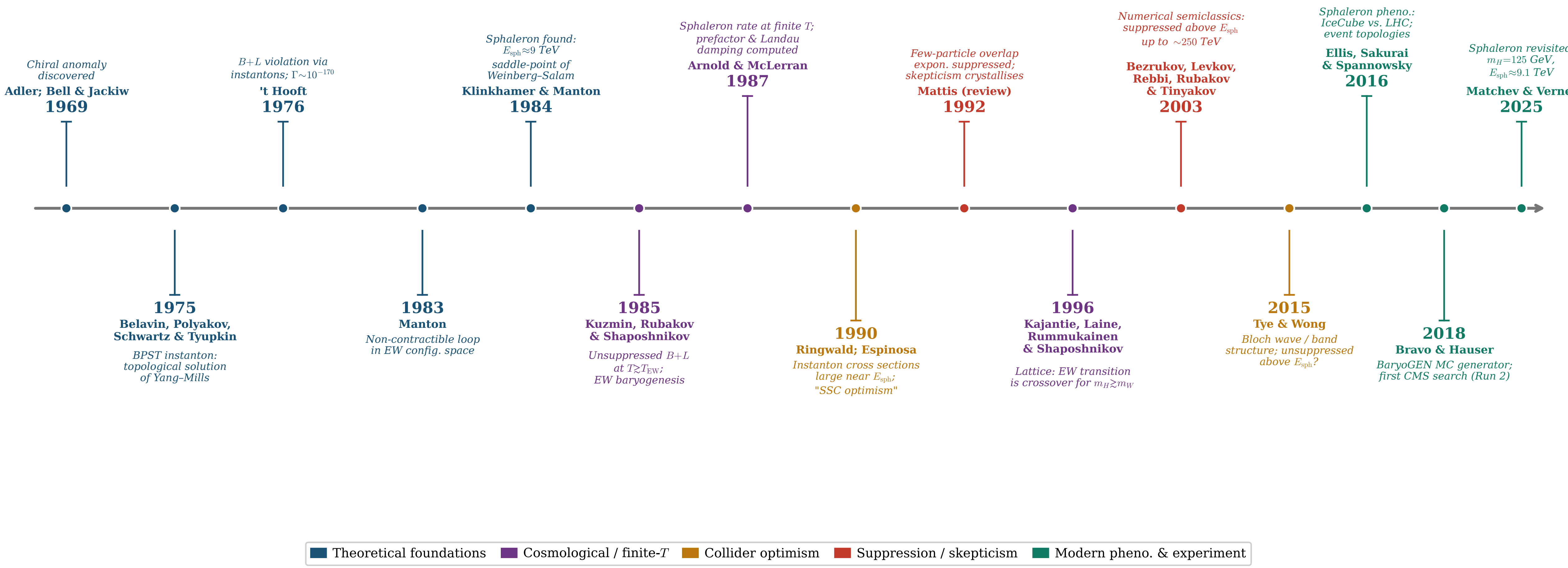
Three logically distinct probelms:

1. **The static field theory problem**: does the elw theory posses a saddle-point configuration between neighbouring vacua, and what is its energy? **(settled)**

2. **The thermal problem**: at finite temperature, how fast do thermal fluctuations hop over the barrier? **(believed to be under good theoretical control)**

3. **The few-particle scattering problem**: can an initial state with only a few hard quanta, such as a collider parton-parton state, efficiently excite the topological transition channel? **(controversial)**

Historical Development of the Sphaleron and $B+L$ Violation



- Anomaly: B+L need not be conserved quantum mechanically
- Topology: the electroweak vacuum sectors are separated by a sphaleron barrier
- Cosmology: thermal crossing of that barrier can wash out or reshape the baryon asymmetry
- Collider/astro: production rate unclear, need improved theoretical methods

The sphaleron energy and size

Full elw action $$S_{\rm EW} = \int \mathrm{d}^4x \; (\mathcal{L}_{\rm bos} + \mathcal{L}_{\rm ferm} + \mathcal{L}_{\rm Yuk})$$

the bosonic piece is $$\mathcal{L}_{\rm bos} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \Phi)^\dagger D^\mu \Phi - \lambda \left(\Phi^\dagger \Phi - \frac{v^2}{2} \right)^2$$

because fermion action is quadratic in the fermion fields one can integrate partition function to $\mathcal{B} \equiv (W, B, \Phi)$

$$Z = \int \mathcal{D}\mathcal{B} \; \exp[\mathrm{i} S_{\rm bos}[\mathcal{B}] + \mathrm{Tr} \log(\mathrm{i} \, \not{D}[\mathcal{B}] - y\Phi)]$$

In leading semiclassical order, i.e. the first term of the saddle point expansion in a small parameter g and resacling the bosonic fields to dimensionless variables gives

$$Z \sim \int \mathcal{D}\mathcal{B} \; \exp\left[\frac{\mathrm{i}}{g^2} S_{\rm bos}[\mathcal{B}] + \mathrm{Tr} \log(\mathrm{i} \, \not{D}[\mathcal{B}] - y\Phi) \right]$$

bosonic part multiplied by large value $1/g^2 \sim 1/\alpha_W$

fermions subleading one-loop corrections

Static energy functional

-> start from bosonic Lagrangian and go to the Hamiltonian

Impose Gauss law and choose temporal gauge (convenient), results in

$$E[W,B,\Phi] = \int \mathrm{d}^3x \left[\frac{1}{4}W^a_{ij}W^a_{ij} + \frac{1}{4}B_{ij}B_{ij} + (D_i\Phi)^\dagger D_i\Phi + \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2\right]$$

This is the functional whose saddle points we are looking for

In other words: The sphaleron problem is obtained by taking the full electroweak bosonic Hamiltonian, restricting to time-indepemendent, charge-neutral, purely magnetic configurations
-> then looking for saddle-point configuration that connects neighbouring topological sectors

Topology to the scene

For E to be finite, i.e. the theory to be well-defined, the fields need to be integrable

implying $W^a_{ij} \to 0 \quad B_{ij} \to 0 \quad D_i\Phi \to 0 \quad \Phi^\dagger\Phi \to \frac{v^2}{2}$ as $r \to \infty$

-> the Higgs field approaches a vacuum configuration of fixed norm and the gauge fields approach pure gauge

-> all directions at infinity are equivalent, as all fields have settled into vacuum configurations -> identify all asymptotic directions with one point

Spatial domain on which finite-energy vacuum lives can be treated as a three-sphere rather than a noncompact $\mathbb{R}^3$ ie. $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$

For pure gauge we can write locally

$$W_i = \frac{\mathrm{i}}{g} U(\mathbf{x})\, \partial_i U^{-1}(\mathbf{x}) \quad \text{with} \quad U(\mathbf{x}) \in \mathrm{SU}(2)$$

$$\Phi(\mathbf{x}) = \frac{v}{\sqrt{2}} U(\mathbf{x})\, \Phi_0$$

the vacuum configuration is encoded by some group-valued function $U(\mathbf{x})$

After compactifying space, the vacuum configuration of the $SU(2)$ gauge-Higgs system can be represented by a map

$$U(\mathbf{x}) : S^3 \to \mathrm{SU}(2) \simeq S^3$$

- the domain S^3 is compactified physical space
- the target $SU(2) \simeq S^3$ is the group manifold of $SU(2)$
- the map says that for each point in space, which gauge rotation takes the reference vacuum into the local vacuum orientation

Because the domain and target are topological three-spheres, the map can wrap the domain around the target an integer number of times. This integer is the **degree** or **winding number** of the map.

$$\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$$ third homotopy group

Winding number explicitly: $$n[U] = \frac{1}{24\pi^2} \int \mathrm{d}^3x \, \epsilon^{ijk} \, \mathrm{Tr}\Big[\Big(U^{-1}\partial_i U\Big)\Big(U^{-1}\partial_j U\Big)\Big(U^{-1}\partial_k U\Big)\Big]$$

two maps with different integer values of $n[U]$ cannot be continuously deformed into one another without leaving the vacuum family somewhere along the way

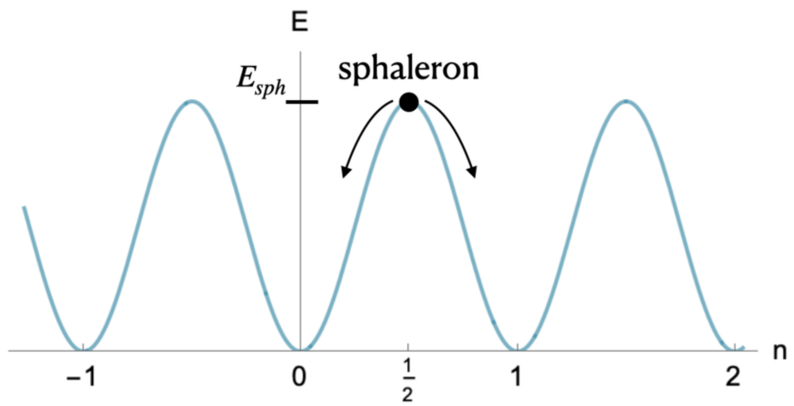
-> the sphaleron sits on the lowest barrier between neighboring winding sectors

Vacuum sectors and Chern-Simons number

The standard way to parametrise the vaccum family is through the Chern-Simons number

$$N_{\rm CS} = \frac{g^2}{32\pi^2}\int \mathrm{d}^3x\, \epsilon^{ijk}\left[W_i^a \partial_j W_k^a + \frac{g}{3}\epsilon^{abc} W_i^a W_j^b W_k^c\right]$$

- $N_{\rm CS}$ is not strictly gauge invariant. Under a **large** gauge transformation it shifts by an integer
- For vacuum configurations, $N_{\rm CS}$ can be chose to equal an integer n; neighbouring vacua differ by $\Delta N_{\rm CS} = 1$



Why a barrier must exist

- If neighbouring vacua belong to same connected component of finite-energy configuration space, one could continuously deform one into other while staying in the vacuum manifold. -> **Topology forbids this**
- Any path connecting sectors with different winding must leave the vacuum manifold somewhere. Once the Higgs field departs from its vacuum magnitude or the gauge fileds cease to be pure gauge the enrgy rises -> **barrier is unavoidable**

→ sphaleron not assumed, it is saddle point of this unavoidable barrier

Anomaly, spectral flow, and baryon number violation

- Classically, the SM conserves baryon number B and lepton number L
- Quantum mechanically, the corresponding currents are anomalous because the elw theory is chiral

$$\partial_\mu j^\mu_B = \partial_\mu j^\mu_L = n_g \frac{g^2}{32\pi^2} W^a_{\mu\nu}\tilde{W}^{a\mu\nu} \quad \text{with} \quad \tilde{W}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}W^a_{\rho\sigma}$$

Thus $\partial_\mu j^\mu_{B+L} = 2n_g \frac{g^2}{32\pi^2} W^a_{\mu\nu}\tilde{W}^{a\mu\nu}$ whereas $B-L$ is anomaly free in the SM

Integrating over space-time between inital and final vacuum

$$\int \mathrm{d}^4x \, \frac{g^2}{32\pi^2} W^a_{\mu\nu}\tilde{W}^{a\mu\nu} = \Delta N_{\mathrm{CS}} \longrightarrow \Delta B = \Delta L = n_g \Delta N_{\mathrm{CS}}$$

For one unit of topological charge $\Delta N_{\mathrm{CS}} = 1$ $\longrightarrow$

$$\Delta B = \Delta L = 3$$

$$\Delta(B+L) = 6$$

$$\Delta(B-L) = 0$$

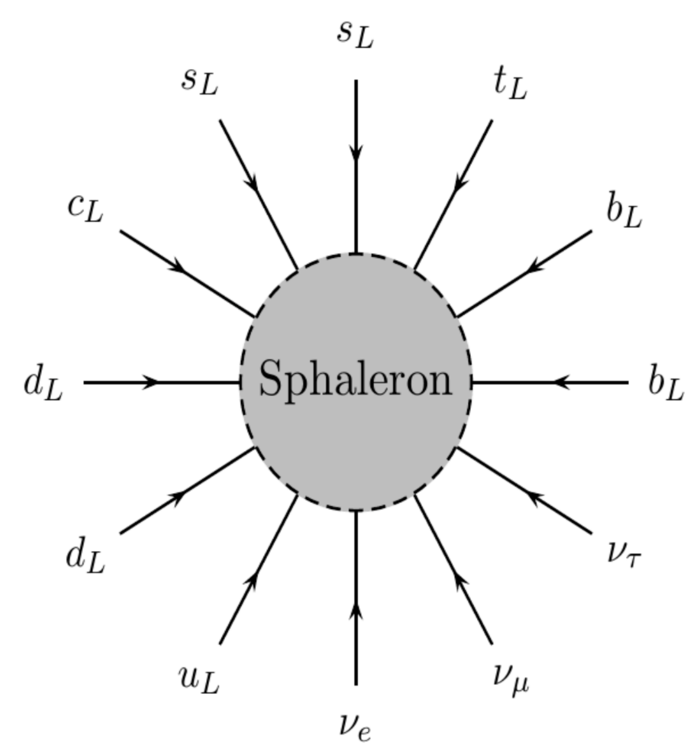
Spectral flow picture

- As the bosonic background moves from one vacuum sector to the next, the one-particle Dirac spectrum of the left-handed fermions change continuously

- Some levels cross zero. The occupation of the Dirac sea therefore changes, and that is the microscopic origin of the net change in baryon and lepton number
  - ➡ the **classical saddle itself is bosonic**
  - ➡ the change in fermion number arises because **femions live in that time-dependent bosonic background**

What final-state fermions are involved?

- For one unit of topological charge, each left-handed electroweak doublet contributes once -> involves 12 left-handed fermions in total



Examples of 't Hooft vertices

$$\mathcal{O}_1 \sim (u_{Lr}\, u_{Lg}\, d_{Lb})\, (c_{Lr}\, c_{Lg}\, s_{Lb})\, (t_{Lr}\, t_{Lg}\, b_{Lb})\, \left(e_L^-\, \mu_L^-\, \tau_L^-\right)$$

$$\mathcal{O}_2 \sim (u_{Lr}\, u_{Lg}\, d_{Lb})\, (c_{Lr}\, s_{Lg}\, s_{Lb})\, (t_{Lr}\, b_{Lg}\, b_{Lb})\, \left(e_L^-\, \nu_{\mu L}\, \nu_{\tau L}\right)$$

Full elw instanton or sphaleron vertex is a sum over many such 12-fermion structures, related by choices of weak-isospin components and colour assignments

How to calculate the sphaleron energy

- **Simplify assuming g'=0**

Only SU(2) gauge field and Higgs doublet remain and solutoin becomes spherically symmetric

The case $g' \neq 0$ can be treated afterwards as a perturbation that slightly deforms the sphaleron and gives it a magnetic dipole moment

It isolates the core topological structure, which lives in the non-Abelian $SU(2)_L$ sector

- **Adopt the Klinkhammer-Manton ansatz**

For spherically symmetric $SU(2)$-Higgs theory, one can write the fields in terms of two radial profile functions $f(r)$ and $h(r)$

$$\Phi(\mathbf{x}) = \frac{v}{\sqrt{2}} h(r)\, U(\hat{\mathbf{x}}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$U(\hat{\mathbf{x}})$ standard hedgehog map on sphere at infinity

$$W_i(\mathbf{x}) = -\frac{2\mathrm{i}}{g} f(r)\, (\partial_i U) U^{-1}$$

where $W_i = W_i^a \tau^a / 2$

→ The full field theory problem (infinity many degrees of freedom) collapses to variational problem for two functions of one variable

- **Introduce dimensionless radius**

It is natural to measure distances in units of the gauge-boson mass scale

define $\xi = gvr$

Since $m_W = \frac{gv}{2}$ the sphaleron core size is of order $r_{\rm core} \sim (gv)^{-1} \sim m_W^{-1}$

- **Reduced energy functional**

Primes denote $\mathrm{d}/\mathrm{d}\xi$

$$E_{\rm sph} = \frac{4\pi v}{g}\int_0^\infty \mathrm{d}\xi \left[4f'^2 + \frac{8}{\xi^2}f^2(1-f)^2 + \frac{1}{2}\xi^2 h'^2 + h^2(1-f)^2 + \frac{\lambda}{4g^2}\xi^2(h^2-1)^2\right]$$

- $4f'^2$ comes from the radial gradient of the gauge profile.
- $\frac{8}{\xi^2}f^2(1-f)^2$ is the gauge-field self-interaction energy.
- $\frac{1}{2}\xi^2 h'^2$ is the Higgs gradient energy.
- $h^2(1-f)^2$ couples the gauge and Higgs profiles.
- $\frac{\lambda}{4g^2}\xi^2(h^2-1)^2$ is the Higgs potential energy.

- **Euler-Lagrange equations**

  Varying the reduced energy functional with respect to f and h gives couple DE

$$f'' = \frac{2}{\xi^2} f(1-f)(1-2f) - \frac{1}{4} h^2 (1-f)$$

$$(\xi^2 h')' = 2h(1-f)^2 + \frac{\lambda}{g^2} \xi^2 h(h^2-1)$$

- **Boundary conditions**

  asymptotics $f(0) = 0$ $f(\infty) = 1$ and $h(0) = 0$ $h(\infty) = 1$

  near the origin, regular solutions behave $f(\xi) \sim a\xi^2$ $h(\xi) \sim b\xi$ $\xi \to 0$

- **why the solution is a saddle, not a minimum**

  If you blindly minimise the energy functional over all admissible fields, you just get the vacuum. The sphaleron exists because one restricts attention to paths in configuration space that **connect different topological sectors**. Along such a path, the maximum energy cannot be avoided.

→ The sphaleron is the minimum over paths of the path maximum

$$E_{\text{sph}} = \min_{\gamma \in \Gamma} M[\gamma] \qquad M[\gamma] = \max_s E[\gamma(s)] \qquad \gamma(s)$$

path connecting neighbouring vacua

- **the numerical solution**

  There is no closed-form analytic solution for the differential equations. One solves them numerically:

  ➡ **Shooting method:** choose the small-ξ coefficients a and b, integrate outward, and tune them until the large-ξ boundary conditions are satisfied

  ➡ **Relaxation:** discretize the interval in ξ and iteratively solve the boundary value problem

  ➡ **Constrained minimisation or minimax algorithms**

  ➡ **ML approach** [Piscopio, MS, Waite '18]

- **Physical scale of the energy**

  The result is commonly written as $$E_{\mathrm{sph}} = \frac{4\pi v}{g}\, B\left(\frac{\lambda}{g^2}\right)$$

  where B is an order-one function determined numerically.

  For Standard Model couplings, one finds $E_{\mathrm{sph}} \sim 9\ \mathrm{TeV}$

Thermal Sphalerons and electroweak Baryogenesis

- **From tunnelling to thermal activation**

At zero temperature, a transition between neighbouring vacua is a tunneling problem. The semiclassical suppression is then controlled by the Euclidean action

$$\Gamma_{T=0} \sim \mathrm{e}^{-S_E^{\mathrm{inst}}}$$

At non-zero temperature the physics changes because the system is described by a thermal ensemble rather than a pure ground state.
A field configuration of energy E appears with Boltzmann weight

$$P[E] \propto \mathrm{e}^{-E/T}$$

If the system explores the configuration space thermally, a transition from one vacuum sector to the next is dominated by the configurations near the top of the barrier (the sphaleron configuration) -> thermal analogue of Arrhenius law form stat. mech.

tot. number of topological transitions per unit time →

$$\frac{\Gamma_{\mathrm{sph}}}{V} \sim A(T) \exp\left(-\frac{F_{\mathrm{barrier}}(T)}{T}\right)$$

← free-energy difference between the vacuum and the barrier top

↑ spatial volume for normalisation

↑ prefactor, incl. fluctuation determinants, zero-mode nomalisation factor etc

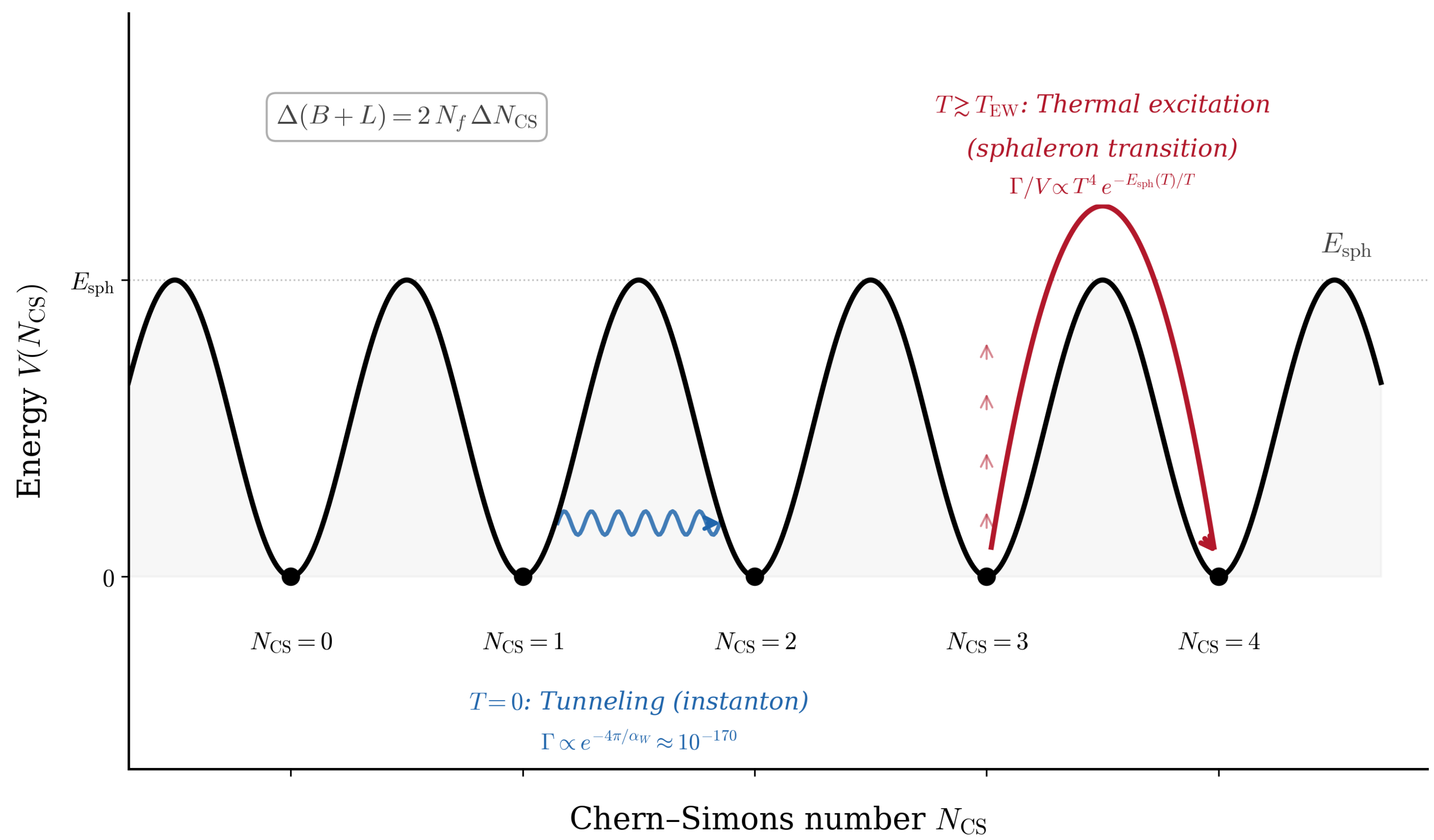
in weakly coupled elw theory one often writes

$$F_{\mathrm{barrier}}(T) \approx E_{\mathrm{sph}}(T)$$

- **Why the sphaleron energy depends on temperature**

At finite temp. the Higgs field no longer sits in the same vacuum as at T=0 - finite temp. eff. potential.

The broken-phase expectation value decreases with temperature and eventually vanishes at electroweak symmetry restoration.



Because the sphaleron energy is set by the elw. scale

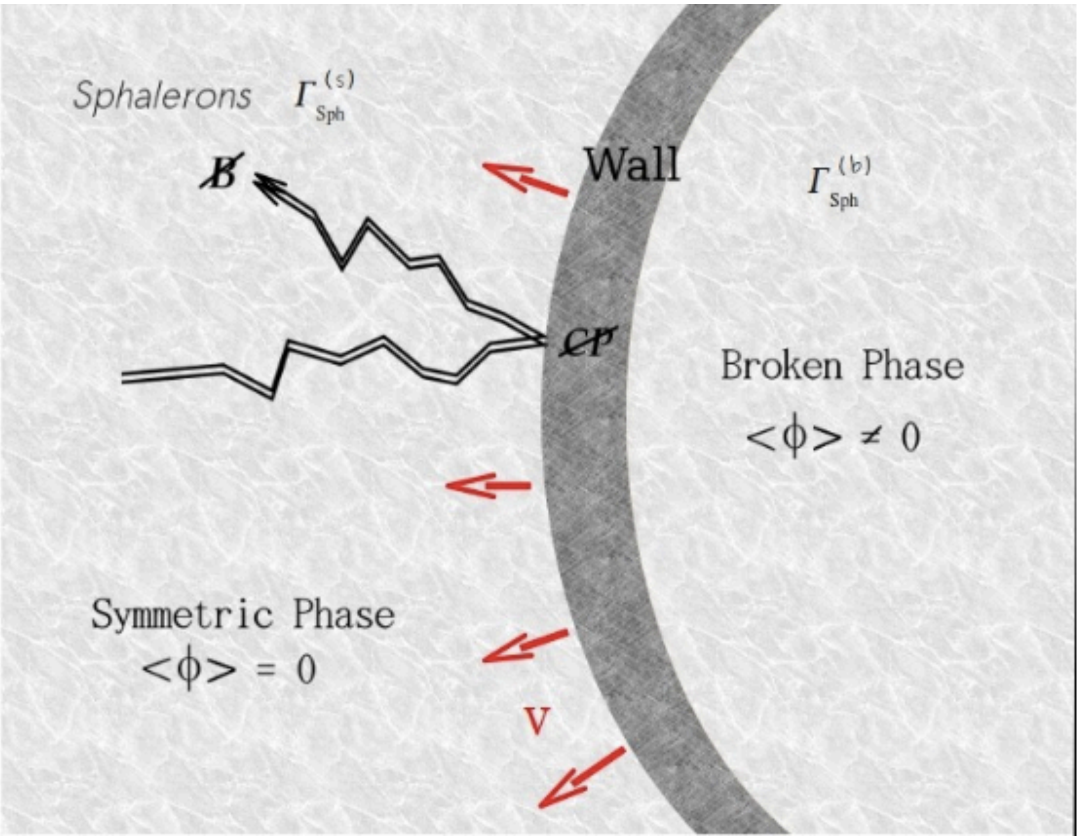
$$E_{sph}(T) \approx \frac{4\pi}{g}\, B(T)\, v(T) \quad \text{with} \quad v(T) \equiv \left\langle \sqrt{2\,\Phi^\dagger \Phi} \right\rangle_T$$

↑

dim-less order-one function. In SM (and nearby extensions) varies more slowly with temp. than $v(T)$ ⟶ $E_{sph}(T) \approx E_{sph}(0)\, \frac{v(T)}{v}$

(useful but not exact)

- **Why this matters for Baryogenesis**



Any mechanism that generates B+L asymmetry before electroweak epoch can be washed out if sphalerons remain in equilibrium afterwards.

→ ELW Baryogenesis requires a strong enough first-order phase transition: once bubbles of broken phase form, sphaleron transitions inside must become suffiencly slow

washout rate $\gamma_{B+L}(T) \sim c\,\frac{\Gamma_{\rm sph}/V}{T^3}$ ⟶ $\frac{dn_{B+L}}{dt} \approx -\gamma_{B+L}(T)\,n_{B+L}$

so, the condition is simply $\gamma_{B+L}(T) < H(T)$ ⟶ $\frac{E_{\rm sph}(T)}{T} \gtrsim \ln\left(\frac{c\,A(T)}{H(T)\,T^3}\right)$ ↙ $\sim 35 - 45$

famous criterium is approximate $\boxed{\frac{v(T)}{T} \gtrsim 1}$ ⟵ $E_{\rm sph}(T) \approx \frac{4\pi}{g} B\,v(T)$

two caveats: (1) $A(T)$ not known, (2) $v(T)$ can be gauge dependent whereas γ_{B+L} is not

(fluctuation prefactor)

- **Symmetric phase** $v = 0$

In unbroken phase there is no Higgs vev and energy barrier between neighboring topological sectors disappears.
Thus the Chern-Simons number now performs a random walk driven by thermal non-Abelian gauge fields

the natural quantity to define is now a diffusion constant

$$\Gamma_{\text{diff}} \equiv \lim_{V,t\to\infty} \frac{\left\langle (N_{\text{CS}}(t) - N_{\text{CS}}(0))^2 \right\rangle}{V\, t}$$

Often referred to (historic) as the "sphaleron rate" in the symmetric phase.
Its however no barrier crossing, it is diffusion

Dimensional argument:

long-distance magnetic sector of hot non-Abelian gauge theory behaves like three-dimensional theory with coupling $g_3^2 \sim g^2 T$

Only nonperturbative length scale is $\ell_{\text{mag}} \sim \frac{1}{g^2 T}$ ⟶ magnetic volume $\ell_{\text{mag}}^3 \sim (g^2 T)^{-3}$

Real-time dynamics $\tau_{\text{mag}} \sim \frac{1}{g^4 T}$

$$\alpha_W \equiv \frac{g^2}{4\pi}$$

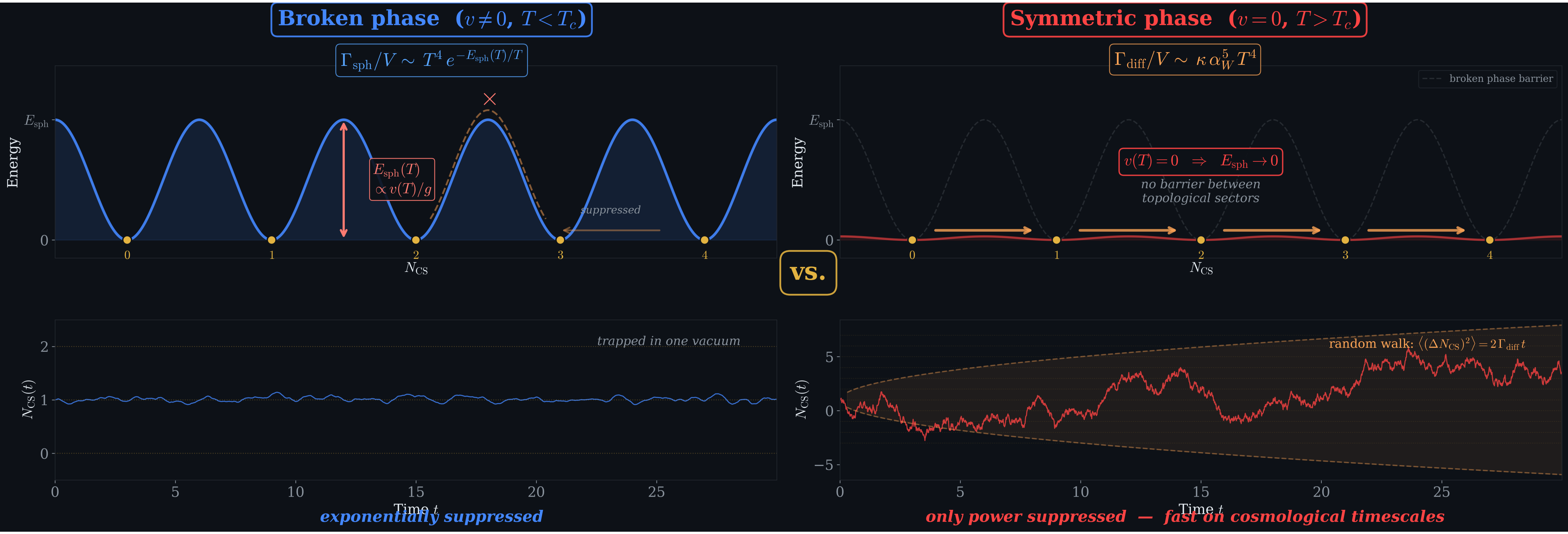
$$\frac{\Gamma_{\text{diff}}}{V} \sim \frac{1}{\ell_{\text{mag}}^3 \, \tau_{\text{mag}}} \sim \frac{\Gamma_{\text{sph}}}{V} \sim \kappa \alpha_W^5 T^4$$

κ from lattice

Contrast with broken phase is sharp:

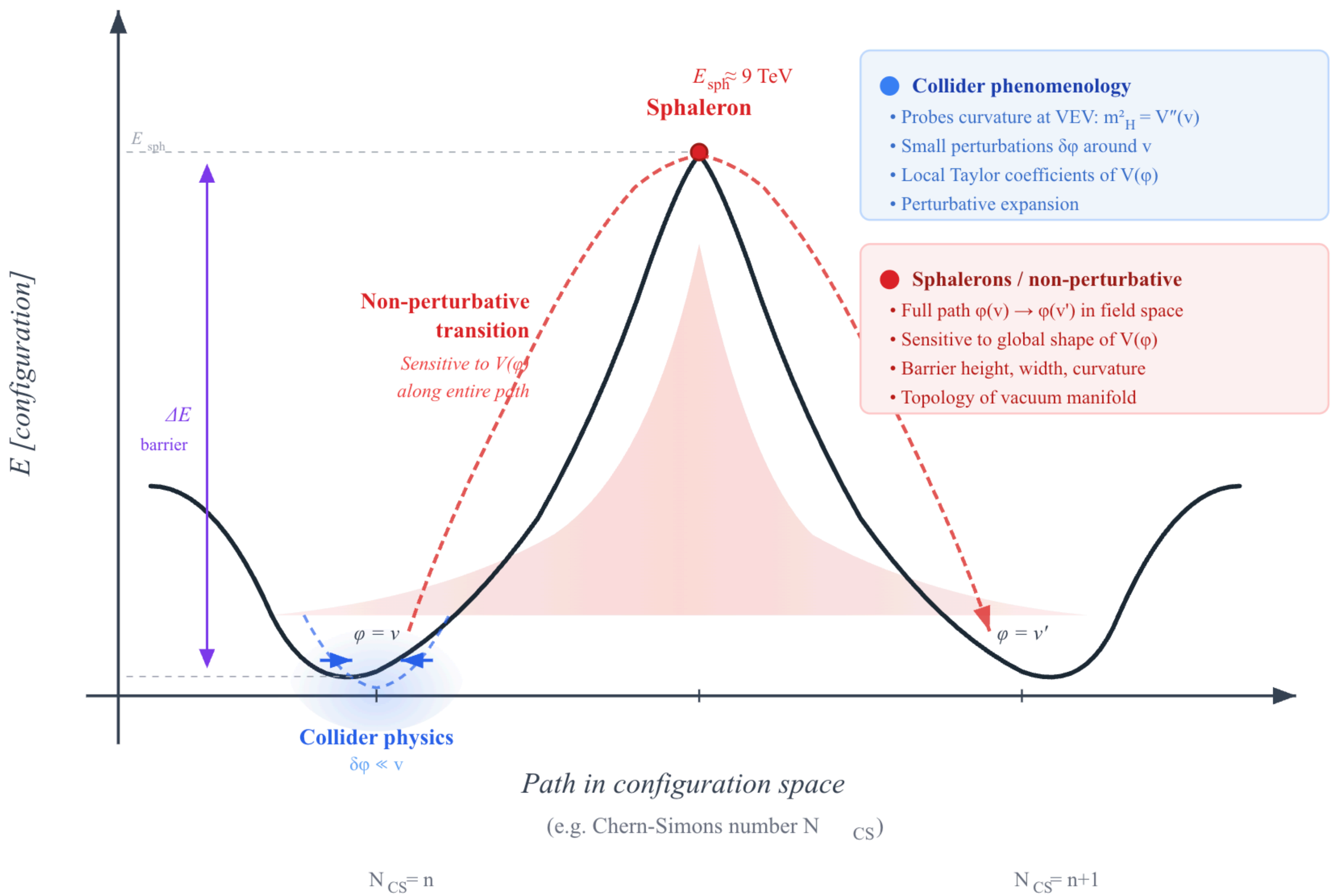
- in the broken phase the rate is exponentially suppressed by $\exp(-E_{\text{sph}}/T)$

- in the symmetric phase, the rate is only power suppressed, so the $B+L$ violation is effectively fast on cosmological time scales



After discovery of Higgs boson the next big goal (holigrail) is to map the potential underlying the Higgs mechanism

Study of Higgs couplings at colliders can only map the potential locally around the VEV



Non-perturbative phenomena allow to study the global structure of the scalar potential, e.g.

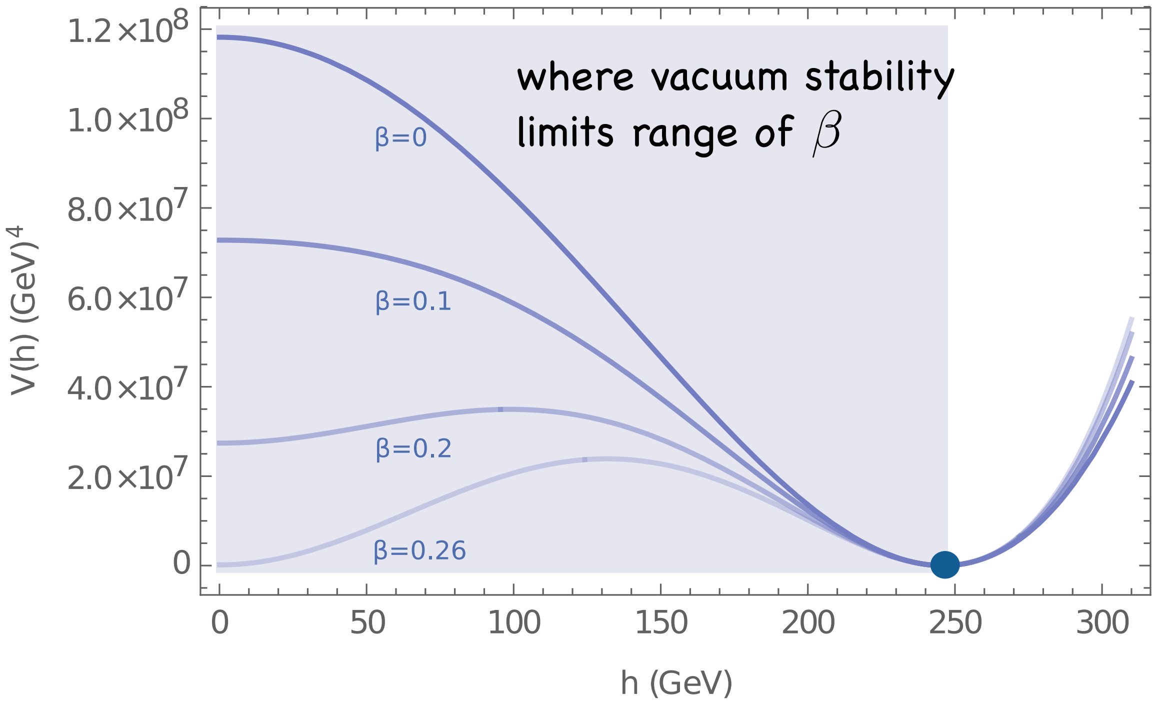
In SM $V(H) = -\frac{m_h^2}{2v^2}\left(H^\dagger H - \frac{v^2}{2}\right)^2$ and $\tilde{V}_{\rm bos}^{\rm SM} = \frac{1}{g^2}\int d^3y \left\{\frac{1}{4g^2}\tilde{W}_{ij}^a\tilde{W}_{ij}^a + \frac{1}{2}D_i\tilde{H}^\dagger D_i\tilde{H} + \tilde{V}(\tilde{H})\right\}$

→ Sphaleron energy in the SM: $E_{\rm sph}^{SM} = 9.11\,{\rm TeV}$ [MS, Tamarit '16]

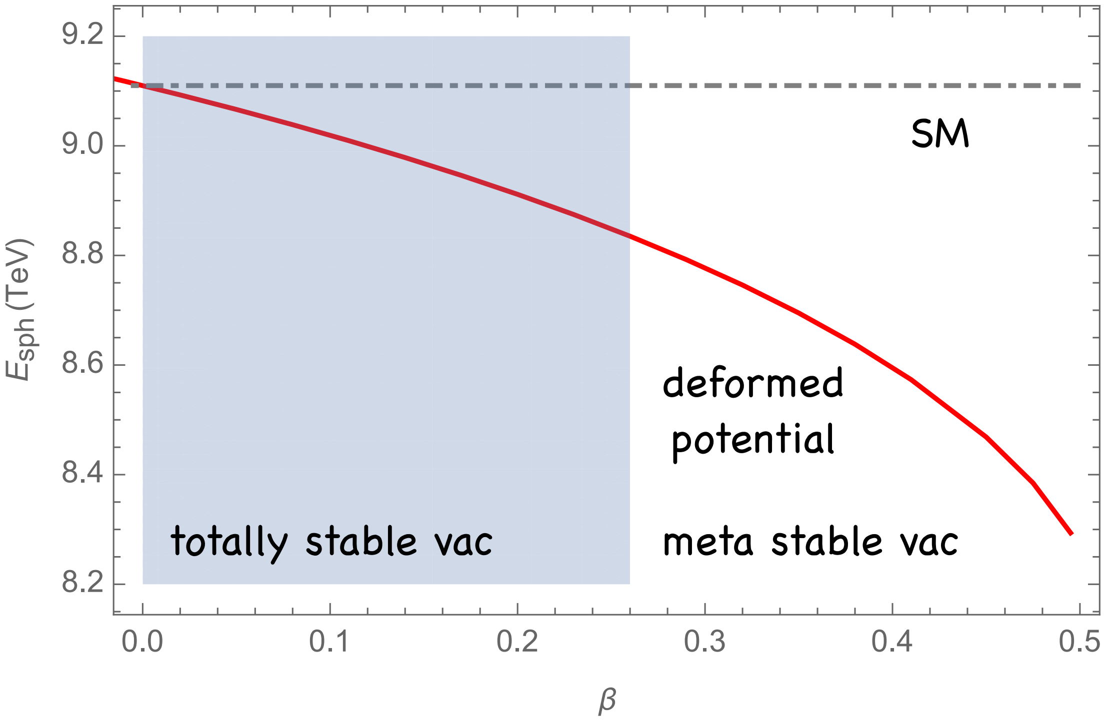
loop-induced eff. quartic

contrib. masses in loop

In deformed potential $V(H) = V_0 + m_H^2 H^\dagger H + (H^\dagger H)^2\left(-\lambda + \beta\log\left[\gamma + \frac{2H^\dagger H}{\phi_0^2}\right]\right)$



→

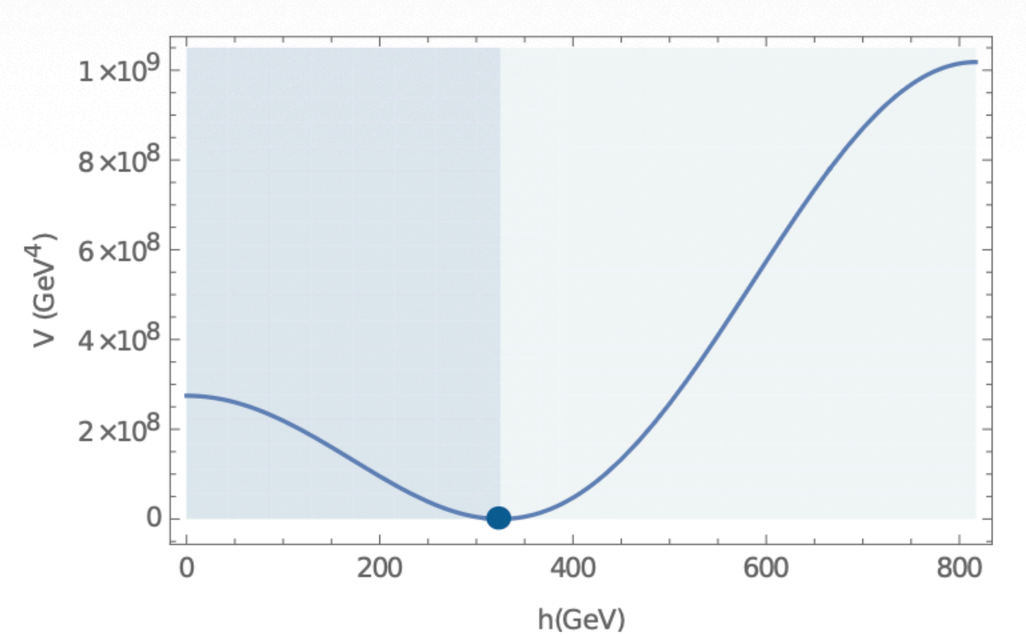


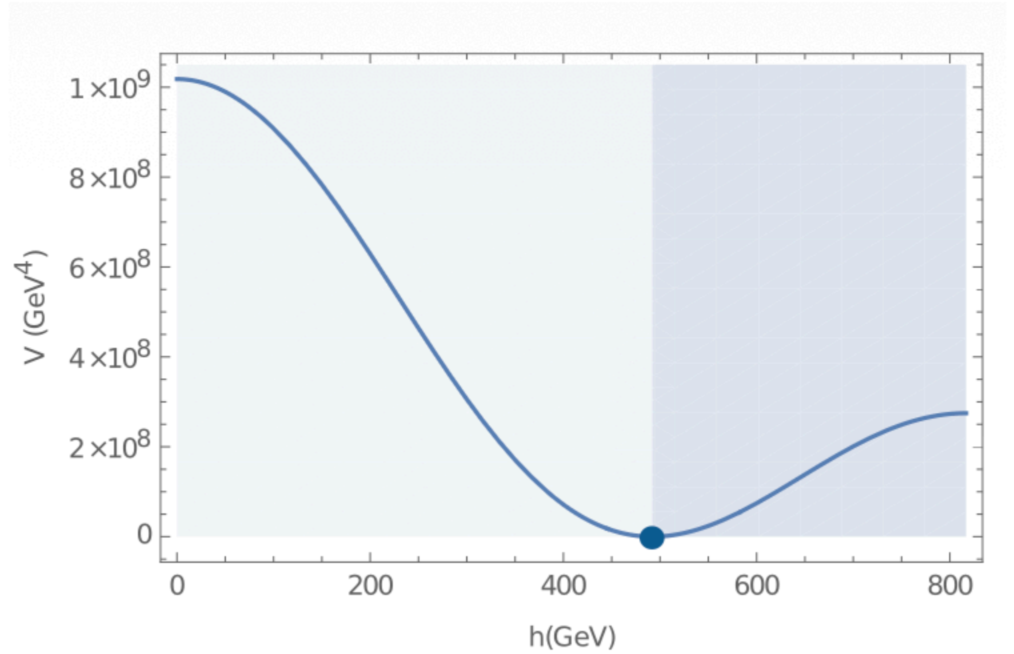
Sphalerons in composite Higgs scenarios

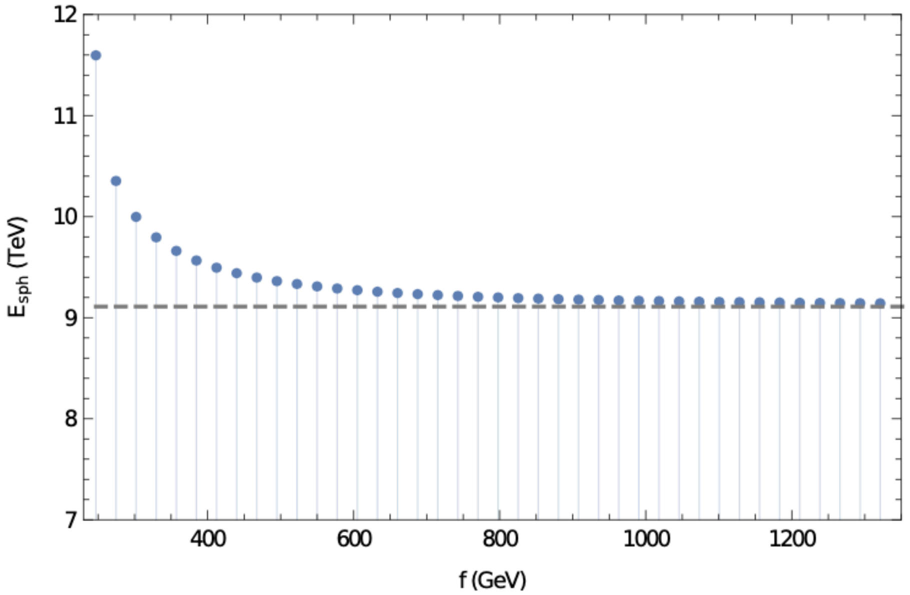
[MS, Tamarit '16]

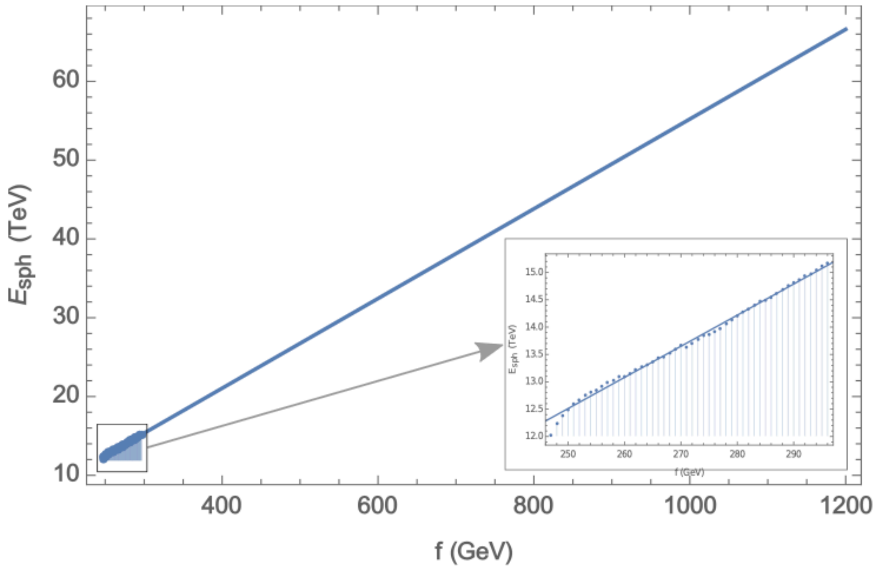
Choose well known standard comp. Higgs scenario $SO(5)/SO(4)$

⟶ results in new second branch of sphalerons (not present in SM)









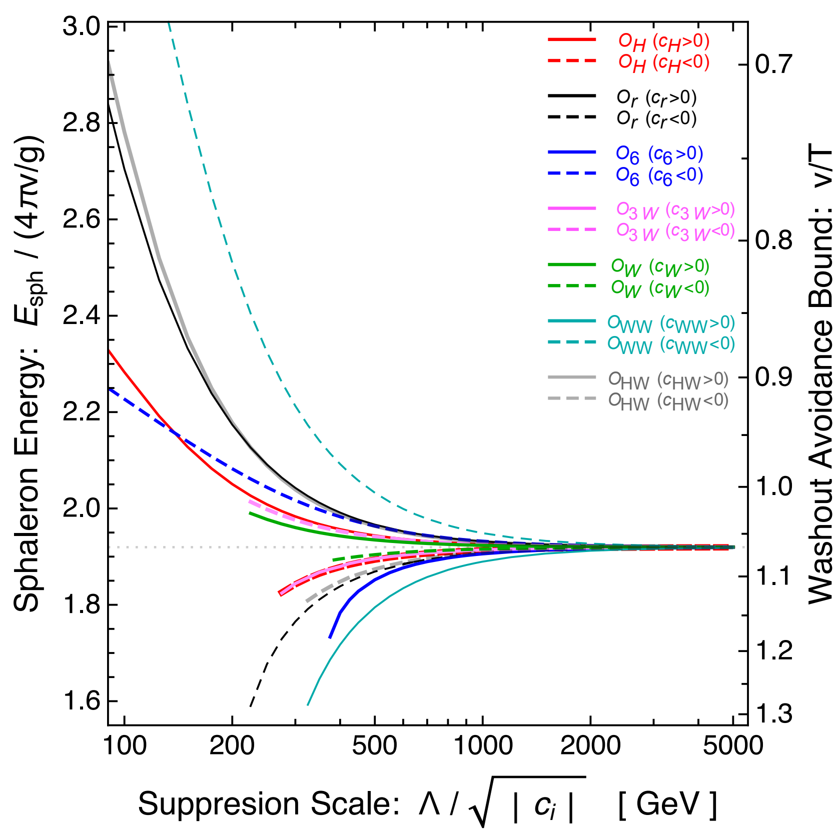
Electroweak sphalerons and dimension-6 operators

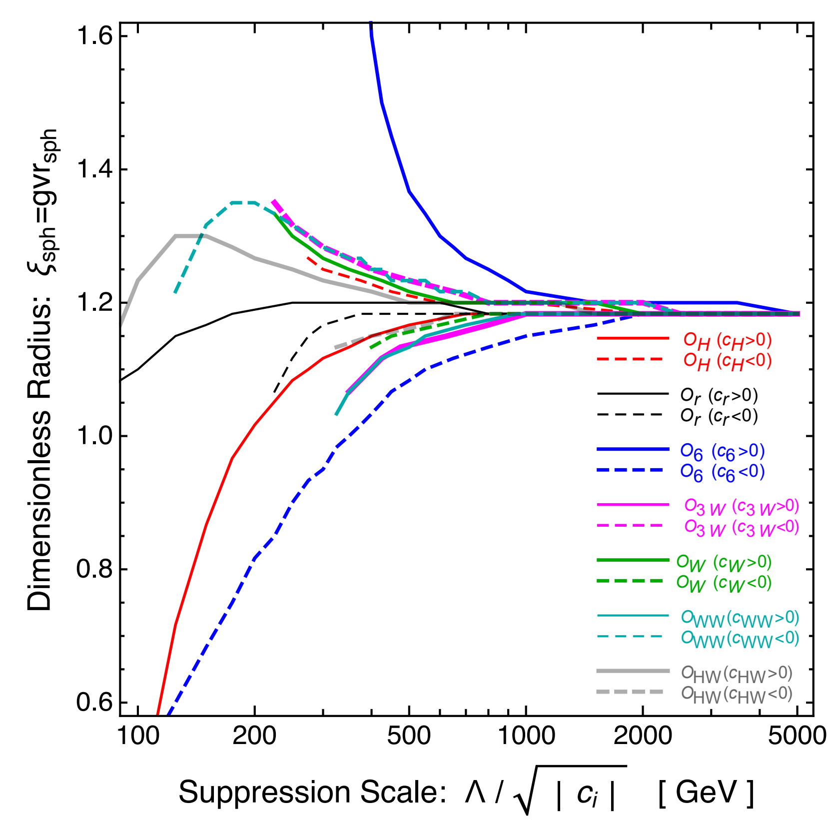
[Gan, Long, Wang '17]

$$\mathcal{L} = (D_\mu H)^\dagger (D^\mu H) - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \mu^2 H^\dagger H - \lambda (H^\dagger H)^2 + \sum_i \frac{c_i}{\Lambda^2} \mathcal{O}_i$$

$$\mathcal{O}_H = \frac{1}{2}\left(\partial^\mu (H^\dagger H)\right)^2 \qquad \mathcal{O}_r = H^\dagger H \, (D_\mu H)^\dagger (D^\mu H) \qquad \mathcal{O}_6 = -\left(H^\dagger H\right)^3$$

$$\mathcal{O}_{2W} = -\frac{1}{2}\left((\hat{D}^\mu W_{\mu\nu})^a\right)^2 \qquad \mathcal{O}_{3W} = \frac{1}{3!} g\, \epsilon^{abc} W_\mu^{a\,\nu} W_{\nu\rho}^b W^{c\,\rho\mu} \qquad \mathcal{O}_{3\tilde{W}} = \frac{1}{3!} g\, \epsilon^{abc} \widetilde{W}_\mu^{a\,\nu} W_{\nu\rho}^b W^{c\,\rho\mu}$$





Sphaleron production

- **Can few-particle scattering produce a sphaleron?**
  - This is a long-standing disputed question.
  - It is not enough to know that a barrier exists and to know its height. A collider prepares a few hard incoming quanta, not a classical coherent field configuration

  ➡ The question is partly kinematic, partly dynamical and partly about quantum overlap

- **Why perturbation theory is not the right language**
  - $B + L$ violation is invisible in ordinary perturbation theory around the trivial vacuum
  - Perturbation theory expands around one vacuum sector and never captures transitions between distinct topological sectors

  ➡ Need non-perturbative methods, first step semi-classical method

• The semi-classical method in a nutshell

In weak coupling, path integrals are dominated by **saddle points**. The non-pert. transition prob. can thus be written schematically as

characteristic elw scale

$$\sigma(E,N) \propto \exp\left(-\frac{4\pi}{\alpha_W} F(\epsilon,\nu)\right) \quad \text{where} \quad \epsilon = \frac{E}{E_0} \qquad \nu = \frac{\alpha_W N}{4\pi} \qquad E_0 \sim \sqrt{6}\,\pi\,\frac{m_W}{\alpha_W} \approx 18\ \text{TeV}$$

suppressio exponent, solved by boundary value problem in complex time

Particle number N introduced, as the semiclassical method is best controlled when $N \sim \dfrac{1}{\alpha_W}$

That may seem to invalidate its use for collider physics, but strategy is:

1. solve the controlled many-particle problem
2. then extrapolate to small N

• Path-integral representation:

$$\mathcal{A}_{i\to f} = \int \mathcal{D}\phi\, \Psi_f^*[\phi(+\infty)] \exp\left(\mathrm{i}S[\phi]\right) \Psi_i[\phi(-\infty)]$$

To calculate probability for fixed E and N one inserts projectors. The conjugate variables are denoted by T and θ.

At saddle-point level the result takes the form $\quad F(E,N) = ET + N\theta - 2\,\mathrm{Im}\,S_{\rm cl}[T,\theta]$

- **The holy-grail function**

The quantity of collider interest is the two-point suppression exponent, defined formally as

$$F_{\rm HG}(E) = \lim_{N\to 0} F(E,N)$$

Then $$\sigma_{2\to{\rm any}}(E) \sim \exp\left(-\frac{4\pi}{\alpha_W} F_{\rm HG}(E)\right)$$

This function is called **holy-grail function.**

If one knew it reliably all the way up to the sphaleron region, the production question would essentially be answered

At energies well below the barrier the semiclassical analysis gives $$F_{\rm HG}(\epsilon) = 1 - \frac{9}{8}\epsilon^{4/3} + \frac{9}{16}\epsilon^2 + \mathcal{O}(\epsilon^{8/3})$$

exp. softens as energy increases

- **When is the method applicable?**
  - When the weak-coupling expansion parameter is small: $\alpha_W \ll 1$
  - Most trustworthy for inclusive probabilities and for initial state with large N
  - Reliably captures existence of exponential suppression at low energies
- **Main limitations**
  - The physically interesting few-particle limit is an extrapolation, not a directly controlled saddle-point calculation
  - Near and above the barrier, the relevant real-time solutions are technically difficult and sensitive to the treatment of unstable modes
  - A small exponent does not automatically mean a large hadronic cross section; parton luminosities, initial-state overlap, and unitarity still matter

Those are the core reasons the scattering question remains open

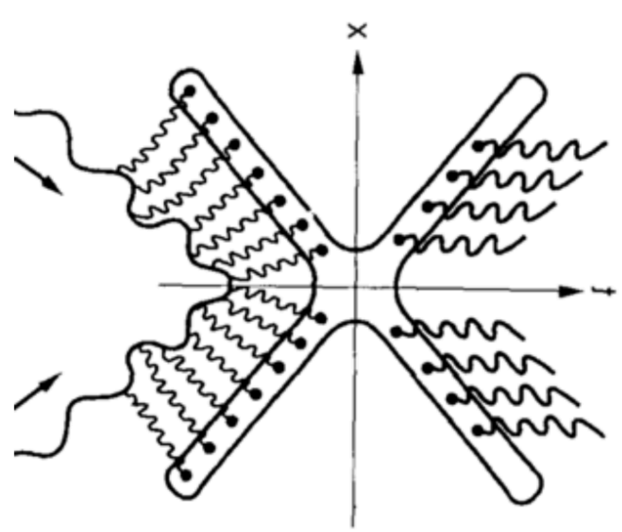
Ringwald, Mattis, and the 1990s debate

The optimistic view (Ringwald)

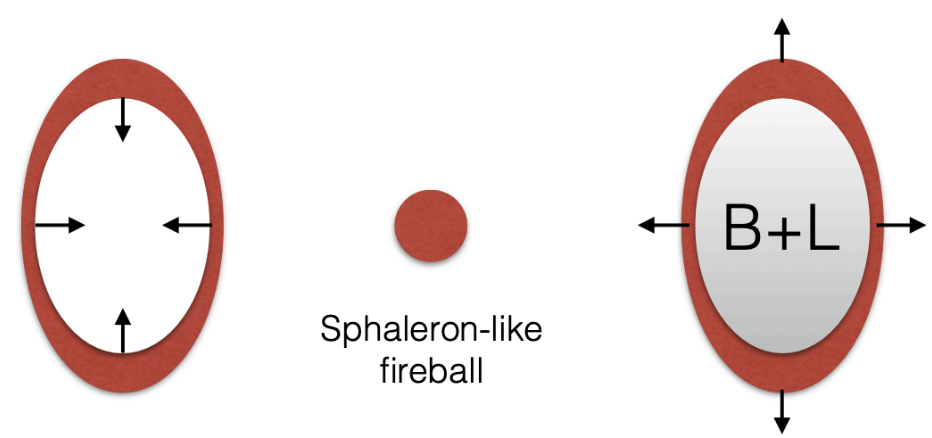
Around 1990, Ringwald and collaborators emphasised that instanton-induced amplitudes come with a very large semiclassical prefactor. Hope was that, as the energy approaches the sphaleron scale, a combination of **phase space** and **prefactor enhancement** might partially compensate the huge exponential suppression

Logic was:

1. the exponent should decrease as one appraoches the barrier
2. the prefactor may be numerically enormous
3. perhaps the rate becomes phenomenologically relevant near tens of TeV



Cartoon of snapshots in time:



The sceptics (Matthis, Arnold, Rubakov, and more)

The sphaleron is a coherent many-field configuraiton. A collider inital state, by contrast, contains only a few hard quanta. Even if total energy is above the barrier, the overlap of the initial state with the field configuration that crosses the barrier may remain exponentially small.

→ not simply “can the collider reach $E_{\rm sph}$

You cannot make a “fish in a collider” [Mattis ’92] -> mismatch between asymptotic state and barrier-crossing configuration

$$2 \text{ hard initial quanta} \rightarrow \text{sphaleron} \rightarrow \pi/\alpha_w \text{ soft final quanta}\,.$$

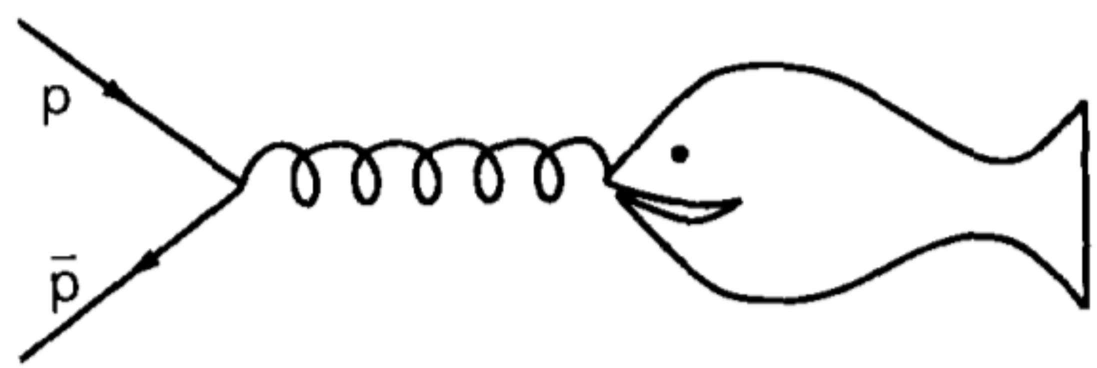


Fig. 3. “You can’t make a fish in a $p\bar{p}$ collider.”

→ sceptics view confirmed by later semiclassical studies

→ Theory methods not necessarily suitable…

[Mattis, '92]

Table 1

Participants in the Santa Fe Workshop on Baryon Number Violation at the SSC (April 1990) (ref. [9]). Shown in parentheses are their answers to the question, "What is the probability that baryon number violation might even in principle be observable in a supercollider?" This poll was conducted by L. Okun.

Ken Aoki (10%) UCLA	Peter Arnold (10%) Argonne National Lab.	Zvi Bern (0.1%) Los Alamos National Lab.
Kevin Cahill (0%) Univ. of New Mexico	James Cline [$\exp(-2\pi/\alpha_w)$] Ohio State University	Mike Cornwall (10%) UCLA
Michael Dugan (10%) M.I.T	Olivier Espinosa (50%) CALTECH	Glennys Farrar [$\exp(-2S_{cl})$] Rutgers University
Haim Goldberg (10%) Northeastern University	Eduardo Guendelman (5%) Los Alamos National Lab.	Peter Herczeg (7%) Los Alamos National Lab.
Robert Jaffe (10^{-2}) M.I.T	David Kosower ($10^{-100\pm10}$) FERMILAB	Olaf Lechtenfeld (10^{-100}) City College, CUNY
Aneesh Monohar (10%) U.C. San Diego	Michael Mattis (5%) Los Alamos National Lab.	Pawel Mazur (10%) UCLA
Larry McLerran (66.7%) Univ. of Minnesota	Emil Mottola (5%) Los Alamos National Lab.	Lev Okun (10%) ITEP
Stuart Raby (5%) Ohio State University	Andreas Ringwald (50%) DESY	Z. Ryzak (3.3%) Harvard University
E.V. Shuryak (10^{-1}) Brookhaven National Lab.	Arkady Vainshtein (30%) Univ. of Minnesota	John Vergados (1%) Univ. of Ioannina
M. Voloshin (90%) Univ. of Minnesota	Geoffrey West ($O(\alpha)$%) Los Alamos National Lab.	Laurence Yaffe (1%) Univ. of Washington

Very optimistic - and new approach (Tye and Wong)

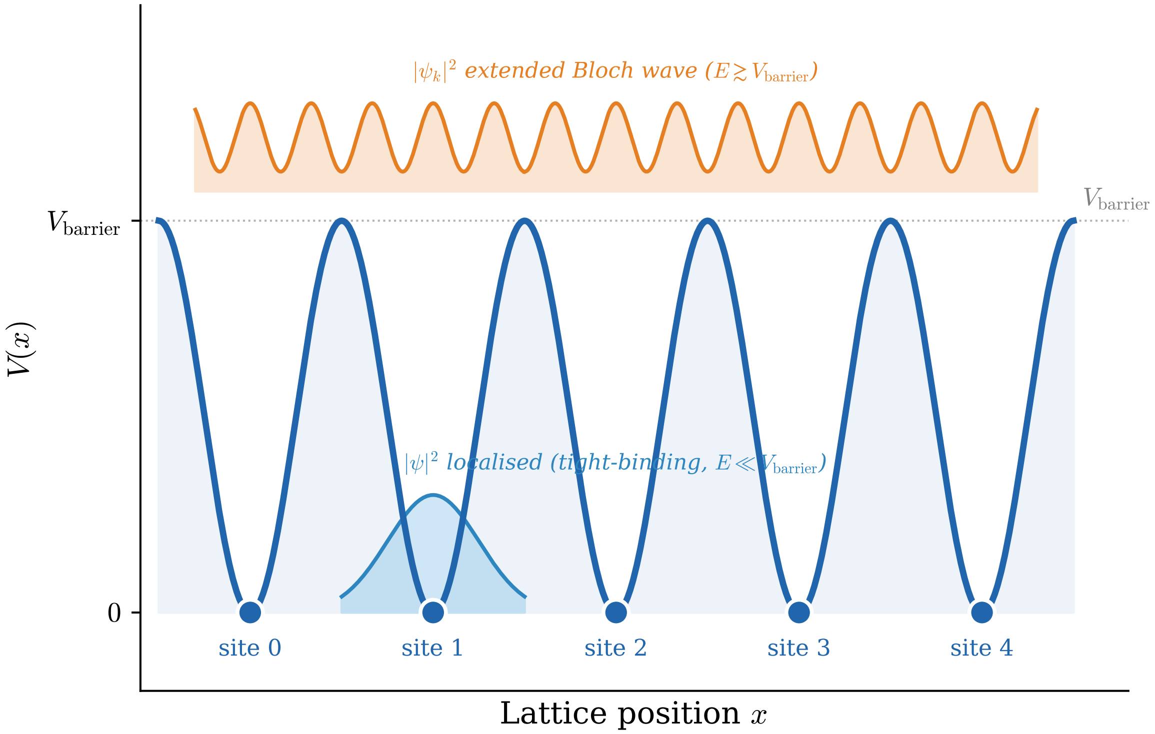
Idea: Promote the Chern-Simons direction itself to a dynamical collective coordinate. Because neighboring vacue repeat periodically in N_{CS} -> obtains effective one-dimensional quantum-mechanical problem in a periodic potential

In ordinary QM, periodic potential leads to Bloch waves and energy bands

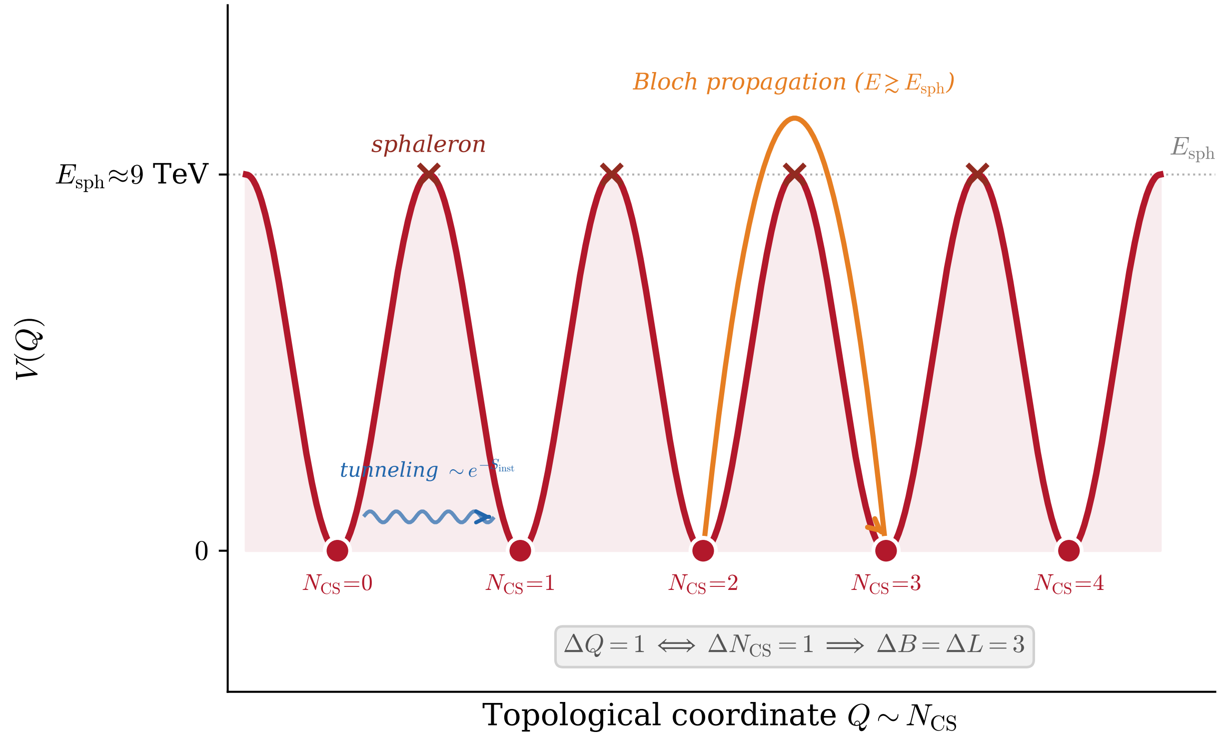
Tye and Wong argue that electroweak topological coordinate should be treated similarly:

- below barrier motion along coordinate is tunneling and suppressed
- near or above barrier, allowed bands can open up
- within an allowed band, motion along collective coordinate need not be exponentially suppressed

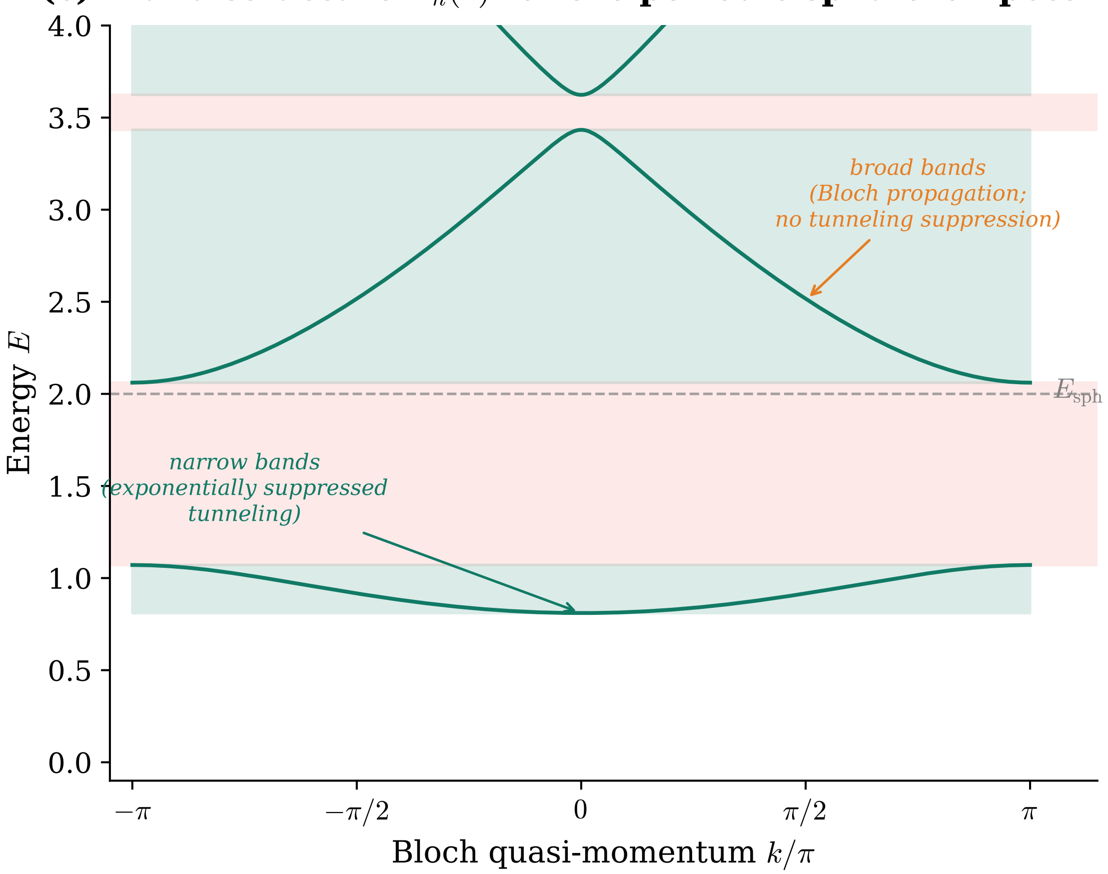
(a) Crystal lattice: electron in periodic potential



(b) Electroweak: $V(Q)$ along Chern–Simons direction



(c) Band structure $E_n(k)$ for the periodic sphaleron potential



(d) The Tye–Wong analogy at a glance

	Crystal lattice	Electroweak vacuum
Coordinate	electron position x	topological coord. $Q \sim N_{\rm CS}$
Potential	ionic potential $V(x)$	$V(Q) = E_{\rm sph} \sin^2(\pi Q)$
Sites	lattice ions	topological vacua ($N_{\rm CS} \in \mathbb{Z}$)
Barrier	inter-atomic $V_{\rm barrier}$	sphaleron $E_{\rm sph} \approx 9$ TeV
Eigenstates	Bloch waves $\psi_k(x)$	Bloch waves $\Psi_k(Q)$
Low E	narrow bands; tight-binding; localized electrons	narrow bands; instanton tunneling; $\Gamma \sim e^{-2S_{\rm inst}}$
High E	broad bands; free propagation; conducting	broad bands; Bloch propagation; $B+L$ violation unsuppressed?

Key insight (Tye & Wong, 2015): Do not estimate $B+L$ violation by treating each inter-vacuum hop as independent tunneling. Solve the periodic problem globally: eigenstates are Bloch waves, transport is efficient within allowed bands.

Why this proposal is attractive

The appeal of the idea is:

- it uses the **periodic vacuum structure** directly (unlike other methods)
- it gives an **intiuitive mechanism for reduced suppression**
- it suggests benchmark **cross sections large** enough to motivate exp. searches

Why the community remains cautious

- reducing full field theory to one collective coordinate may neglect important couplings to orthogonal modes
- Bloch quasi-momentum k is internal quantum number of the effective periodic problem, not the same thing as collider c.o.m momentum
- Even if motion along the collective coordinate is unsuppressed, it is not obvious that a few-particle initial state excites that coordinate with large probability
- Not clear how to go for band picture to established semiclassical field-theory calculations

→ if no final verdict from theory, let nature decide

Sphaleron phenomenology

Characteristic sphaleron processes

Process with very large invariant mass (~9 TeV)

12 left-handed fermions

+

$n_W W^{\pm}$

+

$n_Z Z$

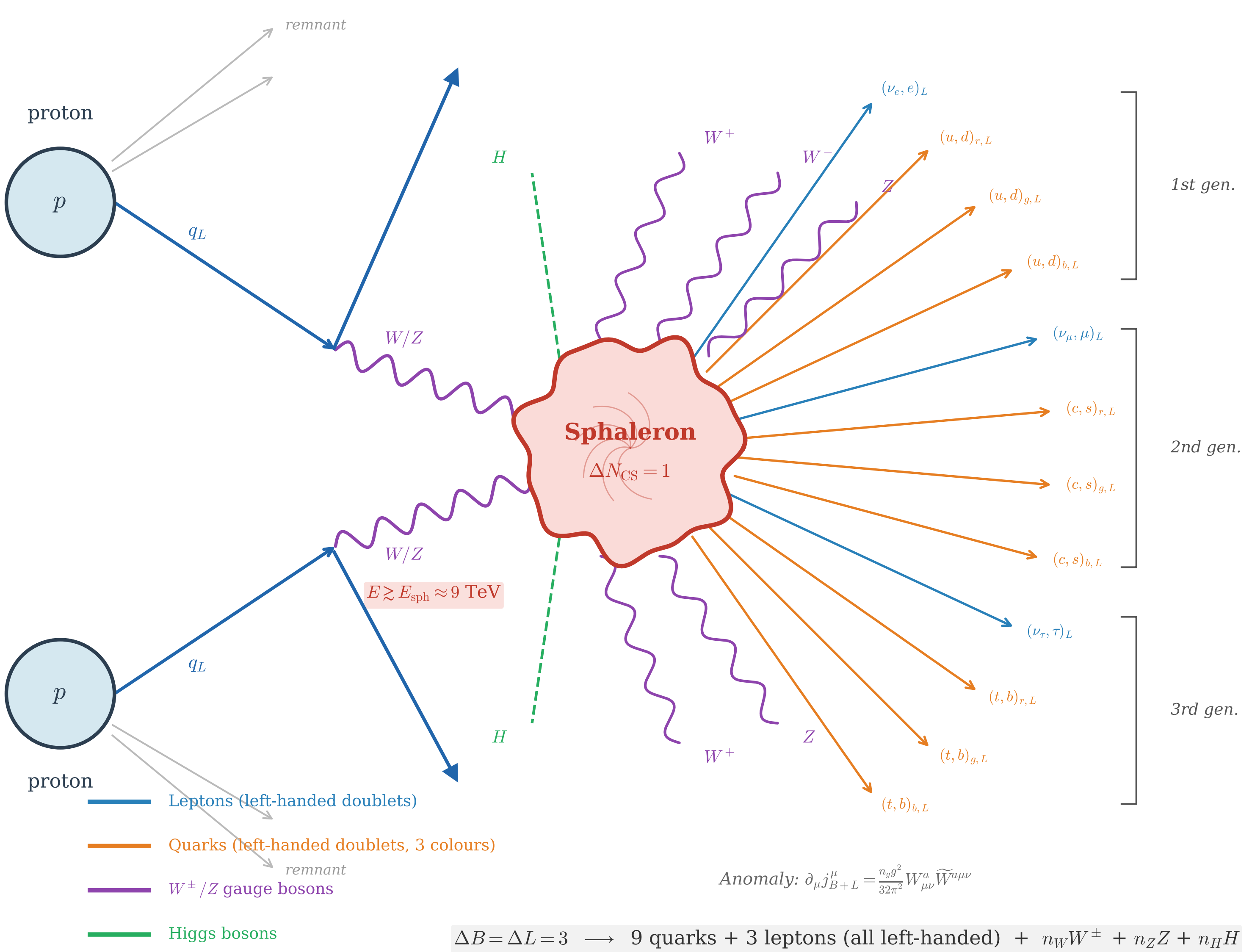
+

$n_H H$

where $n_W, n_Z, n_H \gg 1$

No problem triggering but challenge to fully reconstruct, i.e. measuring the invariant mass precisely

Sphaleron-induced $B+L$ violating process at a pp collider

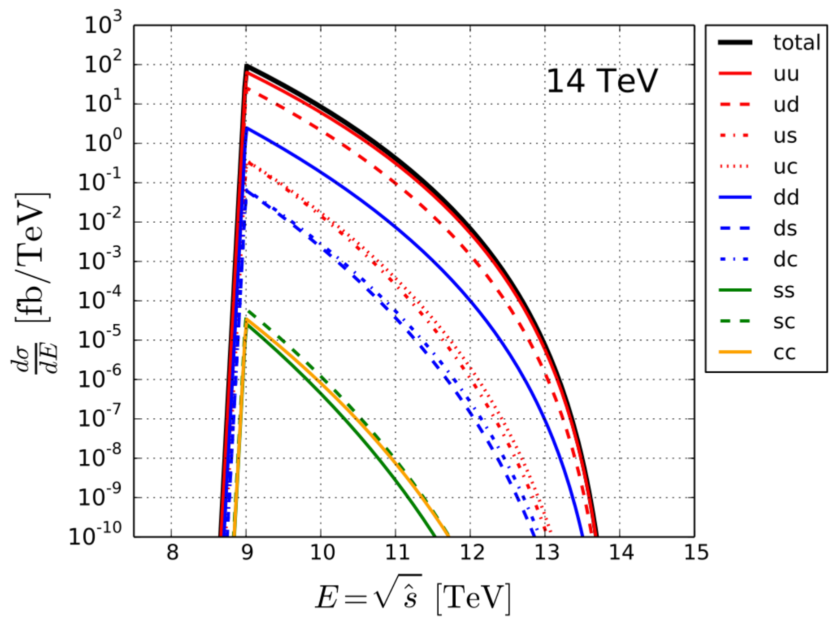


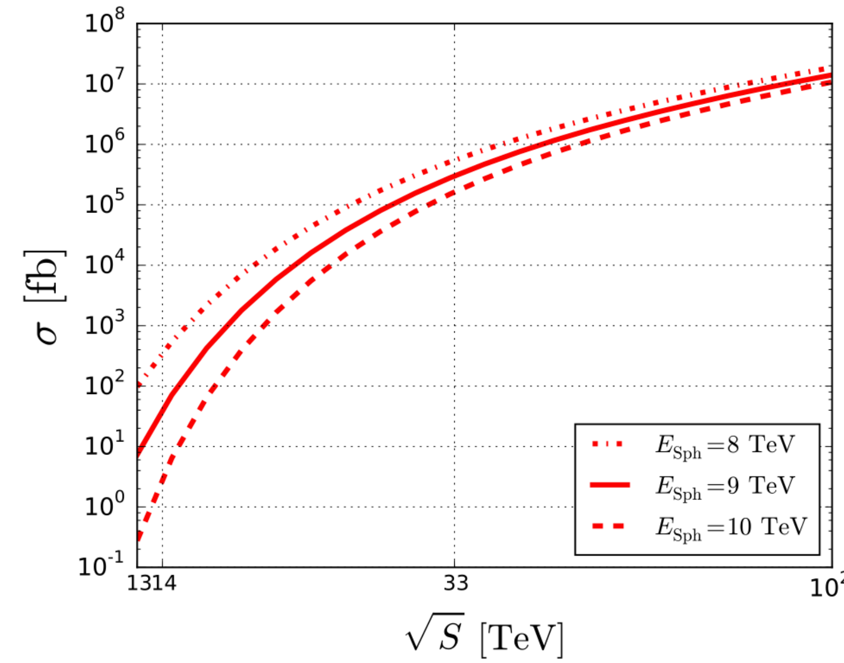
BaryoGEN [Bravo, Hauser '18]

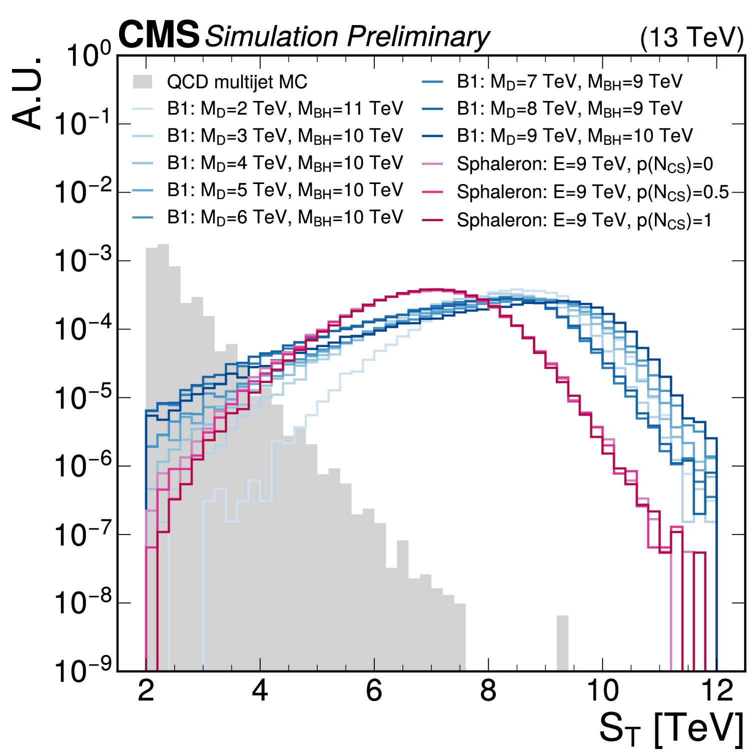
Searches at hadron colliders

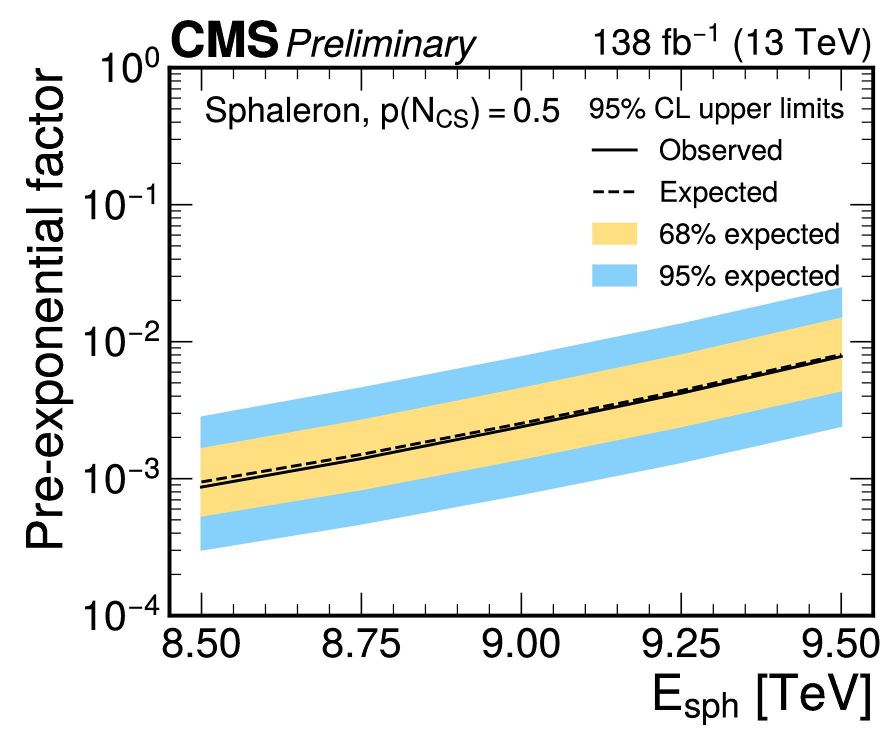
[Ellis, Sakurai '16]

Production cross section according to Tye-Wong mechanism









[CMS-EXO-17-023]

[CMS-PAS-EXO-24-028]

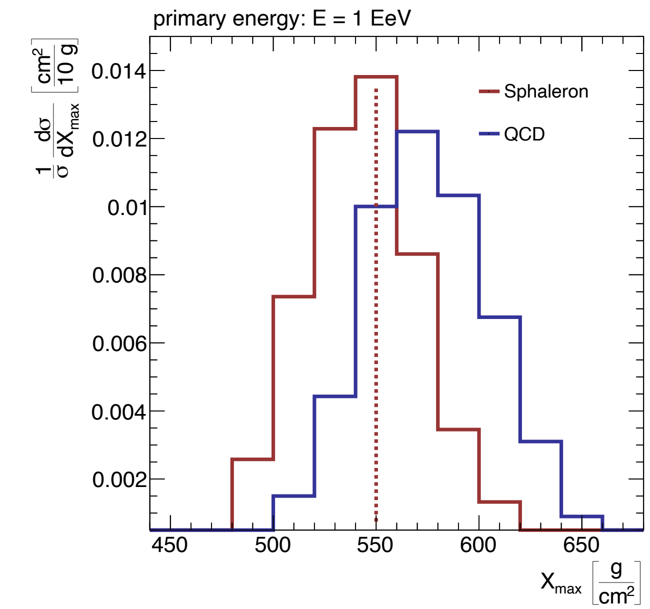
Cosmic Rays - Auger

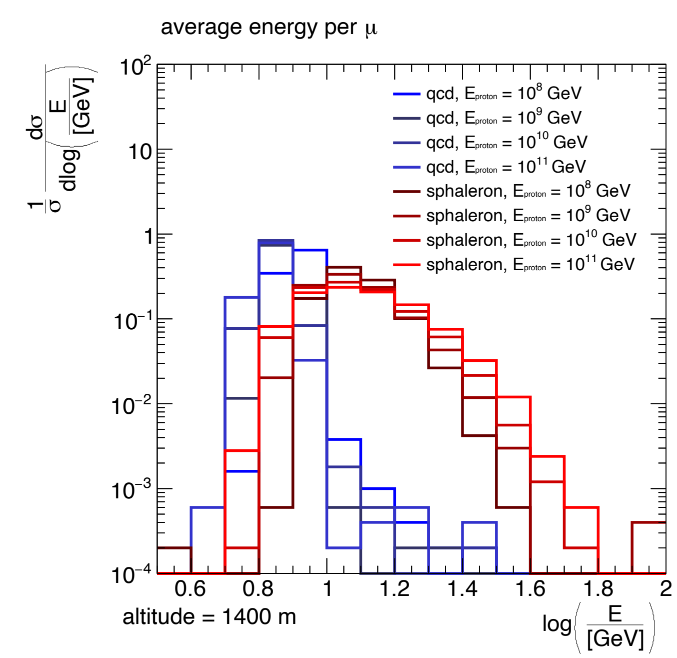
[Brooijmans, Schichtel, MS, '16]

[Schichtel, MS, Waite '19]

- We already have a 100 TeV collider: UHEC colliding with our atmosphere

$$E = 10^{11}\ \text{GeV} \longrightarrow \sqrt{s} \simeq \sqrt{2 m_N E} \simeq 500\ \text{TeV}$$





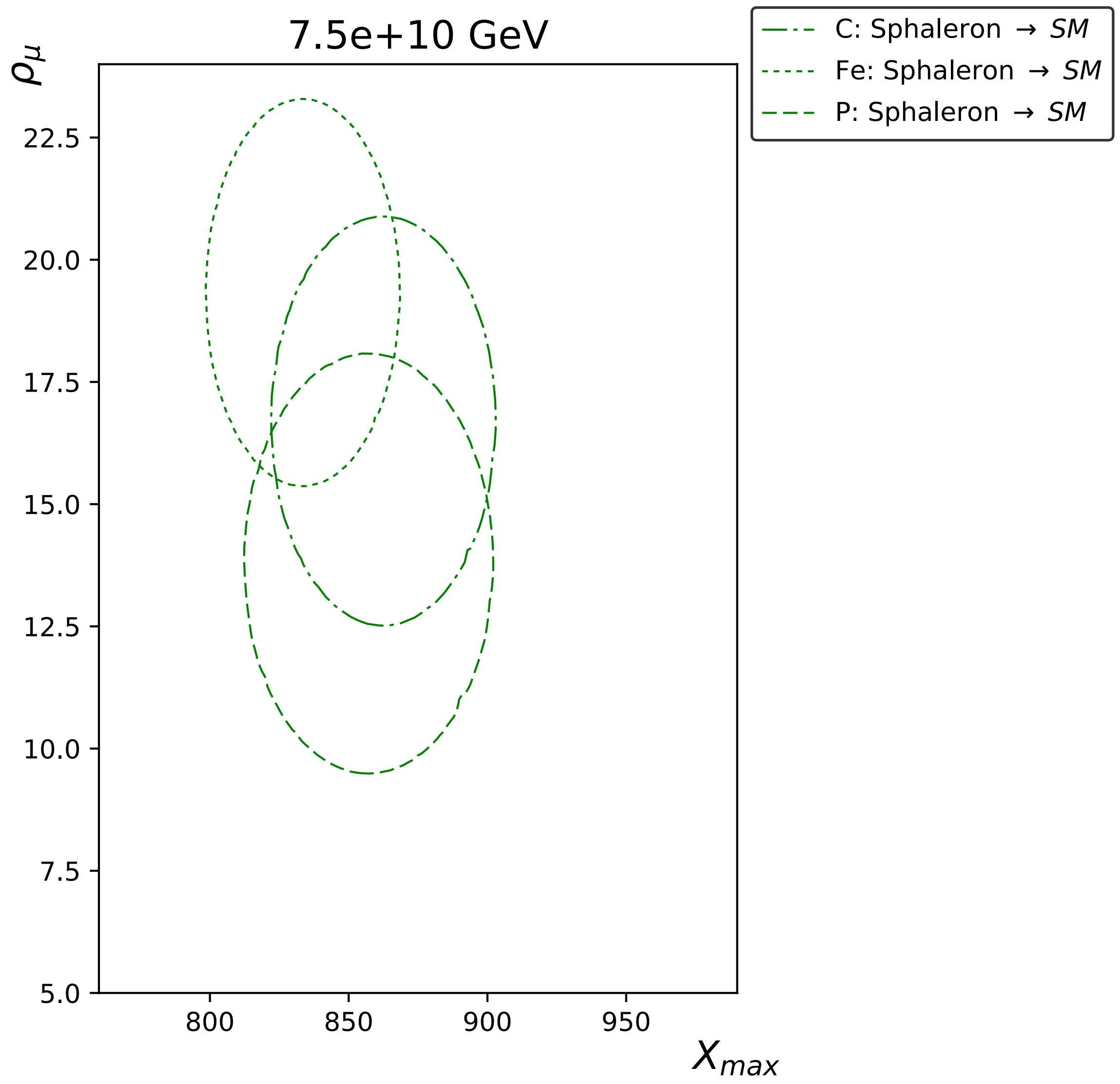
- Observables to discriminate sphalerons from QCD xmax and average energy per muon

Full Event simulation via Herbvi + Herwig + Corsika

Cosmic Rays - Auger

[Brooijmans, Schichtel, MS, '16]

[Schichtel, MS, Waite '19]



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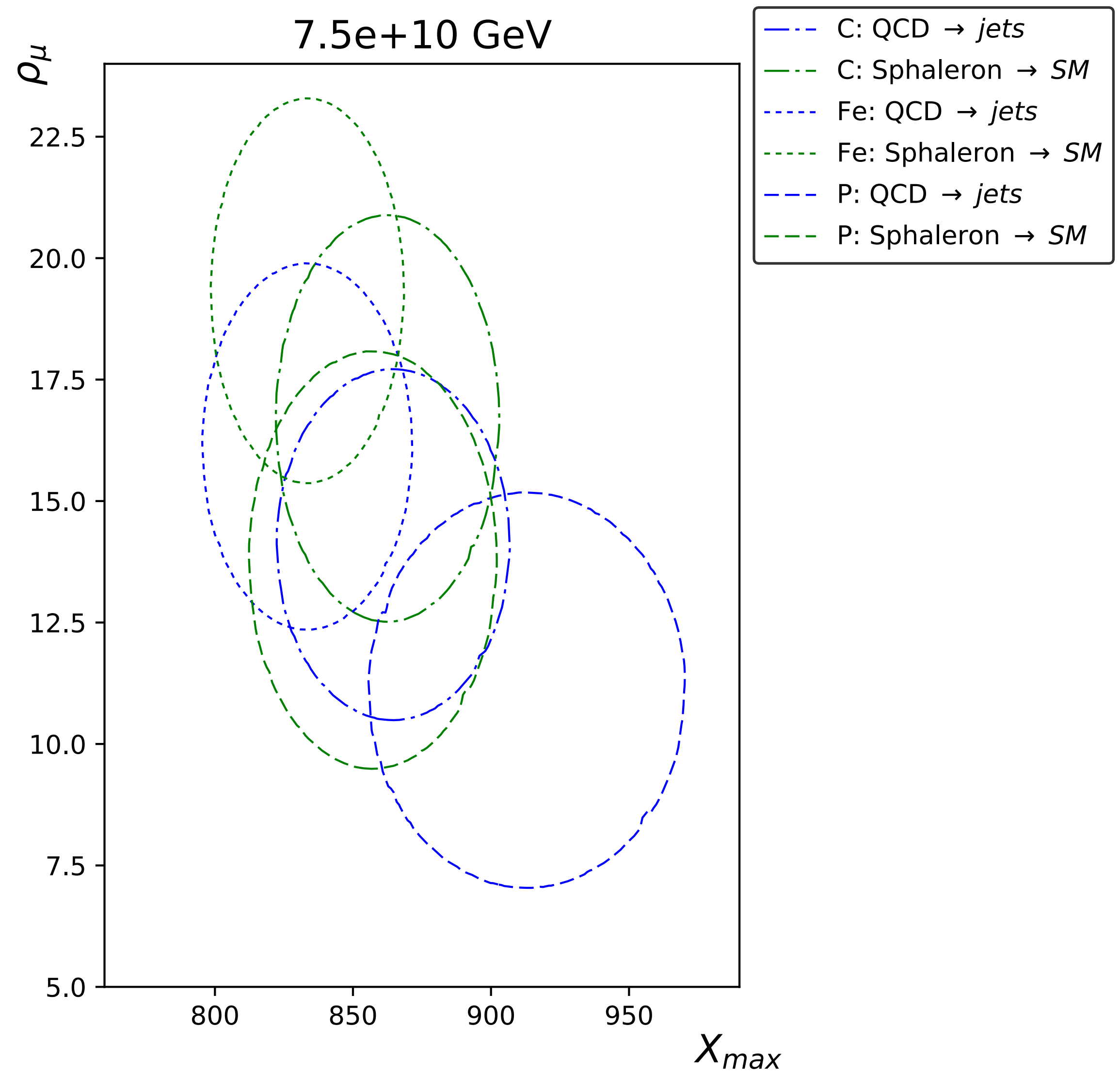

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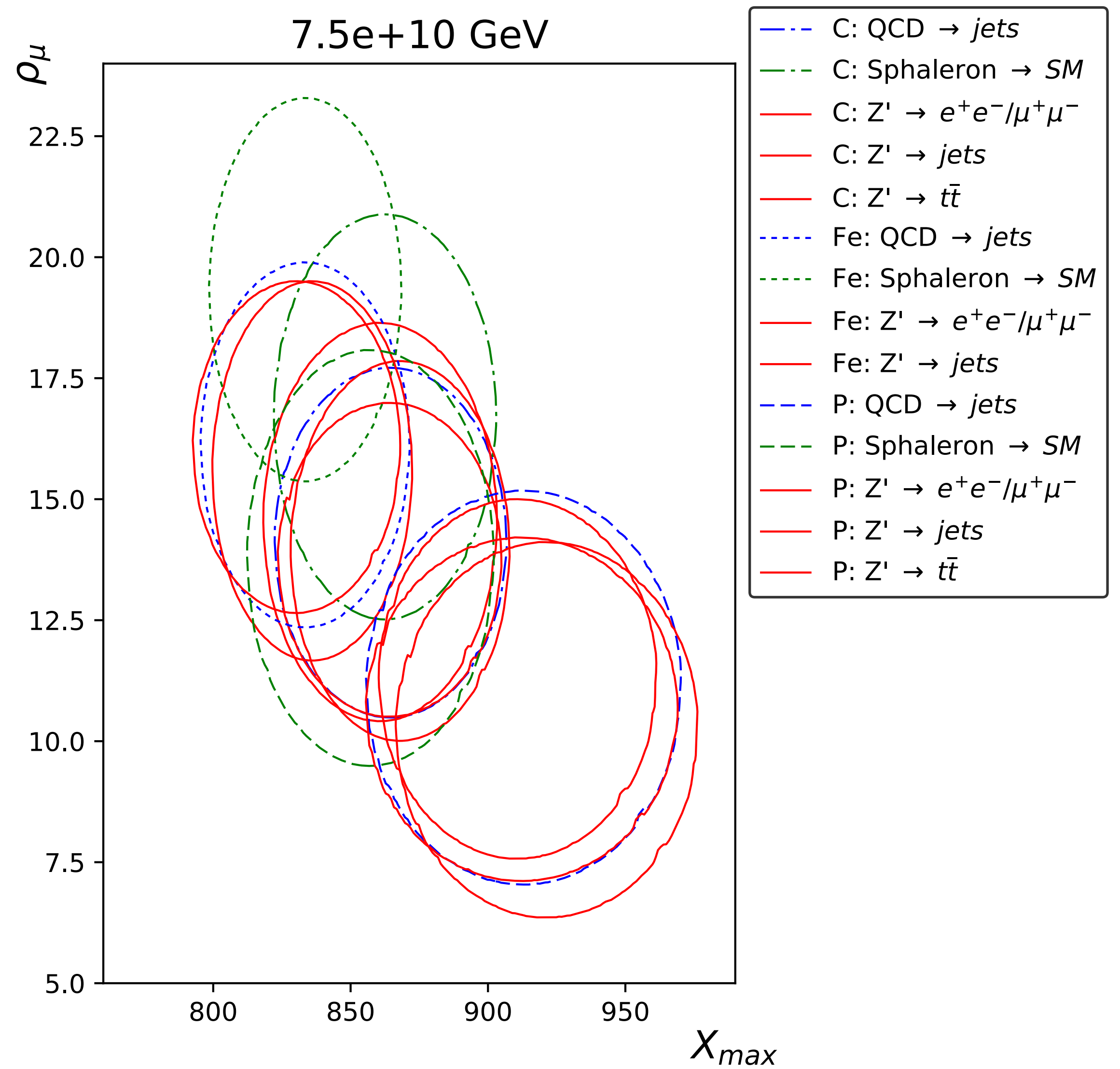

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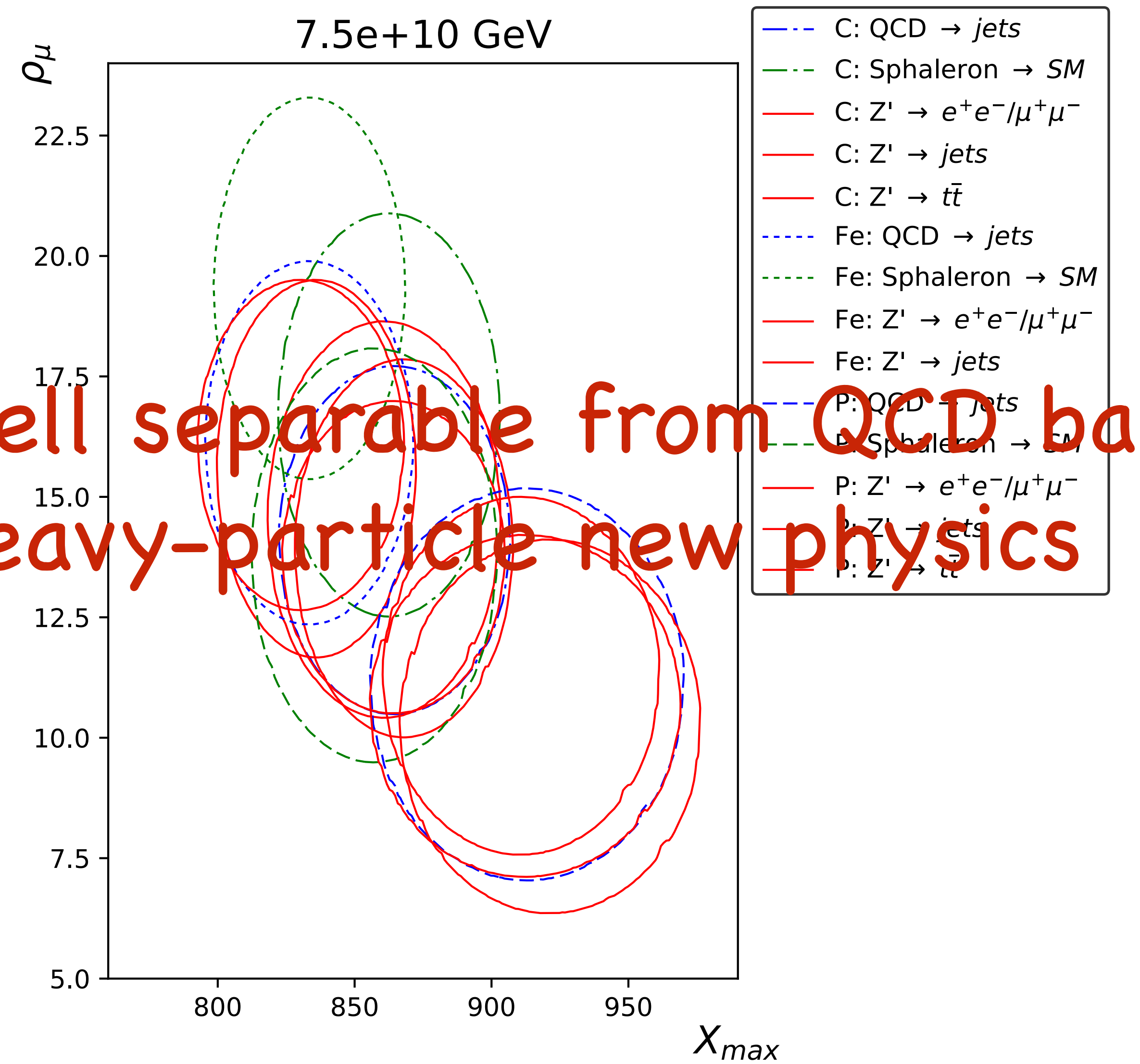


Sphalerons are well separable from QCD backgrounds and heavy-particle new physics

LHC vs Ice Cube

[Ellis, Sakurai, MS '16]

neutrino-nucleon cross section

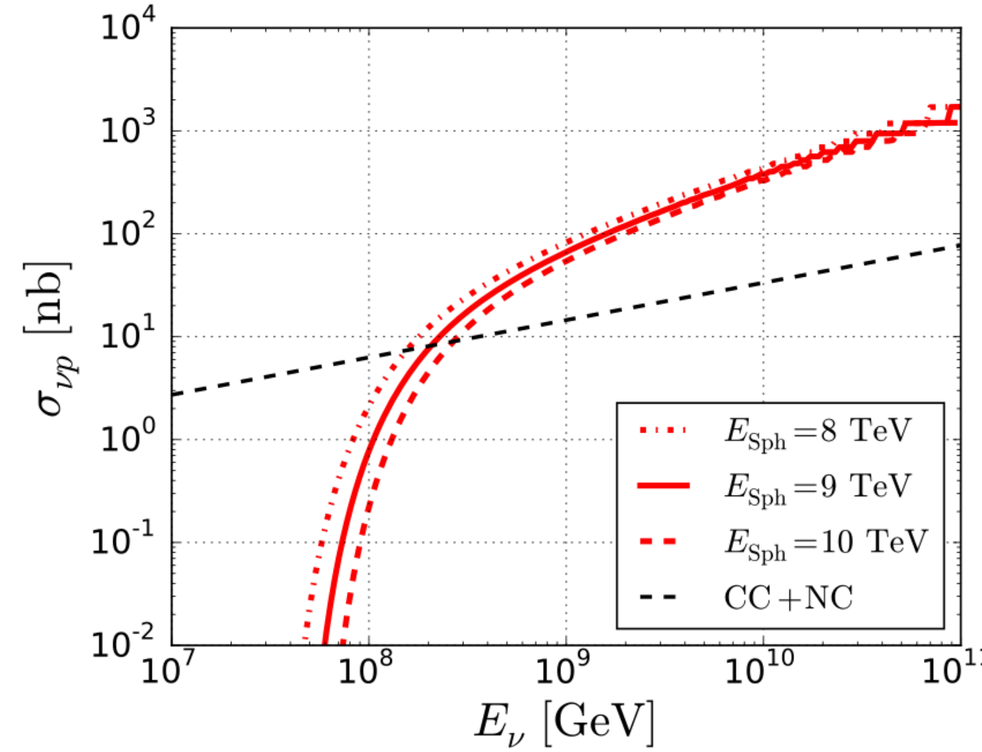
$$\sigma_{\nu N}(E_\nu) = \sum_q \int_0^1 dx\, f_q(x,\mu)\hat{\sigma}_{q\nu}(2x m_N E_\nu)$$

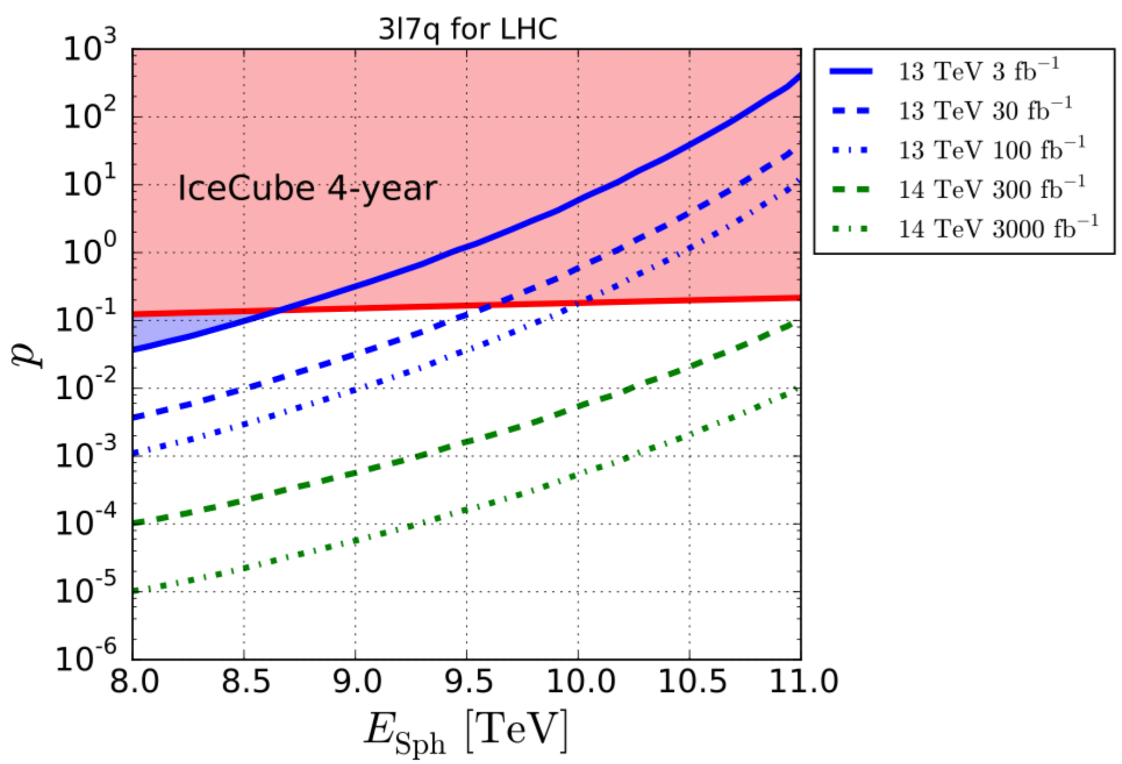
convolute cross sectino with neutrion flux to get event rate

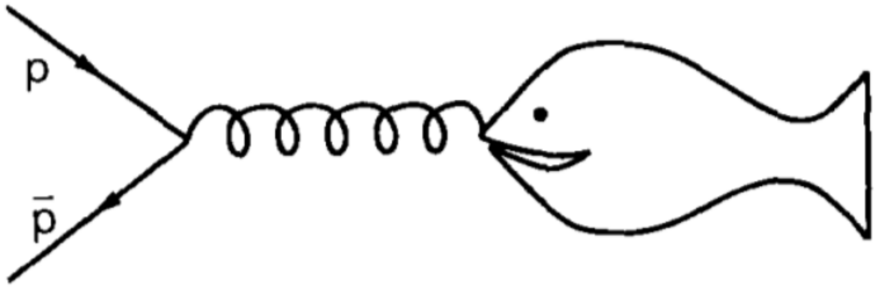
Ice-Cube event rates

$$\frac{dN_{\rm Sph}}{dt} = \int_{E_\nu^{\rm thres}} dE_\nu \int d\Omega \frac{\sigma_{\nu N}^{\rm Sph}(E_\nu)}{\sigma_{\nu N}^{\rm CC/NC}(E_\nu)} A_{\rm eff}(E_\nu) \frac{d^2\Phi}{dE_\nu dt d\Omega}$$

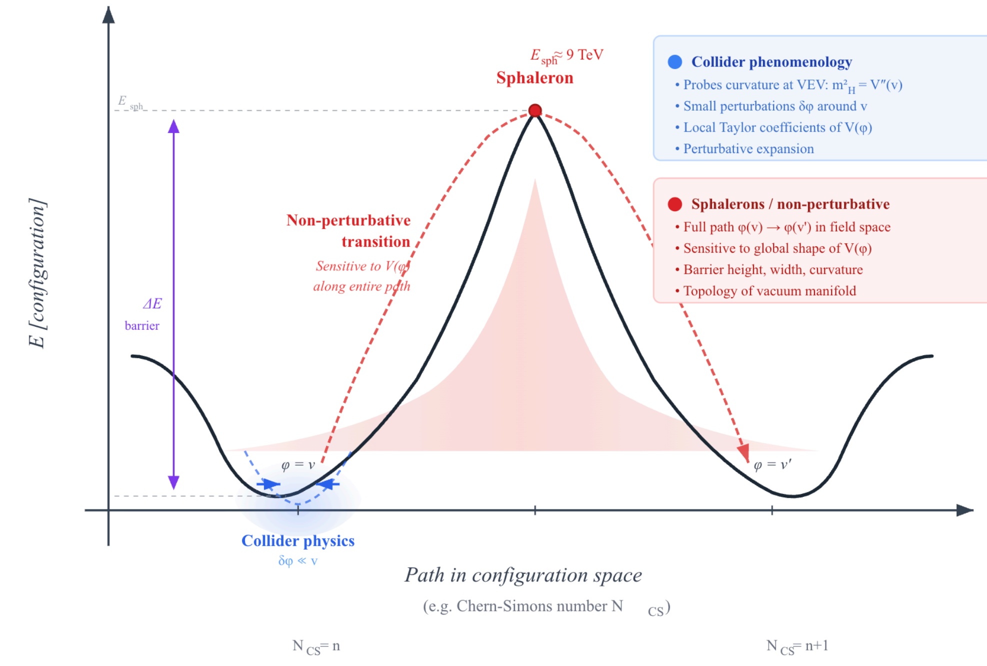
effective neutrino detection area (from Ice Cube)







Summary



- They are a genuinely nonperturbative Standard Model effect. They are not "just another rare process"; they probe field configurations and vacuum structure that ordinary perturbation theory cannot see.

- The precise calculation of their production rate from proton collisions is a long-standing and still open problem. Invites to work on theoretical approaches to improve their description.

- Experimental searches for sphalerons are well-motivated. These processes are striking at colliders, and thus can be observed and separated in cosmic ray experiments (Auger, Ice Cube)