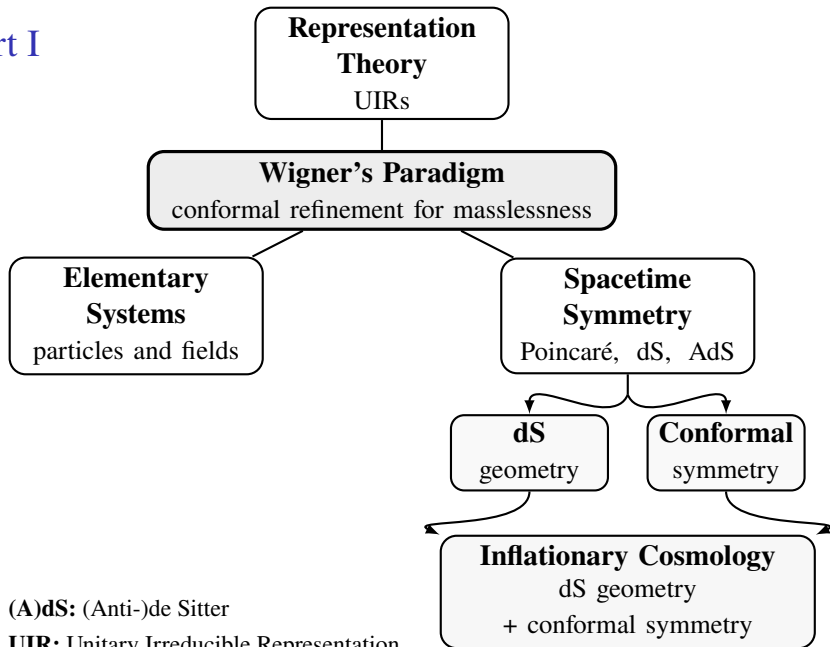


Massless Representations, Conformal Symmetry, and the Legacy of Ivan Todorov

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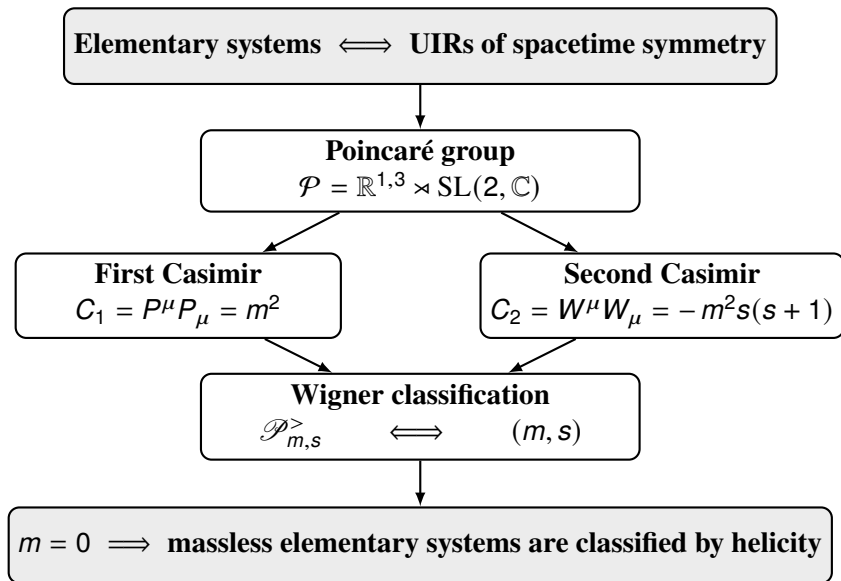


(A)dS: (Anti-)de Sitter

UIR: Unitary Irreducible Representation

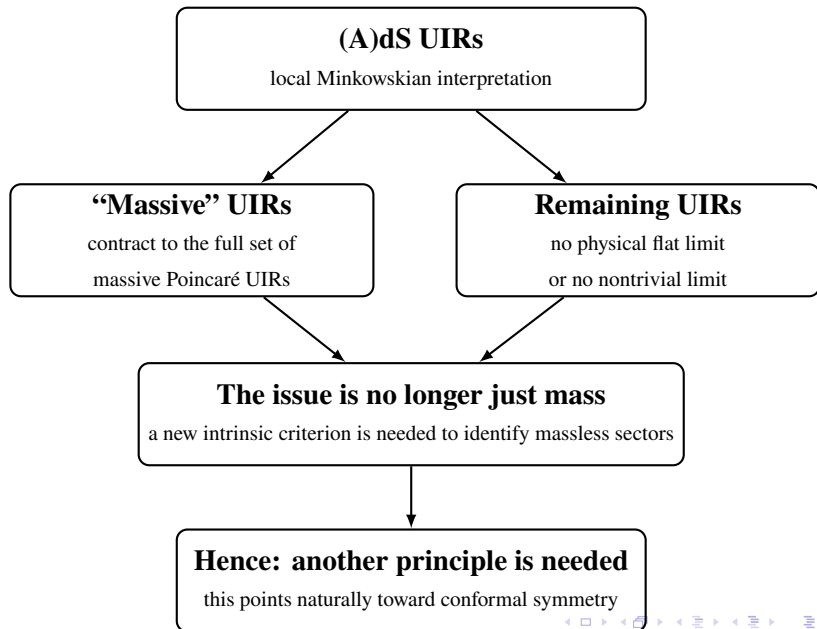
UPEIR: Unitary Positive Energy Irreducible Representation

Wigner's Paradigm in Minkowski Spacetime¹

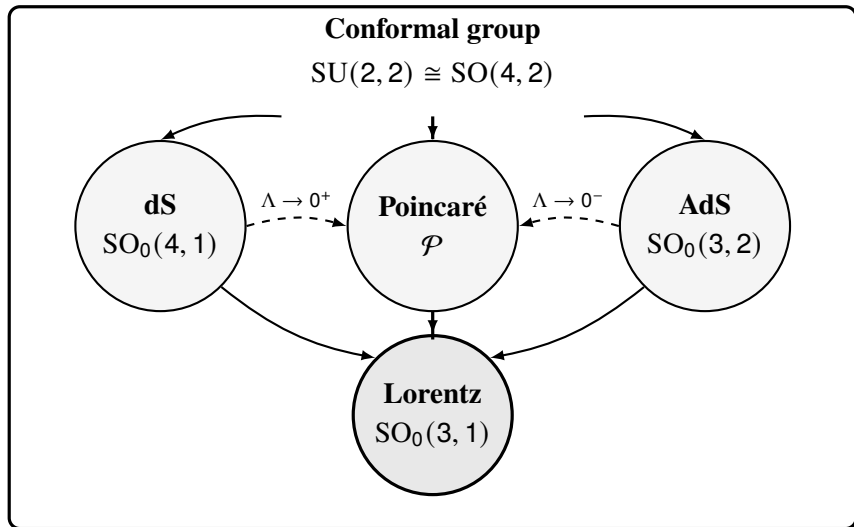


¹E.P. Wigner (1939); Newton–Wigner (1949)

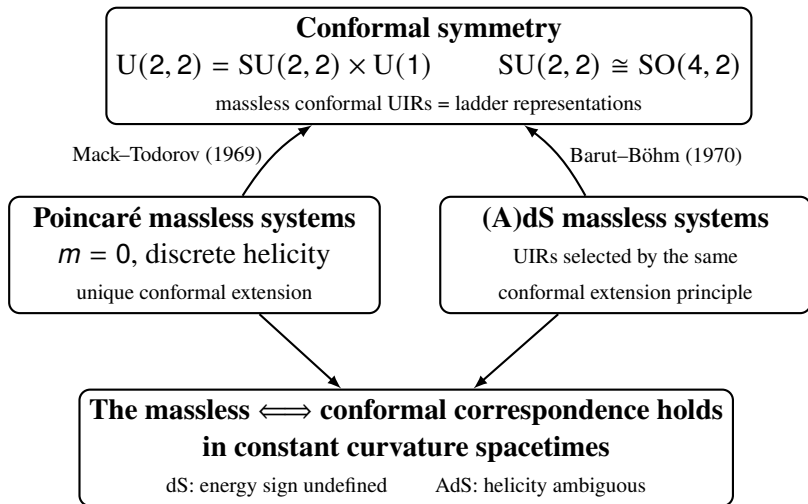
(A)dS UIRs: Beyond the Massive Sector



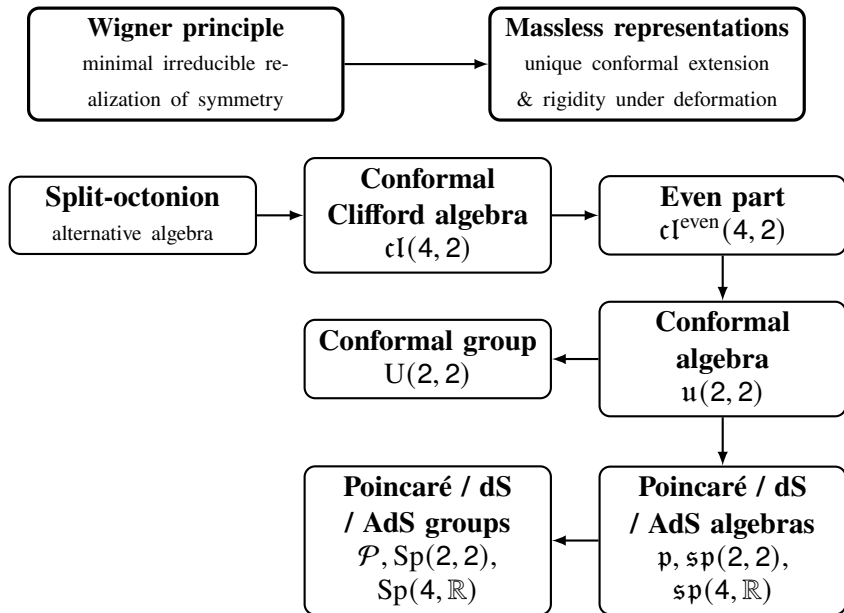
Conformal Unification of Relativistic Symmetry Groups



Masslessness Through Conformal Symmetry



Part II



Composition Algebras and the Split-Octonionic Setting

Composition algebra

finite-dim., real unital algebra, with non-deg., symmetric form $\langle \cdot, \cdot \rangle$

$$x \mapsto x^*, \quad N(x) = x^*x, \quad N(xy) = N(x)N(y)$$

Hurwitz theorem: only four finite-dimensional real composition algebras exist

\mathbb{R}

assoc.

\mathbb{C}

comm., assoc.

\mathbb{H}

non-comm.

\mathbb{O}

alternative

Defining feature of (split-)octonions

$$o = q + \ell p, \quad q, p \in \mathbb{H}, \quad (q + \ell p)^* = q^* - \ell p, \quad N(o) = oo^*$$

$$\ell^2 = -1 \Rightarrow \mathbb{O}, \quad \ell^2 = +1 \Rightarrow \mathbb{O}_S$$

For us: the split choice gives $N(q + \ell p) = N(q) - N(p)$,
hence signature $(4, 4)$ generated by the units
 $\mathbb{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \ell, \ell\mathbf{i}, \ell\mathbf{j}, \ell\mathbf{k}$ underlying the real realization of $\mathfrak{cl}(4, 2)$

Split-Octonion Realization of $\mathfrak{cl}(4, 2)$

Left-regular action of split-octonions

$$A \in \mathbb{O}_S \iff L_A : \mathbb{O}_S \rightarrow \mathbb{O}_S, \quad L_{AX} := AX, \quad \mathbb{O}_S \cong \mathbb{R}^{4,4}.$$

Six distinguished imaginary units

$$v \in \{0 := \iota, 5 := j, 1 := \ell i, 2 := j\ell, 3 := \ell k, 4 := \ell\}, \quad m_v := L_v.$$

Quadratic signature

$$m_{j=0,5}^2 = -\mathbb{1}, \quad m_{j=1,\dots,4}^2 = +\mathbb{1} \\ (4, 2)$$

Clifford relation

$$\{m_\mu, m_\nu\}x = 2\eta_{\mu\nu}x \\ \eta = \text{diag}(-, -, +, +, +, +)$$

Non-associativity \Rightarrow **iterated action** $L_A L_B L_C X := A(B(CX))$

$$L_A L_B \neq L_{AB}, \quad L_A L_B X = A(BX), \quad L_{AB} X = (AB)X$$

$\Rightarrow A \mapsto L_A$ is not an algebra homomorphism

$m_v = L_v \Rightarrow \mathfrak{cl}(4, 2)$ canonical real (Majorana) realization

Emergence of the Conformal Lie Algebra

Step 1: Clifford generators \rightarrow 15 bivectors

$$m_{\mu\nu} := \frac{1}{2}[m_\mu, m_\nu] = m_\mu m_\nu, \quad \mu, \nu \in \{0, 5, 1, 2, 3, 4\}, \quad \mu \neq \nu.$$

Step 2: 15 bivectors \Rightarrow conformal algebra

$$\text{Span}\{m_{\mu\nu}\} \rightarrow \mathfrak{su}(2, 2) \cong \mathfrak{so}(4, 2)$$

Internal sector

$$m_{i=1, \dots, 4} \Rightarrow \mathfrak{cl}(4) \subset \mathfrak{cl}(4, 2),$$

$$m_{ij} = m_i m_j \Rightarrow \mathfrak{so}(4)$$

Orthogonality

$$\text{Tr}(m_{\mu\nu} L_A) = 0$$

Clifford \perp Lie algebra

Trace-orthogonal Cartan basis in the maximal compact sector

$$\mathfrak{so}(2) \oplus \mathfrak{so}(4) \subset \mathfrak{so}(4, 2) \cong \mathfrak{su}(2, 2), \quad \{m_{05}, m_{12}, m_{34}\}$$

The Six-Unit Product, the Volume Form, and Chirality

A remarkable associative product inside a non-associative algebra

$$\iota\left(\mathcal{J}\left(\ell\iota\left(\mathcal{J}\ell\left(\ell\mathbf{k}\left(\ell\right)\right)\right)\right)\right) = \dots = -\mathbf{k}.$$

Non-associative in general, but associative for the ordered six-unit product.



The pseudoscalar of $\text{cl}(4, 2)$, lying in $\text{cl}^{\text{even}}(4, 2)$

$$E = M(4, 2) := m_0 m_5 m_1 m_2 m_3 m_4 = L_i L_j L_{\ell i} L_{j \ell} L_{\ell k} L_{\ell} = L_{-k}$$

$$\text{cl}(4, 2) = \text{Span}\{\mathbb{1}, m_{\mu}, m_{\mu\nu}, \dots, E\}, \quad \text{cl}^{\text{even}}(4, 2) = \text{Span}\{\mathbb{1}, m_{\mu\nu}, \dots, E\}$$

Central $\mathfrak{u}(1)$ generator

$$[m_{\mu\nu}, E] = 0, \quad E \in \mathfrak{u}(1)$$

$$\mathfrak{u}(2, 2) \cong \mathfrak{su}(2, 2) (\cong \mathfrak{so}(4, 2)) \oplus \mathfrak{u}(1)$$

$$\subset \text{cl}^{\text{even}}(4, 2) \subset \text{cl}(4, 2)$$

Chirality $E^2 = -\mathbb{1}$

$$\mathfrak{S} = \mathfrak{S}_+ \oplus \mathfrak{S}_-, \quad E\psi_{\pm} = \pm i\psi_{\pm}$$

two complex-conjugate 4-dimensional semispinors

Exponentiating these generators yields invertible elements of $U(2, 2)$ within the Clifford framework.

Compact Chiral Basis and Invariant Structure

Compact Cartan sector and simultaneous diagonalization

$$\mathfrak{so}(2) \oplus \mathfrak{so}(4) \subset \mathfrak{so}(4, 2) \cong \mathfrak{su}(2, 2), \quad \{m_{05}, m_{12}, m_{34}\}$$

$$[m_{05}, m_{12}] = \dots = [m_{05}, m_{34}] = 0, \quad [m_{05}, E] = \dots = [m_{34}, E] = 0$$

Compact chiral basis

$\{m_{05}, m_{12}, m_{34}, E\} \implies$
simultaneous diagonalization

$$\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-, \quad E|_{\mathfrak{G}_\pm} = \pm i \mathbb{1}$$

$$\mathfrak{G}_\pm = \text{Span}\{n_{\mathbb{1}k_\pm}, n_{05_\pm}, n_{12_\pm}, n_{34_\pm}\}$$

Γ -realization: $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$

$$\Gamma_\mu = U m_\mu U^{-1}, \quad \Gamma_{\mu\nu} = U m_{\mu\nu} U^{-1}$$

$$iD = \omega(4, 2) = U E U^{-1}$$

Invariant Hermitian structure: $\Gamma_0^* = -\Gamma_0, \quad \Gamma_5^* = -\Gamma_5, \quad \Gamma_{i=1, \dots, 4}^* = \Gamma_i$

$$\implies \langle \psi, \phi \rangle_{\text{std}} = \psi^* \phi \quad \not\rightarrow \text{invariant}$$

$$\langle \psi, \phi \rangle_B := \psi^* B \phi, \quad \Gamma_\nu^* B = -B \Gamma_\nu, \quad [B, \omega(4, 2)] = 0$$

$$B = m_{12} m_{43} = \Gamma_{12} \Gamma_{43}, \quad B^* = B, \quad B^2 = \mathbb{1}_8$$

$\implies \Gamma$ -realization + B -form = compact chiral invariant structure

$\text{cI}^{\text{even}}(4, 2) \cong \text{cI}(4, 1)$: 4×4 Chiral Realization

From the conformal Clifford algebra to the dS one

$$\text{cI}^{\text{even}}(4, 2) \cong \text{cI}(4, 1), \quad D_+ = (\mathbb{1} + D)/2, \quad \mathfrak{S}_+ = D_+ \mathfrak{S}$$

Projected generators

$$\gamma_\alpha := \Gamma_{\alpha 5} D_+, \quad \alpha = 0, 1, 2, 3, 4$$

$$\Gamma_{\alpha 5} = \Gamma_\alpha \Gamma_5$$

acts on the 4-dimensional chiral space \mathfrak{S}_+

Clifford / Lie relations

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta},$$

$$\eta_{\alpha\beta} = \text{diag}(-, +, +, +, +)$$

$$[\gamma_{\alpha\beta}, \gamma_{\rho\sigma}] \implies \mathfrak{so}(4, 1)$$

Canonical 4×4 spinorial realization

$$\mathfrak{S}_+ = \text{Span}\{n_{\widehat{1k}_+}, n_{\widehat{05}_+}, n_{\widehat{12}_+}, n_{\widehat{34}_+}\}$$

$$w(4, 1) = \gamma_{05}\gamma_{15}\gamma_{25}\gamma_{35}\gamma_{45} = \omega(4, 2)|_{\mathfrak{S}_+} = i \mathbb{1}$$

Thus, the even conformal Clifford algebra yields an irreducible 4×4 chiral model of $\text{cI}(4, 1)$, with complex structure canonically inherited from conformal chirality.

Positive-Energy Ladder Representations of $\mathfrak{u}(2, 2)$

From conformal semispinors to ladder representations

Chiral semispinors $\varphi \in \mathfrak{S}_+$ and $\tilde{\varphi} = \varphi^* B := \varphi^* B D_+ \in \mathfrak{S}_-$
are promoted to bosonic operators:

$$[\varphi^a, \varphi^{a'}] = 0, \quad [\varphi^a, \tilde{\varphi}_{a'}] = \delta_{a'}^a, \quad [\tilde{\varphi}_a, \tilde{\varphi}_{a'}] = 0$$

Second-quantized realization of $\mathfrak{su}(2, 2) \cong \mathfrak{so}(4, 2)$

$$J_{\mu\nu} = \frac{1}{2} (\tilde{\varphi} \gamma_{\mu\nu} \varphi) \quad \Longrightarrow \quad [J_{\mu\nu}, J_{\lambda\sigma}] = \mathfrak{so}(4, 2) \text{ algebra}$$

Faithful, chirality-preserving action on \mathfrak{S}_\pm

Lowest-weight (positive-energy) representations

"Fock vacuum" $|0^{\text{SO}(4)}\rangle$ ($\text{SO}(4)$ singlet) with

$$H := iJ_{05} = \frac{1}{2} (\tilde{\varphi} i\gamma_{05} \varphi) \geq 1, \quad (H - 1)|0^{\text{SO}(4)}\rangle = 0$$

Creation/annihilation split via projectors P_\pm :

$$P_+ \varphi |0^{\text{SO}(4)}\rangle = 0, \quad \tilde{\varphi} P_- |0^{\text{SO}(4)}\rangle = 0, \quad P_\pm = (\mathbb{1} \pm B)/2$$

Oscillators, Hamiltonian, and Central $u(1)$

B-diagonal chiral basis \Rightarrow oscillator decomposition

$$(n_{\widehat{1k}_+}, n_{\widehat{05}_+}, n_{\widehat{12}_+}, n_{\widehat{34}_+}) \quad \Rightarrow \quad \varphi = \begin{pmatrix} \mathbf{a} \\ \mathbf{b}^* \end{pmatrix}, \quad \tilde{\varphi} = \varphi^* \mathbf{B} = (\mathbf{a}^* \quad -\mathbf{b})$$

$$\mathbf{a} = (\mathbf{a}^A; A = 1, 2), \quad \mathbf{b} = (\mathbf{b}^A; A = 1, 2), \quad [\mathbf{a}^{A'}, \mathbf{a}_A^*] = [\mathbf{b}^{A'}, \mathbf{b}_A^*] = \delta_A^{A'}$$

Fock vacuum

$$\mathbf{a}^A |0^{\text{SO}(4)}\rangle = 0, \quad \mathbf{b}^A |0^{\text{SO}(4)}\rangle = 0$$

$$\langle 0^{\text{SO}(4)} | \mathbf{a}_A^* = 0, \quad \langle 0^{\text{SO}(4)} | \mathbf{b}_A^* = 0$$

Conformal Hamiltonian

$$H = iJ_{05} = \frac{1}{2} (\tilde{\varphi} i\gamma_{05} \varphi) = \frac{1}{2} (\mathbf{a}^* \mathbf{a} + \mathbf{b} \mathbf{b}^*)$$

$$\text{lowest-weight: } (H - 1) |0^{\text{SO}(4)}\rangle = 0$$

Central $u(1)$ charge and quantized representation label

$$C_1 := \tilde{\varphi} \varphi = \mathbf{a}^* \mathbf{a} - \mathbf{b}^* \mathbf{b} - 2 = 2(\lambda - 1) \in \mathbb{Z}, \quad [J_{\mu\nu}, C_1] = 0$$

$$C_1 \in \mathfrak{u}(2, 2) \setminus \mathfrak{su}(2, 2) \iff \text{second-quantized image of } E = M(4, 2)$$

$$\lambda \in \frac{1}{2}\mathbb{Z} \iff \text{IR label (conformal "helicity")}$$

Casimir Invariants and the λ -Parameter

Casimir invariants are polynomial functions of λ

$$C_1 = 2(\lambda - 1) \quad \Longrightarrow \quad J^{\alpha 5} J_{\alpha 5} = \frac{1}{4} C_1^2 + C_1 = \lambda^2 - 1$$

Conformal algebra $\mathfrak{su}(2, 2)$

$$C_2^{\mathfrak{su}(2,2)} = -\frac{1}{2} J^{\mu\nu} J_{\mu\nu} = -3(\lambda^2 - 1)$$

dS subalgebra $\mathfrak{sp}(2, 2)$

$$C_2^{\mathfrak{sp}(2,2)} = -\frac{1}{2} J^{\alpha\beta} J_{\alpha\beta} = -2(\lambda^2 - 1)$$

Barut-Böhm (1970) from $\mathfrak{cl}(4, 2)$

$\mathfrak{cl}(4, 2) \Longrightarrow$ conformal UPEIRs \Longrightarrow dS massless sector

$$C_2^{\mathfrak{sp}(2,2)} = -2(\lambda^2 - 1) \iff \text{dS massless Casimir}$$

$\lambda \in \frac{1}{2}\mathbb{Z} \Rightarrow$ common label for conformal & dS structures

Oscillator Ladder Realization and Lowest-Weight Content

Oscillator realization of $\mathfrak{su}(2, 2)$

$$J_{\mu\nu} := \frac{1}{2} \tilde{\varphi} \gamma_{\mu\nu} \varphi, \quad \varphi = \begin{pmatrix} \mathbf{a} \\ \mathbf{b}^* \end{pmatrix}, \quad J_{\mu\nu} = -(J_{\mu\nu})^*$$

\Rightarrow two oscillator doublets (\mathbf{a}, \mathbf{b}) generate the full ladder algebra

Cartan-Chevalley structure

$$2J_{12} = i(H_1 + H_3), \quad 2J_{34} = i(H_1 - H_3), \quad 2J_{05} = -i(H_1 + 2H_2 + H_3)$$

$$\text{raising: } E_1 = \mathbf{a}_1^* \mathbf{a}^2, \quad E_2 = \mathbf{a}_2^* \mathbf{b}_1^*, \quad E_3 = -\mathbf{b}^1 \mathbf{b}_2^*$$

$$\text{lowering: } F_1 = \mathbf{a}_2^* \mathbf{a}^1, \quad F_2 = -\mathbf{b}^1 \mathbf{a}^2, \quad F_3 = -\mathbf{b}^2 \mathbf{b}_1^*$$

Lowest-weight vector $F_a |\text{LW}_\lambda\rangle = 0$ and compact content

$$\mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(2)) \cong \underbrace{\langle E_1, F_1, \frac{1}{2}H_1 \rangle}_{\mathfrak{su}(2)_L \rightarrow j_L} \oplus \underbrace{\langle E_3, F_3, \frac{1}{2}H_3 \rangle}_{\mathfrak{su}(2)_R \rightarrow j_R} \oplus \underbrace{\langle H = \frac{1}{2}(H_1 + 2H_2 + H_3) \rangle}_{\mathfrak{u}(1) \rightarrow \mathcal{E}}$$

$$|\text{LW}_\lambda\rangle = \begin{cases} (\mathbf{a}_2^*)^{2\lambda} |0\rangle, & \lambda \geq 0 \implies (s = \lambda, 0, \mathcal{E}_o = 1 + \lambda), \\ (\mathbf{b}_1^*)^{-2\lambda} |0\rangle, & \lambda < 0 \implies (0, s = |\lambda|, \mathcal{E}_o = 1 + |\lambda|), \end{cases}$$

Energy Ladder Structure and $\mathfrak{u}(1)$ -Graded Classification

For fixed λ , the positive-energy UPEIR is an energy ladder

$$\mathcal{H}_\lambda^{(1)} = \bigoplus_{\mathcal{E} \geq \mathcal{E}_0 = |\lambda| + 1} \mathcal{H}_{\mathcal{E}}^{(1)}, \quad \mathcal{H}_{\mathcal{E}}^{(1)} := \ker(H - \mathcal{E})$$

$$\mathcal{H}_{\mathcal{E}_0}^{(1)} \xrightarrow{E_a} \mathcal{H}_{\mathcal{E}_0+1}^{(1)} \xrightarrow{E_a} \mathcal{H}_{\mathcal{E}_0+2}^{(1)} \xrightarrow{E_a} \dots$$

Compact content at each level

$\mathcal{H}_{\mathcal{E}=|\lambda|+n}^{(1)}$ carries an $\mathfrak{so}(4)$ -module structure, $\dim \mathcal{H}_{\mathcal{E}}^{(1)} = (2j_L + 1)(2j_R + 1)$

$$j_L(j_L + 1) + j_R(j_R + 1) = \frac{1}{2}(\mathcal{E}^2 + \lambda^2 - 1), \quad j_L, j_R \in \mathbb{N}/2$$

Passing from $\mathfrak{su}(2, 2)$ to $\mathfrak{u}(2, 2)$: central $\mathfrak{u}(1)$ grading

$$\mathfrak{u}(2, 2) = \mathfrak{su}(2, 2) \oplus \mathfrak{u}(1)$$

$$\mathcal{H}^{(1)} = \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{H}_\lambda^{(1)} = \text{Span} \left\{ |\lambda, \mathcal{E}, j_L, j_R \rangle ; \mathcal{E} = |\lambda| + n, j_L, j_R \leq \mathcal{E} - 1 \right\}$$

Each λ -sector yields a ladder UPEIR of $\mathfrak{su}(2, 2)$; together they assemble into the $\mathfrak{u}(1)$ -graded $\mathfrak{u}(2, 2)$ -module.

dS Massless UIRs as SO(4)-Multiplet Towers

Restriction from conformal symmetry to dS symmetry

$$J_{\mu\nu} = \frac{1}{2}(\tilde{\varphi} \gamma_{\mu\nu} \varphi), \quad \mathfrak{sp}(2,2) \cong \mathfrak{so}(4,1) = \text{Span}\{J_{\alpha\beta}\}_{\alpha,\beta=0,\dots,4} \subset \mathfrak{su}(2,2)$$

$$H = iJ_{05} \notin \mathfrak{sp}(2,2) \quad \Rightarrow \quad \text{no global dS energy label } \mathcal{E}$$

Compact content; J_{ij}

$$\mathfrak{so}(4) \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R, \quad (\mathfrak{j}_L, \mathfrak{j}_R)$$

$$\dim(\mathfrak{j}_L, \mathfrak{j}_R) = (2j_L + 1)(2j_R + 1)$$

J_{ij} act within each multiplet.

Non-compact links between multiplets; $J_{0i} \sim (\frac{1}{2}, \frac{1}{2})$

$$(\mathfrak{j}_L, \mathfrak{j}_R) \otimes (\frac{1}{2}, \frac{1}{2}) = (\mathfrak{j}_L \pm \frac{1}{2}, \mathfrak{j}_R \pm \frac{1}{2})$$

J_{0i} generate a tower of SO(4) multiplets.

Massless dS sector: chiral seed multiplet

$$(s, 0) \text{ or } (0, s), \quad s := |\lambda| = 0, \frac{1}{2}, 1, \dots$$

$$(s, 0) \iff \text{helicity } +s, \quad (0, s) \iff \text{helicity } -s$$

$$\Rightarrow \quad \Pi_{s,s}^+ \text{ and } \Pi_{s,s}^- \quad (\text{Dixmier notation - 1961})$$

Massless dS UIRs are generated from a chiral SO(4) seed (no global energy label).

Partially Massless vs. Massless in dS

Positive curvature allows an intermediate gauge regime in dS space

strictly massive \implies partially massless \implies strictly massless

$2s + 1$ d.o.f.

intermediate

2 helicity modes

Strictly massless $\Pi_{s,s}^{\pm}$ ($s > 0$)

UIR level: special dS UIRs with a unique conformal extension.

Field/QFT: gauge invariance; propagation on the light cone; only helicities $\pm s$; hence 2 degrees of freedom.

Partially massless, e.g., $\Pi_{2,1}$

UIR level: higher-spin discrete-series UIRs ($s > 3/2$) that are not strictly massless.

Field/QFT: novel gauge invariance; propagation on the light cone; fewer than $2s + 1$ but more than 2 degrees of freedom.

Key distinction

Massless dS UIRs admit flat/conformal counterparts, whereas **partially massless** elementary systems are **genuinely dS**; in the Wigner/UIR sense, they have no analogue in Poincaré or AdS relativity.

Closing Remarks

This seminar is based on the forthcoming monograph

*Massless Representations in Conformal
Space and Their de Sitter Restrictions*

Jean-Pierre Gazeau, Hamed Pejhan, and Ivan Todorov

to appear with **Cambridge University Press**
in the **Cambridge Monographs on Mathematical Physics** series

In grateful memory of Ivan Todorov

whose scientific vision and lasting influence continue to inspire this work

Thank you for your attention.

