# Variational Quantum Algorithms and Dynamical Lie Algebras


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Department of Mathematics
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Mathematical Quantum Field Theory
Sofia, Bulgaria, May 27, 2026

In memory of Academician Ivan Todorov (1933-2025)

# Conflict of Interest Disclosure

Bojko Bakalov is a co-founder, Chief Scientist, and equity holder of Aqceleration, Inc., a startup company developing quantum algorithms and solutions for quantum error mitigation and characterization.

Aqcelerating the Quantum Stack

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RIVISTA DEL NUOVO CIMENTO VOL. 12, N. 6 1989

**Two-Dimensional Conformal Quantum Field Theory.**

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(ricevuto il 23 Maggio 1988)

*...up-to-dateness is an evanescent quality*
*in a world changing as fast as ours.*
*George Sarton*
Preface to the Dover Edition of
*The Study of the History of Mathematics*
*and The Study of the History of Science*



(*) On leave of absence from the Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia 1784, Bulgaria.

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hep-th/9512160

# A QFT approach to $W_{1+\infty}$

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JAPANESE JOURNAL OF MATHEMATICS SINCE 1924

# Orbifolds of lattice vertex algebras

**Bojko Bakalov · Jason Elsinger · Victor G. Kac · Ivan Todorov**

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Published online: 14 June 2023

# Quantum simulation of massive Thirring and Gross-Neveu models for arbitrary number of flavors

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The study of fermionic quantum field theories is an important problem for realizing the standard model of particle physics on a quantum computer. As a step towards this goal, we consider the massive Thirring and Gross-Neveu models with arbitrary number of fermion flavors, $N_f$, discretized on a spatial one-dimensional lattice of size $L$ in the Hamiltonian formulation. We compute the gate complexity using the higher-order product formula and using block-encoding/qubitization and quantum singular value transformations in the limit of large $N_f$ and $L$. We also prepare the ground states of both models with excellent fidelity for system sizes up to 20 qubits with $N_f = 1, 2, 3, 4$ using the adaptive-variational quantum imaginary time algorithm. In addition, we also classify the dynamical Lie algebras of these relativistic fermionic models and show that they belong to the same isomorphism class. Our work is a concrete step towards the quantum simulation of real-time dynamics of large $N_f$ fermionic quantum field theories models relevant for chiral symmetry breaking, understanding dimensional transmutation, and exploring the conformal window of field theories on near-term and early fault-tolerant quantum computers.

DOI: 10.1103/h7mq-cbz4

*International Journal of Theoretical Physics, Vol. 21, Nos. 6/7, 1982*

# Simulating Physics with Computers

**Richard P. Feynman**

*Department of Physics, California Institute of Technology, Pasadena, California 91107*

*Received May 7, 1981*

Richard Feynman (1918-1988)

# Algorithms for quantum computation: discrete logarithms and factoring

**Publisher: IEEE** Cite This PDF

P.W. Shor All Authors

**Proceedings 35th Annual Symposium on Foundations of Computer Science**

20-22 Nov. 1994

Peter Shor

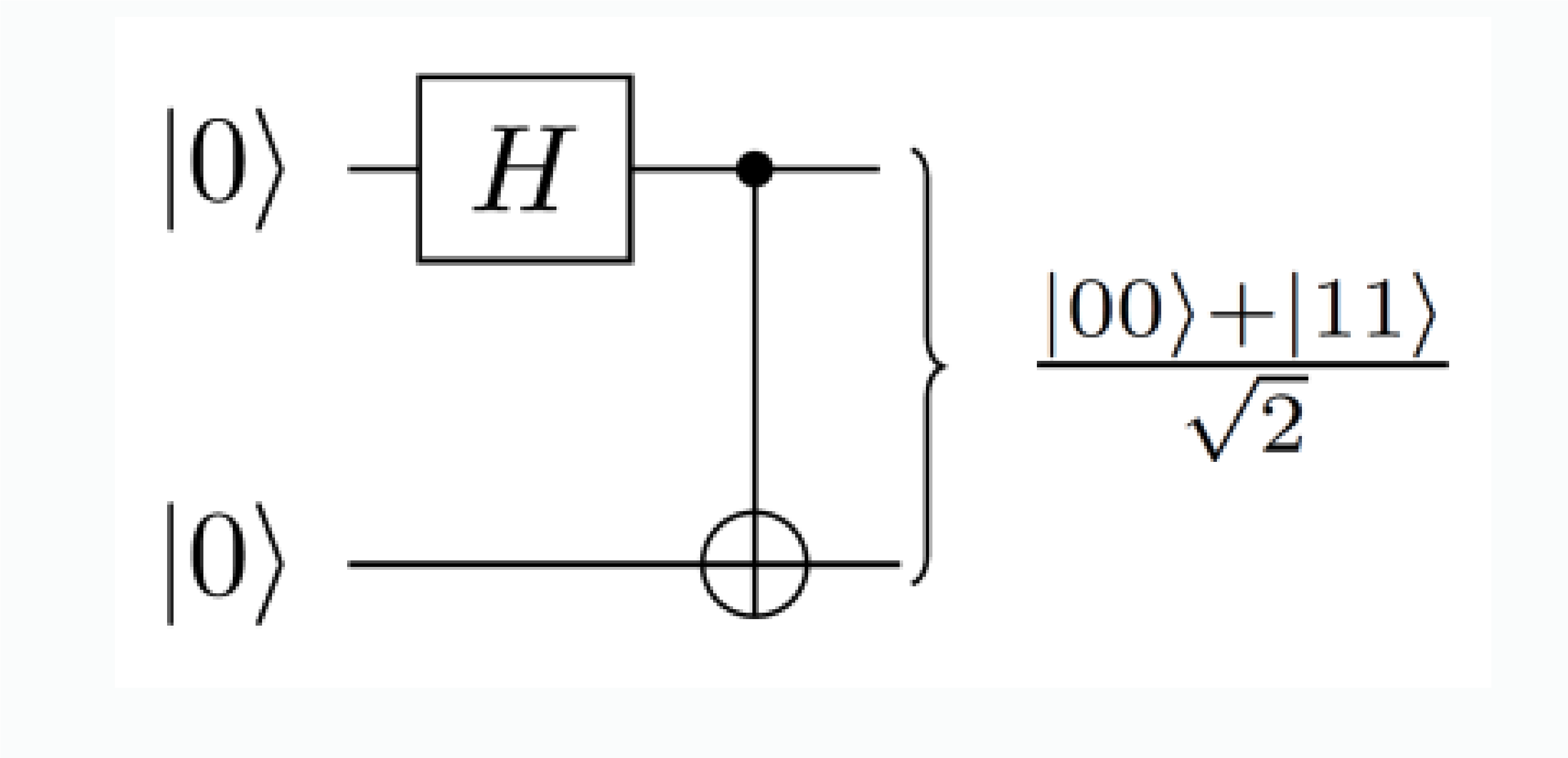
# Quantum Computing Linear Algebra

- An $n$-*qubit* state is a vector $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^N$, $N = 2^n$, with $\langle\psi|\psi\rangle = 1$.
- Hermitian inner product $\langle\varphi|\psi\rangle$ where $\langle\varphi| = |\varphi\rangle^\dagger = \overline{|\varphi\rangle}^T$.
- Schrödinger's equation $(\hbar = 1)$: $$i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$
- $\Rightarrow$ $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, $U(t) = e^{-itH}$ is *unitary*: $U^\dagger = U^{-1}$.
- The **Hamiltonian** $H$ is *Hermitian*: $H^\dagger = H$.

# Quantum Gates and Circuits

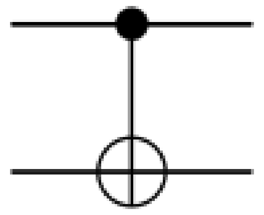

Tony


Io




Bell state, or EPR pair (Einstein-Podolsky-Rosen)

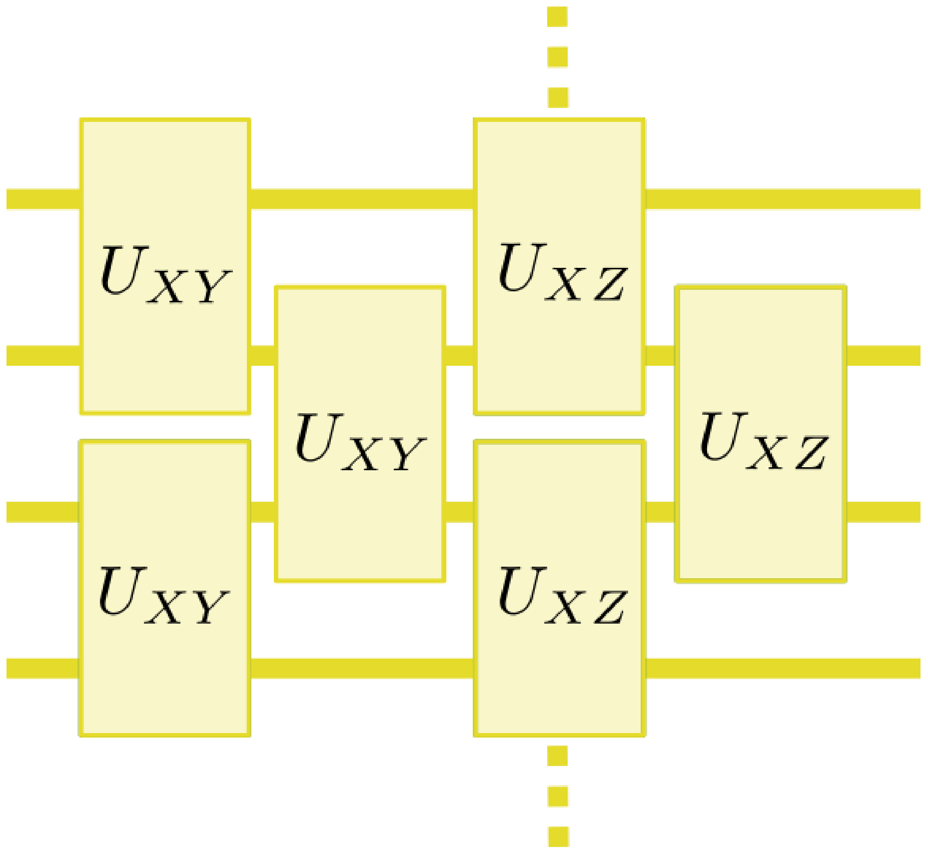
- *entangled* state, the two qubits are correlated.

- $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard gate;
-  $= \mathrm{CNOT} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$ is the controlled-NOT gate;
- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the identity matrix; $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the NOT gate, $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$.

# Dynamical Lie Algebras

**Parametrized Quantum Circuit** (ansatz):

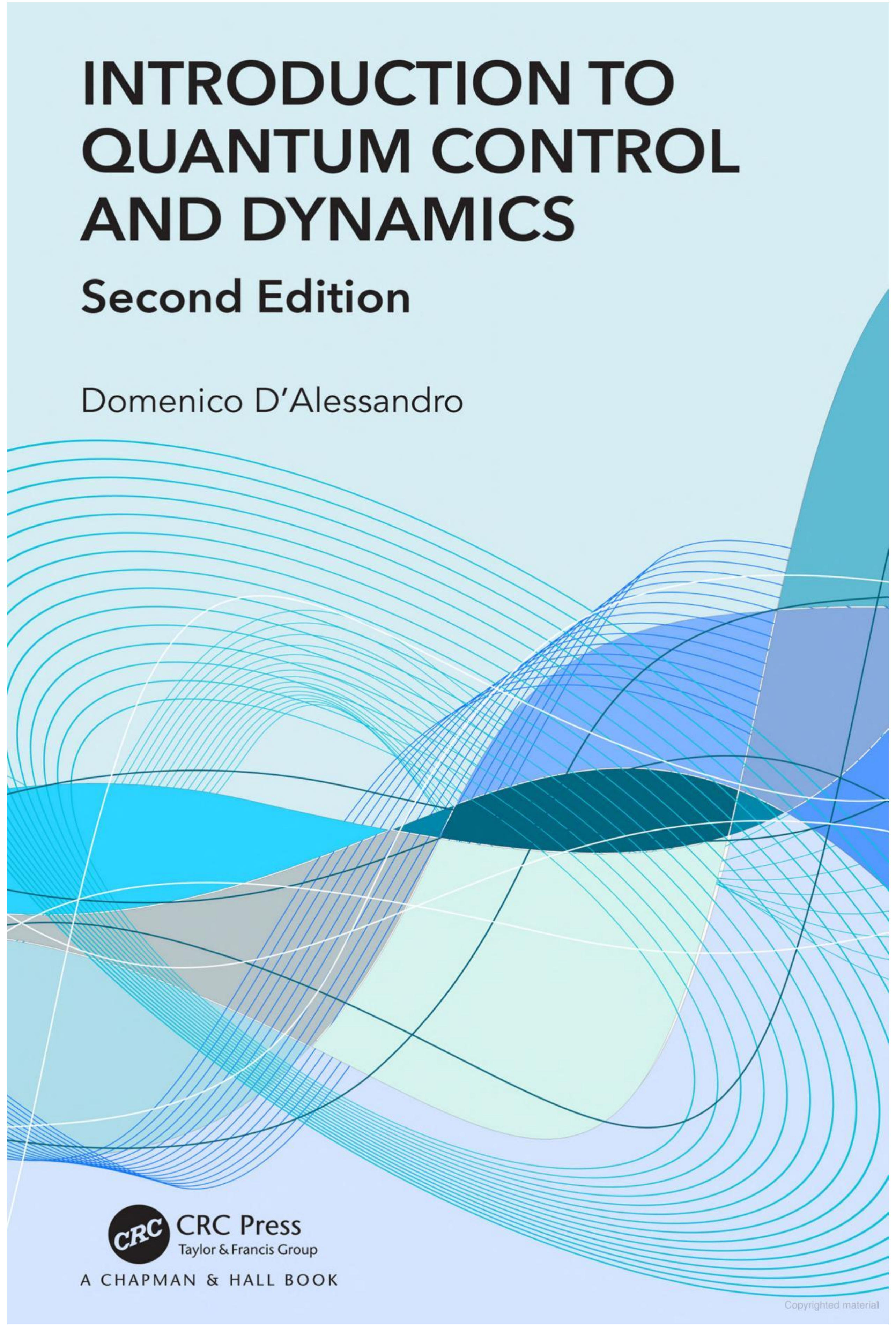
$$U(\theta) = e^{-i\theta_L H_L} \dots e^{-i\theta_2 H_2} e^{-i\theta_1 H_1}$$




The ***dynamical Lie algebra*** (DLA) is the Lie algebra generated by the Hamiltonians:

$$\mathfrak{g} = \langle iH_1, \dots, iH_L \rangle_{\text{Lie}} = \text{span}_{\mathbb{R}}\{[\dots[iH_k, [iH_l, iH_m]]]\}.$$

Then $U(\theta) \in e^{\mathfrak{g}}$ – *dynamical Lie group* $\leq \mathrm{U}(2^n)$.



# The Unitary Group and Lie Algebra

- $\mathrm{SU}(N) = \{A \in \mathbb{C}^{N\times N} \mid A^\dagger A = I,\ \det A = 1\}$ – special unitary group, compact.
- Infinitesimal transformations ⇒ Lie algebra
- $\mathfrak{su}(N) = \{A \in \mathbb{C}^{N\times N} \mid A^\dagger = -A,\ \mathrm{tr}\, A = 0\}$.
- $\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ – **Pauli matrices.**
- $\{X, Y, Z, I\}$ is a basis for the $2 \times 2$ Hermitian matrices.
- The Pauli strings $\{X, Y, Z, I\}^{\otimes n}$ form a basis for $i\mathfrak{u}(2^n)$.

Wolfgang Pauli (1900-1958)

# Examples of DLAs of Hamiltonians

**Example 1: Local Field**

$$H_{\text{loc}} = \sum_{i=1}^{n} X_i$$

$$\mathfrak{g} = \mathfrak{u}(1)^{\oplus n}$$

$\dim \mathfrak{g}$ is linear in $n$.

**Example 2: Transverse Field Ising Model**

$$H_{\text{TFIM}} = \sum_{i=1}^{n-1} Z_i Z_{i+1} + \sum_{i=1}^{n} X_i$$

$$\mathfrak{g} = \mathfrak{so}(2n)$$

$\dim \mathfrak{g}$ is quadratic in $n$.

**Example 3: Heisenberg Model** ($n$ even)

$$H_{\text{Heis.}} = \sum_{i=1}^{n-1} X_i X_{i+1} + Y_i Y_{i+1} + Z_i Z_{i+1}$$

$$\mathfrak{g} = \mathfrak{su}(2^{n-2})^{\oplus 4}$$

$\dim \mathfrak{g}$ is exponential in $n$.

The DLA is generated by the terms of the Hamiltonian.

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## Fixed Depth Hamiltonian Simulation via Cartan Decomposition

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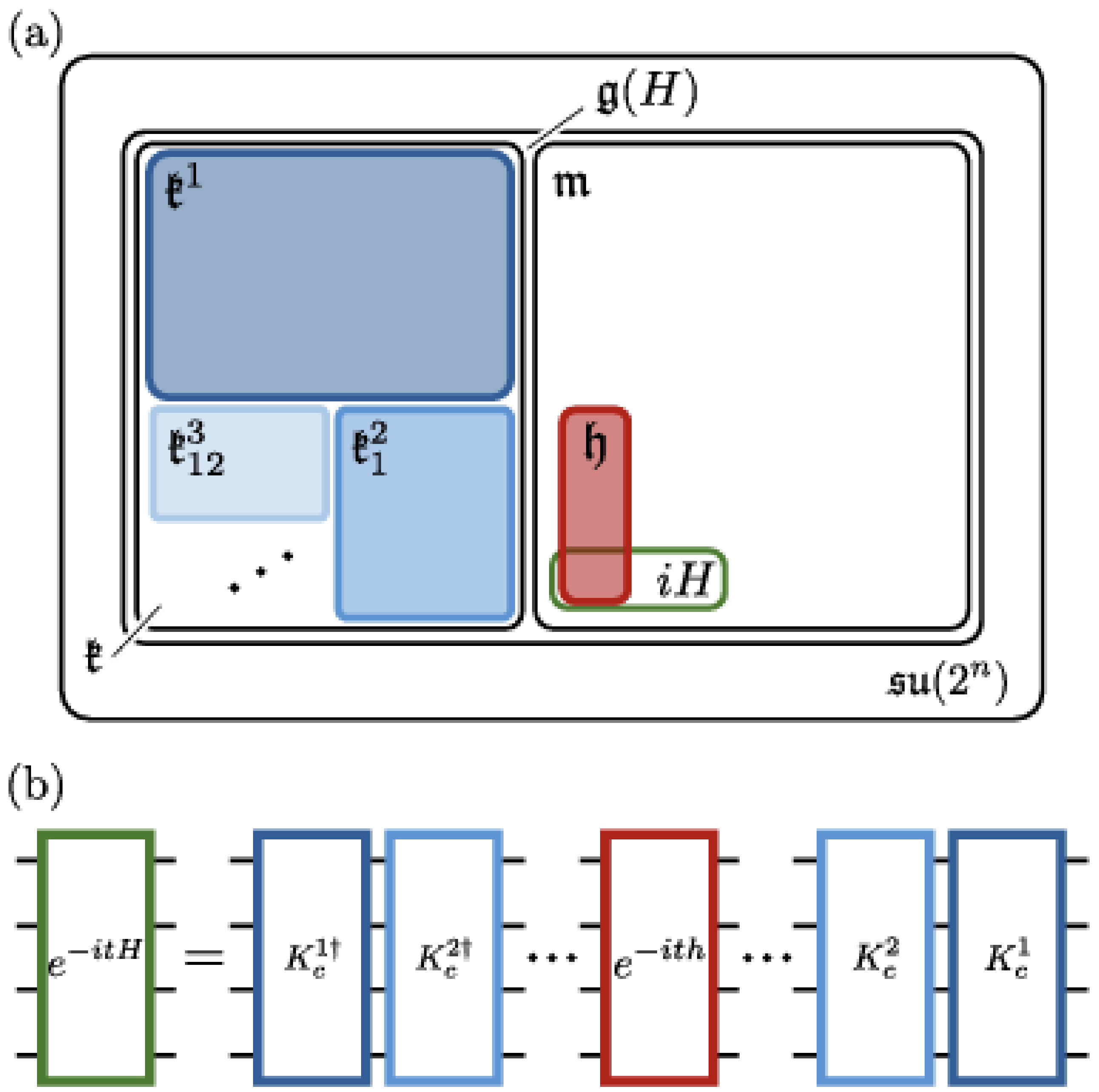
(Received 3 May 2021; revised 9 April 2022; accepted 28 June 2022; published 9 August 2022)

## RedCarD: A Quantum Assisted Algorithm for Fixed-Depth Unitary Synthesis via Cartan Decomposition

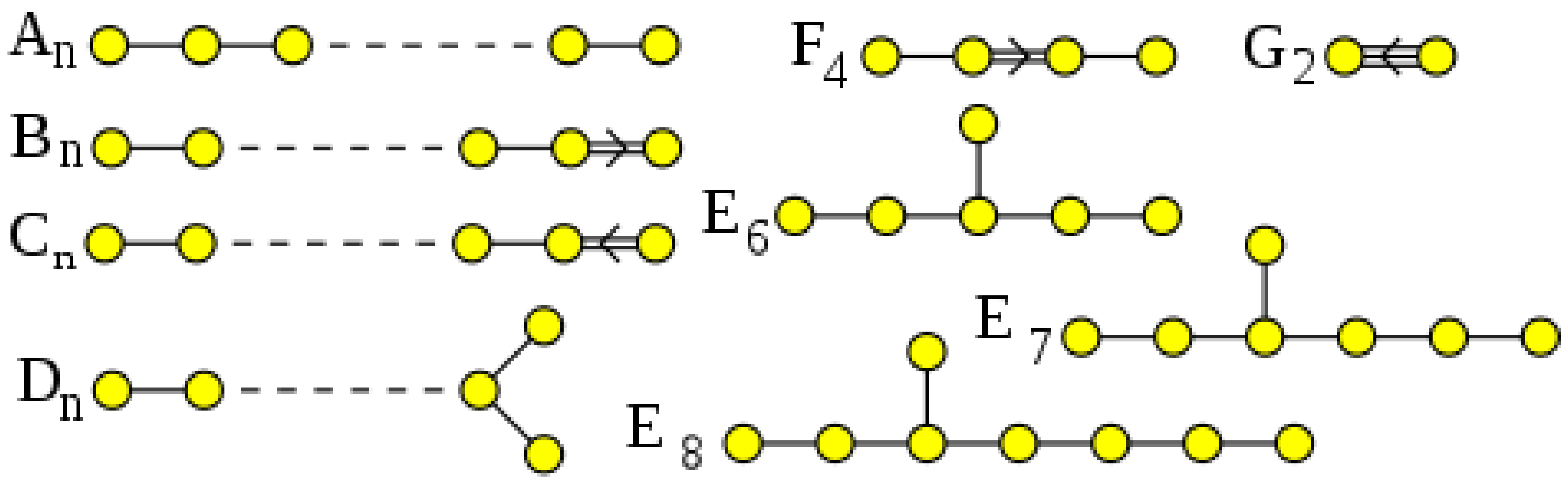
Omar Alsheikh,[1,*] Efekan Kökcü,[2] Bojko N. Bakalov,[3] and Alexander F. Kemper[1,†]



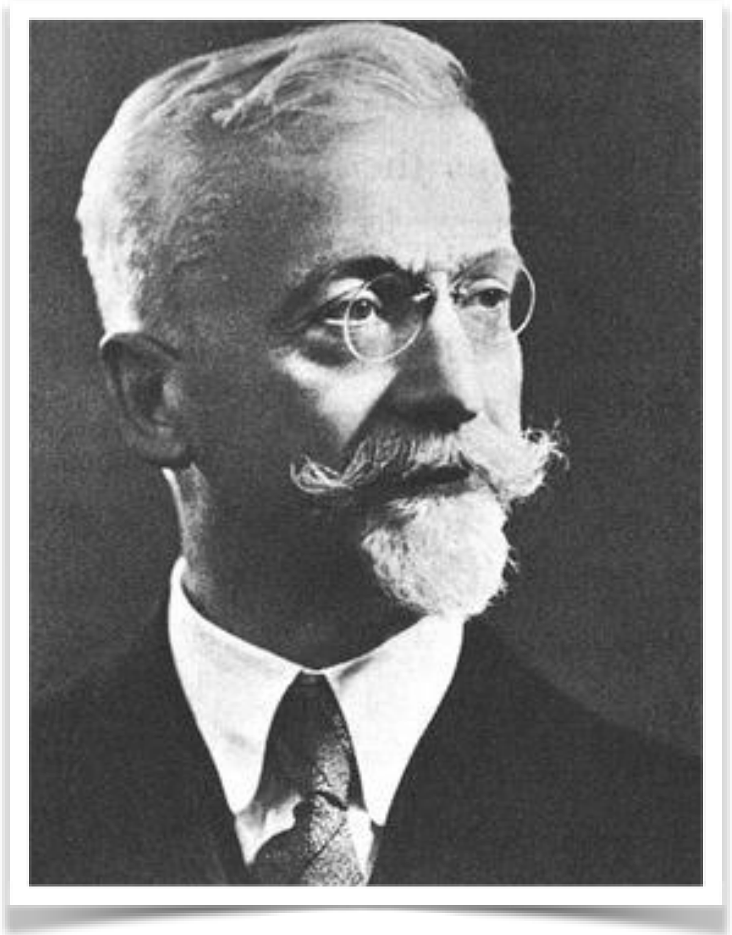


# Classification of Compact Simple Lie Algebras

**Theorem** [É. Cartan, *Ann. Sci. Ec. Norm. Super*. (1914)]
Every compact simple Lie algebra is isomorphic to: $\mathfrak{su}(N)$, $\mathfrak{so}(N)$, $\mathfrak{sp}(N)$ or one of 5 exceptional Lie algebras.




Dynkin diagrams



Élie Cartan (1869-1951)

Eugene Dynkin (1924-2014)

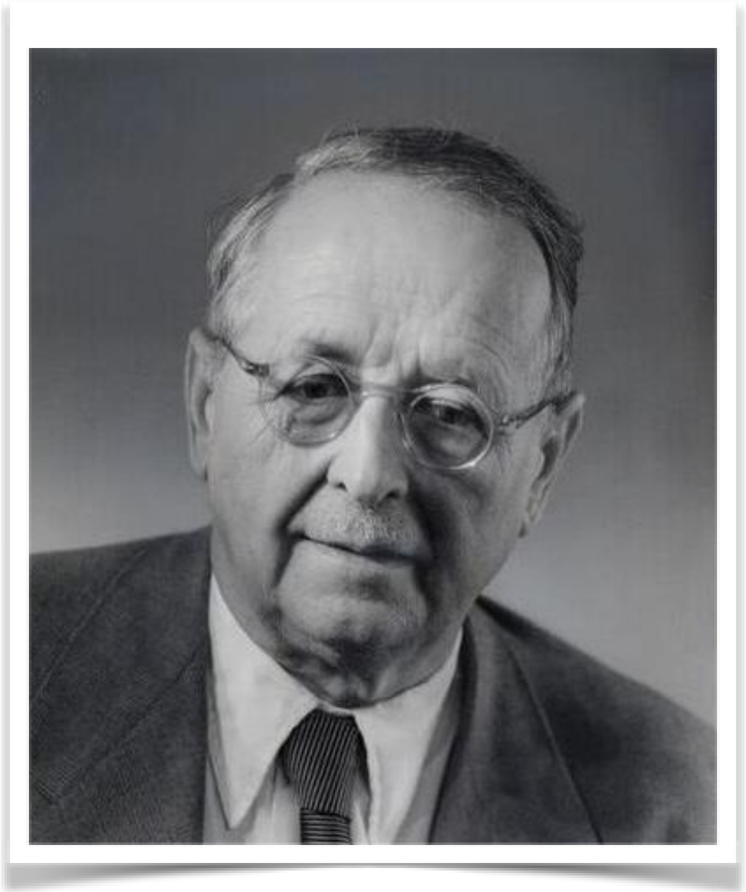
# Can We Classify All DLAs?

- **Proposition** [D'Alessandro 2007; see also *npj Quantum Information* (2024)]
  The DLA is a subalgebra of $\mathfrak{u}(N)$; hence is a direct sum of simple compact Lie algebras and a center.

- An embedding $\mathfrak{g} \hookrightarrow \mathfrak{u}(N) \Leftrightarrow$ unitary representation of $\mathfrak{g}$ on $\mathbb{C}^N$.

- Zeier and Schulte-Herbrüggen [*J. Math. Phys.* 2011] listed all subalgebras of $\mathfrak{su}(2^n)$ for $n \leq 15$ in 95 pages of tables.

- Dynkin (1952) classified all *maximal* subalgebras of simple Lie algebras.



Hermann Weyl (1885-1955)

npj | quantum information

Article

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# Classification of dynamical Lie algebras of 2-local spin systems on linear, circular and fully connected topologies

Check for updates

Roeland Wiersema[1,2,3], Efekan Kökcü[4], Alexander F. Kemper[4] & Bojko N. Bakalov[5]

# Classification of dynamical Lie algebras generated by spin interactions on undirected graphs

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Efekan Kökcü,[1,2,3,a) Roeland Wiersema,[4,5,6] Alexander F. Kemper,[1] and Bojko N. Bakalov[7]

# Classification Theorem

$$\mathfrak{a}_0(n) = \mathrm{span}\{X_jX_{j+1}\}_{1\le j\le n-1} \cong \mathfrak{u}(1)^{\oplus(n-1)}, \quad \dim = n-1,$$

$$\mathfrak{a}_1(n) = \mathrm{span}\{X_iZ_{i+1}\cdots Z_{j-1}Y_j\}_{1\le i<j\le n} \cong \mathfrak{so}(n), \quad \dim = \frac{n(n-1)}{2},$$

$$\mathfrak{a}_2(n) = \mathrm{span}\{X_iZ_{i+1}\cdots Z_{j-1}Y_j\}_{1\le i<j\le n} \oplus \mathrm{span}\{Y_iZ_{i+1}\cdots Z_{j-1}X_j\}_{1\le i<j\le n} \cong \mathfrak{so}(n)\oplus\mathfrak{so}(n), \quad \dim = n(n-1),$$

$$\mathfrak{a}_3(n) \cong \begin{cases} \mathfrak{so}(2^{n-2})^{\oplus 4}, & \dim = 2^{n-1}(2^{n-2}-1), \quad n\equiv 0 \mod 8,\\ \mathfrak{so}(2^{n-1}), & \dim = 2^{n-2}(2^{n-1}-1), \quad n\equiv \pm1 \mod 8,\\ \mathfrak{su}(2^{n-2})^{\oplus 2}, & \dim = 2^{2n-3}-2, \quad n\equiv \pm2 \mod 8,\\ \mathfrak{sp}(2^{n-2}), & \dim = 2^{n-2}(2^{n-1}+1), \quad n\equiv \pm3 \mod 8,\\ \mathfrak{sp}(2^{n-3})^{\oplus 4}, & \dim = 2^{n-1}(2^{n-2}+1), \quad n\equiv 4 \mod 8,\end{cases}$$

$$\mathfrak{a}_4(n) \cong \mathfrak{a}_2(n),$$

$$\mathfrak{a}_5(n) \cong \begin{cases} \mathfrak{so}(2^{n-2})^{\oplus 4}, & \dim = 2^{n-1}(2^{n-2}-1), \quad n\equiv 0 \mod 6,\\ \mathfrak{so}(2^{n-1}), & \dim = 2^{n-2}(2^{n-1}-1), \quad n\equiv \pm1 \mod 6,\\ \mathfrak{su}(2^{n-2})^{\oplus 2}, & \dim = 2^{2n-3}-2, \quad n\equiv \pm2 \mod 6,\\ \mathfrak{sp}(2^{n-2}), & \dim = 2^{n-2}(2^{n-1}+1), \quad n\equiv 3 \mod 6,\end{cases}$$

$$\mathfrak{a}_6(n) \cong \mathfrak{a}_7(n) \cong \mathfrak{a}_{10}(n) \cong \begin{cases} \mathfrak{su}(2^{n-1}), & \dim = 2^{2n-2}-1, \quad n \text{ odd},\\ \mathfrak{su}(2^{n-2})^{\oplus 4}, & \dim = 2^{2n-2}-4, \quad n\ge 4 \text{ even},\end{cases}$$

$$\mathfrak{a}_8(n) \cong \mathfrak{so}(2n-1), \quad \dim = (n-1)(2n-1),$$

$$\mathfrak{a}_9(n) \cong \mathfrak{sp}(2^{n-2}), \quad \dim = 2^{n-2}(2^{n-1}+1),$$

$$\mathfrak{a}_{11}(n) = \mathfrak{a}_{16}(n) = \mathfrak{so}(2^n), \quad \dim = 2^{n-1}(2^n-1), \quad n\ge 4,$$

$$\mathfrak{a}_k(n) = \mathfrak{su}(2^n), \quad \dim = 2^{2n}-1, \quad k = 12,17,18,19,21,22, \quad n\ge 4,$$

$$\mathfrak{a}_{13}(n) = \mathfrak{a}_{20}(n) \cong \mathfrak{a}_{15}(n) \cong \mathfrak{su}(2^{n-1})\oplus\mathfrak{su}(2^{n-1}), \quad \dim = 2^{2n-1}-2,$$

$$\mathfrak{a}_{14}(n) \cong \mathfrak{so}(2n), \quad \dim = n(2n-1),$$

$$\mathfrak{b}_0(n) = \mathrm{span}\{X_i\}_{1\le i\le n} \cong \mathfrak{u}(1)^{\oplus n}, \quad \dim = n,$$

$$\mathfrak{b}_1(n) = \mathrm{span}\{X_i, X_jX_{j+1}\}_{1\le i\le n,\ 1\le j\le n-1} \cong \mathfrak{u}(1)^{\oplus(2n-1)}, \quad \dim = 2n-1,$$

$$\mathfrak{b}_2(n) = \mathfrak{a}_9(n)\oplus\mathrm{span}\{X_1\} \cong \mathfrak{sp}(2^{n-2})\oplus\mathfrak{u}(1), \quad \dim = 2^{n-2}(2^{n-1}+1)+1,$$

$$\mathfrak{b}_3(n) = \mathrm{span}\{X_i, Y_i, Z_i\}_{1\le i\le n} \cong \mathfrak{su}(2)^{\oplus n}, \quad \dim = 3n,$$

$$\mathfrak{b}_4(n) = \mathfrak{a}_{15}(n)\oplus\mathrm{span}\{X_1\} \cong \mathfrak{su}(2^{n-1})\oplus\mathfrak{su}(2^{n-1})\oplus\mathfrak{u}(1), \quad \dim = 2^{2n-1}-1.$$

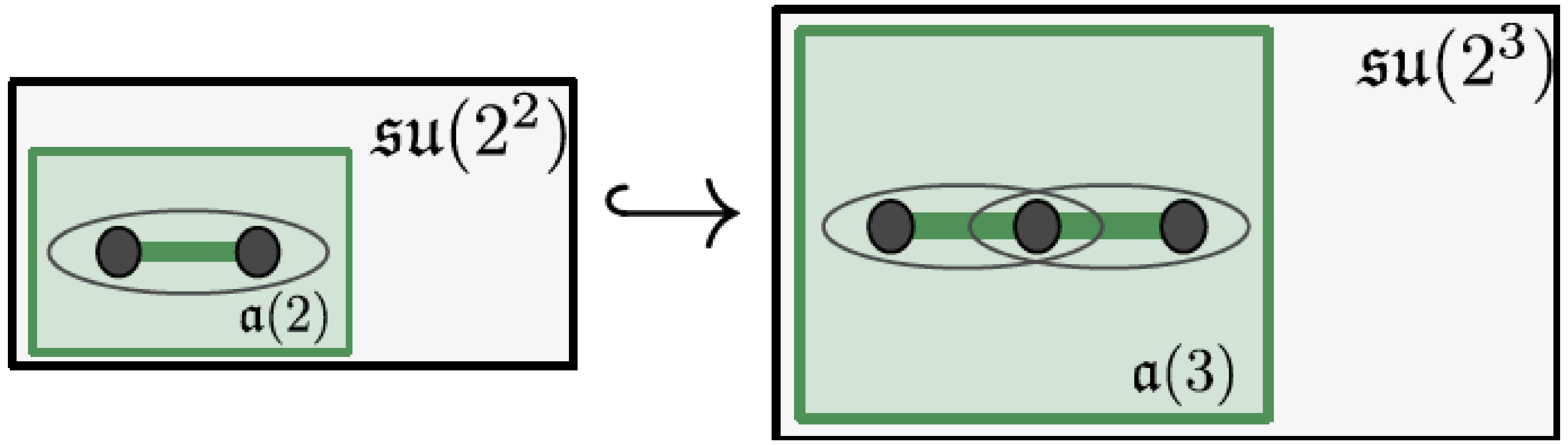
Wiersema-Kökcü-Kemper-Bakalov,
*npj Quantum Information* 2024



| Label | Generating set | Example Model |
|---|---|---|
| $\mathfrak{a}_0$ | *XX* | Ising model |
| $\mathfrak{a}_1$ | *XY* | Kitaev chain |
| $\mathfrak{a}_2$ | *XY*, *YX* | Massless free fermion + magnetic field |
| $\mathfrak{a}_3$ | *XX*, *YZ* | Kitaev chain + Coulomb |
| $\mathfrak{a}_4$ | *XX*, *YY* | XY-model |
| $\mathfrak{a}_5$ | *XY*, *YZ* | |
| $\mathfrak{a}_6$ | *XX*, *YZ*, *ZY* | Massless free fermion + magnetic field + Coulomb |
| $\mathfrak{a}_7$ | *XX*, *YY*, *ZZ* | Heisenberg chain |
| $\mathfrak{a}_8$ | *XX*, *XZ* | Ising model + transverse field |
| $\mathfrak{a}_9$ | *XY*, *XZ* | Kitaev chain + longitudinal field |
| $\mathfrak{a}_{10}$ | *XY*, *YZ*, *ZX* | Heisenberg |
| $\mathfrak{a}_{11}$ | *XY*, *YX*, *YZ* | XY-model + longitudinal field |
| $\mathfrak{a}_{12}$ | *XX*, *XY*, *YZ* | |
| $\mathfrak{a}_{13}$ | *XX*, *YY*, *YZ* | XY-model + longitudinal field |
| $\mathfrak{a}_{14}$ | *XX*, *YY*, *XY* | XY-model + transverse field |
| $\mathfrak{a}_{15}$ | *XX*, *XY*, *XZ* | Ising model + arbitrary field |
| $\mathfrak{a}_{16}$ | *XY*, *YX*, *YZ*, *ZY* | Kitaev chain + longitudinal field |
| $\mathfrak{a}_{17}$ | *XX*, *XY*, *ZX* | Ising model + arbitrary field |
| $\mathfrak{a}_{18}$ | *XX*, *XZ*, *YY*, *ZY* | XY-model + arbitrary field |
| $\mathfrak{a}_{19}$ | *XX*, *XY*, *ZX*, *YZ* | |
| $\mathfrak{a}_{20}$ | *XX*, *YY*, *ZZ*, *ZY* | Heisenberg chain + magnetic field |
| $\mathfrak{a}_{21}$ | *XX*, *YY*, *XY*, *ZX* | XY-model + arbitrary field |
| $\mathfrak{a}_{22}$ | *XX*, *XY*, *XZ*, *YX* | Ising model + arbitrary field |
| $\mathfrak{b}_0$ | *XI*, *IX* | Uncoupled spins |
| $\mathfrak{b}_1$ | *XX*, *XI*, *IX* | Ising model |
| $\mathfrak{b}_2$ | *XY*, *XI*, *IX* | Kitaev chain + longitudinal field |
| $\mathfrak{b}_3$ | *XI*, *YI*, *IX*, *IY* | Uncoupled spins |
| $\mathfrak{b}_4$ | *XX*, *XY*, *XI*, *IX* | Ising model + arbitrary field |

# Proof Strategy

Identify all unique DLAs on 2-qubits and add sites to see how the DLAs extend to large $n$.



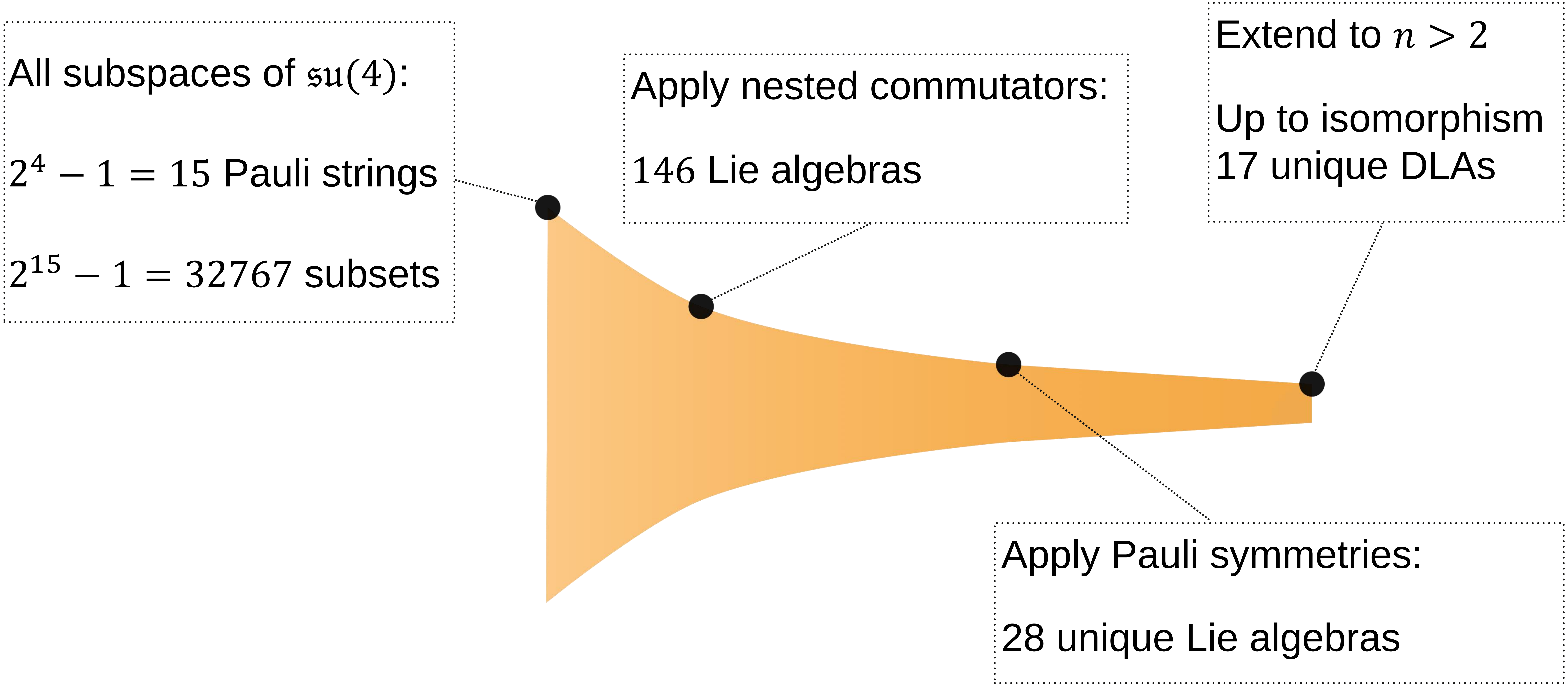

- We find the *commutant* of the DLA: the set $S$ all Pauli strings that commute with the generators of the DLA; hence with the whole DLA.

- We describe the DLA as the fixed points of an involution inside the *centralizer* of $S$.

- An *involution* on a Lie algebra $\mathfrak{g}$ is a map $\theta: \mathfrak{g} \to \mathfrak{g}$ such that $\theta^2 = \mathrm{id}$ and $\theta([A, B]) = [\theta(A), \theta(B)]$ .

- Examples: $\theta(A) = -A^T$ on $\mathfrak{g} = \mathfrak{su}(N)$ has fixed points $\mathfrak{so}(N)$;
  $\theta(A) = JA^TJ$ on $\mathfrak{g} = \mathfrak{su}(2N)$ has fixed points $\mathfrak{sp}(N)$.

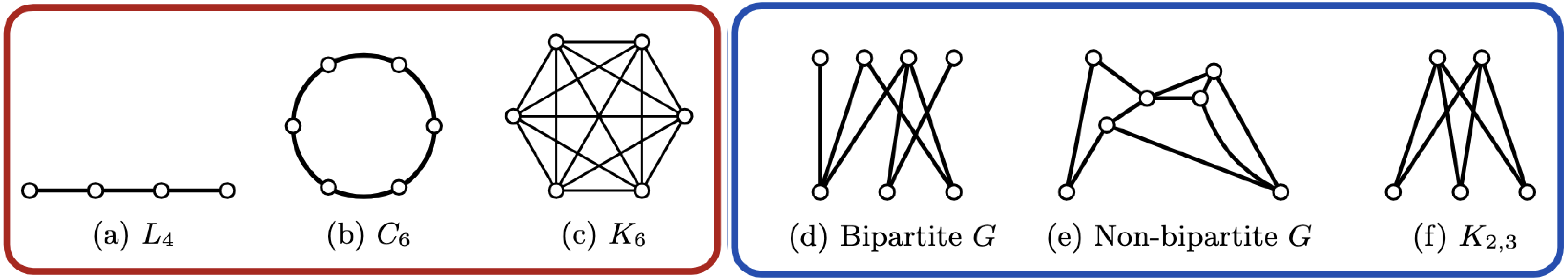
# DLAs on $n$ Sites



# DLAs of Interaction Graphs

- We generate a DLA by placing the Pauli matrices on the vertices of the interaction graph and the length-2 Pauli strings on the edges.

- **Examples** of interaction graphs: (a) line; (b) cycle; (c) complete graph; (f) complete bipartite graph.




[Wiersema-Kökcü-Kemper-Bakalov, *npj Quantum Information* (2024)]
[Kökcü-Wiersema-Kemper-Bakalov, *J. Math Phys.* (2026)]

# DLAs Generated by Pauli Strings

- We define a DLA $\mathfrak{a}^G$ generated by a set of length-2 Pauli strings and single Pauli matrices by placing them on the edges, respectively vertices, of a graph $G$.

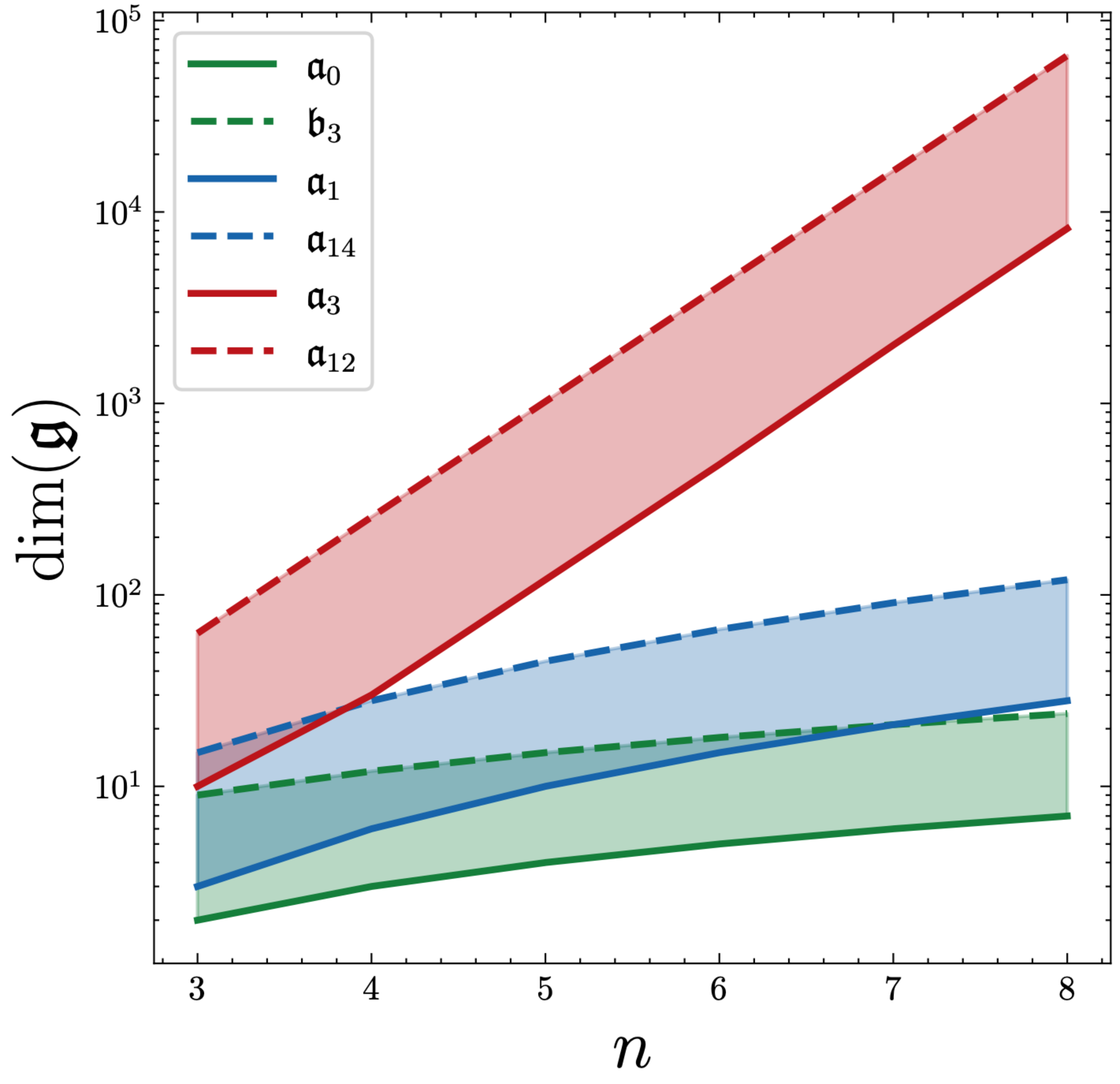
- **Theorem** [Kökcü-Wiersema-Kemper-Bakalov, *J. Math Phys.* (2026)]
  - $\mathfrak{a}_k^G \cong \mathfrak{a}_k^{K_n}$ for $k = 7,16,20,22$ and every graph $G$ with $n$ vertices;
  - $\mathfrak{a}_k^G \cong \mathfrak{a}_k^{K_n}$ for $k = 2,4,6,14$ and every *non-bipartite* graph $G$ with $n$ vertices;
  - $\mathfrak{a}_k^G \cong \mathfrak{a}_k^{K_{l,m}}$ for $k = 2,4,6,14$ and every *bipartite* graph $G$ with $l, m$ vertices.

- Aguilar, Cichy, Eisert, and Bittel [*arXiv:2408.00081*] prove that every DLA generated by Pauli strings is isomorphic to a direct sum of copies of $\mathfrak{su}(N)$, $\mathfrak{so}(N)$, $\mathfrak{sp}(N)$ (no exceptional Lie algebras). They provide a process how to determine the DLA.

# DLA Dimensions

- The scaling of the dimension of these dynamical Lie algebras is either linear, quadratic or exponential in the number of sites $n$.

- The non-exponential ones can be simulated classically – free fermions.



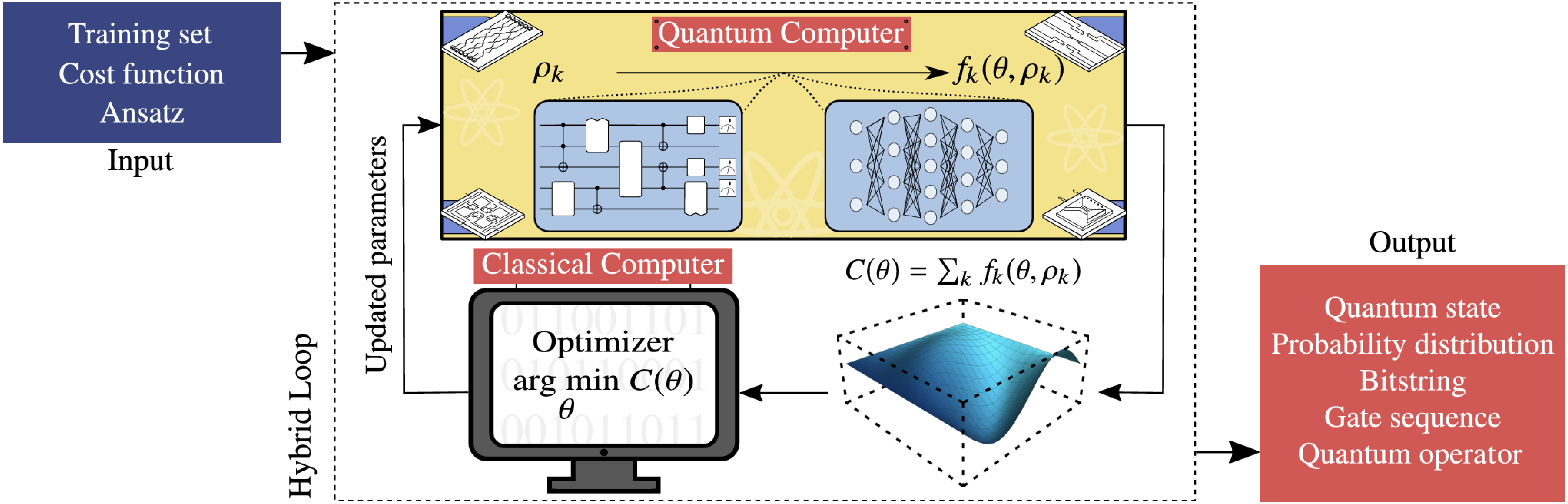

Wiersema-Kökcü-Kemper-Bakalov, *arXiv:2309.05690*
Kökcü-Wiersema-Kemper-Bakalov, *arXiv:2409.19797*
Aguilar, Cichy, Eisert, and Bittel, *arXiv:2408.00081*
Goh, Larocca, Cincio, Cerezo, Sauvage, *arXiv:2308.01432*
M. Cerezo et al., *arXiv:2312.09121*

Review Article | Published: 12 August 2021

# Variational quantum algorithms

M. Cerezo, Andrew Arrasmith, Ryan Babbush, Simon C. Benjamin, Suguru Endo, Keisuke Fujii, Jarrod R. McClean, Kosuke Mitarai, Xiao Yuan, Lukasz Cincio & Patrick J. Coles

*Nature Reviews Physics* **3**, 625–644 (2021) | Cite this article



nature COMMUNICATIONS

ARTICLE

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A variational eigenvalue solver on a photonic quantum processor

Alberto Peruzzo[1,*,†], Jarrod McClean[2,*], Peter Shadbolt[1], Man-Hong Yung[2,3], Xiao-Qi Zhou[1], Peter J. Love[4], Alán Aspuru-Guzik[2] & Jeremy L. O'Brien[1]

## Chemistry Applications

- Finding the ground energy of complex molecules (e.g., Nitrogen fixation catalysts).
- Simulating drug-receptor interactions with exact correlation.
- Discovery of new battery materials and high-temperature superconductors.

## **Variational Quantum Eigensolver** (VQE)

The minimum of the *loss function* $C(\theta) = \langle 0 | U(\theta)^{\dagger} \, H \, U(\theta) | 0 \rangle$

$\geq$ min eigenvalue of $H$. It will approach it when the ansatz $U(\theta)$ is *expressive*.

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# Barren plateaus in quantum neural network training landscapes

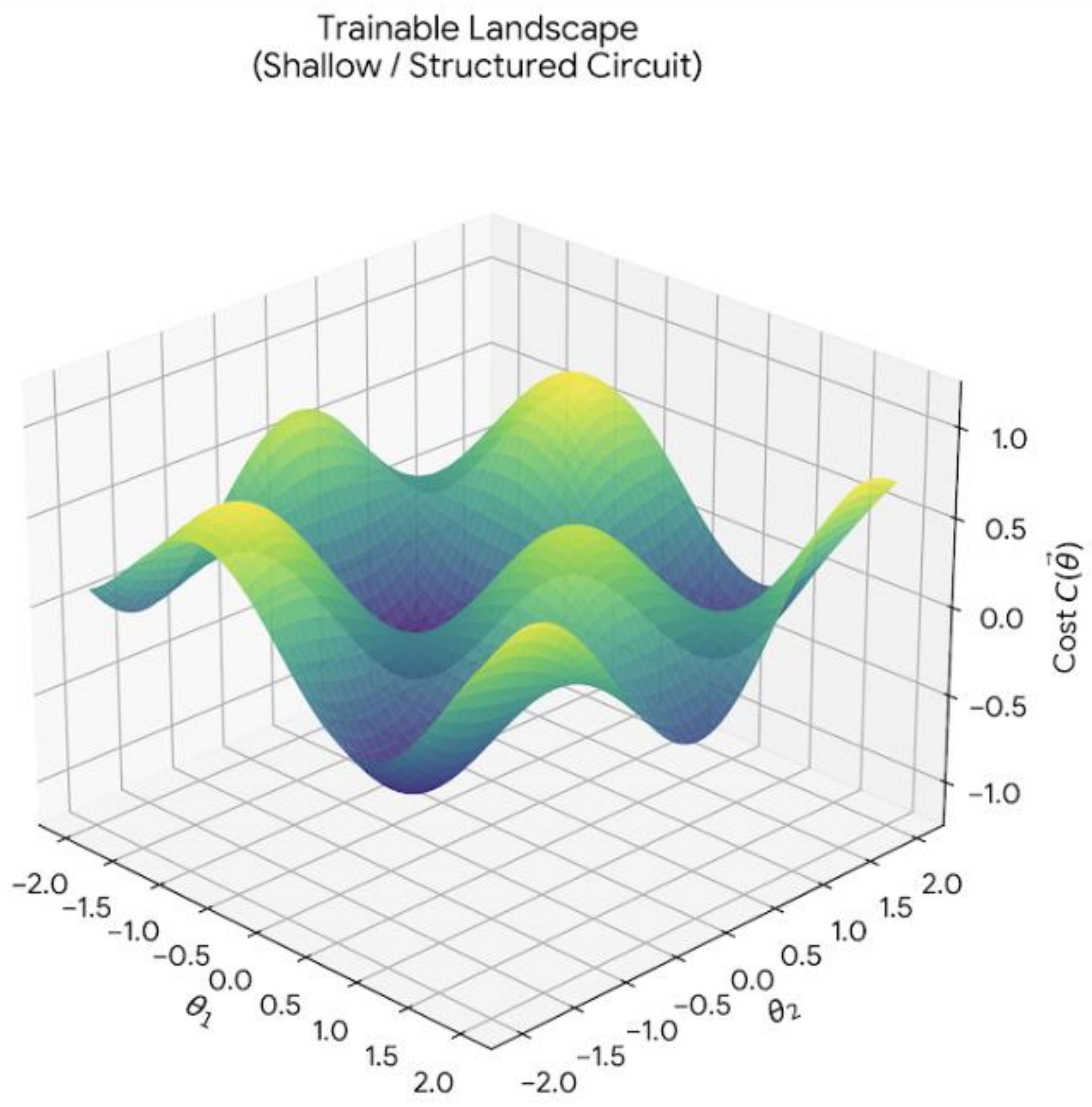
Jarrod R. McClean[1], Sergio Boixo[1], Vadim N. Smelyanskiy[1], Ryan Babbush[1] & Hartmut Neven[1]

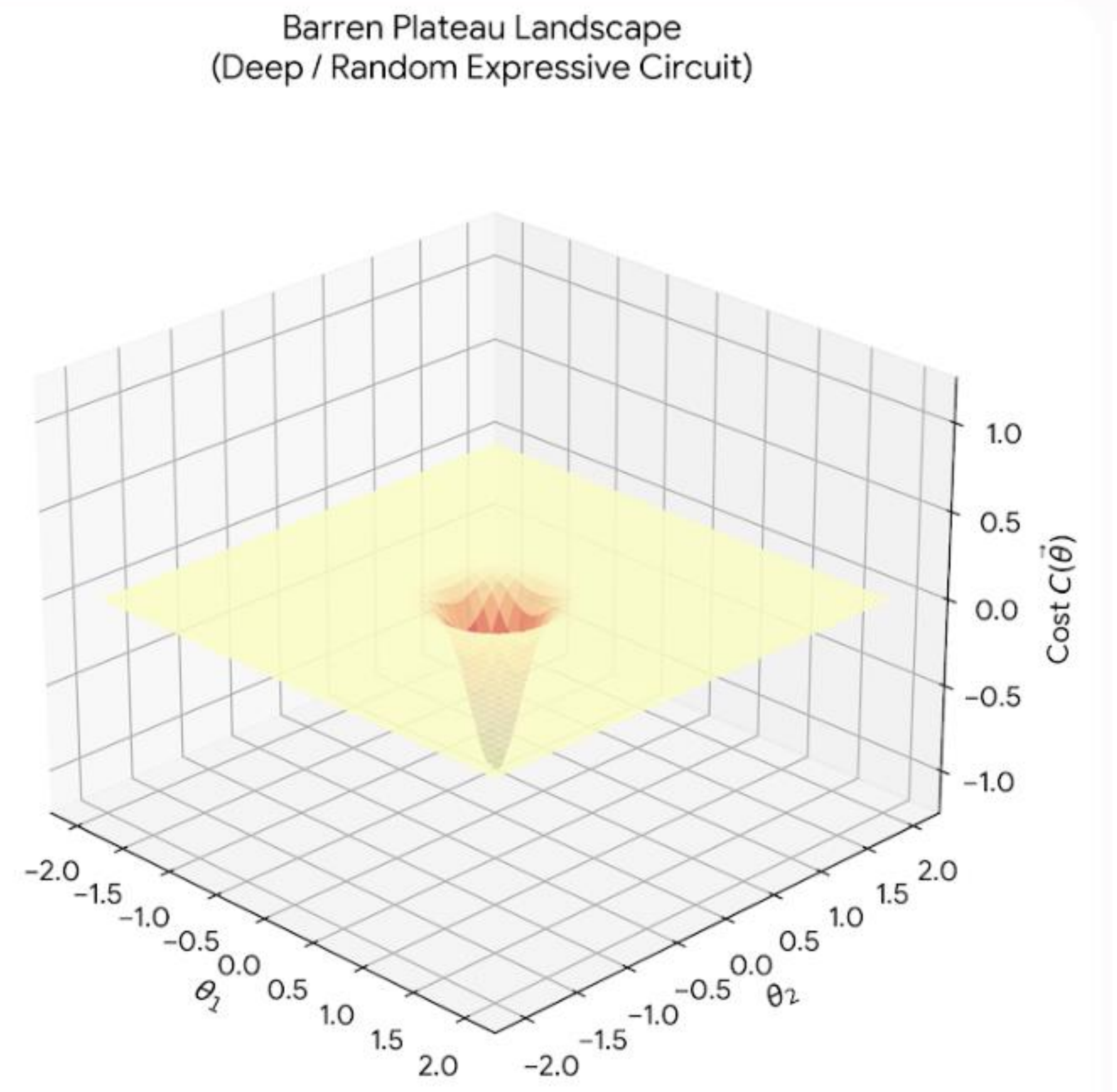
Review Article | Published: 26 March 2025

# Barren plateaus in variational quantum computing

Martín Larocca, Supanut Thanasilp, Samson Wang, Kunal Sharma, Jacob Biamonte, Patrick J. Coles, Lukasz Cincio, Jarrod R. McClean, Zoë Holmes & M. Cerezo

*Nature Reviews Physics* **7**, 174–189 (2025) | Cite this article



# Diagnosing Barren Plateaus with Tools from Quantum Optimal Control

Martin Larocca[1,2], Piotr Czarnik[2], Kunal Sharma[3,2], Gopikrishnan Muraleedharan[2], Patrick J. Coles[2], and M. Cerezo[4,5]

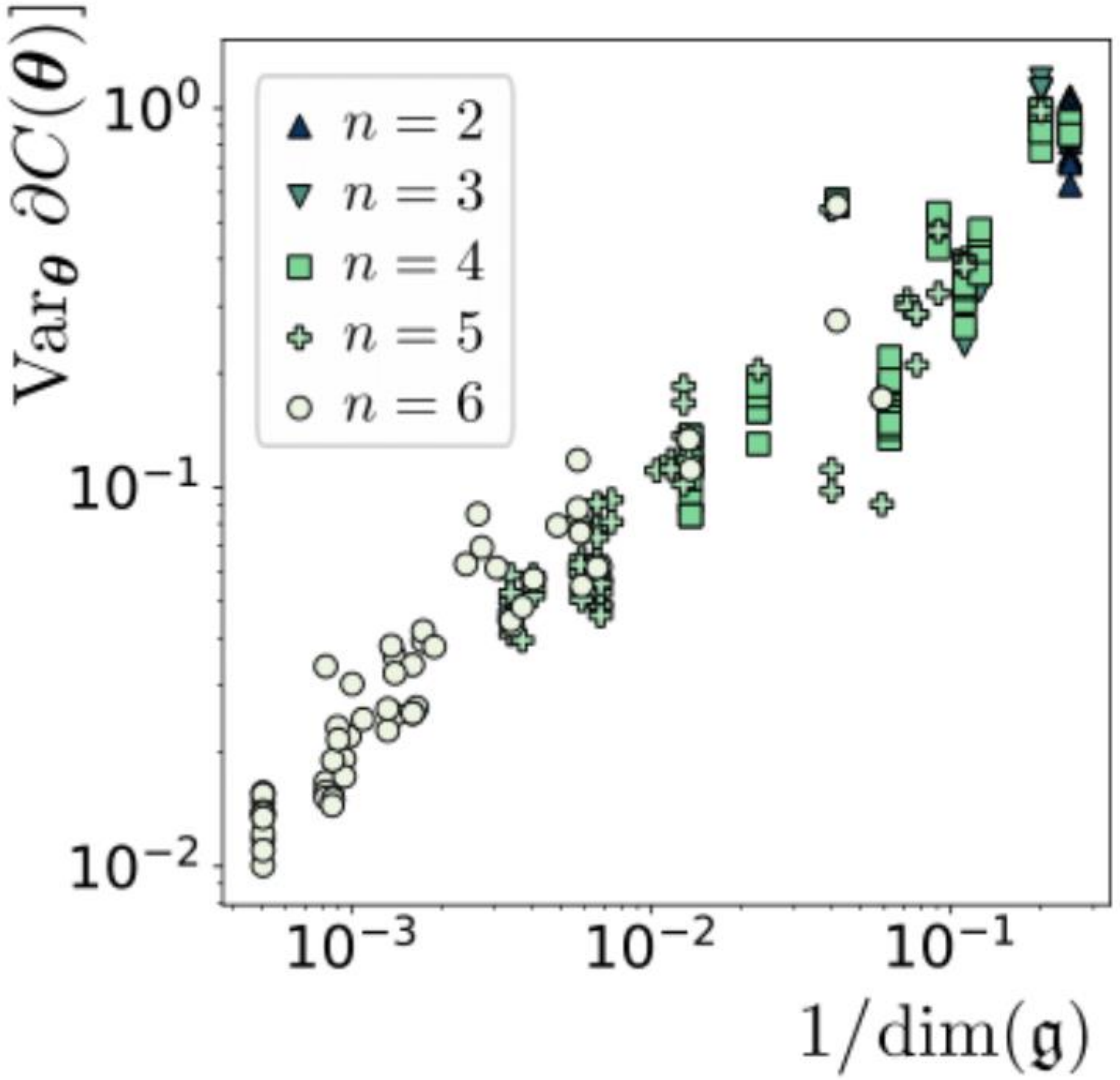
Quantum 6, 824 (2022).

**Conjecture :**

Gradient Scaling

$$\mathrm{Var}_{\boldsymbol{\theta}}[\partial_\mu C(\boldsymbol{\theta})] \in \mathcal{O}\left(\frac{1}{\mathrm{poly}(\dim(\mathfrak{g}_k))}\right)$$

Size of dynamical Lie algebra



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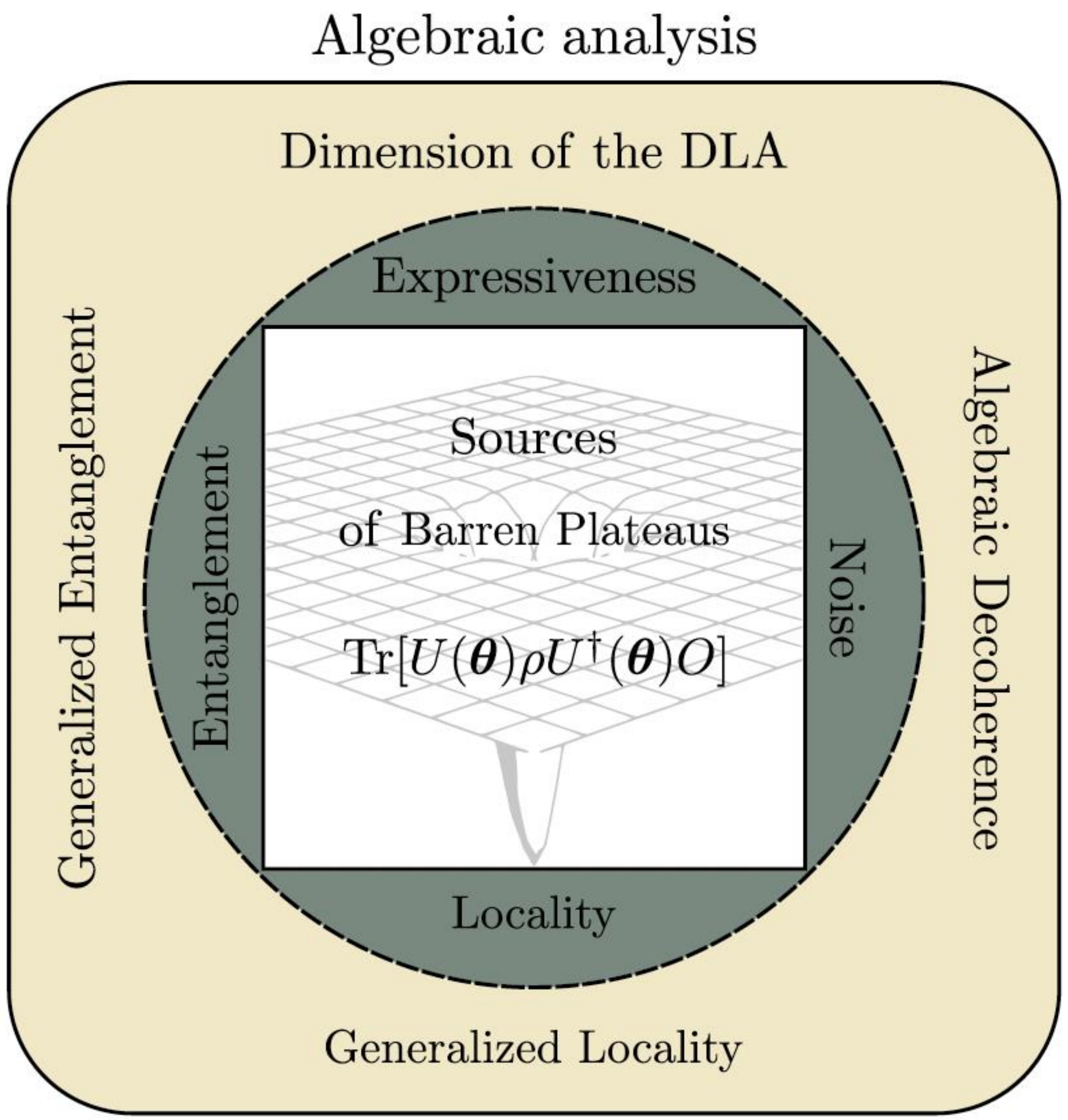
Article

https://doi.org/10.1038/s41467-024-49909-3

# A Lie algebraic theory of barren plateaus for deep parameterized quantum circuits

Michael Ragone[1,9], Bojko N. Bakalov[2,9], Frédéric Sauvage[3], Alexander F. Kemper[4], Carlos Ortiz Marrero[5,6], Martín Larocca[3,7] & M. Cerezo[8]

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Article

https://doi.org/10.1038/s41467-024-49910-w

# Characterizing barren plateaus in quantum ansätze with the adjoint representation

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Enrico Fontana[1,2], Dylan Herman[1], Shouvanik Chakrabarti[1], Niraj Kumar[1], Romina Yalovetzky[1], Jamie Heredge[1,3], Shree Hari Sureshbabu[1] & Marco Pistoia[1]

# Barren Plateaus

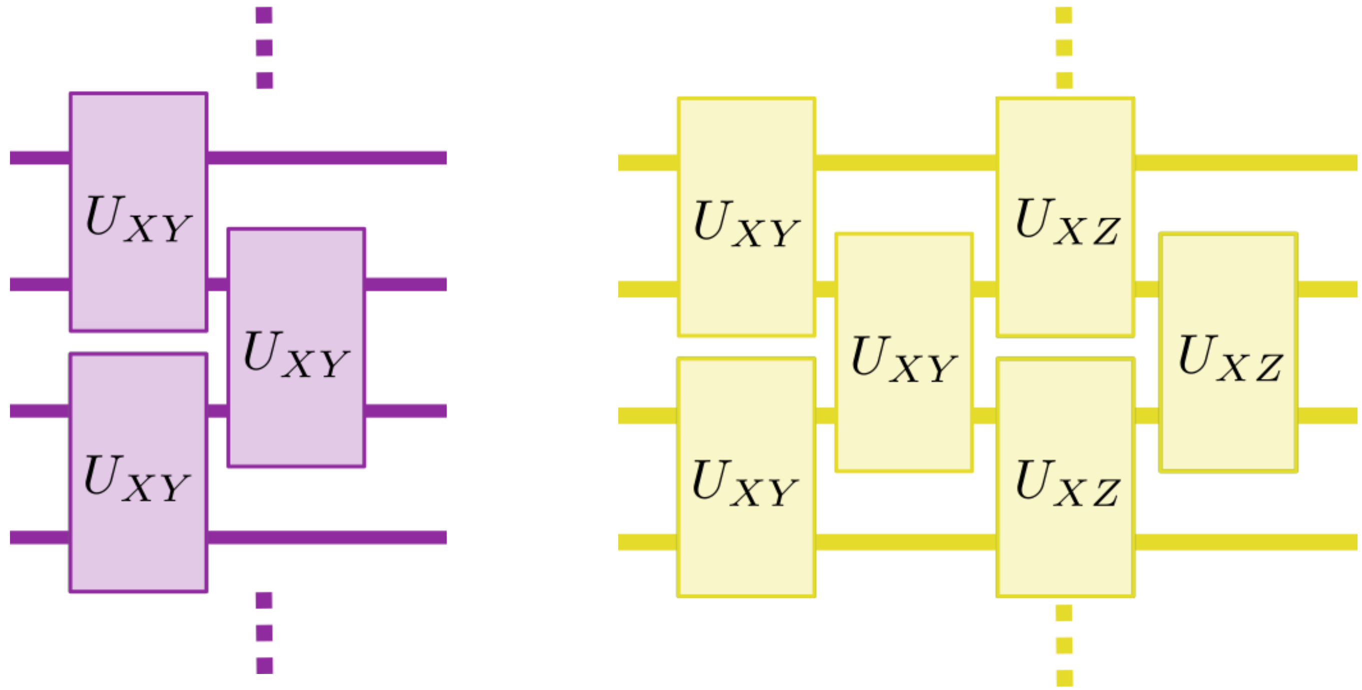
- For a typical loss function in *variational quantum computing*, the loss function and its gradients concentrate (on average) as $\dim \mathfrak{g}$ increases.

- Loss function: $C_\theta(\rho, O) = \mathrm{tr}\big(U(\theta)\rho U(\theta)^\dagger O\big)$.

- Assuming the DLA $\mathfrak{g}$ is simple and $iO \in \mathfrak{g}$:

$$\mathrm{Var}_\theta[C_\theta(\rho, O)] = \frac{\mathrm{tr}\big({\rho_\mathfrak{g}}^2\big)\,\mathrm{tr}(O^2)}{\dim \mathfrak{g}}$$

M. Larocca et al., *Quantum* 6 (2022),
M. Ragone et al., *Nature Comm.* 15 (2024),
E. Fontana et al., *Nature Comm.* 15 (2024).




Parametrized Quantum Circuit

- $U(\theta) = \prod_{l=1}^{L} e^{-i\theta_l H_l}$.

- DLA $\mathfrak{g} = \langle iH_1, \ldots, iH_L \rangle_{\mathrm{Lie}}$.

- Initial state density matrix $\rho = |\xi\rangle\langle\xi|$.

- $\rho_\mathfrak{g}$ is the projection of $\rho$ to $\mathfrak{g}$.

MIT-CTP/4610

# A Quantum Approximate Optimization Algorithm

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Sam Gutmann

arXiv:1411.4028v1 [quant-ph] 14 Nov 2014

# A review on Quantum Approximate Optimization Algorithm and its variants

Kostas Blekos [a,*,1], Dean Brand [b,1], Andrea Ceschini [c,1], Chiao-Hui Chou [d,1], Rui-Hao Li [e,1], Komal Pandya [f,1], Alessandro Summer [g,1]

Physics Reports 1068 (2024) 1–66

# Quantum Approximate Optimization Algorithm (QAOA)

Parametrized Quantum Circuit

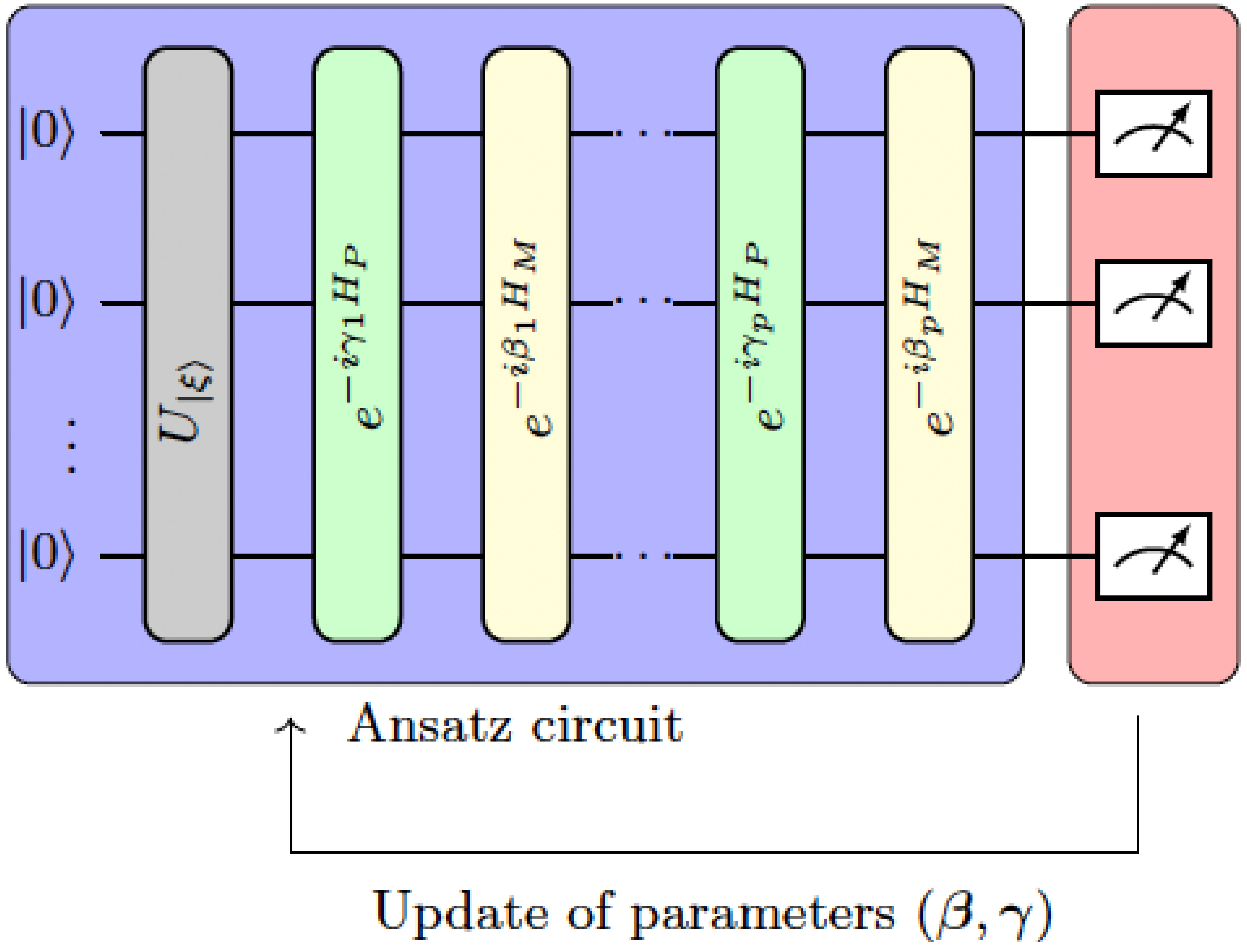
$$U(\beta,\gamma) = \prod_{l=1}^{p} e^{-i\beta_l H_M}\ e^{-i\gamma_l H_P}$$

Problem Hamiltonian $H_P$:

$H_P|x\rangle = F(x)|x\rangle$ for $x \in \{0,1\}^n$

Mixer Hamiltonian $H_M = \sum_{k=1}^{n} X_k$

Initial state $|\xi\rangle = 2^{-n/2} \sum_{x\in\{0,1\}^n} |x\rangle$

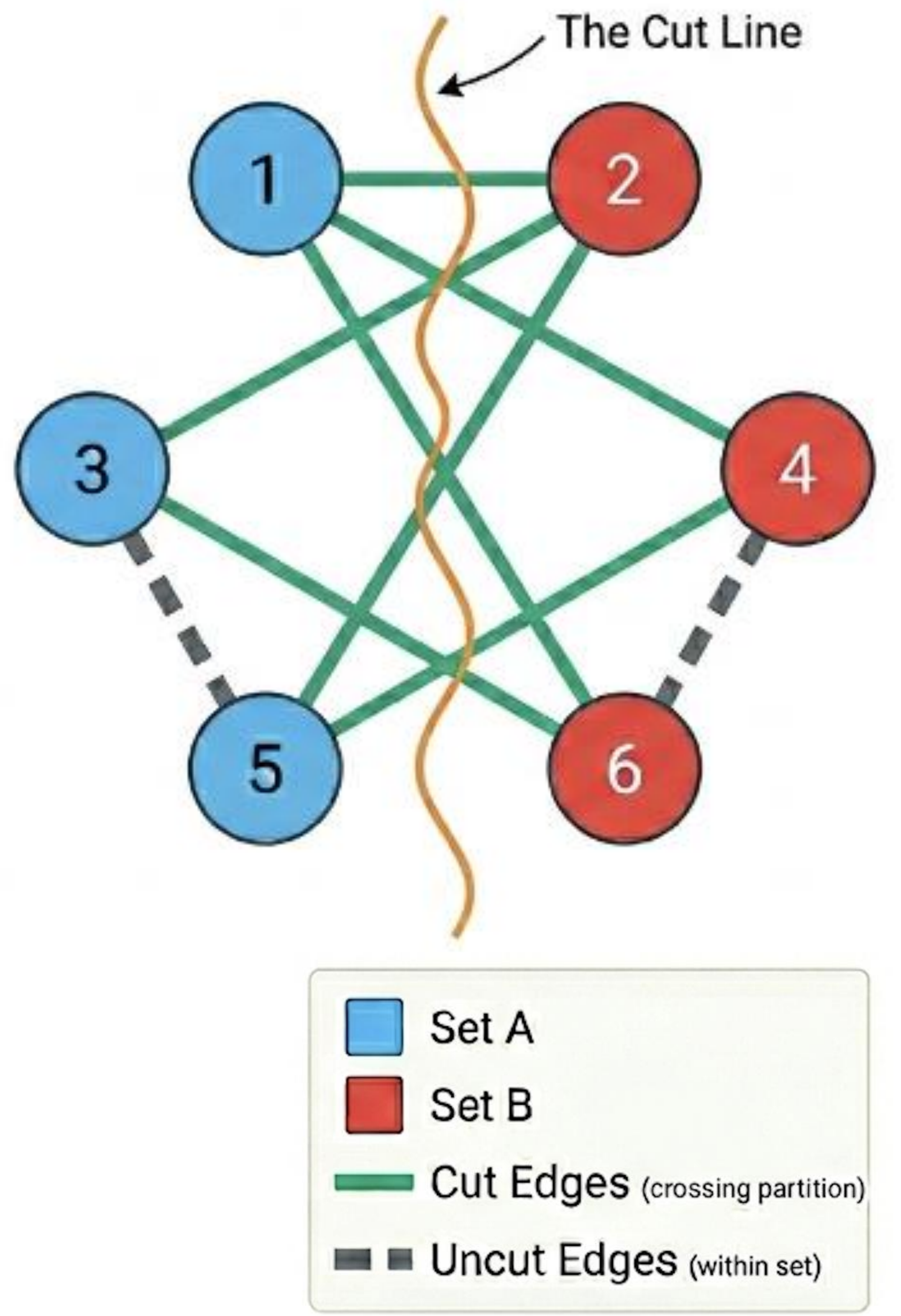



Minimizing the loss function $C(\beta,\gamma) = \langle\xi|U(\beta,\gamma)^\dagger H_P U(\beta,\gamma)|\xi\rangle \rightarrow \min F(x)$.

E. Farhi, J. Goldstone, S. Gutmann, *arXiv:1411.4028*

# QAOA for MaxCut

QAOA is used to solve NP-hard combinatorial **optimization problems** such as Graph MaxCut, Graph Coloring, Graph Vertex Cover, Boolean Satisfiability, Traveling Salesman, Portfolio Optimization, etc.




**MaxCut:** Given a graph $G$ with $n$ vertices, find $z \in \{\pm 1\}^n$ with

$\max \frac{1}{2} \sum_{(j,k) \in E(G)} (1 - z_j z_k) \iff$ ground state of

$H_P = \sum_{(j,k) \in E(G)} Z_j Z_k$ (Ising Hamiltonian)

E. Farhi, J. Goldstone, S. Gutmann, *arXiv:1411.4028*

## On the dynamical Lie algebras of quantum approximate optimization algorithms

Jonathan Allcock[1], Miklos Santha[2,3], Pei Yuan[1], and Shengyu Zhang[1]

arXiv:2407.12587v2 [quant-ph] 4 Mar 2026

PRX QUANTUM 6, 040345 (2025)

## Analyzing the Quantum Approximate Optimization Algorithm: Ansätze, Symmetries, and Lie Algebras

Sujay Kazi,[1,2,3] Martín Larocca,[2,4,*] Marco Farinati,[5] Patrick J. Coles,[6,2] M. Cerezo,[7,†] and Robert Zeier[8,‡]

## Provable avoidance of barren plateaus for the Quantum Approximate Optimization Algorithm with Grover mixers

Boris Tsvelikhovskiy,[1,*] Matthew Nuyten,[2] and Bojko N. Bakalov [2,†]

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[2]*Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA*

arXiv:2509.10424v1 [quant-ph] 12 Sep 2025

## QAOA-MaxCut has barren plateaus for almost all graphs

Rui Mao [*1,2], Pei Yuan [†1], Jonathan Allcock [‡1], and Shengyu Zhang [§1]

[1]Tencent Quantum Laboratory
[2]School of Computer Science and Technology, University of Chinese Academy of Sciences, Beijing 100049, China

arXiv:2512.24577v1 [quant-ph] 31 Dec 2025

# DLA of MaxCut QAOA

**Theorem** [Mao-Yuan-Allcock-Zhang, *arXiv:2512.24577*].

For an Erdős–Rényi random graph $G \sim G(n, \frac{1}{2})$, with probability $\geq 1 - e^{-\Omega(n)}$,

$$\mathfrak{g}_{\mathrm{QAOA}} = \left\langle i \sum_{(j,k)\in E(G)} Z_j Z_k \,,\; i \sum_{l=1}^{n} X_l \right\rangle_{\mathrm{Lie}}$$

$$= \left\langle i Z_j Z_k, i X_l : (j,k) \in E(G), 1 \leq l \leq n \right\rangle_{\mathrm{Lie}} = \mathfrak{g}_{\mathrm{ma-QAOA}}$$

[R. Herrman et al., *Sci. Rep.* (2022)]

**Corollary.** By [Kökcü-Wiersema-Kemper-Bakalov, *J. Math Phys.* (2026)]

$\mathfrak{g}_{\mathrm{ma-QAOA}} = \mathfrak{a}_{14}^{G}$ has $\dim \in O(4^n) \Rightarrow$ MaxCut QAOA has barren plateaus.

# QAOA with Grover Mixer

Replace the standard mixer $H_M = \sum_{k=1}^{n} X_k$ with the *Grover mixer* $G_M = |\xi\rangle\langle\xi|$.
[A. Bärtschi, S. Eidenbenz, *IEEE Conf*. 2020]

Arbitrary initial state $|\xi\rangle$ allows to solve constrained optimization problems by restricting to a *feasible subspace*.

**Theorem** [Tsvelikhovskiy-Nuyten-Bakalov, *arXiv:2509.10424*].

- The DLA is $\boxed{\langle iH_P, iG_M\rangle_{\mathrm{Lie}} \cong \mathfrak{su}(d) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)}$

where $d$ is the number of distinct eigenvalues of $H_P$ when acting on $|\xi\rangle$.

- For $s$-*local* Hamiltonian $H_P$ with integer eigenvalues, $\mathrm{Var}_{\beta,\gamma}[C(\beta,\gamma)] \in \Omega(n^{-2s})$
  $\Rightarrow$ no barren plateaus. (For MaxCut, $s = 2$, $d \leq \binom{n}{2}$.)

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Perspective https://doi.org/10.1038/s41467-025-63099-6

# Does provable absence of barren plateaus imply classical simulability?

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M. Cerezo[1,2], Martin Larocca[3,4], Diego García-Martín[1], N. L. Diaz[1,5], Paolo Braccia[3], Enrico Fontana[6], Manuel S. Rudolph[7], Pablo Bermejo[1,8], Aroosa Ijaz[3,9,10], Supanut Thanasilp[7,11], Eric R. Anschuetz[12,13] & Zoë Holmes[7]

PHYSICAL REVIEW RESEARCH **7**, 033266 (2025)

## Lie-algebraic classical simulations for quantum computing

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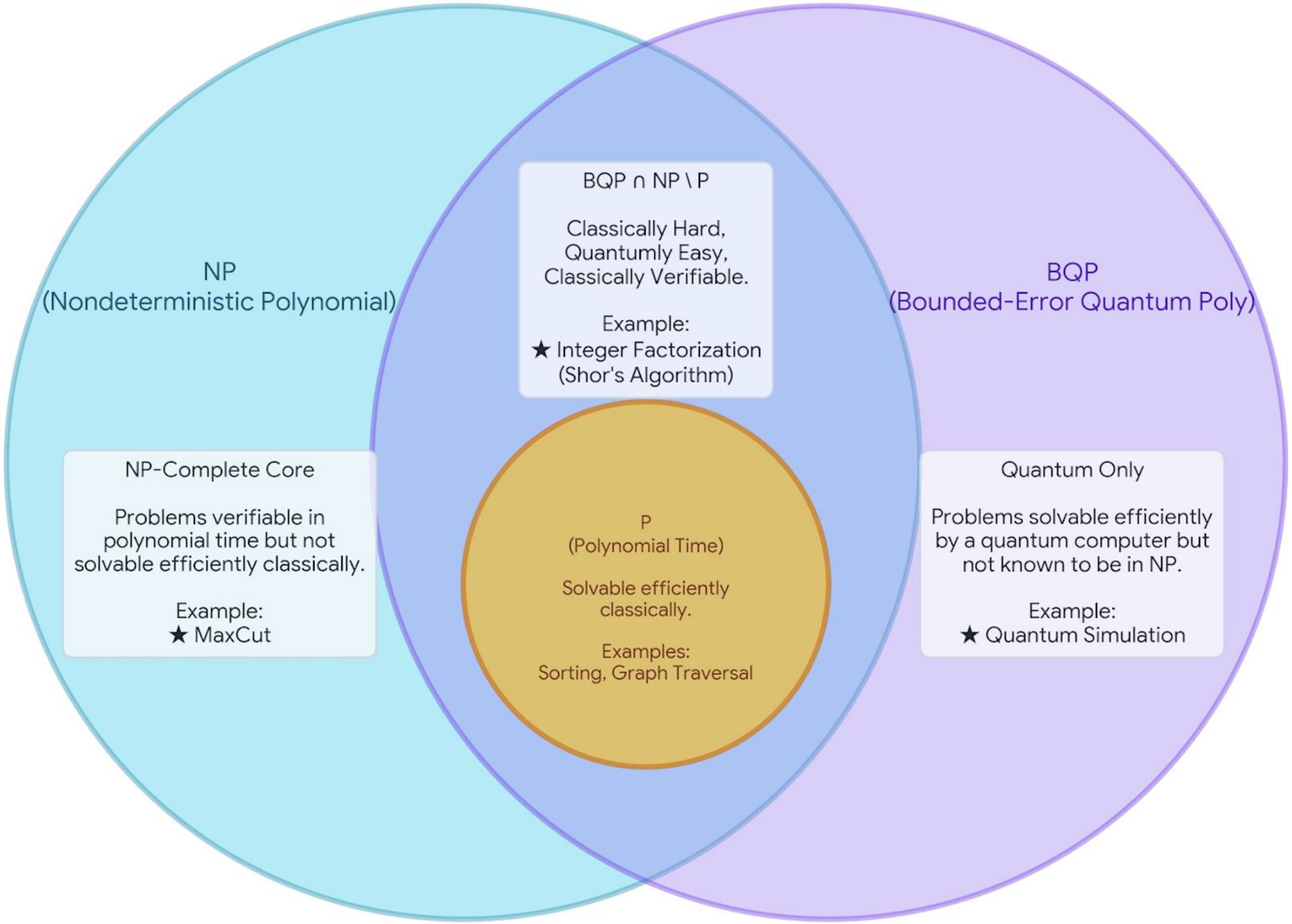
# Conjectured Relationships of Complexity Classes



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