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No-Hair Theorems for Black Holes with Multiple Scalar Fields:
Mathematical and Physical Aspects

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Global properties of stationary black holes in GR

We shall assume that the matter energy-momentum tensor satisfies certain natural energy conditions, such as the weak energy condition, i.e. $T_{\mu\nu}u^{\mu}u^{\nu} \geq 0$ for every future-pointing timelike vector u^{μ}.

A limiting case of the weak energy condition is the null energy condition according to which $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for every future-pointing null vector k^{ν}.

We denote by ξ the stationary Killing vector, $L_{\xi}\, g_{\mu\nu} = 0$, which is asymptotically timelike and is normalized so that $g_{\mu\nu}\xi^{\mu}\xi^{\nu} = -1$ at infinity.
In adapted coordinates $\xi = \frac{\partial}{\partial t}$.

No-hair theorems in General Relativity

The horizon of the black hole will be denoted by H, and we shall assume that H is non-degenerate. We also assume that the domain of outer communication $< M >$ is globally hyperbolic. The topological censorship theorem states that, provided the matter fields satisfy the averaged null energy condition, $< M >$ is a simply connected manifold with boundary, $\partial < M >= H$. Galloway, Schleich, Witt and Woolgar, PRD 1996

As a consequence of the topological censorship theorem, it also follows that the cross-section of the horizon with a partial Cauchy surface, has $\mathbb{S}^{\mathbf{2}}$ topology. (also Hawking, Commun. Math. Phys 1972)

All these results imply that the topology of the domain of outer communications must be $< M > = \boldsymbol{R} \times \boldsymbol{R}^{\mathbf{3}} \backslash \boldsymbol{B}$ where $\boldsymbol{R}^{\mathbf{3}} \backslash \boldsymbol{B}$ is the complement of a 3-dimensional open ball $\boldsymbol{B}$ in Euclidean space $\boldsymbol{R}^{\mathbf{3}}$.

Rigidity theorem

The rigidity theorem relates the global concept of the event horizon to the local notion of a Killing horizon. In simple terms, the theorem states that any stationary black hole is either static or axisymmetric, rotating with a constant angular velocity Ω_H. Hawking, Commun. Math. Phys 1972

Theorem *Let $(\mathcal{M}, g)$ be an analytic, asymptotically flat, stationary black hole space-time. Furthermore, let the energy-momentum tensor of matter satisfy the weak energy condition. Then, the event horizon is a Killing horizon, i.e. there exists a Killing field $\mathcal{K}$ which has an associated Killing horizon coinciding with the event horizon. The Killing field $\mathcal{K}$ either coincides with the stationary Killing field ξ and the spacetime is static, or space-time admits at least one axial Killing field η (i.e. with periodic orbits) such that $\mathcal{K} = \xi + \Omega_H \eta$, where $\Omega_H = const.$*

Horizon: $\boldsymbol{g(K, K) = g_{\mu\nu} K^\mu K^\nu = 0}$

In adapted coordinates $\xi = \frac{\partial}{\partial t}$ and $\eta = \frac{\partial}{\partial \phi}$, so $\mathrm{K} = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$.

Kerr solution

The rotating vacuum black holes in general relativity are described by the two-parameter Kerr solution. The Kerr black hole solution has a stationary, axisymmetric, and asymptotically flat domain of outer communications.

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}dtd\phi$$
$$+\left[\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

Δ and Σ are given by: $\Delta = r^2 + a^2 - 2Mr \quad \Sigma = r^2 + a^2\cos^2\theta \qquad M^2 \geq a^2.$

***M* is the (ADM) mass and *J* = *Ma* is the angular momentum of the black hole**.

The bigger root, $r_+ = M + \sqrt{M^2 - a^2}$ of $\Delta = 0$, corresponds to the event horizon.

No hair conjecture

Physical formulation: Black holes have no hair – *in other words, whatever matter originates the black hole, all information about it, disappears, except for a small set of asymptotically measurable quantities (charges).*

The physical formulation of the conjecture is, however, too broad and vague to be treated mathematically. A natural idealization of the problem is therefore to consider isolated black holes in vacuum (or electrovacuum).

Theorem *The Kerr metric with parameters M and $a = J/M$ is the only black hole solution with $M^2 \geq a^2$, regular event horizon and stationary, asymptotically flat domain of outer communications.*

Carter, PRL 1971; Robinson PRL 1975; Mazur, JPA 1982; Bunting 1983

In vacuum GR, the stationary black holes are fully specified in terms of the conserved asymptotic charges – namely, the mass $\boldsymbol{M}$ and angular momentum $\boldsymbol{J}$.

How the picture changes in higher dimensions?

In higher dimensions the picture changes drastically!

- In higher dimensions we have infinitely many possible horizon topologies. For example, in the 5-dinesional case, the topology of the black holes can be either $\mathbb{S}^3$ (3-sphere), $\mathbb{S}^2 \times \mathbb{S}^1$ (3-ring) or a lens space $L(p, q)$ (with p and q being integers). Hollands and Yazadjiev, Commun. Math. Phys. 2008, 2011

- Vacuum black holes in 5 dimensions ARE NOT fully specified by the mass and the angular momenta, New datum is needed in order to specify the black hole solution uniquely – the so-called interval structure related to the boundary of the factor space $< M >/Iso(M)$ where $Iso(M)$ is the spacetime isometry group.
Hollands and Yazadjiev, Commun. Math. Phys. 2008, 2011; Class. Quant. Grav. 2009

Violations of the no-hair conjecture

How the picture changes in the presence of matter?

- In the presence of matter the black holes **are not,** in general, fully specified by their conserved asymptotic charges like the mass and the angular momentum. There are many counter examples to the no-hair conjecture – Yang-Mills fields, Skyrme fields and scalar fields.

- The topology of black hole horizons is $\mathbb{S}^2$, when the matter satisfies natural physical conditions.

SCALAR FILDS

- Scalar fields are of special interest to modern physics, cosmology and astrophysics. For example, they are among the best candidates for dark matter.

- Sclar fields can be minimally and non-minimally coupled to gravity

Minimally coupled single scalar field

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right)$$

Scalar hair

Field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\nabla_\mu\varphi\nabla_\nu\varphi - g_{\mu\nu}\nabla_\sigma\varphi\nabla^\sigma\varphi - \frac{1}{2}g_{\mu\nu}V(\varphi),$$

$$\nabla_\mu\nabla^\mu\varphi = \frac{1}{4}V'(\varphi).$$

From physical point of view, it is natural to impose $V(\varphi) \geq 0$. Under this condition, the scalar field energy-momentum tensor

$$T^{\varphi}_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\nabla_\alpha\varphi\nabla^\alpha\varphi - \frac{1}{2}g_{\mu\nu}V(\varphi)$$

satisfies all the natural energy conditions, in particular the weak energy condition $T^{\varphi}_{\mu\nu}u^\mu u^\nu \geq 0$.

The rigidity theorem then holds and therefore the stationary black holes in this theory are also axisymmetric and $\mathrm{L}_\xi T^{\varphi}_{\mu\nu} = \mathrm{L}_\eta T^{\varphi}_{\mu\nu} = 0$. As a consequence the scalar field satisfies $L_\xi\varphi = \frac{\partial\varphi}{\partial t} = 0$, $L_\eta\varphi = \frac{\partial\varphi}{\partial\phi} = 0$. In other words the gradient of scalar field is purely spacial.

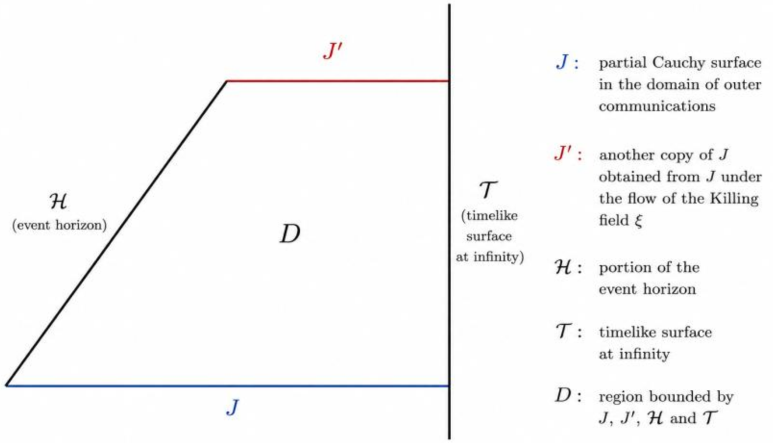
Scalar hair

Theorem: Let us consider the Einstein-scalar field equations with a non-negative potential $V(\varphi) \geq 0$ satisfying the condition $\varphi V'(\varphi) \geq 0$ $(V''(\varphi) \geq 0)$. Then the only black hole solutions to these equations with regular event horizon and stationary and asymptotically flat domain of outer communications is the Kerr family of solutions.

Proof: (Hawking, 1972)

Let us consider a partial Cauchy surface $\mathcal{J}$ in the domain of outer communications and another copy $\mathcal{J}' \neq \mathcal{J}$ of $\mathcal{J}$ obtained from $\mathcal{J}$ under the flow of the Killing field ξ. Let $\mathcal{D}$ be the region bounded by $\mathcal{J}$, $\mathcal{J}'$, a portion of the event horizon H and a timelike 3-surface $\mathcal{T}$ at infinity.

$$\nabla_\mu \left(\varphi \nabla^\mu \varphi\right) = \nabla_\mu \varphi \nabla^\mu \varphi + \frac{1}{4}\, \varphi\, V'(\varphi)$$



Scalar hair

$$\int_{\partial\mathcal{D}} d^3S n_\mu(\varphi\nabla^\mu\varphi) = \int_{\mathcal{D}} d^4x\sqrt{-g}\left[\nabla_\mu\varphi\nabla^\mu\varphi + \frac{1}{4}\varphi V'(\varphi)\right] \qquad \partial\mathcal{D} = \mathcal{J}\cup\mathcal{J}'\cup\mathcal{T}\cup H$$

$$\int_{\partial\mathcal{D}} d^3S n_\mu(\varphi\nabla^\mu\varphi) = \int_{\mathcal{J}} d^3S n_\mu(\varphi\nabla^\mu\varphi) + \int_{\mathcal{J}'} d^3S n_\mu(\varphi\nabla^\mu\varphi)$$
$$+ \int_{\mathcal{T}} d^3S n_\mu(\varphi\nabla^\mu\varphi) + \int_{\mathcal{H}} d^3S n_\mu(\varphi\nabla^\mu\varphi).$$

$\mathcal{J}$ and $\mathcal{J}'$ are isometric copies and have opposite directions. Therefore the surface integral over $\mathcal{J}$ cancels out that over $\mathcal{J}'$. The black hole horizon integral gives no contribution because on the horizon $n^\mu = K^\mu = \xi^\mu + \Omega_H\eta^\mu$ and $n_\mu\nabla^\mu\varphi = K_\mu\nabla^\mu\varphi = \xi_\mu\nabla^\mu\varphi + \Omega_H\eta_\mu\nabla^\mu\varphi = 0$. The surface integral at infinity is also zero because φ is zero there – the asymptotic flatness requires $\varphi_\infty = \varphi_* = 0$. In this way we find that

$$\int_{\mathcal{D}} d^4x\sqrt{-g}\left[\nabla_\mu\varphi\nabla^\mu\varphi + \frac{1}{4}\varphi V'(\varphi)\right] = 0.$$

Scalar hair

In the case $V''(\varphi) \geq 0$, we consider $\nabla_\mu(V'(\varphi)\nabla^\mu\varphi) = V''(\varphi)\nabla_\mu\varphi\nabla^\mu\varphi + \frac{1}{4}(V'(\varphi))^2.$

The additional conditions imposed on the scalar potential lead to certain shortcomings of the above no-hair theorem. For example, the theorem does not cover Higgs-like potentials. From a physical point of view, it would be desirable for the non-negativity of the potential to be the only condition imposed on it. However, no such general result has been proven so far, even 50 years after Hawking's proof.

Only under spherical symmetry can the additional condition on the scalar potential be removed.

Multiscalar hair

Two scalar fields minimally coupled to gravity

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}\,[R\ - 2g^{\mu\nu}\nabla_\mu\varphi^1\,\nabla_\nu\varphi^1\ - 2g^{\mu\nu}\nabla_\mu\varphi^2\,\nabla_\nu\varphi^2\ \ - V(\varphi^1,\varphi^2)\,]$$

To be specific we consider $V(\varphi^1,\varphi^2) = \frac{1}{2}\mu^2\,[(\varphi^1)^2 + (\varphi^1)^2]$ with μ being the mass of the scalar fields.

Equivalently, we can consider one complex scalar field $\Psi = \varphi^1 + i\,\varphi^2$ with a potential $V(\Psi) = |\Psi|$^2.

Not only is it nontrivial, but adding one more scalar field changes the whole picture dramatically!

Multiscalar hair

What is the reason behind the drastic change of the picture?

When there is a single scalar field, one can show that, in the presence of a black hole and under the asymptotic flatness condition, the scalar field shares the symmetries of the spacetime. Thus, if the spacetime is stationary and axisymmetric, the scalar field is stationary and axisymmetric as well.

In the case of two scalar fields, however, they do not necessarily share the symmetries of the geometry. In other words, the scalar fields may depend on the time and on the azimuthal coordinate, while their energy-momentum tensor still remains stationary and axisymmetric. **Therefore, Hawking's arguments can no longer be applied.**

For example, we can have the following simple periodic dependence of the scalar fields on the time and azimuthal coordinate leaving the energy-momentum tensor stationary and axisymmetric: $\Psi = \varphi^1 + i\varphi^2 = \psi(r,\theta)\, e^{i\omega t - im\phi}$, where ω is a real number, ($\omega^2 < \mu^2$) and m is an integer.

Multiscalar hair

It can then be shown numerically that there exist nontrivial rotating black hole solutions with periodic in time scalar hair .

Hod, PRD 2012; Herdeiro and Radu, PRL 2014; Collodel, Doneva and Yazadjiev PRD 2020.

Such hairy black holes can exist only when the frequency of the scalar fields satisfies $\omega^2 < \mu^2$ and is synchronized with the angular frequency of the horizon: $\omega = m\Omega_H$ with m being integer.

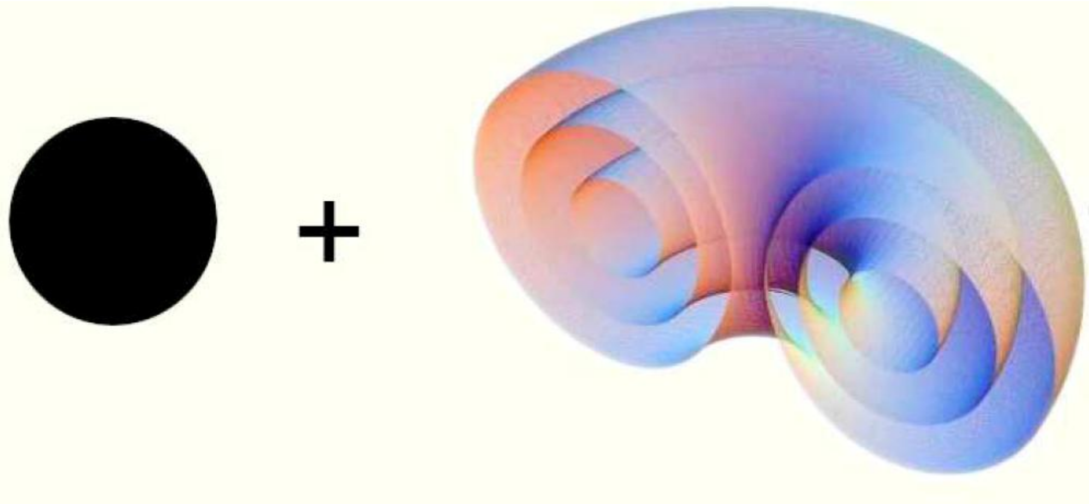
- The first condition is related to the fact that we consider asymptotically flat spacetimes, since only under this condition can bound configurations of scalar fields exist. $\Psi \sim \exp(-\sqrt{\mu^2 - \omega^2}\, r)$
- The second condition is also natural from physical point of view. It prevents the scalar fields from being swallowed by the black hole. In other words, it ensures that the flux of the scalar fields through the horizon vanishes -- which in turn guarantees a stationary configuration.

Multiscalar hair

Conserved current: $j^{\mu} = i\,(\Psi^{*}\nabla^{\mu}\Psi\; - \Psi\nabla^{\mu}\Psi^{*})$

Normal component on the horizon: $j_n = n_{\mu}j^{\mu} = \left(\xi_{\mu} + \Omega_H\eta_{\mu}\right)j^{\mu} = (\omega - m\Omega_H)|\psi(r,\theta)|^2$

How the hairy black holes look like?



Multiscalar hair

How can such black holes with scalar hair be classified mathematically?

- The natural conjecture: Hairy black holes can be uniquely classified in terms of the conserved charges – the mass M, the angular momentum J and the Noether charge Q associated with the conserved current j^{μ} (i.e. the number of bosons).

- The spectrum of hairy black hole solutions is very rich and there exist many solutions with the same M, J, Q.

- Unfortunately, at present there is no rigorous mathematical classification in the general case. The mathematical problem seems to be much more difficult compared even to classification theorems for higher dimensional Einstein equations!

- Fortunately, for some subsectors one can prove non-hair theorems.

Placing the mathematical problem in a more general setting

From physical point of view we want to consider the most general case of arbitrary number of scalar fields and kinetic energy. Doneva and Yazadjiev, PRD 2020

The mathematical picture of the desired generalization is the following.

We consider the 4-dimensional spacetime $(M^4, g_{\mu\nu})$, an N-dimensional Riemannian manifold (E_N, γ_{ab}), the so-called target space, and a map $\varphi : M^4 \to E_N$. The differential $d\varphi$ induces a map between the tangent spaces of M^4 and E_N, $d\varphi : TM^4 \to TE_N$. The norm of the differential $< d\varphi, d\varphi >$ in local coordinate patches on M^4 and E_N is given by:

$$< d\varphi, d\varphi >= g^{\mu\nu}(x)\gamma_{ab}(\varphi(x))\partial_\mu\varphi^a(x)\partial_\nu\varphi^b(x).$$

Multiscalar hair

We shall also consider multi-scalar self-gravitating theories defined by the following action

$$S = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\,[R - 2 < d\varphi, d\varphi > -4V(\varphi)]\,,$$

where R is the Ricci scalar curvature and V(φ) is the scalar fields potential.

The field equations associated with the action and written in local coordinate patches of M^4 and E_N are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\gamma_{ab}(\varphi)\nabla_\mu\varphi^a\nabla_\nu\varphi^b - \gamma_{ab}(\varphi)\nabla_\sigma\varphi^a\nabla^\sigma\varphi^b g_{\mu\nu} - 2V(\varphi)g_{\mu\nu},$$

$$\nabla_\mu\nabla^\mu\varphi^a = -\gamma^a_{cd}(\varphi)\nabla_\mu\varphi^c\nabla^\mu\varphi^d + \gamma^{ab}(\varphi)\frac{\partial V(\varphi)}{\partial\varphi^b},$$

where γ^a_{bc} are the Christoffel symbols with respect to the target space metric $\gamma_{ab}(\varphi)$.

Multiscalar hair

In order to ensure that the energy-momentum tensor of the scalar fields can, in principle, be stationary (time-independent), we assume that the target space metric γ_{ab} admits a Killing field $\boldsymbol{k^a}$ whose flow leaves the potential V(φ) invariant.

The existence of such a Killing field is also very important from a physical point of view – it leads to the existence of a conserved current:

$$j^\mu = k_a \nabla^\mu \varphi^a, \qquad \nabla_\mu j^\mu = 0.$$

In addition we assume that the axis of $\boldsymbol{k^a}$, i.e. the points where $\boldsymbol{k^a = 0}$, is non-empty. This additional requirement is important because, at least, the asymptotic flatness requires $\lim_{\infty} |k| = 0$.

Multiscalar hair

We focus here on static and spherically symmetric spacetimes with a metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 s_{ij}dx^i dx^j,$$

where s_{ij} is the metric on the unit 2D sphere, $s_{ij}dx^i dx^j = d\theta^2 + \sin^2\theta \, d\phi^2$.

The dynamics of scalar fields is confined on the flow of k^a. In more formal language we require $L_\xi \varphi^a = \partial_t \varphi^a = \omega \, k^a$ where $\omega \neq 0$ is a real number. This guarantees that

$$\mathcal{L}_\xi T_{\mu\nu} = -\omega \left(\mathcal{L}_k \gamma_{ab}(\varphi)\right) \left[2\nabla_\mu \varphi^a \nabla_\nu \varphi^b - \nabla_\sigma \varphi^a \nabla^\sigma \varphi^b g_{\mu\nu}\right] + 2\omega \mathcal{L}_k V(\varphi) g_{\mu\nu} = 0.$$

In order to simplify the presentation we shall also assume that the scalar fields do not depend on angular coordinates, i.e. $\partial_i \varphi^a = 0$.

Multiscalar hair

Under the assumptions we made, the dimensionally reduced field equations take the form:

$$\frac{2}{r}e^{-2\Lambda}\Lambda' + \frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) = \omega^2 e^{-2\Phi}|k|^2 + e^{-2\Lambda}\gamma_{ab}(\varphi)\partial_r\varphi^a\partial_r\varphi^b + 2V(\varphi),$$

$$\frac{2}{r}e^{-2\Lambda}\Phi' - \frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) = \omega^2 e^{-2\Phi}|k|^2 + e^{-2\Lambda}\gamma_{ab}(\varphi)\partial_r\varphi^a\partial_r\varphi^b - 2V(\varphi),$$

$$e^{-2\Lambda}\left[\Phi'' + (\Phi' + \frac{1}{r})(\Phi' - \Lambda')\right] = -\frac{1}{2}K - 2V(\varphi),$$

$$\partial_r\left(e^{\Phi-\Lambda}r^2\gamma_{ab}(\varphi)\partial_r\varphi^b\right) = \frac{1}{4}e^{\Phi+\Lambda}r^2\left(\frac{\partial V(\varphi)}{\partial\varphi^a} + \frac{\partial K}{\partial\varphi^a}\right) + \omega^2 r^2 e^{\Lambda-\Phi}k^b\frac{\partial k_a}{\partial\varphi^b},$$

where

$$K = 2\gamma_{ab}(\varphi)\nabla_\mu\varphi^a\nabla^\mu\varphi^b = -2\omega^2 e^{-2\Phi}|k|^2 + 2e^{-2\Lambda}\gamma_{ab}(\varphi)\partial_r\varphi^a\partial_r\varphi^b.$$

Multiscalar hair

The strategy for proving the no-hair theorem is based on a proper divergence identity. Using the dimensionally reduced field equation, after long calculations, one can derive the following divergence identity:

$$\frac{d}{dr}\left[2e^{\Phi+\Lambda}V(\varphi) - e^{\Phi-\Lambda}P^2 - \omega^2 e^{\Lambda-\Phi}|k|^2\right]$$
$$= \frac{1}{r}e^{\Phi+\Lambda}\left[(1+3e^{-2\Lambda})P^2 + \omega^2 e^{2\Lambda-2\Phi}|k|^2(1-e^{-2\Lambda})\right].$$

Here P^2 is defined by $P^2 = \gamma_{ab}(\varphi)\partial_r\varphi^a\partial_r\varphi^b$.

What is important about this identity is the fact that the LHS is total divergence while the RHS is non-negative.

Multiscalar hair

The last step is to integrate the identity from the horizon to infinity and we get

$$-2e^{(\Phi+\Lambda)_h}V(\varphi_h) = \int_{r_H}^{+\infty} dr\frac{1}{r}e^{\Phi+\Lambda}\left[(1+3e^{-2\Lambda})P^2 + \omega^2 e^{2\Lambda-2\Phi}|k|^2(1-e^{-2\Lambda})\right],$$

where we have taken into account that:

$$\lim_{r\to\infty} P^2(r) = 0, \quad \lim_{r\to\infty} V(\varphi) = 0, \quad \lim_{r\to\infty} |k| = 0$$

and

$$\lim_{r\to r_h} e^{\Phi-\Lambda}P^2(r) = \lim_{r\to r_h} e^{\Lambda-\Phi}|k|^2 = 0.$$

The left hand side is non-positive since by assumption $V(\varphi) \geq 0$, while the right hand side is non-negative. Therefore, we can conclude that both sides vanish. Consequently we have that $P^2 = \gamma_{ab}(\varphi)\partial_r\varphi^a\partial_r\varphi^b = 0$ and $\omega^2|k|^2 = \gamma_{ab}(\varphi)\partial_t\partial_t\varphi^b = 0$ everywhere which means that the scalar fields are constant in the domain of outer communications.

Multiscalar hair

Theorem: The static spherically symmetric black holes CAN NOT support time dependent multiscalar hair. The scalar fields must be trivial and the spacetime is isometric to that of the Scharzschild black hole.

Only the rotating black holes can support multiscalar (synchronized) hair of time-periodic fields.

Scalar fields non-minimally coupled to gravity

Consider theories where the scalar field is coupled (non-minimally) to all curvature invariants of second order:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\Big[R - 2\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi) + f_1(\varphi)R^2 + f_2(\varphi)R_{\mu\nu}R^{\mu\nu} + f_3(\varphi)R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + f_4(\varphi)\mathscr{R}*\mathscr{R}\Big] + S_{matter}(A^2(\varphi)g_{\mu\nu},\Psi),$$

$$\mathscr{R}*\mathscr{R} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma}$$

In the general case, this action yields higher-order field equations that are prone to Ostrogradski instability and the appearance of ghosts. Fortunately, there is a particular sector of the theories, namely scalar-Gauss-Bonnet gravity, for which the field equations are of second order as those of GR and the theory is free from ghosts.

Non-minimally coupled scalar field

Scalar-Gauss-Bonnet gravity (vacuum sector)

$$S_{GB} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - 2\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi) + \lambda^2 f(\varphi)\mathscr{R}^2_{GB}\right]$$

$$\mathscr{R}^2_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$$

Field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Gamma_{\mu\nu} = 2\nabla_\mu\varphi\nabla_\nu\varphi - g_{\mu\nu}\nabla_\alpha\varphi\nabla^\alpha\varphi - \frac{1}{2}g_{\mu\nu}V(\varphi),$$

$$\nabla_\alpha\nabla^\alpha\varphi = \frac{1}{4}\frac{dV(\varphi)}{d\varphi} - \frac{\lambda^2}{4}\frac{df(\varphi)}{d\varphi}\mathcal{R}^2_{GB},$$

$$\begin{aligned}\Gamma_{\mu\nu} = & -R(\nabla_\mu\Psi_\nu + \nabla_\nu\Psi_\mu) - 4\nabla^\alpha\Psi_\alpha\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + 4R_{\mu\alpha}\nabla^\alpha\Psi_\nu + 4R_{\nu\alpha}\nabla^\alpha\Psi_\mu \\ & -4g_{\mu\nu}R^{\alpha\beta}\nabla_\alpha\Psi_\beta + 4R^\beta_{\ \mu\alpha\nu}\nabla^\alpha\Psi_\beta\end{aligned}$$

with

$$\Psi_\mu = \lambda^2\frac{df(\varphi)}{d\varphi}\nabla_\mu\varphi.$$

Non-minimally coupled scalar field

Coupling function $\boldsymbol{f(\varphi) = \varphi^2 + const\ \varphi^4 + \ \ldots}$

For such a coupling function the scalar-Gauss-Bonnet gravity is indistinguishable from GR in the weak coupling regime.

Black holes in scalar-Gauss-Bonnet gravity and spontaneous scalarization

For simplicity we shall put $V(\varphi) = 0$.

Using the field equations, one can easily see that the GR black hole solutions are also black hole solutions to scalar-Gauss-Bonnet theory with a trivial scalar field $\varphi = 0$ (due to the fact that $\frac{df}{d\varphi}(0) = 0$).

Non-minimally coupled scalar field

What is important is that, when the curvature of the horizon exceeds a certain critical value depending on the coupling parameter λ, the GR black holes (Schwarzschild and Kerr) become linearly unstable within the bigger scalar-Gauss-Bonnet theory. Doneva and Yazadjiev PRL 2018

This linear instability, called tachyonic instability, produces new fully realistic black holes with scalar hair – the so-called scalarized black holes. Doneva and Yazadjiev PRL 2018

The process of formation of scalarized black holes from pure GR black holes trough tachyonic instability is called spontaneous scalarization.

Spontaneous scalarization is one of the two known dynamical mechanisms for endowing black holes (and other compact objects) with scalar hair without altering the predictions in the weak field limit.

Non-minimally coupled scalar field

In essence, the process of the spontaneous scalarization consists in the following .
A configuration with a trivial scalar field and a spacetime metric that is a solution to Einstein's equations describes all selfgravitating systems except some that exhibit very strong gravity. For the latter case, when the curvature exceeds certain threshold, the zero scalar field configurations become tachyonically unstable. This instability develops until it is eventually quenched by nonlinear effects, forming a new stable configuration (phase) with a nontrivial scalar field and a spacetime metric that no longer solves the pure Einstein's equations.
Doneva, Silva, Sotiriou, Ramazanoglu, Yazadjiev, Rev. Mod. Phys. 2023

Alternatively, the spontaneous scalarization can be viewed as a screening mechanism that forces a scalar field to transition to a trivial configuration in the weak field regime and hence explain why this field has managed to remain undetected in observations so far.

Non-minimally coupled scalar field

The spontaneous scalarization generates, in the general case, infinitely many scalarized black hole solutions (few being linearly stable) which coexist with the GR black holes. This makes the mathematical classification an extremely difficult task.

The rigorous mathematical classification of the spontaneously scalarzied black hole solutions is an open problem.

The good news: We have realistic black holes with scalar hair!

Physical aspects of the black holes with (multi)scalar hair

Astrophysics of scalar-hair

How can we test the existence of scalar fields around black holes by real observations?

We have two main observation channels:

- Electromagnetic channel – the standard astronomy
- Gravitational wave channel – the new born gravitational wave astronomy

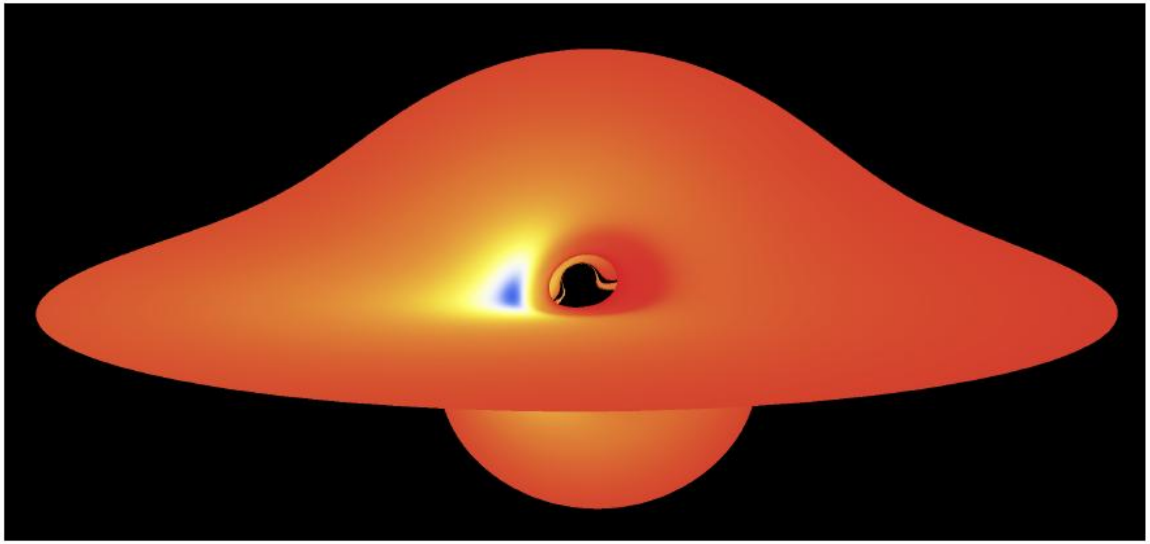
The (multi)scalar hair can change both the electromagnetic and gravitational wave properties of both single and binary black holes leaving in this way observable (measurable) imprints in the electromagnetic and the gravitational wave signals from them.

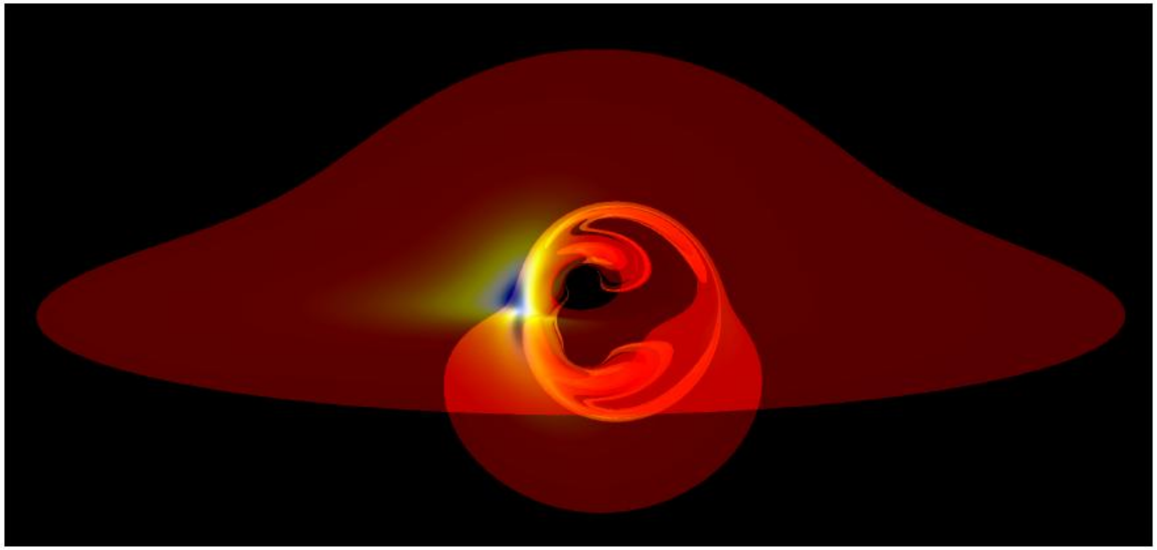
Imprints of the synchronized time-periodic scalar hair in the black hole shadow

Such solutions exist across a broad mass range, from stellar-mass to supermassive black holes, and thus provide a useful theoretical framework for exploring potential observational signatures of scalar hair,

Recent horizon-scale observations by the Event Horizon Telescope (EHT) collaboration have demonstrated the capability to resolve ring-like emission structures around supermassive black holes such as M87* and Sgr A*, opening the possibility of testing strong-field deviations from the Kerr geometry through direct imaging.

Astrophysics of scalar-hair

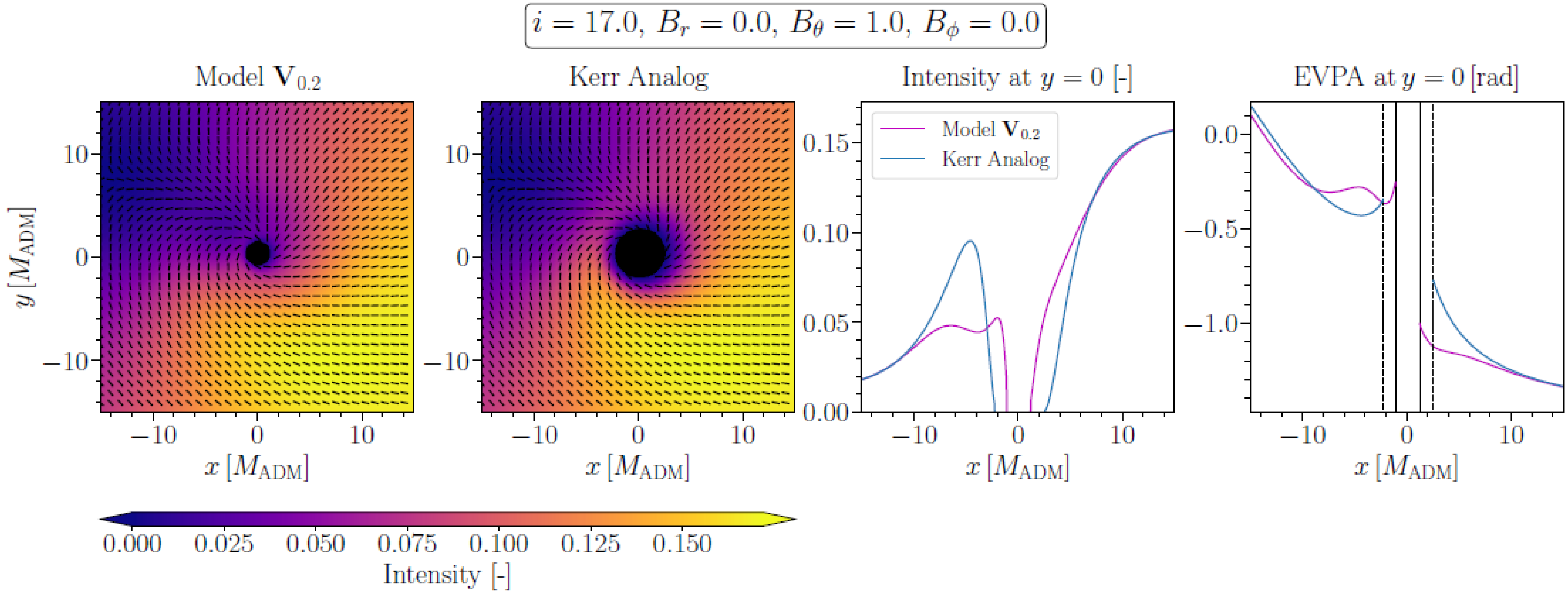




Gyulchev, Deliyski, Yazadjiev, PRD 2026

Astrophysics of scalar-hair

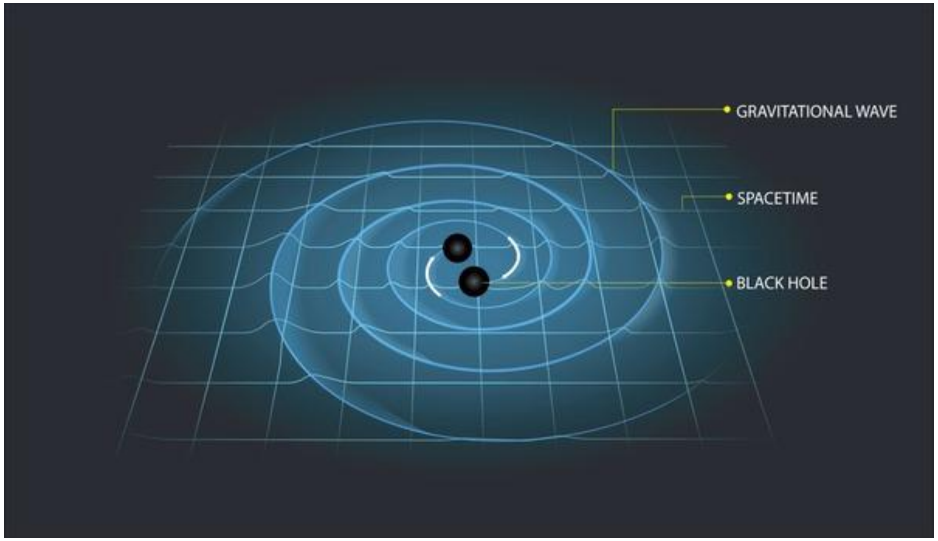
Synchronized scalar hair changes not only the global form of the shadow but also the polarization of the images.



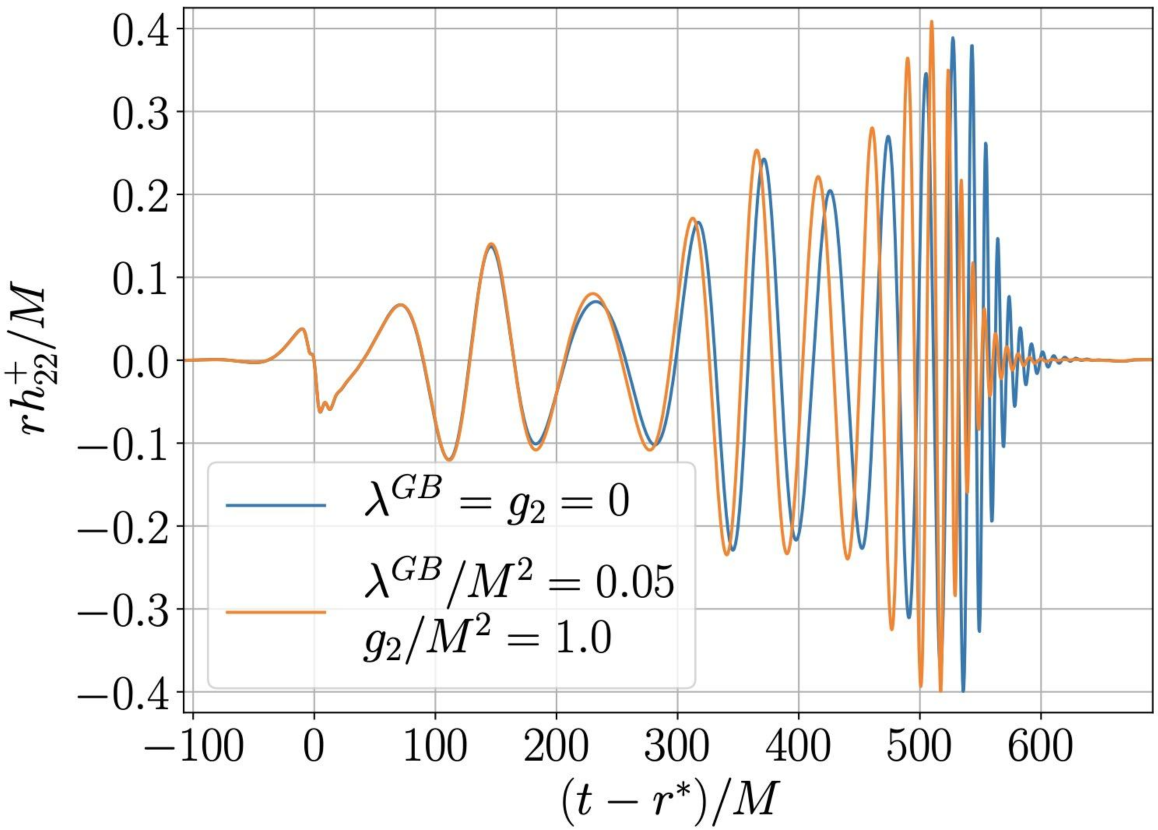
Deliyski, Gyulchev, Doneva, Nedkova, Yazadjiev 2026

Imprints of the scalar hair in the gravitational wave signal from binary black holes

The mass range of scalarized black holes is on the order of solar mass. So the spontaneously generated scalar hair in SGB gravity can be tested with the gravitational wave detectors of LIGO-VIRGO-KAGRA collaboration and the future detectors like the Einstein Telescope.



- Mass ratio **1:1**



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Astrophysics of scalar-hair

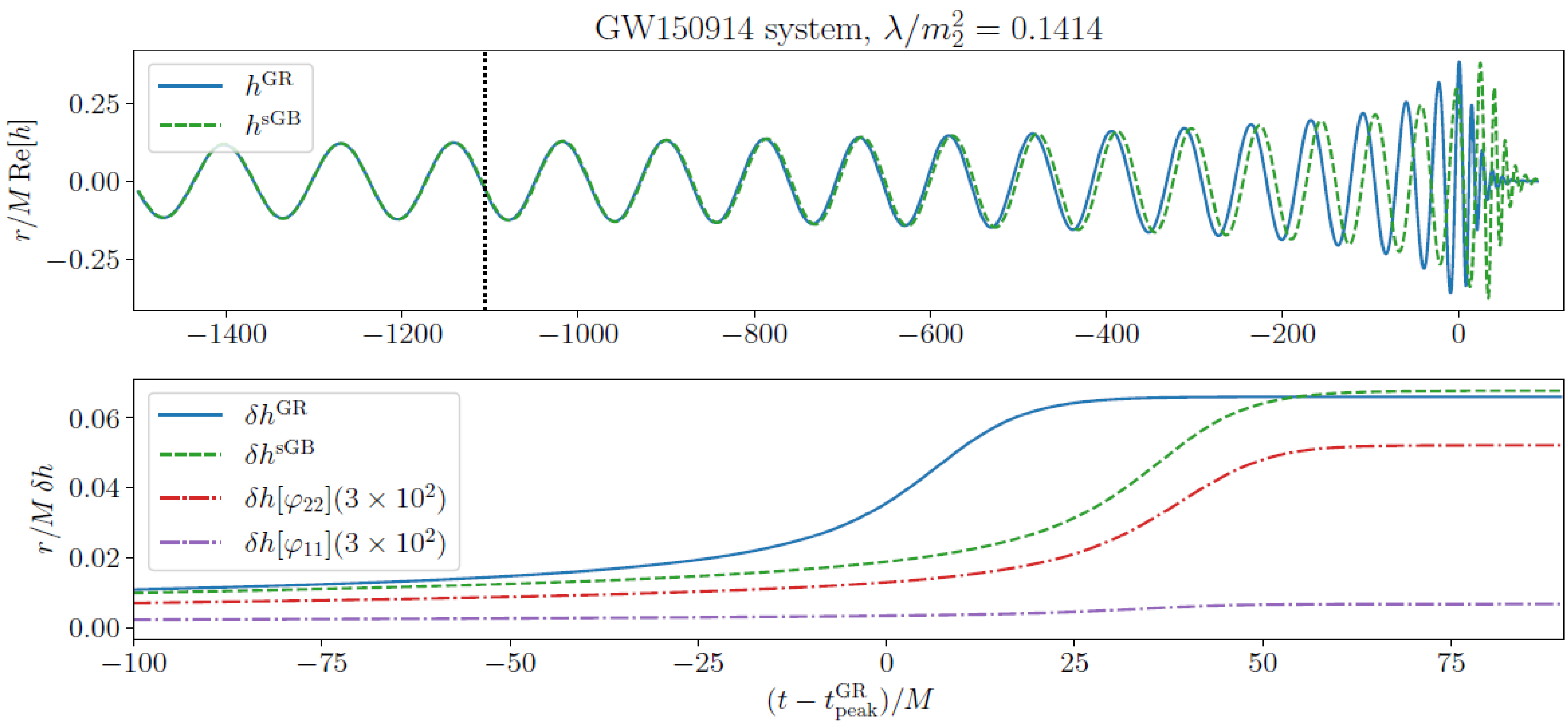

Figure 1: Gravitational waveforms for a near-equal-mass binary in GR and shift-symmetric sGB gravity with $\lambda/m_2^2 = 0.1414$ and mass ratio $q = 1.221$ (GW150914-like). The final memory amplitude is 0.0634 in sGB and 0.0619 in GR at $r = 150M$. As the scalar-induced memory is negative, we show its absolute value. The vertical dotted line indicates the end of the alignment interval (see Ref. [104]).

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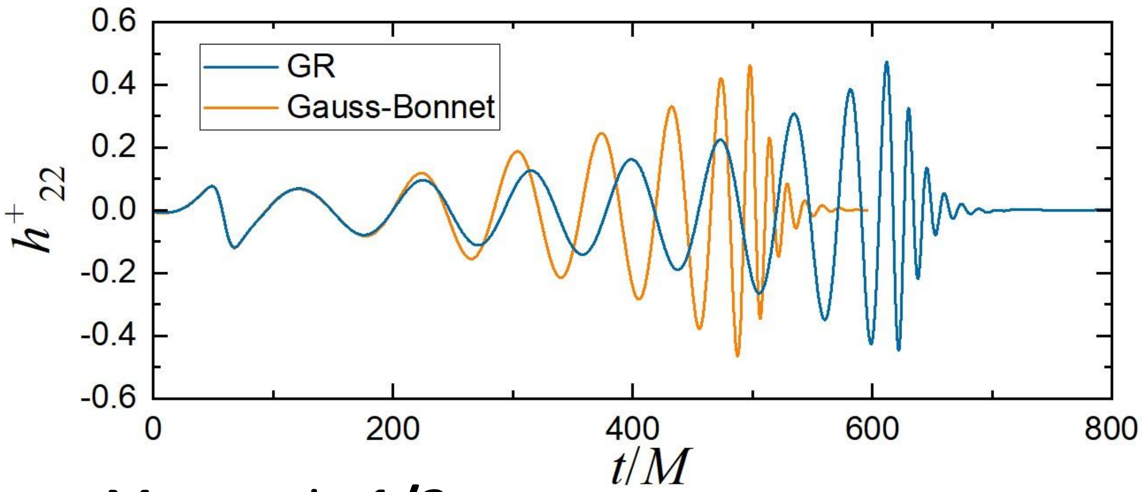
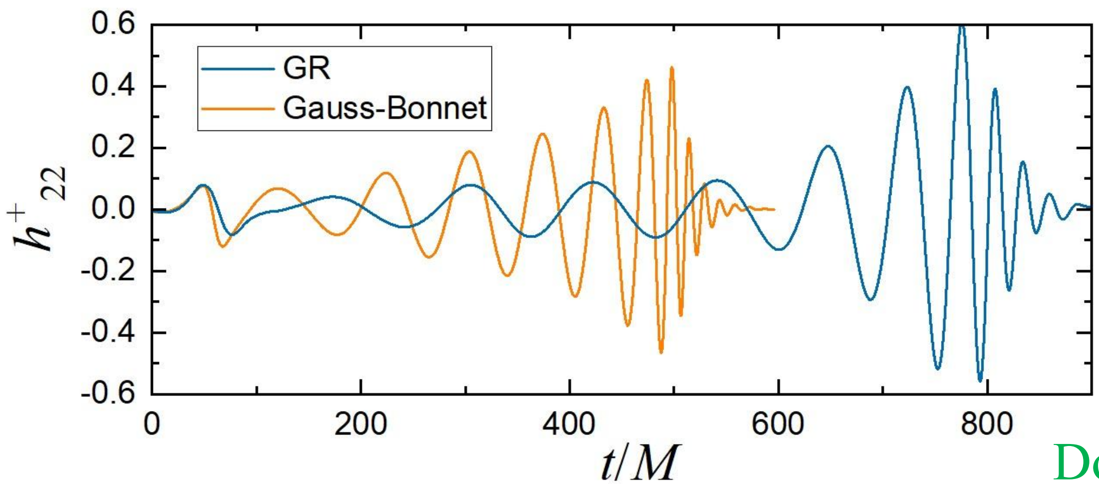
THANK YOU!

Astrophysics of scalar-hair

- Mass ratio **2/3**



- Mass ratio **1/3**



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