Standard Model and Gravity: the role of anomalies and conformal symmetry

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I would like to consider the question:

what happens when we put together gravity and SM as field theories?

A mandatory issue to consider is that of anomalies, so I would like more specifically to answer the question:

what do anomalies have to say about the interaction between gravity and SM? Is there anything useful that we can learn from it?

the answer is: YES

Summary: first half

- Anomalies: chiral and trace
- Consistent and covariant anomalies
- Trace anomalies: odd and even parity
- Basic subdivision: O and NO anomalies
- Obstructions (i.e. non-existence of propagators) and family's index theorem
- NO anomalies (propagators exist)

Partial conclusion of the analysis: the MSM does have O anomalies that need to be canceled

Summary: second half

- A L-R symmetric model
- Connection with cosmology – dark matter
- Conformal invariance
- Trace anomalies and WZ terms
- Renormalization and unitarity

L.B. *Fermions and anomalies in FTs, Springer 2023*

L.B. and S. Giaccari, Arxiv:2412.07470

L.B. *Arxiv:2510.25217*

How to compute anomalies

- Perturbative methods (Feynman diagrams, ...)
- Non-Perturbative (heat kernel like methods: Seeley-Schingwer-DeWitt, Fujikawa,....)
- Family's index theorem (for odd parity anomalies)

Gauge anomalies in chiral theories

(well-known things…)

What are anomalies?

An anomaly is a quantum effect that violates a classical symmetry.
Examples:

- Covariant gauge anomaly. The lagrangian $L = i\bar{\psi}(\not{\partial} + \not{V})\psi$ is invariant under

$$\not{V} \to \not{V} + \gamma_5 \not{D}\lambda, \qquad \psi \to (1 + \gamma_5\lambda)\psi \qquad \lambda = \lambda(x)^a T^a, \qquad D_\mu = \partial_\mu + V_\mu$$

  It follows that the current $j^a_{\mu 5} = \frac{i}{2}\bar{\psi}\gamma_\mu\gamma_5 T^a\psi$ is classically conserved, but, at one loop,

$$[D^\mu j_{\mu 5}]^a = \frac{\hbar}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}\left(T^a F^{\mu\nu}F^{\lambda\rho}\right), \qquad F = dV + \frac{1}{2}[V,V]$$

- Consistent gauge anomaly. The lagrangian $L = i\bar{\psi}_L(\not{\partial} + \not{A})\psi_L$, where $\psi_L = \frac{1+\gamma_5}{2}\psi$ is invariant under

$$A_\mu \to A_\mu + \partial_\mu\lambda + [A_\mu, \lambda], \qquad \psi_L \to \psi_L - \lambda\psi_L$$

  The current $j^a_{\mu L} = \frac{i}{2}\bar{\psi}_L\gamma_\mu T^a\psi_L$ is classically conserved, but

$$[D^\mu j_{\mu L}]^a = \frac{\hbar}{24\pi^2}\epsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left(T^a\partial^\mu(A^\nu\partial^\lambda A^\rho + \frac{1}{2}A^\nu A^\lambda A^\rho)\right) \equiv \mathcal{A}^a$$

WZ consistency conditions

Given the effective action

$$W[V] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int \prod_{i=1}^{n} d^{\mathsf{d}} x_i \, V_{\mu_i}(x_i) \langle 0 | \mathcal{T} j_{\mu_1}(x_1) \dots j_{\mu_n}(x_n) | 0 \rangle_c$$

invariance is expressed via the functional operator $X^a(x)$ defined by

$$X^a(x) = \partial_\mu \frac{\delta}{\delta V^a_\mu(x)} + f^{abc} V^b_\mu(x) \frac{\delta}{\delta V^c_\mu(x)},$$

as follows

$$X^a(x) W[V] = 0$$

In a number of cases this WI is violated

$$X^a(x) W[V] = \mathcal{A}^a(x)$$

Applying $X^b(y)$ to both sides and then inverting the two operations, we find a remarkable relation of group-theoretical nature

$$X^a(x) \mathcal{A}^b(y) - X^b(y) \mathcal{A}^a(x) + f^{abc} \mathcal{A}^c(x) \delta(x-y) = 0,$$

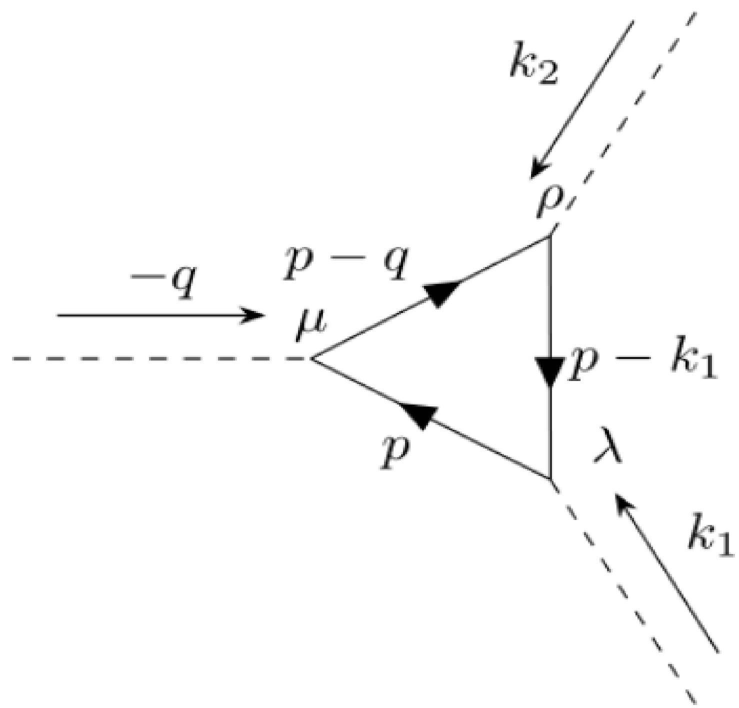
that the anomaly must satisfy. These are the Wess-Zumino (WZ) consistency conditions.

The triangle diagram

The fermion propagator is $\frac{i}{\not p}$ and the vertex $i\gamma_\mu P_L T^a$. The Fourier transform of the three currents amplitude $\langle j_L\, j_L\, j_L\rangle$ is given by

$$\begin{aligned}
\widetilde{F}^{(L)abc}_{\mu\lambda\rho}(k_1,k_2) &= \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\frac{1}{\not p}\gamma_\lambda\frac{1-\gamma_5}{2}T^b\frac{1}{\not p-\not k_1}\gamma_\rho\frac{1-\gamma_5}{2}T^c\frac{1}{\not p-\not q}\gamma_\mu\frac{1-\gamma_5}{2}T^a\right\}\\
&\equiv \mathrm{Tr}(T^aT^bT^c)\widetilde{F}^{(L)}_{\mu\lambda\rho}(k_1,k_2)
\end{aligned}$$

where $q=k_1+k_2$. This is to be contracted with q^μ. It is UV divergent and needs to be regularized. The relevant Feynman diagram is



There is also a divergent quadrangle diagram. The pentagon diagram is UV convergent.

WZ consistency conditions and cohomology

Introduce the FP ghosts $c(x) = c^a(x)T^a$: the gauge trasformations become the BRST transformations

$$\mathfrak{s}V_\mu = D_\mu c, \qquad \mathfrak{s}c = -\frac{1}{2}[c, c],$$

This operation is nilpotent and the functional operator that generate them

$$\mathfrak{s} = \int d^{\mathsf{d}}x \left(\mathfrak{s}V_\mu^a(x)\frac{\partial}{\partial V_\mu^a(x)} + \mathfrak{s}c^a(x)\frac{\partial}{\partial c^a(x)} \right).$$

it is a nilpotent operator: $\mathfrak{s}^2 = 0$. Let us define the integrated anomaly

$$\mathcal{A}_c = \int d^4x c^a(x)\mathcal{A}^a(x)$$

The previous formulae are condensed as follows

$$\mathfrak{s}W[V] = \mathcal{A}_c$$

and the WZ c.c. can be written

$$\mathfrak{s}\mathcal{A}_c = 0$$

$\mathfrak{s}$ is a coboundary operator, $\mathcal{A}_c$ is a cocycle, it represent a cohomology class.

Descent equations and anomalies

There is a time honored formalism to construct all possible solutions of the consistency conditions. Start from an order n symmetric polynomial in some representation of the Lie algebra, $P_n(T^{a_1}, ..., T^{a_n})$, invariant under the adjoint transformations. In many cases these polynomials are symmetric traces of the generators in the corresponding representation

$$P_n(T^{a_1}, ..., T^{a_n}) = Str(T^{a_1}...T^{a_n}) = d^{a_1...a_n}$$

Let $V = V_\mu dx^\mu = V_\mu^a T^a dx^\mu$ and $F = dV + V \wedge V = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, Then one can construct the 2n-form

$$\Delta_{2n}(V) = P_n(F, F, \ldots F)$$

Then

$$P_n(F, F, \ldots F) = d\left(n \int_0^1 dt\, P_n(V, F_t, \ldots, F_t)\right) = d\Delta_{2n-1}^{(0)}(V)$$

where $V_t = tV$ and its curvature $F_t = dV_t + \frac{1}{2}[V_t, V_t]$. This is the first of a sequence of equations that can be shown to hold

$$\begin{aligned}
&\Delta_{2n}(V) - d\Delta_{2n-1}^{(0)}(V) = 0 \\
&\mathfrak{s}\Delta_{2n-1}^{(0)}(V) - d\Delta_{2n-2}^{(1)}(V, c) = 0 \\
&\mathfrak{s}\Delta_{2n-2}^{(1)}(V, c) - d\Delta_{2n-3}^{(2)}(V, c) = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Anomaly cancellation

First mechanism:

Both covariant and consistent anomalies in 4d are proportional to the tensor

$$t^{abc} = Str(T^a T^b T^c)$$

For antisymmetric T^a this tensor vanishes identically. This is true, for example, for the Lorentz group $SO(4)$.

Second mechanism:

The anomaly coefficiemts for the various species sum up to zero.

Third mechanism:

Introducing WZ terms (which requires new fields) or by means of the Green-Schwarz mechanism.

Minimal Standard Model

Three families of massless quarks and leptons

G/*fields*	$SU(3)$	$SU(2)$	$U(1)$
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$\frac{1}{6}$
u_R	3	1	$\frac{2}{3}$
d_R	3	1	$-\frac{1}{3}$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	$-\frac{1}{2}$
e_R	1	1	-1

The second column specifies the representations of $SU(3)$, the third the ones of $SU(2)$ and the last is the list of $U(1)$ representations, denoted by the corresponding hypercharge eigenvalue. The hypercharge is defined by

$$Y = Q - T_3, \qquad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

here Q is the electromagnetic charge and T_3 the third generator of $SU(2)$ in the doublet representation.

Conjugate chiral spinors

A frequent alternative notation is to use the Lorentz covariant conjugates $(u_R)^c, (d_R)^c$ and $(e_R)^c$ instead of u_R, d_R and e_R, in order to collect all the fields in a unique left-handed multiplet.

The symbol $(\psi_R)^c$ (for instance u_R) can be rewritten as

$$(\psi_R)^c = \gamma^0 C \psi_R^* = \gamma^0 C P_R^* \psi^* = P_L \gamma^0 C \psi^* = P_L \psi^c = (\psi^c)_L.$$

Inserted into the kinetic term, it gives

$$\sqrt{g}\,\overline{(\psi^c)_L}\,\gamma^\mu(\partial_\mu + \frac{1}{2}\omega_\mu)(\psi^c)_L = \sqrt{g}\,\overline{(\psi_R)^c}\,\gamma^\mu(\partial_\mu + \frac{1}{2}\omega_\mu)(\psi_R)^c = \sqrt{g}\,\overline{\psi_R}\,\gamma^\mu(\partial_\mu + \frac{1}{2}\omega_\mu)\psi_R$$

In this case one has to reverse the signs of the $U(1)$ charges and replace the representation 3 of $SU(3)$ of u_R, d_R with the $\bar{3}$ of $(u_R)^c, (d_R)^c$.

MSM: gauge anomalies

- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)}$: there are two left-handed and two right-handed triplet, whose anomalies cancel one another.
- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)}$, which vanishes because the tensor d^{abc} vanishes in general for the Lie algebra $\mathfrak{su}(2)$.
- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(2)$ generators in two doublet representations. These traces are non-vanishing because $\mathrm{tr}(T^a T^b) \sim \delta^{ab}$, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charges, whose total value is $6\left(\frac{1}{6}\right) - 2(\frac{1}{2}) = 0$.
- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(3)$ left triplet generators and two right triplet generators. These traces are again non-vanishing, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charge, whose total value is $3\left(2\left(\frac{1}{6}\right) - \frac{2}{3} + \frac{1}{3}\right) = 0$.
- $T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)}$, in this case the tensor is proportional to the overall sum of the charge products: $6\left(\frac{1}{6}\right)^3 - 3\left(\frac{2}{3}\right)^3 - 3\left(-\frac{1}{3}\right)^3 + 2\left(-\frac{1}{2}\right)^3 - (-1)^3 = 0$.

Trace anomalies

When a metric is involved

When a metric is present at least two symmetries are involved: diffeomorphisms

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

and Weyl transformations

$$\delta_\omega g_{\mu\nu} = 2\,\omega\, g_{\mu\nu}$$

The effective action is

$$W[g] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{2^n n!} \int \prod_{i=1}^{n} d^{\mathsf{d}} x_i \sqrt{g(x_i)} h^{\mu_i \nu_i}(x_i) \langle 0 | \mathcal{T} T_{\mu_1\nu_1}(x_1) \dots T_{\mu_n\nu_n}(x_n) | 0 \rangle$$

and the Ward identity for diffeomorphims takes the form

$$\delta_\xi\, W[g] = 0$$

This relation may be violated at one-loop,

$$\delta_\xi\, W[g] = \mathcal{A}_\xi$$

The term in the RHS is linear in ξ and, since $\delta_\xi^2 = 0$, it satisfies the consistency condition

$$\delta_\xi\, \mathcal{A}_\xi = 0$$

Weyl transformation and trace anomalies

The invariance under Weyl transformations, or conformal invariance, is expressed by

$$0 = \delta_\omega W[g] = \int d^{\mathsf{d}}x \, \frac{\delta W[g]}{\delta g^{\mu\nu}(x)} \delta_\omega g^{\mu\nu}(x) = -\int d^{\mathsf{d}}x \, \omega(x) \langle\langle T_{\mu\nu}(x) \rangle\rangle g^{\mu\nu}(x)$$

Since ω is a generic infinitesimal function, this implies

$$\langle\langle T_{\mu\nu}(x) \rangle\rangle g^{\mu\nu}(x) = 0$$

where

$$\langle\langle T_{\mu\nu}(x) \rangle\rangle = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \int \prod_{i=1}^{n} d^{\mathsf{d}}x_i \sqrt{g(x_i)} h^{\mu_i \nu_i}(x_i) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_n\nu_n}(x_n) | 0 \rangle$$

But we may have violations of this classical invariance

$$\delta_\omega W[g] = \mathcal{A}_\omega$$

Due to the nilpotence of δ_ω the consistency condition

$$\delta_\omega \mathcal{A}_\omega = 0$$

must be satisfied.

Possible trace anomalies

The possible non-trivial cocycles of δ_ω with vanishing diffeomorphism partner in 4d are well-known, they take the form

$$\Delta[g,\omega] = \int d^4x \sqrt{g}\, \omega\, T[g], \qquad \delta_\omega \Delta[g,\omega] = 0$$

where the density $T[g](x)$ can be the quadratic Weyl density

$$\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$

the Gauss-Bonnet (or Euler) density,

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

and the Pontryagin density,

$$P = \frac{1}{2}\left(\varepsilon^{\mu\nu\mu'\nu'} R_{\mu\nu\lambda\rho} R_{\mu'\nu'}{}^{\lambda\rho}\right).$$

Other possible cocycles have densities

$$T_e[V] = F_{\mu\nu}F^{\mu\nu}, \qquad T_o[V] = \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu}F_{\lambda\rho}.$$

Weyl fermion coupled to a metric

Consider again

$$S = \int d^4x\, \sqrt{g}\, i\overline{\psi_R}\gamma^\mu \left(\partial_\mu + \frac{1}{2}\omega_\mu\right)\psi_R$$

where $\psi_R = P_R\psi$, $P_R = \frac{1+\gamma_5}{2}$. The action can be rewritten as

$$S = \int d^4x\, \sqrt{g}\, \left[\frac{i}{2}\overline{\psi_R}\gamma^\mu \overset{\leftrightarrow}{\partial}_\mu \psi_R - \frac{1}{4}\epsilon^{\mu abc}\omega_{\mu ab}\overline{\psi_R}\gamma_c\gamma_5\psi_R\right]$$

where it is understood that the derivative applies to ψ_L and $\overline{\psi_L}$ only. We have used the relation $\{\gamma^a, \Sigma^{bc}\} = i\,\epsilon^{abcd}\gamma_d\gamma_5$.

The classical e.m. tensor is

$$T_{\mu\nu} = \frac{i}{4}\overline{\psi_R}\gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi_R + \{\mu \leftrightarrow \nu\}$$

This theory is invariant under diffeomorphisms $\delta_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$ and Weyl transformations $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$. As a consequence, classically,

$$\nabla^\mu T_{\mu\nu}(x) = 0, \qquad T^\mu_\mu(x) = 0$$

on shell.

Weyl fermion *cont.*

The fermion propagator is

$$\frac{i}{\not p + i\epsilon}$$

and there is only one graviton-fermion-fermion (V_{ffg}) vertex given by

$$\frac{i}{8}\left[(p+p')_\mu \gamma_\nu + (p+p')_\nu \gamma_\mu\right]\frac{1+\gamma_*}{2}$$

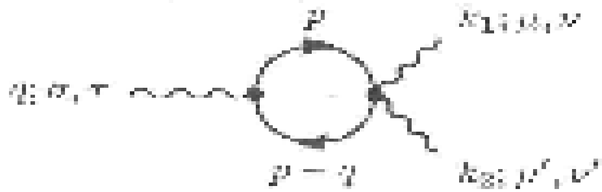
where p and p' are the two graviton momenta, and a two-fermion-two-graviton vertex (V^ϵ_{ffhh}) is

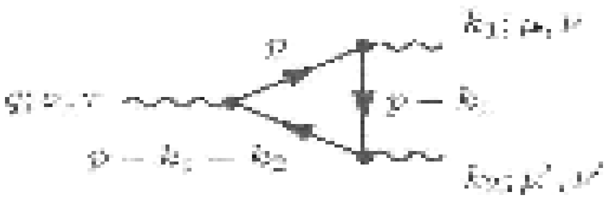
$$V^\epsilon_{ffhh}: \qquad \frac{1}{64} t_{\mu\nu\mu'\nu'\kappa\lambda}(k-k')^\lambda \gamma^\kappa \frac{1+\gamma_5}{2}$$

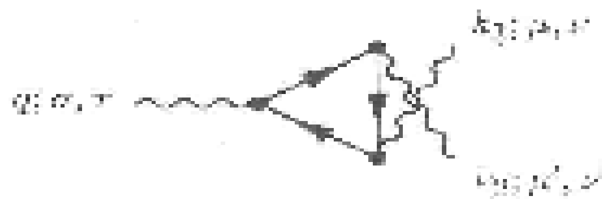
where

$$t_{\mu\nu\mu'\nu'\kappa\lambda} = \eta_{\mu\mu'}\epsilon_{\nu\nu'\kappa\lambda} + \eta_{\nu\nu'}\epsilon_{\mu\mu'\kappa\lambda} + \eta_{\mu\nu'}\epsilon_{\nu\mu'\kappa\lambda} + \eta_{\nu\mu'}\epsilon_{\mu\nu'\kappa\lambda}$$

Relevant diagrams







The triangle diagram for the e. m. trace

$$
\begin{aligned}
\widetilde{T}_{\mu\nu\mu'\nu'}(k_1,k_2) &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\Big\{ \frac{i}{8}\left[(2p-k_1)_\mu \gamma_\nu + (\mu \leftrightarrow \nu)\right] \left(\frac{1+\gamma_5}{2}\right) \frac{i}{(\not p - \not k_1)} \\
&\times \frac{i}{8}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu' \leftrightarrow \nu')\right] \left(\frac{1+\gamma_5}{2}\right) \\
&\times \frac{i}{(\not p - \not k_1 - \not k_2)} (2\not p - \not k_1 - \not k_2) \left(\frac{1+\gamma_5}{2}\right) \frac{i}{\not p} \Big\}.
\end{aligned}
$$

After regularization it becomes

$$
\begin{aligned}
\widetilde{T}_{\mu\nu\mu'\nu'}(k_1,k_2) &= -\frac{1}{256}\int \frac{d^4p}{(2\pi)^4} \int \frac{d^{n-4}\ell}{(2\pi)^{n-4}} \mathrm{Tr}\Big\{ \left[\frac{\not p + \not \ell}{p^2-\ell^2}(2p-k_1)_\mu \gamma_\nu + (\mu \leftrightarrow \nu)\right] \\
&\times \frac{(\not p + \not \ell - \not k_1)}{(p-k_1)^2 - \ell^2}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu' \leftrightarrow \nu')\right] \\
&\times \frac{(\not p + \not \ell - \not k_1 - \not k_2)}{(p-k_1-k_2)^2-\ell^2} (2\not p + 2\not \ell - \not k_1 - \not k_2) \left(\frac{1+\gamma_5}{2}\right) \Big\}
\end{aligned}
$$

The result

We use the expansion $g_{\mu\nu}(x) \approx \eta_{\mu\nu} + h_{\mu\nu}(x)$. The triangle diagrams eventually give

$$\langle T^{\mu}_{\mu}(x)\rangle = \frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h^{\tau}_{\nu}\,\partial_\lambda\partial_\tau h^{\sigma}_{\rho} - \partial_\mu\partial_\sigma h^{\tau}_{\nu}\,\partial_\lambda\partial^\sigma h_{\tau\rho}\right)$$

The corresponding covariant expression is the trace anomaly

$$\langle T^{\mu}_{\mu}(x)\rangle = \frac{i}{768\pi^2}\,\frac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau}$$

Therefore

$$e = \frac{1}{1536\pi^2}$$

... other calculations are possible!

The previous result corresponds, at the lowest order, to

$$\langle 0|\mathcal{T}T^{\mu}_{\mu}(x)T_{\mu'\nu'}(y)T_{\alpha\beta}(z)|0\rangle^{(odd)}$$

One can also compute

$$\eta^{\mu\nu}\langle 0|\mathcal{T}T_{\mu\nu}(x)T_{\mu'\nu'}(y)T_{\alpha\beta}(z)|0\rangle^{(odd)}$$

and get

$$\eta^{\mu\nu}\langle 0|\mathcal{T}T_{\mu\nu}(x)T_{\mu'\nu'}(y)T_{\alpha\beta}(z)|0\rangle^{(odd)} = 0$$

This is because

$$\langle 0|\mathcal{T}T_{\mu\nu}(x)T_{\mu'\nu'}(y)T_{\alpha\beta}(z)|0\rangle^{(odd)} = 0$$

for algebraic reasons!

So, what is the right result?

Ambiguities in trace anomaly calculations

The true reason for these contradictory results is due to the ambiguities in the (perturbative) calculation of the trace anomaly

There are four sources of ambiguity:

1) The divergent integrals
2) The very definition of trace anomaly
3) The cohomological ambiguity (Weyl + diffeomorphisms)
4) The \`new’ ambiguity

Ambiguities

1. Loop integration for a Feynman diagram is UV divergent $\longrightarrow$ Solution: choose a regularization scheme

2. Ambiguous definition of trace anomaly $\longrightarrow$ Solution: Define it as:

$$T(x) = g^{\mu\nu}(x)\langle\langle T_{\mu\nu}(x)\rangle\rangle - \langle\langle T^{\mu}_{\mu}(x)\rangle\rangle$$

3. Cohomological ambiguity: $\longrightarrow$ Solution: Diffeomorphims must be conserved

4. 'New' ambiguity: $\longrightarrow$ Solution: Go to higher loops or use a non-perturbative method

Trace anomaly: definition

The (gravity) effective action is defined by

$$W[h] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{2^n n!} \int \prod_{i=1}^{n} d^4x_i \sqrt{g(x_i)} h^{\mu_i\nu_i}(x_i) \langle 0|\mathcal{T} T_{\mu_1\nu_1}(x_1)\ldots T_{\mu_n\nu_n}(x_n)|0\rangle$$

The one-loop one-point function is defined by

$$\langle\langle T_{\mu\nu}(x)\rangle\rangle = \sum_{n=0}^{\infty} \frac{i^{n}}{2^n n!} \int \prod_{i=1}^{n} d^4x_i \sqrt{g(x_i)} h^{\mu_i\nu_i}(x_i) \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1)\ldots T_{\mu_n\nu_n}(x_n)|0\rangle$$

with $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. The trace anomaly is

$$T(x) = g^{\mu\nu}(x)\langle\langle T_{\mu\nu}(x)\rangle\rangle - \langle\langle T_{\mu}^{\mu}(x)\rangle\rangle$$

To the lowest order

$$T^{(odd)}(x) = \eta^{\mu\nu}\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1) T_{\mu_2\nu_2}(x_2)|0\rangle - \langle 0|\mathcal{T} T_{\mu}^{\mu}(x) T_{\mu_1\nu_1}(x_1) T_{\mu_2\nu_2}(x_2)|0\rangle + \ldots$$

Since $T_{\mu}^{\mu}(x) = 0$ on shell, this is the quantization of 0.

The 'new' ambiguity

Let us apply the same procedure to the odd-parity trace anomaly in 4D. The two amplitudes in question are

$$
\begin{aligned}
\widetilde{T}_{\mu\nu\mu'\nu'}(k_1,k_2) \quad = \quad & \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big\{\frac{i}{8}\left[(2p-k_1)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\right]\left(\frac{1+\gamma_5}{2}\right)\frac{i}{(\not p-\not k_1)} \\
& \times \frac{i}{8}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu'\leftrightarrow\nu')\right]\left(\frac{1+\gamma_5}{2}\right) \\
& \times \frac{i}{(\not p-\not k_1-\not k_2)}(2\not p-\not k_1-\not k_2)\left(\frac{1+\gamma_5}{2}\right)\frac{i}{\not p}\Big\}.
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{T}'_{\mu\nu\mu'\nu'}(k_1,k_2) \quad = \quad & \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big\{\frac{i}{8}\left[(2p-k_1)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\right]\frac{i}{(\not p-\not k_1)} \\
& \times \frac{i}{8}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu'\leftrightarrow\nu')\right] \\
& \times \frac{i}{(\not p-\not k_1-\not k_2)}(2\not p-\not k_1-\not k_2)\left(\frac{1+\gamma_5}{2}\right)\frac{i}{\not p}\Big\}.
\end{aligned}
$$

Once regularized they lead to different results (like in the previous case).

Solution for the ‘new’ ambiguity

Does it mean that we cannot compute the trace anomaly? No, it only means that this problem cannot be solved at the lowest perturbative order.

Possible solutions:

- go to higher order (four-point, five-point, ... amplitudes)
- use non-perturbative methods (heat kernel methods)

Trace anomaly in chiral theories with non-pertubative methods.

Idea: use Bardeen’s method also for trace anomalies

Bardeen's method for chiral anomaly

Consider a theory of Dirac fermions coupled a vector V_μ and an axial A_μ gauge potentials, both valued in a Lie algebra with T^a. The action is

$$S[V, A] = i \int d^4x\, \overline{\psi}\left(\partial\!\!\!/ + V\!\!\!/ + \gamma_5 A\!\!\!/\right)\psi$$

It is invariant under two sets of gauge transformations

$$\begin{cases} V_\mu \longrightarrow V_\mu + D_{V\mu}\alpha \\ A_\mu \longrightarrow A_\mu + [A_\mu, \alpha], \\ \psi \longrightarrow (1-\alpha)\psi \end{cases} \qquad \begin{cases} V_\mu \longrightarrow V_\mu + [A_\mu, \beta] \\ A_\mu \longrightarrow A_\mu + D_{V\mu}\beta \\ \psi \longrightarrow (1+\gamma_5\beta)\psi \end{cases}$$

where $D_{V\mu} = \partial_\mu + [V_\mu, \;\cdot\;]$ and $\alpha = \alpha^a(x)T^a, \beta = \beta^a(x)T^a$.

As a consequence there are two covariantly conserved currents, $j_\mu = j^a_\mu T^a$ and $j_{5\mu} = j^a_{5\mu}T^a$, where $j^a_\mu = \overline{\psi}\gamma_\mu T^a\psi$ and $j^a_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5 T^a\psi$.

After quantization one finds:

$$[D^\mu_V j_\mu]^a + [A^\mu, j_{5\mu}]^a = 0$$

while the axial conservation becomes anomalous:

$$\begin{aligned} [D^\mu_V j_{5\mu}]^a + [A^\mu, j_\mu]^a \;&=\; \frac{1}{4\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\Bigg[T^a\Bigg(\frac{1}{4}F^{\mu\nu}_V F^{\lambda\rho}_V + \frac{1}{12}F^{\mu\nu}_A F^{\lambda\rho}_A - \frac{1}{6}F^{\mu\nu}_V A^\lambda A^\rho \\ &\quad - \frac{1}{6}A^\mu A^\nu F^{\lambda\rho}_V - \frac{2}{3}A^\mu F^{\nu\lambda}_A A^\rho - \frac{1}{3}A^\mu A^\nu A^\lambda A^\rho\Bigg)\Bigg] \end{aligned}$$

where $F^{\mu\nu}_V = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu]$, and $F^{\mu\nu}_A = \partial^\mu A^\nu - \partial^\nu A^\mu + [V^\mu, A^\nu] + [A^\mu, V^\nu]$.

Bardeen's method, *cont.*

From this expression we can derive two results in particular. Setting $A_\mu = 0$ we get the covariant anomaly

$$[D_V^\mu j_{5\mu}]^a = \frac{1}{16\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left(T^a F_V^{\mu\nu} F_V^{\lambda\rho}\right)$$

which is the covariant anomaly.

Taking the chiral limit $V \to \frac{V}{2}, A \to \frac{V}{2}$ we get

$$[D_{V\mu} j_L^\mu]^a = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[T^a \partial^\mu \left(V^\nu \partial^\lambda V^\rho + \frac{1}{2} V^\nu V^\lambda V^\rho\right)\right]$$

where $j^a_{L\mu} = \overline{\psi}_L \gamma_\mu T^a \psi_L$, here $\psi_L = \frac{1+\gamma_5}{2}\psi$, which is the consistent non-Abelian gauge anomaly.

The advantage of this method is that we work with Dirac fermions (as opposed to Weyl fermions).

The Schwinger-DeWitt method

To represent the determinant of the Dirac operator one can use the DeWitt method based of the Schwinger proper time. One introduces the vacuum-to-vacuum amplitude

$$\langle x, s|x', 0\rangle = \langle x|e^{i\mathcal{F}s}|x'\rangle,$$

where $\mathcal{F}_x$ is a differential quadratic operator, in the ordinary case the Dirac square

$$\mathcal{F}_x = \nabla_\mu g^{\mu\nu}\nabla_\nu - \frac{1}{4}R + \mathcal{V},$$

where $\mathcal{V} = \Sigma^{ab}e^\mu_a e^\nu_b\left(\partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]\right)$. It follows that the functional determinant W can be represented as

$$W = -\frac{1}{4}\int_0^\infty \frac{ds}{is}\mathsf{Tr}\left(e^{i\mathcal{F}s}\right) + \text{const} \equiv \mathsf{L} + \text{const},$$

where L is the relevant effective action

$$\mathsf{L} = \int d^{\mathsf{d}}x\, L(x) \equiv \int d^{\mathsf{d}}x\,\langle x|\mathsf{L}|x\rangle,$$

The trick is to use the (heat kernel) differential equation

$$i\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = -\mathcal{F}_x\langle x, s|x', 0\rangle,$$

The DWS method: the results

In conclusion the Pontryagin Weyl anomaly for left-handed Weyl fermion

$$\mathcal{T}'(x) = \frac{i}{1536\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}.$$

We can easily compute also the ABJ-like anomaly, by taking the limit the (vector) limit $g_{\mu\nu} \to g_{\mu\nu}, f_{\mu\nu} \to 0$. The result is the anomaly

$$\mathcal{T}_5(x) = \frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}.$$

which coincides with the perturbative result found above.

There are plenty of different anomalies: gauge and trace, even- and odd-parity, consistent and covariant.

- Are they all on the same footing and of the same importance?

- Why some are dangerous and other are not?

- Do they have to be cancelled?

The methods considered so far do not allow us to answer these questions!
We need the family's index theorem

The Dirac operator and its index

Consider a gauge field theory on a *Euclidean* spacetime M. The Dirac operator

$$\not{\mathcal{D}} = \not{D} + i\not{V}$$

acts on the tensor product of a spinor bundle $S^{\pm}_{\mathbb{C}}$ with a vector bundle E corresponding to a representation ρ of the structure group G of $\mathsf{P}(X, \mathsf{G})$: $S^{\pm}_{\mathbb{C}} \otimes E$.

The relevant connection is the spin connection plus a gauge connection $V = V_\mu dx^\mu$ valued in the representation ρ of the Lie algebra of G with antihermiten generators. The Dirac operator acts on the space of sections of $S^{\pm}_{\mathbb{C}}(E) \equiv S^{\pm}_{\mathbb{C}} \otimes E$ (i.e. on the spinor fields) and maps it to itself. Accordingly it splits into

$$\not{\mathcal{D}} = \begin{pmatrix} 0 & \not{\mathcal{D}}^- \\ \not{\mathcal{D}}^+ & 0 \end{pmatrix}$$

The index of $\not{\mathcal{D}}^+$ is defined by

$$Ind\, \not{\mathcal{D}}^+ = dim(ker\, \not{\mathcal{D}}^+) - dim(ker\, \not{\mathcal{D}}^-)$$

Atiyah and Singer showed that

$$ch\left(ind(\not{\mathcal{D}}^+)\right) = \int_{\mathsf{M}} ch(\mathcal{V}) \cdot \hat{A}(T\mathcal{Q})$$

Here $\mathcal{Q} \equiv \frac{\mathcal{A}}{\mathcal{G}}$ is the orbit space of connections,

The family's index theorem

The Atiyah-Singer family's index theorem says that the index of the Dirac operator for a Weyl fermion is given

$$c_1\left(ind(\not{D}^+)\right) = \int_M ch(\mathcal{V}) \cdot \hat{A}(T\mathcal{Q})\Big|_{\mathsf{d},2}$$

where $\mathcal{V}$ is the gauge bundle and $T\mathcal{Q}$ is the tangent bundle to the moduli space of the theory. ch is the Chern character

$$ch(\mathcal{V}) = r + \frac{i}{2\pi}\mathrm{tr}\, F + \frac{i^2}{2(2\pi)^2}\mathrm{tr}\, F^2 + \frac{i^3}{3!(2\pi)^3}\mathrm{tr}\, F^3 + \ldots$$

and $\widehat{A}$ is the $\widehat{A}$-genus

$$\widehat{A}(X) \;\; = \;\; 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\,\mathrm{tr}\, R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}\,\mathrm{tr}\,(R^2)^2 + \frac{1}{360}\,\mathrm{tr}\, R^4\right]$$

$c_1\left(ind(\not{D}^+)\right)$ represents an obstruction to the invertibility of $\not{D}^+$. I.e. $c_1\left(ind(\not{D}^+)\right) \neq 0$ the Weyl fermion propagator does not exist!

The Pontryagin class $\mathrm{tr}R^2$ and Chern class $\mathrm{tr}F^2$ are obstructions to the existence of the fermion propagator.

Here is the conundrum: anomalies are dangerous when they prevent the existence of the fermion propagators.

I call type O the anomalies whose family's index is nonvanishing

I call type NO all the others

The family's index of a self-adjoint operator vanishes identically.

Examples: the Dirac operator for Dirac fermions; the Maxwell operator (after gauge fixing)….

Anomaly recap

- Where do they appear:
  - in divergence of current or of e.m. tensor
  - in trace of e.m. tensor
- Local Anomalies are of two types:
  - type O
    - prevent existence of propagators,
    - dangerous : must be canceled
  - type NO
    - no obstruction for propagators,
    - need not be canceled
- In what theories:
  - type O — only in chiral theories
  - type NO — in any theory
- Cancellation:
  - type O
    - A : group theoretical (unavailable for trace anomalies)
    - B : coefficient matching (unlikely for trace anomalies)
    - C : Wess – Zumino terms or Green – Schwarz mechanisms
  - type NO
    - in general not required
    - for even trace anomalies
      - B : coefficient matching unlikely
      - C : with Wess – Zumino terms

Thus we have to find all the type O anomalies in the SM coupled to gravity and make sure that they vanish.

They are:

- chiral gauge anomalies (they vanish)
- gravitational anomalies (absent)
- mixed gauge-gravity anomalies
- chiral trace anomalies

Mixed gauge-gravity anomaly

The pure gravitational anomalies in the divergence of the e.m. tensor vanish identically, but there is a mixed gauge-gravity chiral anomaly! Consider a right-handed fermion ψ_R coupled both to a metric and to an Abelian gauge field V_μ.

The relevant current $j_\mu = \bar{\psi}_R \gamma_\mu \psi_R$ is classically conserved, but after quantization

$$\partial^\mu j_{R\mu} = \frac{1}{1536\pi^2} \, \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}.$$

Its integrated form is $\sim \Delta_G(\lambda, g) = \int d^4x \lambda \, \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}$, which is a diffeomorphism-invariant (trivially) consistent Abelian gauge cocycle. This cocycle can take different forms, for instnace $\Delta_G(\lambda, g)$ is equivalent to

$$\Delta_{\mathsf{d}}(\xi, g, V) = \int d^4x \sqrt{g} \, \varepsilon^{\mu\nu\lambda\rho} \mathrm{tr} \left(\partial_\mu \Xi \, \Gamma_\nu \right) F_{\lambda\rho} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, Ξ represents the matrix $\Xi_\tau{}^\sigma = \partial_\tau \xi^\sigma$ and Γ_μ represents the matrix $\Gamma^\tau_{\mu\sigma}$. Nevertheless this anomaly vanishes in the MSM coupled to a metric:

- $\Sigma \times \Sigma \times T^{\mathrm{u}(1)}$, the trace $\mathrm{tr}\left(\Sigma^{ab}\Sigma^{cd}\right)$ is non-vanishing, but it is multiplied by the total $U(1)$ charge: $6\left(\frac{1}{6}\right) - 3\left(\frac{2}{3}\right) - 3\left(-\frac{1}{3}\right) + 2\left(-\frac{1}{2}\right) + 1 = 0$.

The addition of sterile neutrinos does not alter this conclusion.

Gauge-induced trace anomalies

Consider the action of a Dirac fermion coupled to a metric and an Abelian vector field

$$S = \int d^4x \sqrt{g}\, i\overline{\psi}\gamma^\mu \left(D_\mu + \frac{1}{2}\omega_\mu - iV_\mu\right)\psi$$

with the usual notation. The vector current is $j_\mu = \overline{\psi}\gamma_\mu\psi$ and the stress-energy tensor

$$T_{\mu\nu} = \frac{i}{4}\overline{\psi}\gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi + \{\mu \leftrightarrow \nu\}, \qquad \nabla_\mu = D_\mu + \frac{1}{2}\omega_\mu - iV_\mu$$

With the same methods (both pertubative and non-perturbative) we get

$$g^{\mu\nu}\langle\langle T_{R\mu\nu}(x)\rangle\rangle\Big|_{odd} - \langle\langle g^{\mu\nu}T_{R\mu\nu}(x)\rangle\rangle\Big|_{odd} = -\frac{i}{96\pi^2}\varepsilon_{\mu\nu\lambda\rho}\,\partial^\mu V^\nu(x)\partial^\lambda V^\rho(x)$$

The non-Abelian version of this result is

$$\mathcal{A}_\omega^{(odd,R)} = -\frac{i}{384\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}(F^{\mu\nu}F^{\lambda\rho})$$

Trace anomalies in the SM

MSM trace-gravity

- The SM multiplet, when coupled to gravity, produces an overall non-vanishing (imaginary) coefficient for the Pontryagin density in the trace anomaly.

This breakdown is naturally avoided if we add to the above MSM multiplet a right-handed sterile neutrino.

MSM trace-gauge

- We have six units of the anomaly $\sim \mathrm{tr}F^2$ with curvature $F \equiv F^{\mathfrak{su}(3)}$ and six units with opposite sign. Therefore the MSM multiplet is free of these anomalies.
- We have instead 4 units of the same anomaly with gauge field $F \equiv F^{\mathfrak{su}(2)}$ and positive sign, computed in the doublet representation of $\mathfrak{su}(2)$.
- Finally we have a $U(1)$ gauge-induced trace anomaly with vanishing total coefficient: $6\left(\frac{1}{6}\right)^2 - 3\left(\frac{2}{3}\right)^2 - 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2 - (-1)^2 = 0$

The addition of sterile neutrinos does not change these conclusions.
The $SU(2)$ gauge-induced odd trace anomalies do not cancel in the MSM.

Summary: second half

- A L-R symmetric model
- Axial-complex analysis
- Conformal invariance
- Trace anomalies: even parity
- WZ terms
- Renormalization and unitarity
- Connection with cosmology

A chirally symmetric model

The left-handed multiplet is

$\mathsf{G}/fields$	$SU(3)$	$SU(2)$	$U(1)$
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$\frac{1}{6}$
$(u_R)^c$	$\bar{3}$	1	$-\frac{2}{3}$
$(d_R)^c$	$\bar{3}$	1	$\frac{1}{3}$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	$-\frac{1}{2}$
$(e_R)^c$	1	1	1
$(\nu_R)^c$	1	1	0

This multiplet couples to a left gravitational metric and connection, and to the $SU(3)_L \times SU(2) \times U(1)_L$ gauge fields.

A chirally symmetric model

The right-handed multiplet is

$\mathsf{G}/fields$	$SU(3)$	$SU(2)$	$U(1)$
$\begin{pmatrix} u' \\ d' \end{pmatrix}_R$	3	2	$\frac{1}{6}$
$(u'_L)^c$	$\bar{3}$	1	$-\frac{2}{3}$
$(d'_L)^c$	$\bar{3}$	1	$\frac{1}{3}$
$\begin{pmatrix} \nu'_e \\ e' \end{pmatrix}_R$	1	2	$-\frac{1}{2}$
$(e'_L)^c$	1	1	1
$(\nu'_L)^c$	1	1	0

coupled to a right gravitational metric and connection. This multiplet couples to the $SU(3)_R \times SU(2) \times U(1)_R$ gauge fields.

All O-type anomalies cancel.

The Weyl fermions of $\mathcal{T} = \mathcal{T}_L + \mathcal{T}_R$

Fermion kinetic action

$$S_f^{(+)} \equiv S_{fR} \; = \; \int d^4x \left(\sqrt{g}\, i\overline{\psi'_R}\gamma^a e_a^\mu \left(\mathcal{D}_\mu^{(+)} + \frac{1}{2}\omega_\mu \right) \psi'_R \right) (\widehat{x})$$

where ψ'_R represents the right-handed multiplet, and

$$\mathcal{D}_\mu^{(+)} = \partial_\mu + \mathsf{g}_X^+ X_\mu^{(+)} + \mathsf{g}_W W_\mu + \mathsf{g}_B^+ B_\mu^{(+)}$$

$\omega_\mu = \omega_\mu^{ab}\Sigma_{ab}$ is the spin connection corresponding to the metric g and Σ_{ab} the anti-hermitean Lorentz generators. For the left sector

$$S_f^{(-)} \equiv S_{fL} \; = \; \int d^4x \left(\sqrt{g}\, i\overline{\psi_L}\gamma^a e_a^\mu \left(\mathcal{D}_\mu^{(-)} + \frac{1}{2}\omega_\mu \right) \psi_L \right) (\widehat{x})$$

where ψ_L represents the left-handed multiplet, and

$$\mathcal{D}_\mu^{(-)} = \partial_\mu + \mathsf{g}_X^- X_\mu^{(-)} + \mathsf{g}_W W_\mu + \mathsf{g}_B^- B_\mu^{(-)}$$

The symbols $X_\mu^{(\pm)}, W_\mu, B_\mu^{(\pm)}$ refer to the $SU(3)_{R/L}, SU(2)$ and $U(1)_{R/L}$ potentials, respectively.

The Weyl fermions of $\mathcal{T} = \mathcal{T}_L + \mathcal{T}_R$, cont.

Symbols such as $(\psi_R)^c$ (for instance $(u_R)^c, (d_R)^c, ...$) can be rewritten as

$$(\psi_R)^c = \gamma^0 C \psi_R^* = \gamma^0 C P_R^* \psi^* = P_L \gamma^0 C \psi^* = P_L \psi^c = (\psi^c)_L.$$

Inserted into the kinetic term, this gives

$$\int d^4x \sqrt{g}\, \overline{(\psi^c)_L}\, \gamma^\mu (\partial_\mu + \frac{1}{2}\omega_\mu)(\psi^c)_L = \int d^4x \sqrt{g}\, \overline{\psi_R}\, \gamma^\mu (\partial_\mu + \frac{1}{2}\omega_\mu)\psi_R$$

The gauge sector of $\mathcal{T} = \mathcal{T}_L + \mathcal{T}_R$

The SU(2) gauge field action has the usual form

$$S_g^{SU(2)} = -\frac{1}{4\mathrm{g}^2}\int d^4x\,\sqrt{g}\,\mathrm{tr}\left(g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'}\right)$$

where $F_{\mu\nu} = dV + \frac{1}{2}[V, V]$ is the curvature of the SU(2) gauge field.

For the groups $SU(3)_L \times SU(3)_R$ and $U(1)_L \times U(1)_R$ we have instead $S_g^{(+)} + S_g^{(-)}$ with

$$S_g^{(\pm)} = -\frac{1}{4\mathrm{g}_\pm^2}\int d^4x\,\sqrt{g}\,\mathrm{tr}\left(g^{\mu\mu'}g^{\nu\nu'}F^{(\pm)}_{\mu\nu}F^{(\pm)}_{\mu'\nu'}\right)$$

where $F^{(\pm)}_{\mu\nu} = dV^{(\pm)} + \frac{1}{2}[V^{(\pm)}, V^{(\pm)}]$ and $F^{\pm}$ denotes the curvatures of the $SU(3)_R$ and $U(1)_R$, and $SU(3)_L$ and $U(1)_L$ potentials, respectively.

The gravity and Higgs sector of $\mathcal{T} = \mathcal{T}_L + \mathcal{T}_R$

The action for the metric is the usual EH action with different cosmological constants in the left and right sector

$$S_{EH}^{(\pm)} = -\frac{1}{2\kappa}\int d^4x\,\sqrt{g}\,(R + \mathfrak{c}_\pm)$$

R is the Ricci scalar, κ the gravitational constant and $\mathfrak{c}_\pm$ the left/right cosmological constant.

In the MSM we need also a couple $H_\pm$ of complex scalar fields, which minimally couple to the metric $g_{\mu\nu}$ and are a doublet under $SU(2)$. The corresponding actions in the two sectors are given by

$$S_d^{(\pm)} = \int d^4\widehat{x}\,\sqrt{g}\left[g^{\mu\nu}\mathcal{D}_\mu H_\pm^\dagger \mathcal{D}_\nu H_\pm - M_\pm^2 H_\pm^\dagger H_\pm - \frac{\lambda_\pm}{4}\left(H_\pm^\dagger H_\pm\right)^2\right]$$

where $\mathcal{D}_\mu = \partial_\mu - i\mathsf{g}W_\mu$, and W_μ is the $SU(2)$ gauge field.

The Yukawa couplings $\mathcal{T} = \mathcal{T}_L + \mathcal{T}_R$

For SU(2) doublets we have

$$S_{YdL} = \frac{y^{-}_{H_d}}{2} \int d^4x \sqrt{g} \left(\overline{\psi_{dL}}\, H_{d-} \chi_{sR} \right) + h.c.$$

where ψ_{dL} is a left-handed SU(2) doublet, H_{d-} is also an $SU(2)$ doublet, conjugate to the ψ_{dL} one, while χ_{sR} is a right-handed singlet, all of them belonging to $\mathcal{T}_L$. Similarly, for $\mathcal{T}_R$,

$$S_{YdR} = \frac{y^{+}_{H_d}}{2} \int d^4\widehat{x} \sqrt{g} \left(\overline{\psi'_{dR}}\, H_{d+} \chi'_{sL} \right) + h.c.$$

Let us write $S_f = S_f^{(+)} + S_f^{(-)}$, $S_g = S_g^{SU(2)} + S_g^{(+)} + S_g^{(-)}$, $S_{EH} = S_{EH}^{(+)} + S_{EH}^{(-)}$, $S_d = S_d^{(+)} + S_d^{(-)}$ and $S_Y = S_{YdL} + S_{YdR}$. Then for the total action of our model minimally coupled to gravity we can tentatively set

$$S = S_f + S_g + S_{EH} + S_d + S_Y$$

This theory is invariant under $SU(2)$, as well as $SU(3)_L \times SU(3)_R$ and $U(1)_L \times U(1)_R$, gauge transformations. It is also invariant under diffeomorphisms and local Lorentz transformations.

The left sector of T represents the SM. So, what is the right sector?

-it interact weakly with the left sector (only through the weak and gravitational force)
-It has a structure similar to the left sector, but an independent evolution.

CDM may be made of the SM baryons, like MACHOs (massive astrophysical compact halo objects) or primordial black holes. The non-baryonic dark matter might be made of particles: massive neutrinos, axions, or weakly interacting particles present in supersymmetric models: neutralinos (a mixture of supersymmetric partners), fotinos, Binos,... popularly denoted by the acronim WIMPs. WIMPS may achieve the miracle of Ωdm =0.25.

The right sector may represent dark matter

Standard model and Weyl invariance

In the above theory the fermion action, the gauge actions and the Yukawa couplings are conformal invariant.

The scalar actions and the EH action are not conformal invariant.

It is possible to render the whole action conformal invariant by introducing a scalar field, φ, the dilaton.

Weyl geometry

In an ordinary gravitational background geometry the Weyl transformation is given by

$$g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}$$

The Christoffel symbols transform as

$$\Gamma^{\lambda}_{\mu\nu} \to \Gamma^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\mu}\,\partial_\nu\omega + \delta^{\lambda}_{\nu}\,\partial_\mu\omega - g_{\mu\nu}g^{\lambda\rho}\partial_\rho\omega$$

We can construct Weyl-invariant Christoffel symbols as follows

$$\widetilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \left(\delta^{\lambda}_{\mu}\,\partial_\nu\varphi + \delta^{\lambda}_{\nu}\,\partial_\mu\varphi - g_{\mu\nu}g^{\lambda\rho}\partial_\rho\varphi\right)$$

where the field φ (a dilaton) under Weyl transforms as

$$\varphi \to \varphi + \omega$$

We can construct Weyl invariant Ricci tensor

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} + 3D_\nu S_\mu - D_\mu S_\nu + g_{\mu\nu} D\cdot S + 2S_\mu S_\nu - 2g_{\mu\nu} S\cdot S$$

where $S_\mu = \partial_\mu\varphi$. Important: φ is dimensionless !

Weyl geometrization

One possible Weyl invariant action for gravity and a scalar field is

$$
\begin{aligned}
S^{(c)}_{EH+s} \;&=\; \frac{1}{2\kappa}\int d^4x\sqrt{g}\,\left(e^{-2\varphi}+\zeta\Phi^2\right)\left(\widetilde{R}+\mathfrak{c}\,e^{-2\varphi}\right)\\
&\quad+\frac{1}{2}\int d^4x\,\sqrt{g}\,\left[g^{\mu\nu}\mathsf{D}_\mu\Phi\mathsf{D}_\nu\Phi-m^2e^{-2\varphi}\Phi^2-\frac{\lambda}{4}\Phi^4\right]
\end{aligned}
$$

where

$$\widetilde{R}=R+6\,\left(D\cdot\partial\varphi-\partial\varphi\cdot\partial\varphi\right)$$

and

$$\mathsf{D}_\mu\Phi=\left(\partial_\mu+\partial_\mu\varphi\right)\Phi$$

Now this has to be done for $\mathcal{T}$ and transform it into $\mathcal{TW}$.

Embedding $\mathcal{T}$ in Weyl geometry

S_g, S_f, S_Y remain the same, while $S_{EH} + S_d$ become

$$\begin{aligned} S^{(c)}_{EH+d} \;=\; & \frac{1}{2}\int d^4x\,\mathrm{tr}\left[\frac{1}{\kappa_\pm}\sqrt{g}\left(e^{-2\varphi_\pm} + \zeta_{h\pm}H^\dagger_\pm H_\pm + \zeta_\pm \Phi^2_\pm\right)\left(\widetilde{R} + \mathfrak{c}_\pm\, e^{-2\varphi_\pm}\right)\right] \\ & + \int d^4x\,\mathrm{tr}\left\{\sqrt{g}\left[g^{\mu\nu}\left(\mathsf{D}_\mu H_\pm\right)^\dagger\left(\mathsf{D}_\nu H_\pm\right) + M^2_\pm e^{-2\varphi_\pm}H^\dagger_\pm H_\pm - \frac{\lambda_h}{4}\left(H^\dagger_\pm H_\pm\right)^2\right]\right\} \end{aligned}$$

where $\mathsf{D}^\pm_\mu = \partial_\mu + \partial_\mu\varphi_\pm - i\mathsf{g}W_\mu$. To these we can add

$$S_C = \frac{1}{\eta}\int d^4x\sqrt{g}\,C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$$

$C_{\mu\nu\lambda\rho}$ is the Weyl tensor (${C^\mu}_{\nu\lambda\rho}$ is Weyl invariant), and

$$S_Q = \int d^4x\,\sqrt{g}\,Q^2$$

where $Q = \Box\varphi - \partial_\mu\varphi\partial^\mu\varphi + \frac{1}{6}R$, which under a Weyl transformation becomes $Q \to e^{-2\omega}Q$, so both S_C and S_Q are Weyl invariant.

Why Weyl invariance?

Two main reasons:

- the cosmological constant problem
- renormalization and unitarity

+

- a philosophical prejudice: the *apeiron* of Anaximander

The cosmological constant problem

There is a longstanding problem to unify gravity with the SM, due to the relation between the cosmological constant and the energy of the vacuum; more precisely the vacuum energy density due to gravitation is represented by

$$\rho_\Lambda = \frac{\mathfrak{c}}{2\kappa}$$

The observed value is

$$|\rho_\Lambda^{(obs)}| \sim 2 \times 10^{-10} erg/cm^3 \tag{1}$$

The trouble is that when we put together in a unique theory gravity and matter, the matter field theory comes with its own vacuum energy. The latter is always a divergent quantity and can be estimated only using different cutoffs.

- using QCD scale one finds $\rho_{vac}^{QCD} \sim 1.6 \times 10^{36} erg/cm^3$,
- using the electroweak scale one finds $\rho_{vac}^{EW} \sim 3 \times 10^{47} erg/cm^3$,
- using the Planck scale, one gets $\rho_{vac}^{Pl} \sim 2 \times 10^{110} erg/cm^3$

In any case the gap with (1) is gigantic, and one is obliged to imagine another unknown entry in the above calculations to fill in the gap.

The cosmological constant problem *cont.*

... but, the fermion and gauge parts of the action, as well as the Yukawa coupling and the quartic scalar couplings, are unaffected by Weyl transformations. On the contrary the other terms in $S^{(c)}$, and in particular the cosmological constant term may undergo gigantic changes under a Weyl transformation.

If we choose, for instance, a 'gauge' $\varphi = 0$ we reproduce the just mentioned unnatural situation, but if we choose a sufficiently negative value for φ, for instance a 'gauge' $\varphi \approx -25$ or a similar one, the effective cosmological constant takes on a value for which the gravitational vacuum energy may be comparable with the value of the vacuum energy of the theory, whatever it may be.

The point is that we are able to quantize field theories only via a perturbative series. Therefore, for instance, the smallness of the measured cosmological constant disappears compared to the quantum corrections of the SM. Simply it does not make much sense to juxtapose matter and gravity (if the cosmological constant represents its vacuum energy) in the same quantum theory. However, the theory $\mathcal{TW}$ (the Weyl invariant $\mathcal{T}$ theory) is conformal invariant. Therefore we can quantize it at the scale (i.e. the 'gauge') where the perturbative approach makes sense, and transfer the quantized results (renormalization and unitarity) to the other scales.

The drama of quantum gravity

The problem of gravity quantization is very simply said. If we consider only the EH action the vertices and the graviton propagators (after gauge fixing) in the UV go asymptotically like

$$\sim p^2, \qquad \text{and} \qquad \sim \frac{1}{p^2}$$

respectively, i.e. the divergence in the UV grow with the number of loops, the theory is non-renormalizable.

If we add a term with quartic derivatives like S_C, then propagators and vertices in the UV go like

$$\sim p^4, \qquad \text{and} \qquad \sim \frac{1}{p^4}$$

respectively. Then the diagrams divergence in the UV stabilizes and the theory is perturbatively renormalizable.

But there is a price. Disregarding the tensorial factor the graviton propagator is proportional to the inverse of $\alpha\Box + \beta\Box^2$, i.e. the inverse of $-\alpha p^2 + \beta p^4$, which can be written as follows

$$\frac{1}{-\alpha p^2 + \beta p^4} = \frac{1}{p^2(-\alpha + \beta p^2)} = -\frac{1}{\alpha}\left(\frac{1}{p^2} - \frac{1}{p^2 - \frac{\alpha}{\beta}}\right)$$

This inevitably introduces a quadratic pole with negative residue, corresponding to a negative norm state, which is likely to violate unitarity.

Is like having to navigate between Skylla and Charybdis

Gauge fixing

A gauge fixing action for diffeomorphisms is

$$S_{g.f.}^{(diff)} = -\frac{1}{2\sqrt{\kappa}} \int d^4x \, g_{\mu\nu} \left(\partial^\mu b^\nu + \partial^\nu b^\mu\right) - \frac{\alpha_0}{2} \int d^4x \, \eta_{\mu\nu} b^\mu b^\nu$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$. It is accompanied by a FP

$$S_{FP} = -\frac{1}{2\sqrt{\kappa}} \int d^4x \left(\left(\partial^\mu \overline{\xi}^\nu + \partial^\nu \overline{\xi}^\mu \right) \delta_\xi g_{\mu\nu} + 4\overline{\xi}^\mu \partial_\mu \delta_\xi \varphi \right)$$

where ξ^μ are the ghost fields and $\overline{\xi}^\mu$ the antighost', with BRST transform

$$\begin{aligned}
\delta_\xi g_{\mu\nu} &= \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\lambda \xi^\mu g^{\lambda\nu} + \partial_\lambda \xi^\nu g^{\mu\lambda} \\
\delta_\xi \varphi &= \xi^\lambda \partial_\lambda \varphi \\
\delta_\xi \xi^\mu &= \xi^\lambda \partial_\lambda \xi^\mu \\
\delta_\xi \bar{\xi}^\mu &= -b^\mu \\
\delta_\xi b^\mu &= 0
\end{aligned}$$

As is well known $\delta_\xi^2 = 0$.

Gauge fixing, *cont.*

But we have to fix also the residual conformal gauge symmetry. The relevant gauge fixing term we choose is:

$$S^{(c)}_{g.f.} = -\int d^4x \left(b\,\widetilde{\Box}\varphi - \frac{\beta_0}{2} b^2 \right)$$

The corresponding FP term is

$$S^{(c)}_{FP} = -\int d^4x\, \overline{\omega}\, \delta_\omega \widetilde{\Box}\varphi$$

In both formulae $\Box$ is the condormal-covariant D'Alambertian, i.e. $\widetilde{\Box} = g^{\mu\nu}\left(\partial_\mu\partial_\nu - \widetilde{\Gamma}^\lambda_{\mu\nu}\partial_\lambda\right)$

We complete the BRST set of transformations by

$$\delta_\xi b = \xi^\lambda \partial_\lambda b + b\,\partial_\lambda \xi^\lambda, \quad \delta_\xi \omega = \xi^\lambda \partial_\lambda \omega, \quad \delta_\xi \overline{\omega} = \xi^\lambda \partial_\lambda \overline{\omega} + \overline{\omega}\,\partial_\lambda \xi^\lambda$$

and

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \qquad \delta_\omega \varphi = \omega, \qquad \delta_\omega b_\mu = 0, \qquad \delta_\omega \overline{\omega} = -b \qquad \delta_\omega b = 0, \qquad \delta_\omega \omega = 0$$

We have $(\delta_\xi + \delta_\omega)^2 = 0$.

Propagators

Graviton propagators

$$\langle h\,h\rangle^{(2)} = \frac{i}{\frac{p^2}{8\kappa}+\frac{p^4}{2\eta}}P^{(2)}, \qquad \langle h\,h\rangle^{(1)} = -i\frac{4\alpha_0\kappa}{p^2}P^{(1)}, \qquad \langle h\,h\rangle^{(0)} = i\left(\frac{2}{\frac{p^4}{\gamma}-\frac{p^2}{2\kappa}}-\frac{6\beta_0}{p^4}\right)P^{(0)}$$

$$\langle h\,h\rangle^{\overline{(0)}} = i\left(\frac{4\beta_0}{p^4}+\frac{2\alpha_0\kappa}{p^2}\right)\overline{P}^{(0)}, \qquad \langle h\,h\rangle^{(T)} = -i\frac{4\sqrt{3}\beta_0}{p^2}T^{(0)}$$

Other propagators are

$$\langle h_{\mu\nu}b_\alpha\rangle = \frac{2\sqrt{\kappa}}{p^2}\left(p_\mu\eta_{\nu\alpha}+p_\nu\eta_{\mu\alpha}-\omega_{\mu\nu}p_\alpha\right), \qquad \langle h_{\mu\nu}\,\varphi\rangle = i\frac{\beta_0}{p^4}\left(\theta_{\mu\nu}+2\omega_{\mu\nu}\right)$$

$$\langle h_{\mu\nu}\,b\rangle = -\frac{2i}{p^2}\left(\theta_{\mu\nu}+2\omega_{\mu\nu}\right)$$

and

$$\langle\varphi\,\varphi\rangle = -i\frac{\beta_0}{2p^4}, \qquad \langle b\,\varphi\rangle = \frac{i}{p^2}, \qquad \langle b_\alpha\,b_\beta\rangle = 0, \qquad \langle b_\alpha\,\varphi\rangle = 0$$

$$\langle b_\alpha\,b\rangle = 0, \qquad \langle b\,b\rangle = 0$$

For the ghosts

$$\langle\overline{\xi}_\mu,\xi_\nu\rangle = -i\frac{\sqrt{\kappa}}{p^2}\left(\theta_{\mu\nu}+\frac{1}{2}\omega_{\mu\nu}\right), \qquad \langle\overline{\omega}\,\omega\rangle = \frac{i}{p^2}$$

One should prove that propagators and vertices satisfy the Lowenstein conditions for BPHZL renormalization.

But the problem of Skylla and Charybdis does not exist if you can fly

Conformal symmetry, hopefully, makes you fly

Conformal invariance can be implemented in a classical field theory by simply adding a scalar field.

At quantum level we meet the issue of conformal (trace) anomalies. This raises two problems:

- Finding all of them
- Cancelling them

- We know that odd-parity anomalies do cancel
- Even parity anomalies are not dangerous, but if we want to preserve Weyl invariance we must cancel them.

Possible trace anomalies

The possible non-trivial cocycles of δ_ω with vanishing diffeomorphism partner in 4d are well-known, they take the form

$$\Delta[g,\omega] = \int d^4x\,\sqrt{g}\,\omega\,T[g], \qquad \delta_\omega \Delta[g,\omega] = 0$$

where the density $T[g](x)$ can be the quadratic Weyl density

$$\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$

the Gauss-Bonnet (or Euler) density,

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

and the Pontryagin density,

$$P = \frac{1}{2}\left(\varepsilon^{\mu\nu\mu'\nu'}R_{\mu\nu\lambda\rho}R_{\mu'\nu'}{}^{\lambda\rho}\right).$$

Other possible cocycles have densities

$$T_e[V] = F_{\mu\nu}F^{\mu\nu}, \qquad T_o[V] = \varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}.$$

How do we cancel even trace anomalies?

- The first mechanism (anomaly polynomial vanishes identically) is not available

- The second mechanism (vanishing overlap of various species) is extremely unlikely

- We are left with the WZ terms.

WZ terms

Any trace anomaly can be written in the form

$$\mathcal{A}_\omega[g,f] = \int d^{\mathsf{d}}x\,\sqrt{g}\,\omega\,F[g,f]$$

where $g = \{g_{\mu\nu}\}$ is the metric, ω is the Weyl transformation parameter $\delta_\omega g_{\mu\nu} = 2\omega\, g_{\mu\nu}$, f denotes any other field and F is a local function of g and f

The corresponding WZ term is defined as follows

$$\mathcal{W}_{WZ}[\sigma,g,f] = \int_0^1 dt \int d^{\mathsf{d}}x\,\sqrt{g(t)}F[g(t),f(t)]\,\sigma$$

in terms of the dimensionless field σ, with $\delta_\omega\sigma = -\omega$. Moreover $g_{\mu\nu}(t) = e^{2\sigma t}g_{\mu\nu}$ and $f(t) = e^{-y\,t\,\sigma}f$,, and $y=0$ a gauge field, $y = \frac{\mathsf{d}-2}{2}$ for a scalar field.

We have

$$\delta_\omega \mathcal{W}_{WZ}[\sigma,g,f] = -\mathcal{A}_\omega[g,f]$$

For instance the WZ term for the anomaly with density $\sim F_{\mu\nu}F^{\mu\nu}$ takes the simple form

$$\mathcal{W}_{WZ}[\sigma,g,V] \sim \int d^4x\sqrt{g}\,\sigma\,F_{\mu\nu}F^{\mu\nu}$$

There is no need to introduce new fields:

just replace σ with $-\varphi$

A possible mechanism for unitarity

One of the term with four derivatives we have added in order to guarantee renormalizability is

$$S_C = \frac{1}{\eta} \int d^4x \sqrt{g}\, C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$$

$C_{\mu\nu\lambda\rho}$ is the Weyl tensor ($C^{\mu}{}_{\nu\lambda\rho}$ is invariant under Weyl transformations). This term is conformal invariant.

One of the anomalies that appear in theories with gravity is

$$\mathcal{A}_\omega = a \int d^4x \sqrt{g}\, \omega\, C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$$

The constant a depends on the model.

The corresponding WZ term is

$$\mathcal{W}_{WZ} = a \int d^4x \sqrt{g}\, \varphi\, C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$$

In this way the theory recovers conformal invariance, that is any two values of φ are equivalent. Therefore suppose we choose $\varphi = -\frac{1}{a\,\eta}$. Then S_C and $\mathcal{W}_{WZ}$ cancel out exactly. If we can do the same with S_Q (which we can with an analogous trick), any quartic derivative terms disappear. Therefore neither the corresponding physical ghosts are there. It means that unitarity is at hand.

In other words there is a configuration of the theory in which unitarity is possible. This field configuration is equivalent to all the others. It appears the quartic derivative terms are simply an escamotage to prove renormalizability, but they do not seem to spoil unitarity.

Therefore in any classically conformal invariant theory, where Weyl invariance is broken at one-loop by trace anomalies, we can restore invariance by adding suitable WZ terms.

For our SM+Gravity theory there may be problems with unitarity: the spectrum very likely contains ghost particles (negative norm states)

Pottel-Sibold (2023-2024), Oda-Saake (2020), Oda (2022-2024)

However conformal anomalies and WZ terms might solve this problems

THANKS