Is the Standard Model Exceptional?

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Based on:

1) arxiv:1808.08110 (M. Dubois-Violette + I. Todorov, "Exceptional Quantum Geometry and particle physics II")

2) arXiv:2006.16265 (LB, "The Standard Model, The Exceptional Jordan algebra, + Triality")

3) Work in progress with John Baez + Endre Bokor

Questions:

1) Why $G_{SM} = [SU(3)_c \times SU(2)_L \times U(1)_Y] / \mathbb{Z}_6$?

2) Why
$$\rho^{(1)}_{SM} = \overset{q_L}{(3, 2, \tfrac{1}{6})} \oplus \overset{\bar{d}_R}{(\bar{3}, 1, \tfrac{1}{3})} \oplus \overset{\bar{u}_R}{(\bar{3}, 1, -\tfrac{2}{3})}$$
$$\oplus \underset{l_L}{(1, 2, -\tfrac{1}{2})} \oplus \underset{\bar{e}_R}{(1, 1, 1)} \oplus \underset{\bar{\nu}_R}{(1, 1, 0)} \quad ?$$

3) Why 3 generations

$$\rho_{SM} = \rho^{(1)}_{SM} \oplus \rho^{(1)}_{SM} \oplus \rho^{(1)}_{SM} \quad ?$$

Old Hints:

1) Grand Unification

i) $SU(5) \longrightarrow G_{SM}$

$\Lambda\mathbb{C}^5 \longrightarrow \rho^{(1)}_{SM}$

ii) $Spin(10) \longrightarrow G_{SM}$

$16 \longrightarrow \rho^{(1)}_{SM}$

Old Hints:

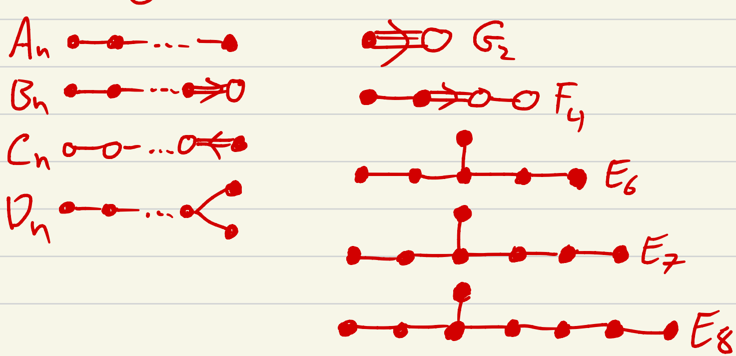
1) Grand Unification

i) $SU(5) \longrightarrow G_{SM}$ $\quad$ ii) $Spin(10) \longrightarrow G_{SM}$

$\Lambda \mathbb{C}^5 \longrightarrow \rho^{(1)}_{SM}$ $\quad$ $16 \longrightarrow \rho^{(1)}_{SM}$

2) Exceptional?

Killing-Cartan:



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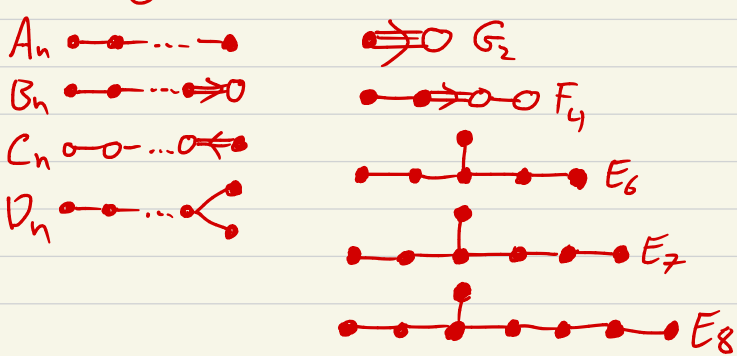
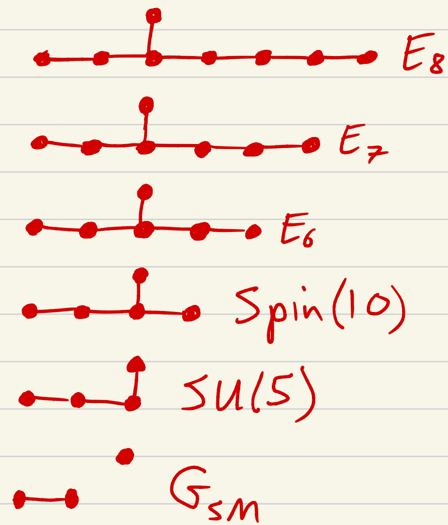




New Hint:

Dubois-Violette + Todorov (+ Baez) (2018)

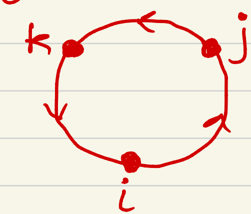
$$h_3(\mathbb{O}) \longleftrightarrow G_{SM}$$

What about $\mathfrak{g}_{SM}$?

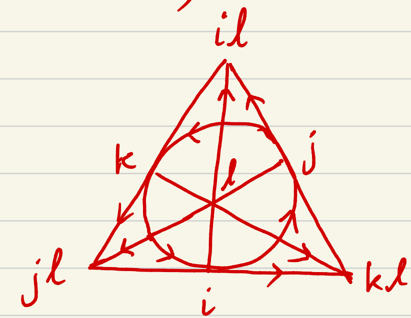
4 Normed Division Algebras: $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (Hurwitz)

$\mathbb{C}$: $z = a_0 + a_1 i$

$\mathbb{H}$: $q = a_0 + a_1 i + a_2 j + a_3 k$



$\mathbb{O}$: $x = a_0 + a_1 i + a_2 j + a_3 k + a_4 \ell + a_5 i\ell + a_6 j\ell + a_7 k\ell$



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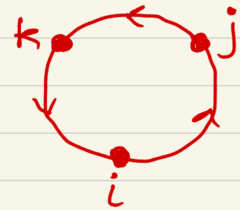

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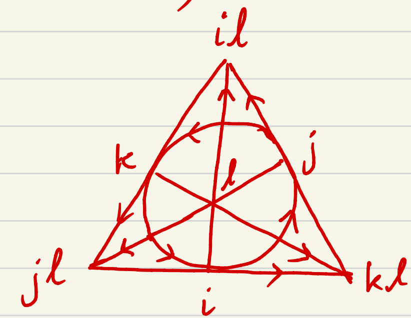

Euclidean Jordan Algebras (Jordan, Wigner, von Neumann)

1) $J\mathrm{spin}(n)$
2) $h_n(\mathbb{R})$
3) $h_n(\mathbb{C})$
4) $h_n(\mathbb{H})$

$h_3(\mathbb{O})$ ("Exceptional Jordan Algebra")

$$Y = \begin{pmatrix} \alpha_1 & x_3 & x_2^* \\ x_3^* & \alpha_2 & x_1 \\ x_2 & x_1^* & \alpha_3 \end{pmatrix} \in h_3(\mathbb{O})$$

$$Y_1 \circ Y_2 = \frac{1}{2}\{Y_1, Y_2\} = \frac{1}{2}(Y_1 Y_2 + Y_2 Y_1)$$

G_{SM} from $h_3(\mathbb{O})$

$y \in h_3(\mathbb{O})$

$$G = F_4 = \{\gamma \mid \det(y) = \det(\gamma y),\ \langle y_1 | y_2 \rangle = \mathrm{Tr}[y_1 \circ y_2] = \langle \gamma y_1 | \gamma y_2 \rangle \}$$

$H_1 = Spin(9)$: preserves $\Pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow$ 2+1 split $\left(\begin{array}{cc|c} \alpha_1 & x_3 & x_2^* \\ x_3^* & \alpha_2 & x_1 \\ \hline x_2 & x_1^* & \alpha_3 \end{array}\right)$

(a pt in $\mathbb{O}P^2$)

$H_2 = [SU(3) \times SU(3)]/\mathbb{Z}_3$: preserves $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ split

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$H_1 \cap H_2 = G_{SM} = [SU(3) \times SU(2) \times U(1)]/\mathbb{Z}_6$ (Dubois-Violette + Todorov, 2018)

Baez's Interpretations: i) fix M_{9+1} in $h_3(\mathbb{O})$ and M_{3+1} in M_{9+1}.

ii) fix $\mathbb{O}$-bit in $\mathbb{O}$-trit and $\mathbb{C}$-bit in $\mathbb{O}$-bit.

Puzzle:

- SM rep is "complex" – i.e. ρ_{SM} and $\bar{\rho}_{SM}$ are inequivalent
- SM is chiral,
- So usually consider G_{GUT} with complex reps:

$SU(n)$ $(n \geq 3)$

$SO(4n+2)$ or $Spin(4n+2)$

E_6

- But in $h_3(\mathbb{O})$, G_{SM} arises as subgroup of F_4 and $Spin(9)$ which have no complex reps.

G_{LR} from $h_3^{\mathbb{C}}(\mathbb{O})$

$y \in h_3^{\mathbb{C}}(\mathbb{O})$

$$\tilde{G} = E_6 = \{ \gamma \mid \det(y) = \det(\gamma y) \,,\ \langle y_1 | y_2 \rangle = \mathrm{Tr}[\bar{y}_1 \circ y_2] = \langle \gamma y_1 | \gamma y_2 \rangle \}$$

$$\tilde{H}_1 = Spin(10) : \text{preserves } \Pi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow 2+1 \text{ split } \left(\begin{array}{cc|c} \alpha_1 & x_3 & x_2^* \\ x_3^* & \alpha_2 & x_1 \\ \hline x_2 & x_1^* & \alpha_3 \end{array}\right) \leftarrow \text{preserved by } Spin(10)\times U(1)$$

(a pt in $(\mathbb{C}\otimes\mathbb{O})P^2$)

$$\tilde{H}_2 = [SU(3)\times SU(3)\times SU(3)]/\mathbb{Z}_3 \ : \ \text{preserves } \mathbb{O} = \mathbb{C}\oplus\mathbb{C}^3 \text{ split}$$

$$\tilde{H}_1 \cap \tilde{H}_2 = G_{LR} = [SU(3)\times SU(2)_L \times SU(2)_R \times U(1)]/\mathbb{Z}_6$$

$$\begin{pmatrix} x_2^* \\ x_1 \end{pmatrix} \xrightarrow{G_{LR}} \rho_{LR} = \underbrace{(3,2,1,\tfrac{1}{6})}_{q_L} \oplus \underbrace{(\bar{3},1,2,-\tfrac{1}{6})}_{\bar{q}_R} \oplus \underbrace{(1,2,1,-\tfrac{1}{2})}_{l_L} \oplus \underbrace{(1,1,2,\tfrac{1}{2})}_{\bar{l}_R}$$

Jordan pairs (Ottmar Loos) ← building on work by Jacobsen, McCrimmon on quadratic J. algebras

- Jordan pair: (V_+, V_-)
- Triple product: $\{\_,\_,\_\}: V_\pm \times V_\mp \times V_\pm \to V_\pm$
- Define linear map $D(x,y): V_\pm \to V_\pm$

$$\{x,y,z\} = D(x,y)z$$

Axioms:

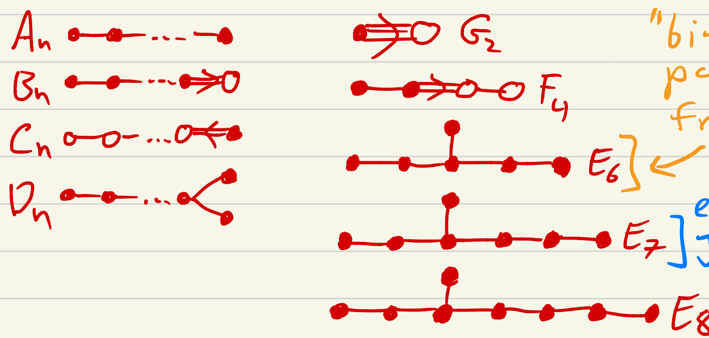
1) symmetry: $\{x,y,z\} = \{z,y,x\}$
2) D as derivation: $D(x,y)\{u,v,w\} = \{D(x,y)u, v, w\} + \{u, D(x,y)v, w\} + \{u, v, D(x,y)w\}$

Jordan pairs are equivalent to 3-graded Lie algebras

Forthcoming work w/ J. Baez + E. Bokor

- Can rephrase QM in terms of Jordan pairs.
- This is a slight generalization because one simple J. Pair has no corresponding J. algebra: the "bi-Cayley pair" $(V^+, V^-) = ((\mathbb{C}\times\mathbb{O})^2, (\mathbb{C}\otimes\mathbb{O})^2)$

4 families of J. algebras come from here



"bi-Cayley" pair comes from here

exceptional J. algebra comes from here

$$e_6 = \underset{L_{-1}}{\overline{16}} \oplus \underset{L_0}{[spin(10)\oplus u(1)]} \oplus \underset{L_1}{16}$$

$\leftarrow$ same structure as one SM generation.

Triality

- $\mathrm{der}(A) = \{ \delta \mid \delta(ab) = \delta(a)b + a\delta(b) \quad \forall\ a, b \in A \}$

- $\mathrm{tri}(A) = \{ (T_1, T_2, T_3) \mid T_1(ab) = T_2(a)b + aT_3(b) \quad \forall\ a, b \in A \}$

$\mathbb{K}$	$\mathrm{tri}(\mathbb{K})$
$\mathbb{R}$	–
$\mathbb{C}$	$u(1) \oplus u(1)$
$\mathbb{H}$	$su(2) \oplus su(2) \oplus su(2)$
$\mathbb{O}$	$so(8)$

$$(T_1, T_2, T_3) \in \mathrm{tri}(\mathbb{K})$$

$$(\overline{T_3}, \overline{T_1}, T_2) \in \mathrm{tri}(\mathbb{K})$$

Magic Square

Freudenthal + Tits
Adams
Gursey
Ramond
Barton + Sudbery

$$m(\mathbb{K},\tilde{\mathbb{K}}) = \mathrm{tri}(\mathbb{K}) \oplus \mathrm{tri}(\tilde{\mathbb{K}}) \oplus (\mathbb{K}\otimes\tilde{\mathbb{K}})_1 \oplus (\mathbb{K}\otimes\tilde{\mathbb{K}})_2 \oplus (\mathbb{K}\otimes\tilde{\mathbb{K}})_3$$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$so(3)$	$su(3)$	$sp(3)$	f_4
$\mathbb{C}$	$su(3)$	$su(3)$ $+su(3)$	$su(6)$	e_6
$\mathbb{H}$	$sp(3)$	$su(6)$	$so(12)$	e_7
$\mathbb{O}$	f_4	e_6	e_7	e_8

symmetries of $h_3(\mathbb{K}\otimes\tilde{\mathbb{K}})$ and $(\mathbb{K}\otimes\tilde{\mathbb{K}})P^2$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$so(2)$	$so(3)$	$so(5)$	$so(9)$
$\mathbb{C}$	$so(3)$	$so(4)$	$so(6)$	$so(10)$
$\mathbb{H}$	$so(5)$	$so(6)$	$so(8)$	$so(12)$
$\mathbb{O}$	$so(9)$	$so(10)$	$so(12)$	$so(16)$

symmetries of $h_2(\mathbb{K}\otimes\tilde{\mathbb{K}})$ and $(\mathbb{K}\otimes\tilde{\mathbb{K}})P^1$

$$m(\mathbb{K},\tilde{\mathbb{K}}) = \overbrace{\mathrm{tri}(\mathbb{K})\oplus \mathrm{tri}(\tilde{\mathbb{K}}) + (\mathbb{K}\otimes\tilde{\mathbb{K}})_1}^{\text{bosonic}} + \overbrace{(\mathbb{K}\otimes\tilde{\mathbb{K}})_2 + (\mathbb{K}\otimes\tilde{\mathbb{K}})_3}^{\text{fermionic}}$$

Lie brackets:

- $\mathrm{tri}(\mathbb{K})\oplus \mathrm{tri}(\tilde{\mathbb{K}})$ forms a Lie subalgebra
- $[\mathrm{tri}(\mathbb{K})\oplus \mathrm{tri}(\tilde{\mathbb{K}}), (\mathbb{K}\otimes\tilde{\mathbb{K}})_i] \in (\mathbb{K}\otimes\tilde{\mathbb{K}})_i$
- $[(\mathbb{K}\otimes\tilde{\mathbb{K}})_1, (\mathbb{K}\otimes\tilde{\mathbb{K}})_2] \in (\mathbb{K}\otimes\tilde{\mathbb{K}})_3$ etc.
- $[(\mathbb{K}\otimes\tilde{\mathbb{K}})_i, (\mathbb{K}\otimes\tilde{\mathbb{K}})_i] \in \mathrm{tri}(\mathbb{K})\oplus \mathrm{tri}(\tilde{\mathbb{K}})$

$$m(\mathbb{C},\mathbb{O}) = e_6 = \underbrace{\mathrm{spin}(10)\oplus u(1)}_{\text{bosonic}} + \underbrace{(\mathbb{C}\otimes\mathbb{O}) + (\mathbb{C}\otimes\mathbb{O})}_{\text{fermionic}}$$

(16-dim Weyl spinor of Spin(10))

$m(\mathbb{C},\mathbb{O}) \downarrow$ $\mathbb{O} \to \mathbb{C}\oplus\mathbb{C}^3$

bosonic $\downarrow$ G_{LR}

fermionic $\downarrow$ $\mathfrak{g}_{LR}$

3 generations: (some comments/speculations)

$$m(\mathbb{K},\tilde{\mathbb{K}}) = tri(\mathbb{K}) \oplus tri(\tilde{\mathbb{K}}) + (\mathbb{K}\otimes\tilde{\mathbb{K}})_1 + (\mathbb{K}\otimes\tilde{\mathbb{K}})_2 + (\mathbb{K}\otimes\tilde{\mathbb{K}})_3$$

1) $m(\mathbb{C},\mathbb{O}) = e_6$: 3 overlapping generations?

$$\left(\begin{array}{cc|c} \alpha_1 & x_3 & x_2^* \\ x_3^* & \alpha_2 & x_1 \\ \hline x_2 & x_1^* & \alpha_3 \end{array}\right) \qquad \begin{pmatrix} x_2^* \\ x_1 \end{pmatrix} \quad \begin{pmatrix} x_3^* \\ x_2 \end{pmatrix} \quad \begin{pmatrix} x_1^* \\ x_3 \end{pmatrix} \Bigg\} \text{ related by large Bogoliubov transformations?}$$

2) $m(\mathbb{H},\mathbb{O}) = e_7$: right counting to describe 3 gen (internal d.o.f.)

3) $m(\mathbb{O},\mathbb{O}) = e_8$: right counting to describe 3 gen (internal + spacetime d.o.f.)

iv) $A_\mu \longrightarrow$ Lie-algebra valued 1-form

$\longrightarrow$ derivation-valued 1-form (Dubois-Violette)

$\longrightarrow$ triality-valued 1-form (naturally acts on "3 generations")

$\longrightarrow$ triality-valued superconnection (following Quillen, modelled on Ramond's magic square)

Is the standard model (or its LR-symmetric extension) a Yang-Mills theory of such a generalized connection?

Summary

i) Complexifying $h_3(\mathbb{O}) \to h_3^{\mathbb{C}}(\mathbb{O})$ incorporates SM fermions.

ii) Consequence: SM $\to$ LR-symmetric extension

iv) Alternative: Jordan algebras $\to$ Jordan pairs (bi-Cayley pair)

iii) Rephrase above construction: from $m(\mathbb{R},\mathbb{O}) \to m(\mathbb{C},\mathbb{O})$

iv) superalgebra decomposition of $m(\mathbb{K},\widetilde{\mathbb{K}})$ agrees w/ Spin(10) GUT

v) Thoughts about 3 generations