MATHEMATICAL QUANTUM FIELD THEORY

Conference in memory of Ivan Todorov
Sofia, 25-30 May 2026

A Matrix Model for Higher-Genus Fuss-Catalan Numbers

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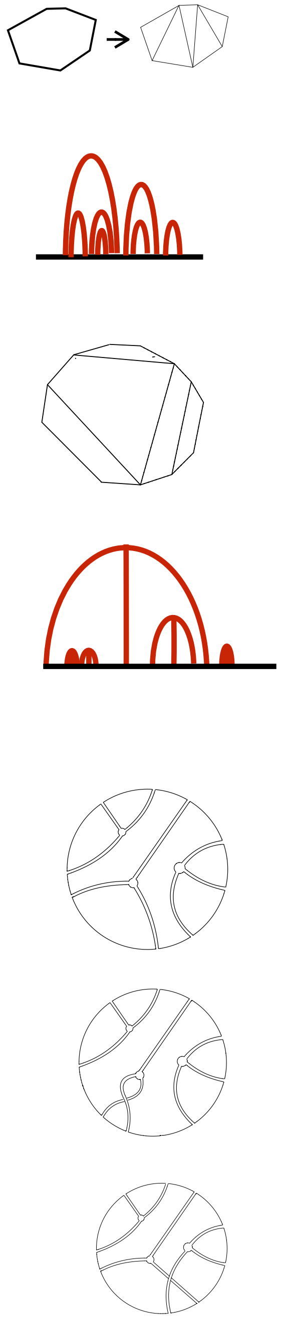
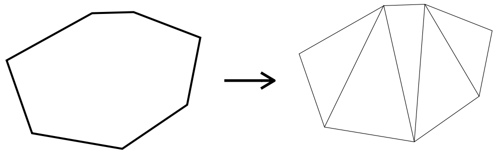
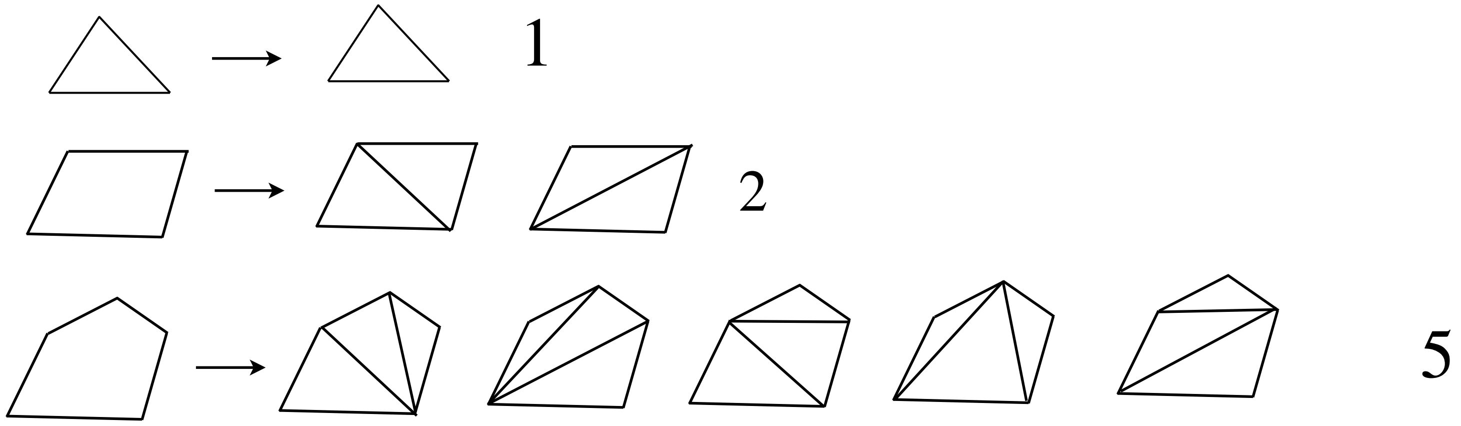


1a. The classical Catalan numbers

- Euler's problem 1751 (St Petersburg)

triangulations of a convex $(n+2)-$gon





- Catalan 1838 (Ecole Polytechnique, Paris)

balanced parentheses of length $2n$ (()())((())(()))(). (Chord or rainbow diagrams)

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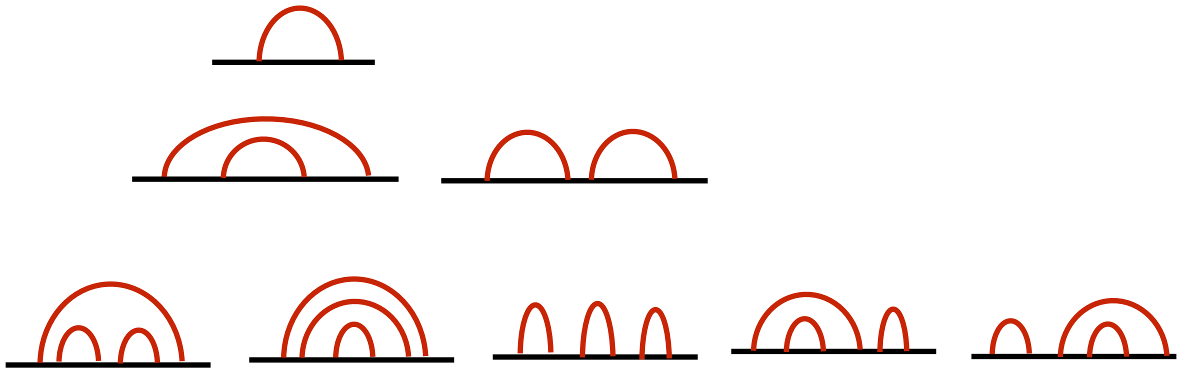
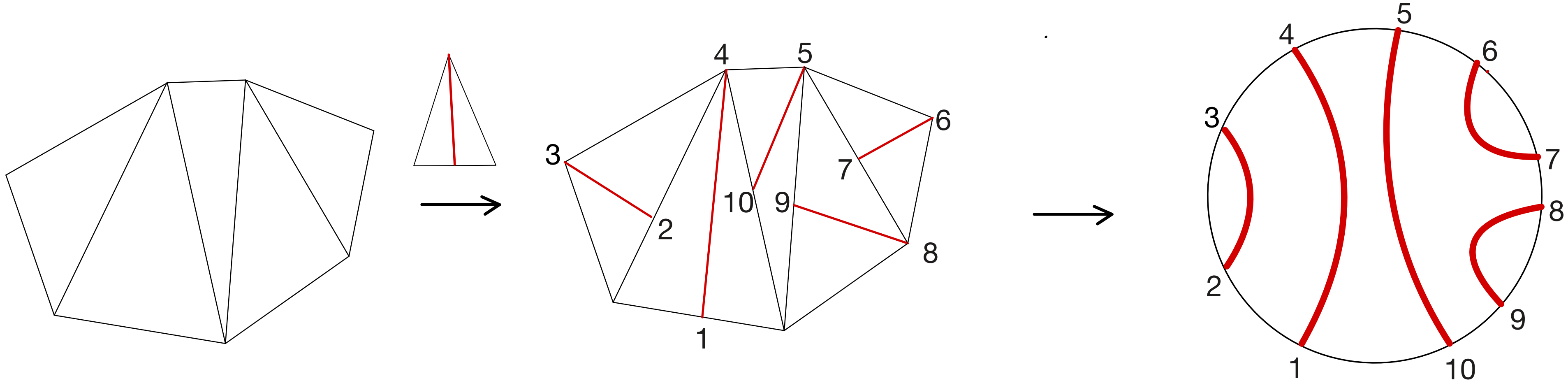
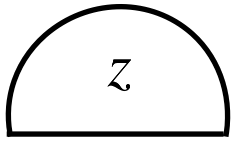


= ways to obtain a sphere by identifying pairwise edges of a 2n-gon

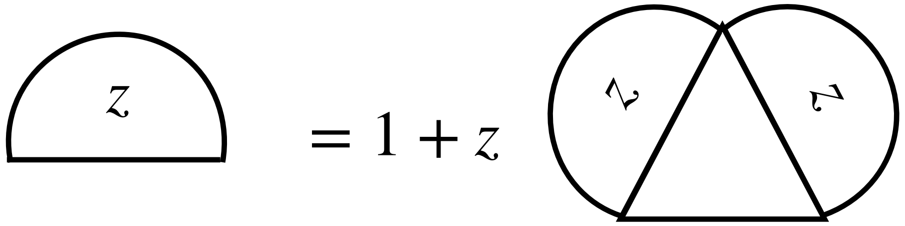
Polygon dissections. ↔ chord diagrams



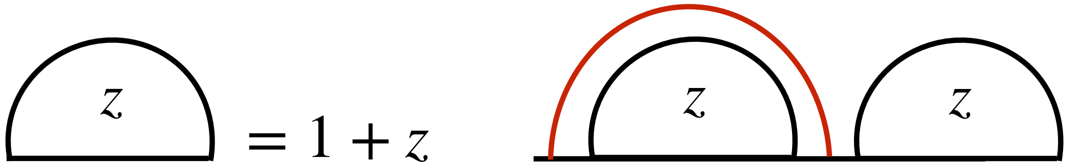
Same generating function:



$$\equiv f(z) = \sum_{n\geq 0} C_p(n)\; z^n$$



Polygon dissection (Euler)



Rainbow diagrams (Catalan):

$$f = 1 + z\, f^2$$

General solution (Catalan sequence):

$$C_n = \frac{1}{2n+1}\binom{2n+1}{n}, \qquad n \geq 0,$$

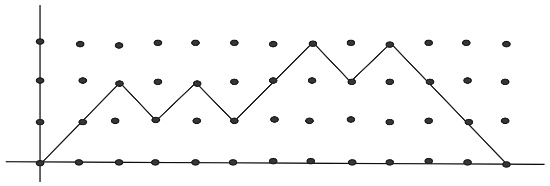
— one of the most celebrated sequences in mathematics

C_n counts:

— triangulations of a convex $(n+2)-$gon (Euler's problem 1751);

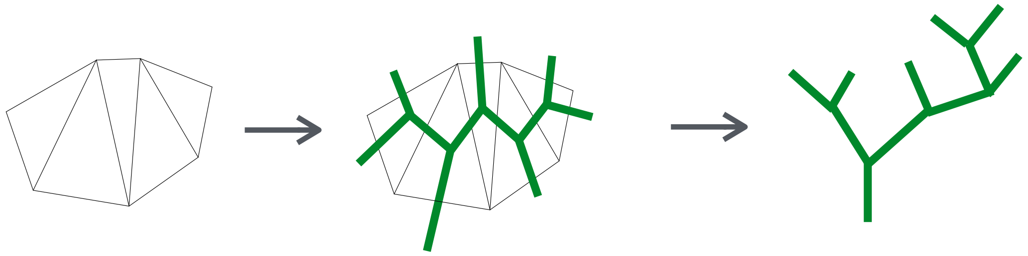
— balanced parentheses of length $2n$ (Catalan 1838)

—Dyck paths,



— non-crossing pair partitions,

—- binary rooted trees,



… over 60 other interpretations [R.P. Stanley, "Enumerative Combinatorics" (2011)]

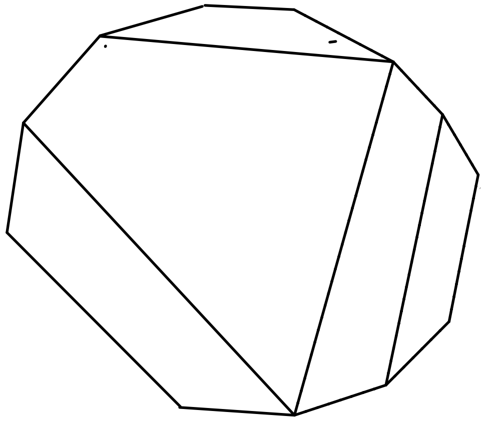
1b. Fuss-Catalan: Euler’s problem generalised

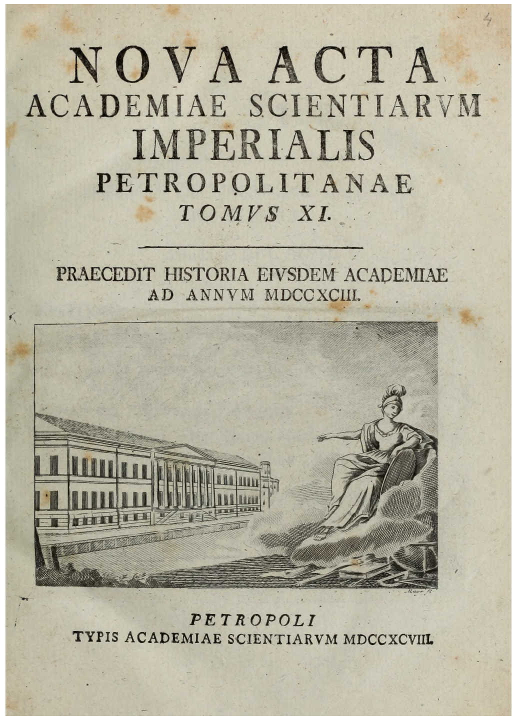
More than 40 years **before** Catalan rediscovered Euler’s problem,

Nicolaus Fuss, Euler’s secretary since 1773 (recommended by Bernoulli), From 1790: professor of mathematics at the Marine Corps in Petersburg

formulated and solved in 1791 a more general problem:

“Find the number $C_p(n)$ of ways to dissect a convex $(n(p-1)+2)$-gon into n $(p+1)$-gons”





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N.Fuss,

“Solutio quaestionis, quot modis polygonum n laterum in polygona m laterum, per diagonales resolvi queat”,

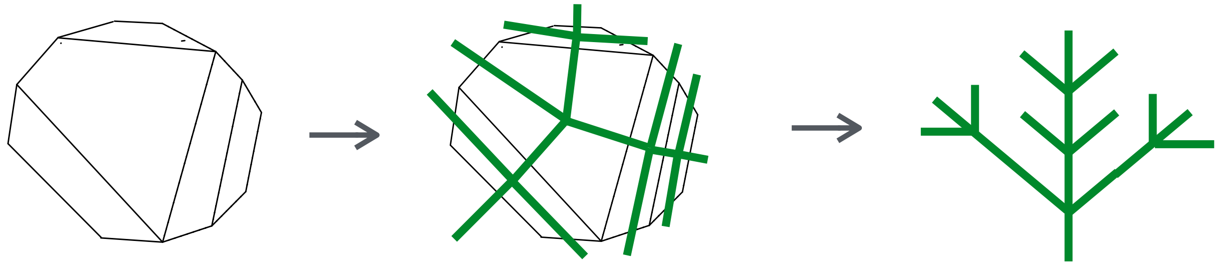
Nova Acta Academiae Scientiarum Imperialis Petropolitanae,
9:243–251, 1791

1b. Fuss-Catalan: Euler's problem generalised

Nicolaus Fuss, 1791:

$C_p(n)$ = # of ways to dissect a convex $(n(p-1)+2)$-gon into n $(p+1)$-gons

—geometrically: # planar p-ary trees with n internal nodes; each internal node has p children

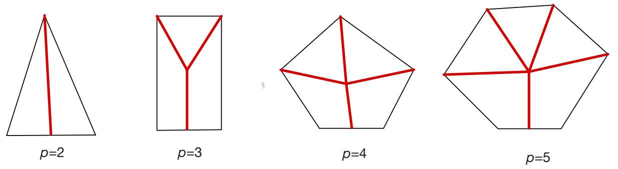


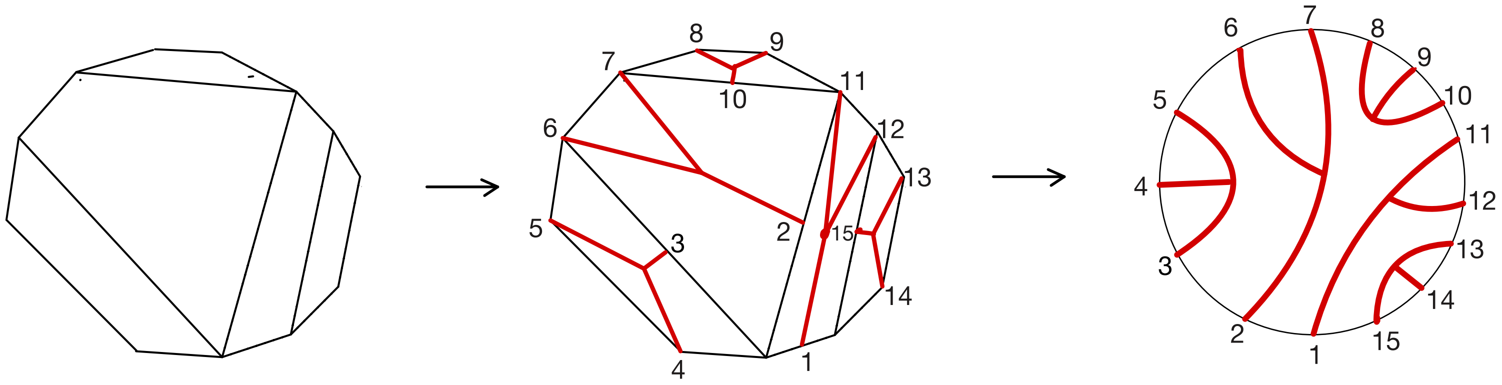
$p = 3$

$$C_3(n) = 1,\ 3,\ 12,\ 55,\ 273,\ \ldots$$

$$\boxed{C_p(n) = \frac{1}{pn+1}\binom{np+1}{n}, \qquad p \geq 2 \qquad \text{(Fuss-Catalan Numbers)}}$$

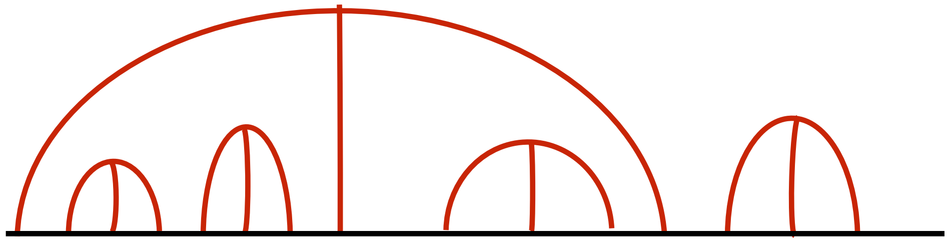
1d. Hyperedge diagram picture: p>1 generalisation of Catalan





— ***p*-hyperedge diagram picture (*p*-valent)**

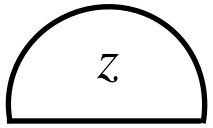
Number of on-crossing partitions of $\{1,2,\ldots,pn\}$ on a circle into n blocks of size p.



= ways to obtain a sphere by identifying groups of p edges of a pn-gon by p-valent hyperedges generalising the 2-valent chords.

1d. Hyperedge diagram picture: p>1 generalisation of Catalan

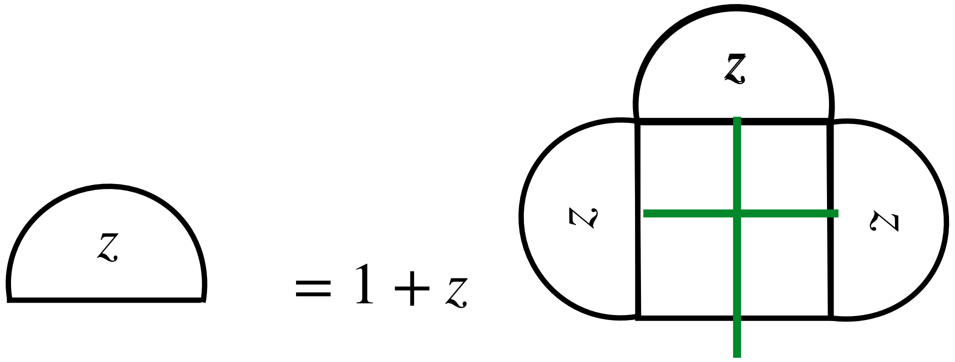
Equation for the generating function:



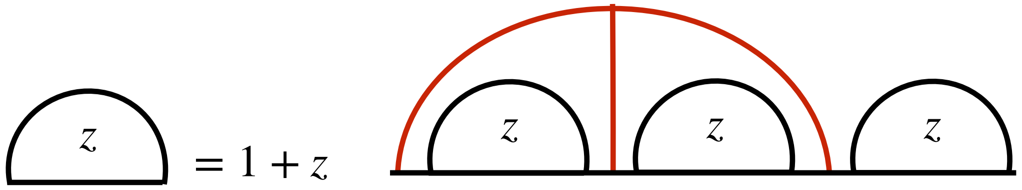
$$f(z) = \sum_{n\geq 0} C_p(n)\ z^n$$

$$f = 1 + z\, f^p$$

E.g. for $p = 3$: cubic equation



Tree picture



Hyperedge picture

1c. Summary: Two combinatorial pictures for $C_p(n)$:

Euler-Fuss:

Count $(p+1)$***-valent rooted planar trees with*** n ***internal nodes***

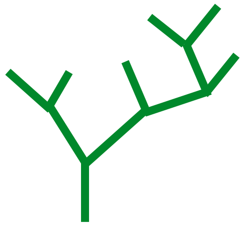
Catalan (generalised for $p \geq 2$):

Count p***-valent hyperedge diagrams***

= non-crossing partitions of $\{1,2,\ldots,pn\}$ on a circle into n blocks of size p.

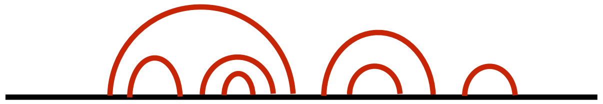
= ways to obtain a sphere by identifying groups of p edges of a pn-gon by p-valent hyperedges generalising the 2-valent chords.

$p = 2$



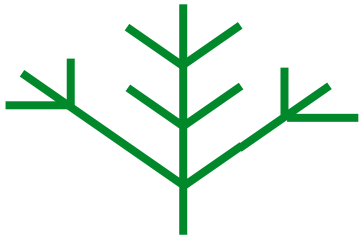
Trivalent trees

⇔



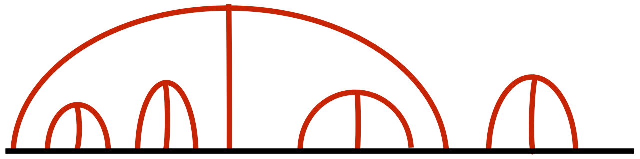
Chordes or rainbow diagrams

$p = 3$



4-valent trees

⇔



3-valent hyperedges

Only the hyperedge picture admits a natural higher-genus extension:

= ways to obtain a genus-g surface by identifying groups of p edges of a pn-gon by p-valent hyperedges generalising the 2-valent chords.

2a. Higher-Genus Fuss-Catalan: two inequivalent definitions

At $p = 2$: the ways to identifying sides of a $2n$-gon by chords to obtain a Riemann surface of genus g :

$$C_2^{(g)}(n) \;=\; \varepsilon_g(n)$$ (Harer-Zagier numbers)

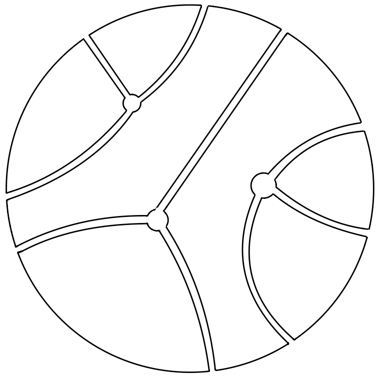
For $p \geq 3$: two inequivalent definitions

1) Counting partitions by genus

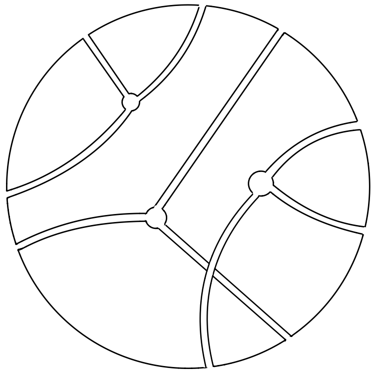
imposes a *restricted-crossing* (rc) constraint: edges from each p-vertex cannot cross [Zuber, 2023].

2) Counting planar maps

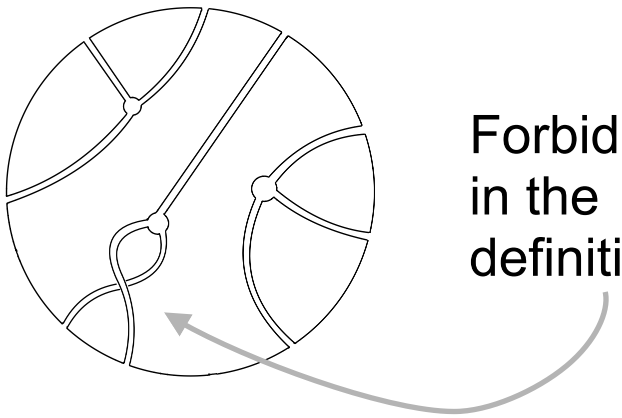
Edges → ribbon graphs with p-valent vertices. Sum over all ribbon graphs without restriction.



$g = 0$



$g = 1$



$g = 1$

Forbidden
in the
definition 1.

This talk: definition 2 throughout.

2b. Reminder: $p = 2$ (Harer-Zagier)

Harer & Zagier (1986) reduced the computation of the Euler characteristic in the moduli space of curves $\chi(\mathcal{M}_g)$ to counting pair-identifications of a $2n$-gon.

Itzykson and Zuber (1990) reformulated this as a Gaussian matrix integral:

$$\sum_{g \le n/2} N^{1+n-2g}\, \varepsilon_g(n) \;=\; \langle \mathrm{tr} X^n \rangle_N \;=\; \frac{1}{\mathcal{Z}_N} \int dX\, \mathrm{tr} X^n\, e^{-\frac{1}{2}\mathrm{tr} X^2}.$$

Harer-Zagier polynomial:

$$\frac{\langle \mathrm{tr} X^n \rangle_N}{(2n-1)!!} \;=\; \frac{1}{2} \oint \frac{dy}{2\pi i}\, y^{-n-2} \left(\frac{1+y}{1-y}\right)^N \;=\; \sum_{k=1}^{\min(N,n+1)} 2^{k-1} \binom{n}{k-1} \binom{N}{k}.$$

Question. Is there an analogous matrix integral for $C_p^{(g)}(n)$ at $p \geq 3$?

3. The Fuss-Catalan matrix model

Two Hermitian $N \times N$ matrices X, Y, partition function

$$\mathcal{Z}_{N,p} = \int dX\, dY\, \exp\Big[-N\,\mathrm{tr}\Big(XY - \tfrac{1}{p} Y^p\Big)\Big], \qquad p \geq 2.$$

Main claim:

The $1/N$-expansion of the one-trace expectation value

$$\frac{1}{N^{1+(p-1)n}} \Big\langle \mathrm{tr} X^{pn} \Big\rangle_{N,p} = \sum_{g \geq 0} C_p^{(g)}(n)\, N^{-2g}$$

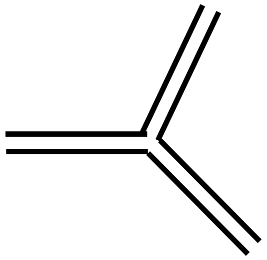
generates the higher-genus Fuss--Catalan numbers $C_p^{(g)}(n)$ for all $p \geq 2$.

At $p = 2$: reproduces Gaussian / Harer--Zagier.

3a. Feynman rules

Treat $\frac{1}{p}\mathrm{tr}Y^p$ as a perturbation of the Gaussian $\mathrm{tr}XY$:

Y-vertex from $\frac{1}{p}\mathrm{tr}Y^p$ → $\dfrac{1}{p}\delta_{i_1,j_2}\delta_{i_2,j_3}\ldots\delta_{i_p,j_1}$: p-valent hyper-vertex

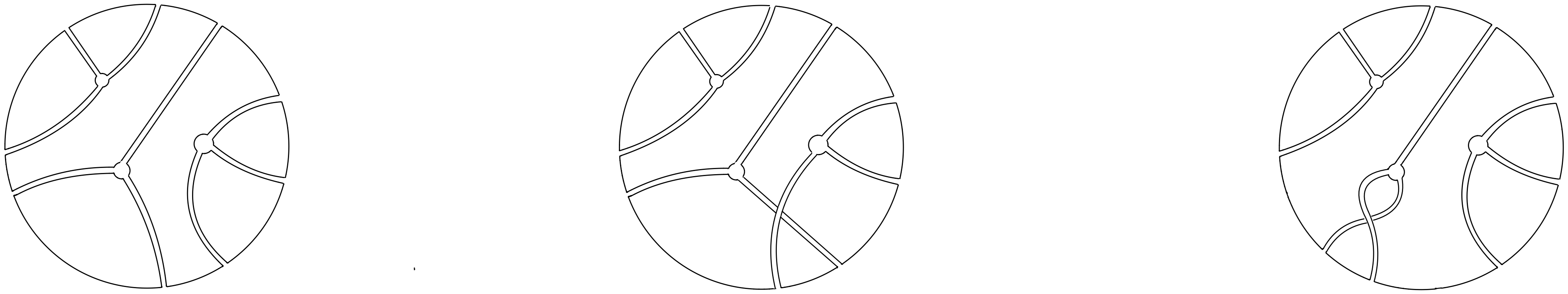


X-cycle from $\mathrm{tr}(X^{pn})$: cyclic word with pn half-edges;

XY-propagator from $\mathrm{tr}(XY)$: $\langle X_{ij}\,Y_{kl}\rangle = \delta_{il}\delta_{jk}$.

Performing all pn contractions: each Wick pairing produces a ribbon graph in which p points of the X-cycle are joined into a Y-hyperedge.

This is exactly the p-valent hyperedge picture for higher-genus Fuss-Catalan, generalising the chord-diagram of $p = 2$.



3b. ’t Hooft double lines and the genus

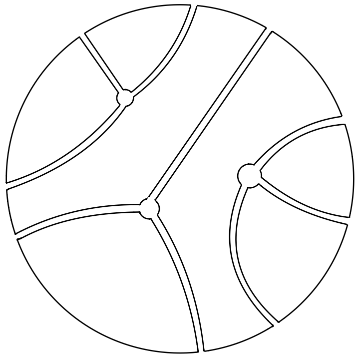
Each ribbon graph from Wick contractions has $V = n$ (p-valent vertices), $E = pn$ (propagators), F (faces from closed index loops). Counting factors of N from $\mathrm{tr}(XY)$ and the action:

$$\frac{1}{N^{(p-1)n+1}} N^F = N^{V-E+F} = N^{2-2g}/N = N^{-2g}.$$

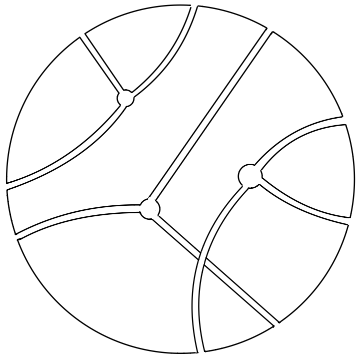
Hence $\langle \mathrm{tr} X^{pn} \rangle / N^{(p-1)n+1}$ generates ribbon graphs weighted by N^{-2g}:

$$\boxed{C_p^{(g)}(n) = \#\{\text{ribbon graphs of genus } g \text{ with } n \ p\text{-valent vertices and } pn \text{ edges}\}}$$

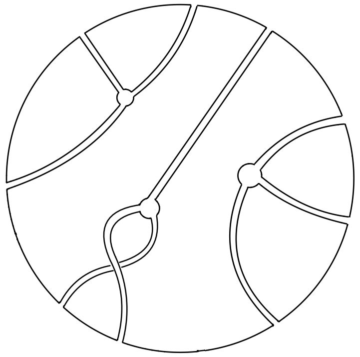
$$=$$



$$+\frac{1}{N^2}$$



$$+\frac{1}{N^2}$$



$$+\dots$$

3c. Spectral curve

Going to eigenvalues and applying Harish Chandra-Itzykson-Zuber (HCIZ) formula,

$$Z_{N,p} = [\text{volume of } U(N)] \times \int \prod_{j=1}^{N} dx_j dy_j \; \Delta(x)\Delta(y) e^{\sum_i (-x_i y_i + \frac{1}{p} y_i^p)}, \quad \Delta(z) = \prod_{j<k} (z_j - z_k).$$

Large-N saddle point yields meromorphic functions $X(y), Y(x)$ mutually inverse with

$$x = X(y) = y^{p-1} + \frac{1}{y} + o(1/y^2), \qquad y = Y(x) = \frac{1}{x} + o(1/x^2)\,.$$

From here one reconstructs the Spectral curve:

$$\boxed{xy \;=\; 1 + y^p}$$

— encodes the $g = 0$ data of the Fuss--Catalan problem.

— obtained setting $z = x^{-p}$ and $f = x\, y$ from the generating-function equation $f = 1 + z f^p$.

— The spectral curve is the $g = 0$ data; higher g come from topological recursion, but we will obtain them directly.

4. Exact identities (sum rules) from the matrix model

4a: exponential spectral density

To evaluate the expectation value of $\mathrm{tr}(X^{np})$, we use Brézin--Hikami contour integral representation of the exponential 1-point density

$$e_p(s \mid N) \;:=\; \left\langle \mathrm{tr}\, e^{sX} \right\rangle_{N,p} \;=\; \sum_{k\geq 0} \frac{s^k}{k!} \left\langle \mathrm{tr}(X^k) \right\rangle_{N,p}$$

— For the Gaussian measure, $p = 2$, $e_p(s \mid N)$ can be represented as a single contour integral (Brézin—Hikami, 2008)

$$e_2(s \mid N) \;=\; \frac{e^{s^2/2}}{s} \oint_0 \frac{du}{2\pi i}\, e^{us} \left(1+\tfrac{s}{u}\right)^N \qquad (p = 2)\,.$$

The integrand has a pole of order N at $u = 0$ from the factor $(1 + s/u)^N$, and the exponential is entire function of u (and of s).

— To solve the FC MM, we derived a $p \geq 2$ generalisation of Brézin--Hikami formula:

$$\boxed{e_p(s \mid N) \;=\; \frac{1}{s} \oint_{u=0} \frac{du}{2\pi i}\; \exp\!\left[\tfrac{1}{p}\left((u+s)^p - u^p\right)\right] \left(1 + \frac{s}{u}\right)^N \qquad (p \geq 2)}$$

4b. Extracting $\langle \mathrm{tr} X^{pn} \rangle_{N,p}$

Substitute $u = sw$ and expand in $q = s^p$:

$$\langle \mathrm{tr} X^{pn} \rangle_{N,p} = \frac{(pn)!}{p^n n!} \left[w^{N-1} \right] \left((w+1)^p - w^p \right)^n (w+1)^N .$$

Change variables $w = 1/t$:

$$\sum_{g \geq 0} C_p^{(g)}(n)\, N^{-2g} = \frac{(pn)!}{p^n n!\, N^{(p-1)n+1}} \oint_0 \frac{dt}{2\pi i}\, t^{-pn-2} \left((1+t)^p - 1 \right)^n (1+t)^N .$$

Expand

$$\left((1+t)^p - 1 \right)^n = \sum_{r=n}^{pn} T_p^{(n)}(r)\, t^r$$

with

$$T_p^{(n)}(r) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{pk}{r} .$$

Main result:

$$\boxed{\sum_{g \geq 0} C_p^{(g)}(n)\, N^{-2g} = \frac{(pn)!}{p^n n!\, N^{(p-1)n+1}} \sum_{r=n}^{pn} T_p^{(n)}(r) \binom{N}{pn+1-r} .}$$

Multiplying by $N^{(p-1)n+1}$, the LHS is a polynomial of degree $(p-1)n+1$ in N that can be analytically continued and expanded in $1/N^2$. At $p = 2$: reproduces the Harer--Zagier polynomial.

5. Genus g Fuss-Catalan numbers

Define the ratios

$$N_p^{(g)}(n) \;:=\; \frac{C_p^{(g)}(n)}{C_p^{(0)}(n)}, \qquad p \geq 2,\, n \geq 1,\, g \geq 1.$$

Theorem [master formula for the ratio]:

$$\boxed{N_p^{(g)}(n) \;=\; \big[(p-1)n+1\big]_{2g}\, \big[t^{2g}\big]\, G_n(t)}$$

where $[a]_k := a(a-1)\cdots(a-k+1)$, and

$$G_n(t) \;:=\; e^t\, \frac{[h(pt)]^n}{[h(t)]^{pn+2}}, \qquad h(t) = \frac{e^t-1}{t}\,.$$

Consequences:

— $N_p^{(g)}(n)$ is a polynomial in n of degree $\leq 3g$

— $N_p^{(g)}(0) = 0$ for $g \geq 1$

— All coefficients of this polynomial are computable in closed form.

5a. **Structure of** $N_p^{(g)}(n)$

Computation gives

$$N_p^{(g)}(n) \;=\; \sum_{k=1}^{3g} (p-1)^k \, Q_k(p) \, n^k$$

with $Q_k(p)$ a polynomial in p of degree $\leq 2g-1$.

- The leading n^{3g} coefficient is for all $g \geq 1$ and $p \geq 2$

$$Q_{3g}(p) \;=\; \frac{1}{g\,!}\left(\frac{p}{24}\right)^g$$

-- The linear term in n is

"Bernoulli identity": $$Q_1(p) \;=\; -\,\frac{B_{2g}}{2g}\,.$$

Closed form for $g = 1{,}2{,}3$:

$g = 1$: $$N_p^{(1)}(n) = \frac{p-1}{24}\, n\Big[p(p-1)^2n^2 + (p-1)(p-2)n - 2\Big]$$

$$Q_1 = -\frac{1}{12} = -\frac{B_2}{2}$$
$$Q_2 = \frac{p-2}{24}$$
$$Q_3 = \frac{p}{24}$$

$g = 2$

$$N_p^{(2)}(n) = \sum_{k=1}^{6}(p-1)^k\, Q_k\, n^k,$$

$$\begin{aligned}
Q_1 &= \frac{1}{120} = -\frac{B_4}{4},\\
Q_2 &= -\frac{p^3+p^2+11p+6}{1440},\\
Q_3 &= \frac{p^3+6\,p^2+11\,p-24}{2880},\\
Q_4 &= \frac{4p^3-p^2+44\,p+24}{5760},\\
Q_5 &= -\frac{p\,(p^2+6\,p+11)}{2880},\\
Q_6 &= \frac{1}{2!}\,\left(\frac{p}{24}\right)^2.
\end{aligned}$$

$g = 3$

$$N_p^{(3)}(n) = \sum_{k=1}^{9}(p-1)^k\, Q_k\, n^k,$$

$$\begin{aligned}
Q_1 &= -\frac{1}{252} = -\frac{B_6}{6},\\
Q_2 &= \frac{4p^5+4p^4+25p^3+25p^2+151p+130}{30240},\\
Q_3 &= -\frac{52p^5+178p^4+451p^3+1081p^2+1963p-900}{362880},\\
Q_4 &= -\frac{20p^5-71p^4-36p^3-421p^2+755p+1000}{241920},\\
Q_5 &= \frac{40p^5+103p^4+222p^3+628p^2+1510p+432}{290304},\\
Q_6 &= -\frac{48p^5+398p^4+825p^3+2400p^2+1812p+160}{967680},\\
Q_7 &= \frac{16p^4+394p^3+1353p^2+2368p+604}{2903040},\\
Q_8 &= -\frac{2p^2+17p+12}{138240},\\
Q_9 &= \frac{1}{3!}\,\left(\frac{p}{24}\right)^3
\end{aligned}$$

Universal features:

$$N_p^{(g)}(0) = 0$$
$$Q_1 = -B_{2g}/(2g),$$
$$Q_{3g} = (1/g!)(p/24)^g$$

5.3 Interpretation in terms of p-spin complex curves:

Top coefficient: p-spin intersection number

Top coefficient = intersection number in the the moduli space of p-spin curves.

$$\left[n^{3g}\right] N_p^{(g)}(n) \;=\; (p-1)^{3g}\, Q_{3g}(p) \;=\; p^g (p-1)^{2g}\, \langle \tau_{3g-2,0} \rangle^{(g)}_{p\text{-spin}}, \quad \langle \tau_{3g-2,0} \rangle^{(g)}_{p\text{-spin}} \;=\; \frac{(p-1)^g}{24^g\, g\,!}\,.$$

The factor $p^g(p-1)^{2g}$ is the matrix-model normalisation;
trivial ($= p^g$) only at $p = 2$. At $p = 2,\ g = 1$: $\langle \tau_{1,0} \rangle_2^{(1)} = \frac{1}{24}$ (Witten).

Bottom coefficient: Euler characteristic

The linear in n coefficient expected to be related to the orbifold Euler characteristic of the moduli space of p-spin curves.

Bottom coefficient (Bernoulli identity): For all $g \geq 1$,

$$\left[n^1\right] N_p^{(g)}(n) \;=\; (p-1)\, Q_1(p) \;=\; -\,(p-1)\,\frac{B_{2g}}{2g} \;=\; \chi\big(\overline{\mathcal{M}}^{1/p}_{g,1}\big)\,.$$

The $(p-1)$ prefactor reflects the orbifold structure of the p-spin cover $\overline{\mathcal{M}}^{1/p}_{g,1} \to \overline{\mathcal{M}}_{g,1}$.

At $p = 2$ this is the classical Harer—Zagier/Penner formula $\chi(\overline{\mathcal{M}}_{g,1}) = -\,B_{2g}/(2g)$.

The **intermediate** $Q_k(p)$, $1 < k < 3g$, are expected to encode p-spin Hodge integrals.
At $g = 1$: a 3-term polynomial in n already interpolates between the Euler characteristic (n^1) and the $\langle \tau_{1,0} \rangle$ intersection (n^3).

Conjecture: The middle $Q_k(p)$ arise as linear combinations of p-spin Hodge integrals involving ψ-classes and the Witten class.

Two-trace correlator at genus zero:

Brézin--Hikami extends to the multi-trace correlators.
At leading order, $C_p^{(0)}(n_1, n_2)$ has a remarkably simple form:

Genus-0 two-trace:

$$C_p^{(0)}(n_1, n_2) \;=\; (p-1)\,\frac{n_1 n_2}{n_1 + n_2}\binom{pn_1}{n_1}\binom{pn_2}{n_2}$$

— A comment on the first definition of higher genus FC:

6. Matrix model for "counting partitions by genus":

p pairs of hermitian matrices $\{X_a, Y_a\}, \quad a = 1,\ldots,p$

$$Z_{N,p} = \int dX_1 dY_1 \ldots dX_p dY_p \; e^{-S(X,Y)}$$

$$S(X,Y) = \sum_{a=1}^{p} \mathrm{tr} X_a Y_a - \frac{1}{p} \mathrm{tr}[Y_1 Y_2 \ldots Y_p]$$

$$C_p^{(g)}(n)\Big|_{\text{partitions}} = \left\langle \mathrm{tr}[(X_p \ldots X_2 X_1)^n] \right\rangle$$ — order of the legs of the p-vertex frozen

Does not diagonalise by HCIZ, relative $U(N)$ rotations of matrices appear non-trivially.

Much more complicated problem!

Summary

1. A two-matrix model with action

$$S(X,Y) = \mathrm{tr}(XY) - \frac{1}{p}\mathrm{tr}Y^p$$

generates the higher-genus Fuss--Catalan numbers $C_p^{(g)}(n)$

as the $1/N$-expansion of $\langle \mathrm{tr}X^{pn}\rangle$

2. Exact sum rule from a higher-Airy Brézin—Hikami contour integral; reduces to Harer—Zagier at $p = 2$.

3. Master formula:

$$N_p^{(g)}(n) := C_p^{(g)}(n)/C_p^{(0)}(n)$$

is a polynomial of degree $3g$ in n with no constant term.

— Top coefficient ↔ one-point p-spin Witten--Kontsevich intersection number.

— Bottom coefficient ↔ Euler characteristic $\chi\left(\overline{\mathcal{M}}_{g,1}^{1/p}\right) = -(p-1)\frac{B_{2g}}{2g}$

Thank you!