

Vogel universality and beyond

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Based on joint works with S.Krivosos, R.Mkrtchyan and A.Provorov

**Mathematical Quantum Field Theory,
Conference in memory of Ivan Todorov,
Bulgarian Academy of Sciences, Sofia, May 26-30, 2026.**

Hecke algebraic properties of dynamical R -matrix Application to related quantum matrix algebra

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(Received 26 February 1998; accepted for publication 5 June 1998)

The quantum dynamical Yang–Baxter (or Gervais–Neveu–Felder) R -matrix $\hat{R}(p)$, where p stands for a set of mutually commutative parameters, provides a family of $SL(n)$ -type solutions of this equation provides a new Hecke algebra. We define quantum antisymmetrizers, introduce the determinant and compute the inverse quantum matrix for matrix type $\hat{R}(p)a_1a_2 = a_1a_2\hat{R}$. It is pointed out that such a quantum matrix in the operator realization of the chiral zero modes of the WZNW in *American Institute of Physics*. [S0022-2488(99)02801-7]

Quantum matrix algebra for the $SU(n)$ WZNW model

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Received 13 January 2003, in final form 2 April 2003

Published 7 May 2003

Online at stacks.iop.org/JPhysA/36/5497

Abstract

The zero modes of the chiral $SU(n)$ WZNW model give rise to an *intertwining quantum matrix algebra* \mathcal{A} generated by an $n \times n$ matrix $a = (a_i^j)$, $i, j = 1, \dots, n$ (with noncommuting entries) and by rational functions of n commuting elements q^{α_i} satisfying $\prod_{i=1}^n q^{\alpha_i} = 1$, $q^{\alpha_i} a_j^i = a_j^i q^{\alpha_i + \delta_j^i}$. We study a generalization of the Fock space (\mathcal{F}) representation of \mathcal{A} for generic q (q not a root of unity) and demonstrate that it gives rise to a *model* of the quantum universal enveloping algebra $U_q = U_q(\mathfrak{sl}_n)$, with each irreducible representation entering \mathcal{F} with multiplicity 1. For an integer $\tilde{u}(n)$ height h ($=k+n \geq n$) the complex parameter q is an even root of unity, $q^h = -1$, and the algebra \mathcal{A} has an ideal \mathcal{I}_k such that the factor algebra $\mathcal{A}_k = \mathcal{A}/\mathcal{I}_k$ is finite dimensional. All physical U_q modules—of shifted weights satisfying $p_{1n} = p_1 - p_n < h$ —appear in the Fock representation of \mathcal{A}_k .

1 Introduction

- Vogel parameters and Vogel map
- Achievements of the Vogel universality

2 Sets of representations appeared in the decomposition of $\mathfrak{ad}^{\otimes k}$

3 Split Casimir operator (SCO)

- Universal description of Lie algebras using the SCO technique
- Universal characteristic identities for \widehat{C} for all simple LA

4 Char. identities for 3,4 -split Casimir operators in $\mathfrak{ad}^{\otimes 3}$ and $\mathfrak{ad}^{\otimes 4}$

5 Beyond Vogel universality

- Universal dimensions for representations in the decomposition $\square \otimes Y_n$
- The case of $\square \times Y_1 = \square \times \mathfrak{ad}$

6 Conclusion

1. It was observed by P. Vogel [The Universal Lie Algebra, Preprint (1999)], in his investigation of the Vassiliev theory of knot invariants, that for all simple Lie algebras \mathfrak{g} , the square $\mathfrak{ad}^{\otimes 2}$ of the adjoint representation \mathfrak{ad} is decomposed in a uniform way (for all simple Lie algebras)

$$\mathfrak{ad}^{\otimes 2} \equiv \mathfrak{g}^{\otimes 2} = \underbrace{\mathfrak{ad} + X_2}_{\wedge(\mathfrak{g}^{\otimes 2})} + \underbrace{1 + Y_2 + Y_2' + Y_2''}_{\text{sym}(\mathfrak{g}^{\otimes 2})},$$

where $\mathfrak{ad} = \mathfrak{g} = Y_1$ and X_2, Y_2, Y_2', Y_2'' are Casimir subreps in $\mathfrak{ad}^{\otimes 2}$.

Moreover, it was shown that the dimensions of $\mathfrak{ad}, Y_2 = Y_2(\alpha), Y_2' = Y_2(\beta), Y_2'' = Y_2(\gamma)$ and X_2 are rational homogeneous "symmetric" functions of three universal Vogel parameters α, β, γ .

Even earlier P. Deligne [1996], A.M. Cohen and R. de Man [1996] consider the universality of the decomposition $\text{ad}^{\otimes k} |_{k=2,3,4}$ only for exceptional Lie algebras, and then P. Vogel [1999], J.M. Landsberg, L. Manivel [2006], M. Avetisyan, API, S. Krivonos and R. Mkrtychyan [2024] proved that decompositions of $\text{ad}^{\otimes k}$ exhibit universal behavior up to degree $k = 4$ for all simple Lie algebras. The conjecture is that, for all simple Lie algebras, all Casimir subrepresentations appeared in the decomposition of $\text{ad}^{\otimes k}$ exhibit universal behavior.

2. In the decomposition of the totally symmetric part of $\mathfrak{g}^{\otimes k}$ the Cartan k -th power $Y_k(\alpha)$ of the adjoint representations will always arise (the highest weight of Y_k is $k\lambda_{\text{ad}}$, where λ_{ad} is the highest weight of $\text{ad} = Y_1$). The universal formula for the dimensions of all representations $Y_k = Y_k(\alpha)$ is known [J.M. Landsberg, L. Manivel (2006)]

$$\dim Y_k(\alpha) = \frac{(1 - (2k - 1)\hat{\alpha})}{k! \hat{\alpha}^k (1 - (k - 1)\hat{\alpha})} \prod_{\ell=0}^{k-1} \frac{(\ell \hat{\alpha} - 1)(\hat{\beta} + \ell \hat{\alpha} - 1)(\hat{\gamma} + \ell \hat{\alpha} - 1)}{(\hat{\beta} - \ell \hat{\alpha})(\hat{\gamma} - \ell \hat{\alpha})}, \quad (1)$$

and in particular we have

$$\dim Y_1 = \dim \mathfrak{g} = \frac{(\hat{\alpha} - 1)(\hat{\beta} - 1)(\hat{\gamma} - 1)}{\hat{\alpha} \hat{\beta} \hat{\gamma}}. \quad (2)$$

Our homogeneous parameters $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are related to the standard Vogel parameters α, β, γ by

$$\hat{\alpha} = \frac{\alpha}{2t}, \quad \hat{\beta} = \frac{\beta}{2t}, \quad \hat{\gamma} = \frac{\gamma}{2t}, \quad (t := \alpha + \beta + \gamma), \quad (3)$$

so that $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 1/2$.

3. Vogel parameters and other characteristics of the simple Lie algebras

	sl_N	so_N	$sp_{N=2r}$	g_2	f_4	e_6	e_7	e_8
α	-2	-2	-2	-2	-2	-2	-2	-2
β	2	4	1	10/3	5	6	8	12
γ	N	$N-4$	$\frac{(N+4)}{2}$	8/3	6	8	12	20
$t = \frac{C_{(2)}(\text{ad})}{2}$	N	$N-2$	$\frac{(N+2)}{2}$	4	9	12	18	30
$\hat{\alpha} = \frac{\alpha}{2t}$	$-\frac{1}{N}$	$-\frac{1}{N-2}$	$-\frac{2}{N+2}$	-1/4	-1/9	-1/12	-1/18	-1/30
$\hat{\beta} = \frac{\beta}{2t}$	$\frac{1}{N}$	$\frac{2}{N-2}$	$\frac{1}{N+2}$	5/12	5/18	1/4	2/9	1/5
$\hat{\gamma} = \frac{\gamma}{2t}$	1/2	$\frac{N-4}{2(N-2)}$	$\frac{N+4}{2(N+2)}$	1/3	1/3	1/3	1/3	1/3
$\dim \square$	N	N	N	7	26	27	56	248
$\dim \mathfrak{g}$	$N^2 - 1$	$\frac{N(N-1)}{2}$	$\frac{N(N+1)}{2}$	14	52	78	133	248
$c_{(2)}^{(\square)}$	$\frac{N^2-1}{2N^2}$	$\frac{(N-1)}{2(N-2)}$	$\frac{(N+1)}{2(N+2)}$	1/2	2/3	13/18	19/24	1
$c_{(2)}^{(\square)} \cdot \dim \square$	$\frac{\dim \mathfrak{g}}{2N}$	$\frac{\dim \mathfrak{g}}{(N-2)}$	$\frac{\dim \mathfrak{g}}{(N+2)}$	$\frac{\dim \mathfrak{g}}{4}$	$\frac{\dim \mathfrak{g}}{3}$	$\frac{\dim \mathfrak{g}}{4}$	$\frac{\dim \mathfrak{g}}{3}$	$\dim \mathfrak{g}$

Here \square denotes the defining representation. For the normalization $\alpha = -2$, the value $t = \alpha + \beta + \gamma$ is equal to the dual Coxeter number h^\vee .

4. Since all universal formulas are homogeneous symmetric functions of the Vogel parameters, one can consider all simple Lie algebras as points on the 2d plane $\mathcal{P}_{(\alpha=-2)}$ in 3d space of the Vogel parameters (α, β, γ) . More precisely they are points in \mathbb{RP}^2/S_3 (the Vogel map).

Before we show this Vogel map, we note that the condition $\hat{\gamma} = 1/3 \Leftrightarrow 2t = 3\gamma$, for exceptional Lie algebras, defines the line $(\gamma + 4) = 2\beta$ on the plane

$\mathcal{P}_{(\alpha=-2)} \in \mathbb{R}^3$.

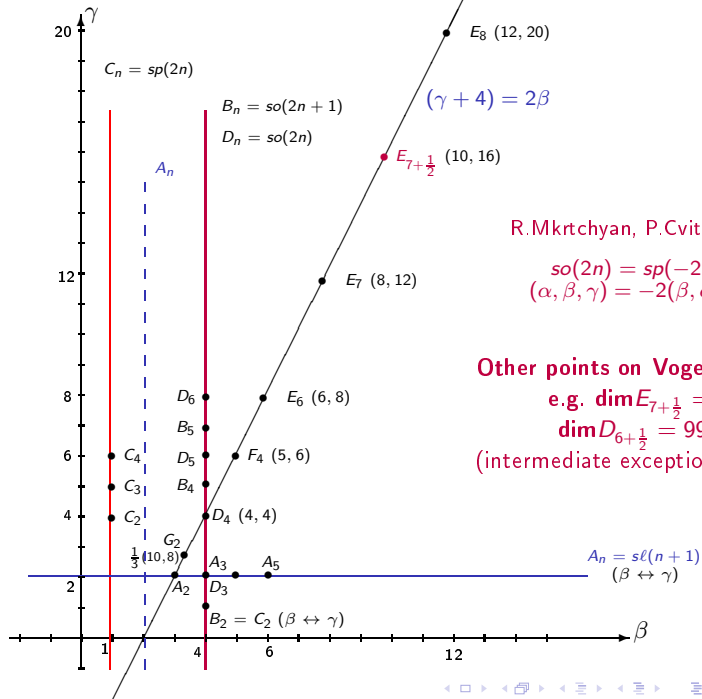
Remarkable fact is that the points of Lie algebras $sl(3)$ and $so(8)$ are also on this line.

Type	Lie algebra	α	β	γ	$t = h^\vee = \alpha + \beta + \gamma$	$\hat{\gamma} = \frac{\gamma}{2t}$
A_2	$sl(3)$	-2	2	3	3	$\frac{\beta}{2t} = 1/3$
D_4	$so(8)$	-2	4	4	6	1/3
G_2	\mathfrak{g}_2	-2	10/3	8/3	4	1/3
F_4	\mathfrak{f}_4	-2	5	6	9	1/3
E_6	\mathfrak{e}_6	-2	6	8	12	1/3
E_7	\mathfrak{e}_7	-2	8	12	18	1/3
E_8	\mathfrak{e}_8	-2	12	20	30	1/3

Unified description of all simple LA by means of 3 parameters (α, β, γ) leads to the conjecture of existing the **universal LA** and to some new approach to classify Lie algebras.

Vogel map (1999)

$$\alpha = -2$$



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$$so(2n) = sp(-2n) \Leftrightarrow (\alpha, \beta, \gamma) = -2(\beta, \alpha, \gamma)$$

Other points on Vogel map
 e.g. $\dim E_{7+\frac{1}{2}} = 190$,
 $\dim D_{6+\frac{1}{2}} = 99, \dots$
 (intermediate exceptional line)

Some other achievements of the Vogel universality.

1. The generating function of universal eigenvalues for higher Casimir operators $C_{\text{ad}}^{(k)}$ (for all simple Lie Algebras) in adjoint representations was obtained [R.Mkrtchyan, A.Sergeev, and A.Veselov, Casimir eigenvalues for universal Lie algebra, J.Math.Phys. 53.10(2012) 102106]

$$\hat{C}(z) = \left(\frac{1}{\dim(\mathfrak{g})} \left(\frac{1}{1+z} + \frac{\dim Y_2(\alpha)}{1+\frac{z\alpha}{2t}} + \frac{\dim Y_2(\beta)}{1+\frac{z\beta}{2t}} + \frac{\dim Y_2(\gamma)}{1+\frac{z\gamma}{2t}} \right) + \frac{1}{1+\frac{z}{2}} + \frac{1}{2} \dim(\mathfrak{g}) - \frac{3}{2} \right) = \sum_{k=0}^{\infty} C_{\text{ad}}^{(k)}(\alpha, \beta, \gamma) z^k .$$

2. Formula for volumes of simple compact Lie groups G [R.Mkrtchyan and A.Veselov, 2014]:

$$\text{Vol}(G) = (2^{3/2}\pi)^{\dim \mathfrak{g}} e^{-\Phi(\alpha, \beta, \gamma)} ,$$

where $\Phi(\alpha, \beta, \gamma) = \int_0^\infty dz \frac{F(z/t)}{z(e^z-1)}$.

3. R.Mkrtchyan and A.Veselov calculated the generating function for $p_k = \sum_{\mu \in \Phi(\mathfrak{g})} (\mu, \rho)^k$, where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$,

$$F(z) = \sum_{k=1} p_k \frac{z^k}{k!} = \frac{\operatorname{sh} \frac{z}{4}(\alpha - 2t) \operatorname{sh} \frac{z}{4}(\beta - 2t) \operatorname{sh} \frac{z}{4}(\gamma - 2t)}{\operatorname{sh} \frac{z}{4}\alpha \operatorname{sh} \frac{z}{4}\beta \operatorname{sh} \frac{z}{4}\gamma} - \dim \mathfrak{g} .$$

The first term in rhs is the deformation of the universal formula for $\dim \mathfrak{g}$ (it follows from $F(z)|_{z=0} = 0$).

4. Besides, in many papers [A.Mironov, R.Mkrtchyan, A.Morozov, A.Sleptsov, M.Bishler, a.o., 2016-2026] the universal formulas for some types of knot polynomials and their deformations were constructed which valid for all simple Lie algebras simultaneously.

Now the description of the sets of representations appeared in the decomposition of $\text{ad}^{\otimes k}$ in \mathfrak{sl}_N and \mathfrak{so}_N cases is in order.

Case of \mathfrak{sl}_N . It is known that any irrep T of \mathfrak{sl}_N can be numerated by a pair of the Young diagrams $\mu \vdash m, \lambda \vdash n$ and we use notation $T = (\mu, \lambda)$. The example Λ/λ is shown as

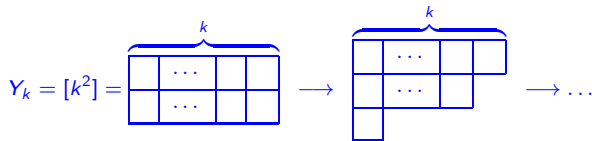
$$\Lambda/\lambda = N \left\{ \begin{array}{cccc|ccc} & & & & \mu & \mu & \mu \\ & & & & \mu & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ & & & & & & \\ & & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & & & \\ \lambda & \lambda & \lambda & \lambda & & & \end{array} \right. \leftrightarrow \begin{array}{l} \Lambda/\lambda = [7, 5, 4^{N-4}, 2] \leftrightarrow \\ (\mu, \lambda) = ([3, 1], [4, 2]) . \end{array} \quad (4)$$

The representations (μ, λ) and (λ, μ) are contravariant (dual) to each other and it means that $\dim(\mu, \lambda) = \dim(\lambda, \mu)$.

Our statement is that all irreducible subrepresentations arising in the decomposition of $\text{ad}^{\otimes k}$ are numerated by the pairs of Young diagrams (μ, λ) such that $\mu \vdash m, \lambda \vdash m$, i.e. $|\mu| = |\lambda| = m$ and $m \leq k$.

The set of subrepresentations in $\mathfrak{ad}^{\otimes k}$ in the case of \mathfrak{so}_N .

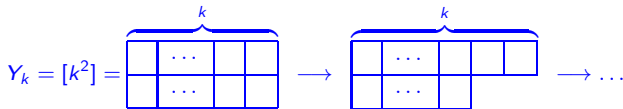
Conjecture. For the Lie algebra $\mathfrak{so}(N)$ and sufficiently large N , all irreps, which appear in the decomposition of $\mathfrak{ad}^{\otimes k}$, are associated with the Young diagrams $\lambda \vdash 2m$ with $m \leq k$ and with the number of columns not bigger than k . All these diagrams are obtained from the diagram



by allowable shifting of boxes to the left, e.g.

$$[k^2] \rightarrow [k, k-1, 1] \rightarrow [k, k-2, 2] \rightarrow [k, k-2, 1^2] \rightarrow \dots$$

and allowable¹ cancelation of pairs of boxes in one row.



¹Here the word allowable means that as a result of the cancelations we again obtain Young diagrams but with a reduced number of boxes.

Split Casimir operator [API, V.A.Rubakov, Theory of groups and symmetries: finite groups, Lie groups, and Lie algebras, WS, 2018]

Let X_a ($a = 1, \dots, \dim(\mathfrak{g})$) be basis elements of the simple Lie algebra \mathfrak{g} with defining relations $[X_a, X_b] = X_d X_{ab}^d$. The Cartan-Killing metric is

$$g_{ab} = \text{Tr}(\text{ad}X_a \cdot \text{ad}X_b) = X_{ac}^d X_{ad}^c, \quad (5)$$

and the quadratic Casimir operator is

$$c_{(2)} = g^{ab} X_a X_b \in \mathcal{U}(\mathfrak{g}), \quad (6)$$

where g^{ab} is inverse to the Cartan-Killing metric (5). The element (6) is central in $\mathcal{U}(\mathfrak{g})$ and its image in any irreducible representation T has the form

$$T(c_{(2)}) = c_{(2)}^{(T)} I_T,$$

where $c_{(2)}^{(T)}$ is the eigenvalue of the operator $c_{(2)}$ and I_T is the unit operator in the space of the irrep T . By definition we have $c_{(2)}^{(\text{ad})} = 1$. To simplify the notation, we denote $C \equiv c_{(2)} = g^{ab} X_a X_b$. The split Casimir operator is

$$\hat{C} \equiv \hat{C}_{(12)} = g^{ab} X_a \otimes X_b.$$

[I.Todorov and L.Hadjiivanov, Quantum Groups and Braid Group Statistics in Conformal Current Algebra Models (EDUFES, Vitória, 2010), CERN Preprint No. CERNPH-TH-2009-050.]

The split Casimir operator $\hat{C}_{(12)} = g^{ab} X_a \otimes X_b$ is generalized to the k -split Casimir operator

$$\hat{C}_{(1\dots k)} := \sum_{i < j}^k \hat{C}_{ij} \equiv \sum_{i < j}^k g^{ab} (X_a)_i (X_b)_j \in \mathcal{U}(\mathfrak{g})^{\otimes k}, \quad (7)$$

where

$$\hat{C}_{ij} := g^{ab} (I^{\otimes(i-1)} \otimes X_a \otimes I^{\otimes(j-i-1)} \otimes X_b \otimes I^{\otimes(k-j)}) \equiv g^{ab} (X_a)_i (X_b)_j,$$

The k -split Casimir operator (7) is obtained by the action of the comultiplication

$$\hat{C}_{(1\dots k)} = \frac{1}{2} (\Delta^{k-1}(C) - \sum_{i=1}^k C_i),$$

where $C_\ell := I^{\otimes(\ell-1)} \otimes C \otimes I^{\otimes(k-\ell)}$, and Δ is the comultiplication in $\mathcal{U}(\mathfrak{g})$:

$$\begin{aligned} \Delta(X_a) &= X_a \otimes I + I \otimes X_a, \quad \Delta(I) = I \otimes I, \\ \Delta^{k-1} &:= (\Delta \otimes \underbrace{I \otimes \dots \otimes I}_{k-2}) \cdots (\Delta \otimes I) \Delta. \end{aligned}$$

In particular ($k = 2$)

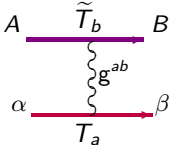
$$\Delta(C) = g^{ab} \Delta(X_a) \Delta(X_b) = C \otimes I + I \otimes C + 2 \underbrace{g^{ab} X_a \otimes X_b}$$

The split Casimir operators $\hat{C}_{1,\dots,k}$ appears in many **applications**, namely in the RT, in the theory of integrable systems, in the nonabelian gauge theories, ...

Interpretations and applications of the split Casimir operator (SCO)

1.) Colour factors in the nonabelian gauge theories

Let T and \tilde{T} be two representations of \mathfrak{g} . We visualize the image of SCO in the representation $T \otimes \tilde{T}$ as:

$$(T_\beta^\alpha \otimes \tilde{T}_B^A) \hat{C} = g^{ab} (T_a)_\beta^\alpha (\tilde{T}_b)_B^A =$$


where

$$\begin{aligned} \hat{C} &= g^{ab} X_a X_b, \\ (T_a)_\beta^\alpha &= T_\beta^\alpha(X_a), \quad (\tilde{T}_b)_B^A = \tilde{T}_B^A(X_b), \\ \alpha, \beta &= 1, \dots, \dim T, \quad A, B = 1, \dots, \dim \tilde{T}. \end{aligned}$$

Thus, SCO $g^{ab} (T_a)_\beta^\alpha (\tilde{T}_b)_B^A$ is the colour factor for the exchange by gluon between two matter particles in the representations T and \tilde{T} .

2.) Hamiltonian for non-local Heisenberg type spin chain

The k -SCO

$$\hat{C}_{(1\dots k)} := \sum_{i < j}^k \hat{C}_{ij} \equiv \sum_{i < j}^k g^{ab} (X_a)_i (X_b)_j$$

is the Hamiltonian for the generalized spin model with k sites, where every "spin" interact with spins of all other sites. Since the element $\hat{C}_{(1\dots k)}$ is ad-invariant, it commutes with the Hamiltonians of local open and closed spin chains:

$$\mathcal{H}_0^{(k)} = \sum_{i=1}^{k-1} \hat{C}_{ii+1}, \quad \mathcal{H}_1^{(k)} = \sum_{i=1}^{k-1} \hat{C}_{ii+1} + \hat{C}_{k1} \quad \Rightarrow \quad [\mathcal{H}_\alpha^{(k)}, \hat{C}_{(1\dots k)}] = 0.$$

(for $\mathfrak{g} = \mathfrak{su}_2$ they are Hamiltonians for the Heisenberg spin chains).

3) Kohno-Drinfeld Lie algebra and k -split Casimir operator

The elements $\hat{C}_{ij} := g^{ab} (X_a)_i (X_b)_j$ satisfy defining relations for Kohno-Drinfeld Lie algebra

$$[\hat{C}_{ij}, \hat{C}_{in} + \hat{C}_{jn}] = 0, \quad [\hat{C}_{ij}, \hat{C}_{n\ell}] = 0 \quad (i, j \neq n, \ell),$$

and k -split Casimir operator $\hat{C}_{(1\dots k)} := \sum_{i < j}^n \hat{C}_{ij}$ (the Hamiltonians for nonlocal spin chain) is a central element of the Kohno-Drinfeld Lie algebra.

4) Higher order Casimir operators $C^{(k)}$ (for $k > 2$) are constructed via 2-split Casimir operator $\widehat{C}_{(12)}$ [S.Okubo, J. Math. Phys. 18(1977) 2382; A.P.I. and V.A. Rubakov, Theory of Groups and Symmetries, WS (2018)]. Indeed, define

$$(\widehat{C}_{(12)})^k = X_{a_1} \cdots X_{a_k} \otimes X^{a_1} \cdots X^{a_k} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}),$$

then we take *ad*-representation in the second factor and then take trace

$$C^{(k)} = \text{Tr}_2((I \otimes \text{ad})\widehat{C}^k) = X_{a_1} \cdots X_{a_k} \underbrace{\text{Tr}(\text{ad}(X^{a_1} \cdots X^{a_k}))}_{d^{a_1 \cdots a_k}} \in \mathcal{U}(\mathfrak{g}).$$

5) Since the k -split CO $\widehat{C}_{1, \dots, k}$ of the Lie algebra \mathfrak{g} commutes with its action $\Delta^{k-1}\mathfrak{g}$, then this operator can be used **for the decomposition the tensor product of k irreducible representations** $T^{(\Lambda_i)}$ of the algebra \mathfrak{g} :

$$T^{(\Lambda_1)} \otimes \cdots \otimes T^{(\Lambda_k)} = \sum_{\Lambda} T^{(\Lambda)},$$

where Λ, Λ_i – are the highest weights of the representations.

To decompose the tensor product of k irreducible representations $T^{(\Lambda_i)}$ of the algebra \mathfrak{g} :

$$T^{(\Lambda_1)} \otimes \dots \otimes T^{(\Lambda_k)} = \sum_{\Lambda} T^{(\Lambda)} \quad (8)$$

(Λ, Λ_i – are the highest weights of the representations) with the help of SCO, we deduce a formula for the eigenvalues $\hat{c}_{(1\dots k)}^{(\Lambda)}$ of the k -split CO

$\hat{C}_{\Lambda_1, \dots, \Lambda_k} = (T^{(\Lambda_1)} \otimes \dots \otimes T^{(\Lambda_k)}) \hat{C}_{(1\dots k)}$ which acts on the subspace $V_{\Lambda} \subset V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_k}$ of the irrep Λ :

$$\hat{c}_{(1\dots k)}^{(\Lambda)} = \frac{1}{2} \left(c_{(2)}^{(\Lambda)} - \sum_{i=1}^k c_{(2)}^{(\Lambda_i)} \right). \quad (9)$$

Here $c_{(2)}^{(\Lambda)}, c_{(2)}^{(\Lambda_i)}$ are eigenvalues of the quadratic Casimir operator (6) on the spaces $V_{\Lambda}, V_{\Lambda_i}$ of irreps with the highest weights Λ, Λ_i . In the representation of \mathfrak{g} with the highest weight Λ the quadratic Casimir operator C has the eigenvalue

$$C_{(2)}(\Lambda) = (\Lambda, \Lambda + 2\rho), \quad (10)$$

where (\cdot, \cdot) is a scalar product in the root space of \mathfrak{g} and ρ is the Weyl vector for the simple Lie algebra \mathfrak{g} of the rank r : $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{f=1}^r \lambda(f)$.

Note that the eigenvalues $\hat{c}_{(1\dots k)}^{(\Lambda)}$ can take the same value for different irreducible representations $T^{(\Lambda)}$ in the decomposition

$$T^{(\Lambda_1)} \otimes \dots \otimes T^{(\Lambda_k)} = \sum_{\Lambda} T^{(\Lambda)} .$$

We denote by a_i the eigenvalues $\hat{c}_{1\dots k}^{(\Lambda)}$ that take mismatched values (the index $i = 1, 2, \dots, M$ numerates these eigenvalues). Then the k -split Casimir operator (7) in the representation $T^{(\Lambda_1)} \otimes \dots \otimes T^{(\Lambda_k)}$ satisfies the characteristic identity

$$\prod_{i=1}^M (\hat{C}_{\Lambda_1, \dots, \Lambda_k} - a_i) = 0 , \quad (11)$$

which leads to the definition of projectors

$$P_{a_j} = \prod_{i \neq j} \frac{(\hat{C}_{\Lambda_1, \dots, \Lambda_k} - a_i)}{(a_j - a_i)} . \quad (12)$$

The images of these projectors are invariant subspaces $V_{a_j} \subset V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_k}$ with dimensions

$$\dim V_{a_j} = \text{Tr}_{1,2,\dots,k}(P_{a_j}) . \quad (13)$$

As we mentioned above, the invariant subspaces V_{a_j} can be reducible since the quadratic Casimir operator can take the same eigenvalue $\hat{c}_{1\dots k}^{(\Lambda)}$ in different irreducible representations $T^{(\Lambda)}$. When we consider the decomposition of $\mathfrak{ad}^{\otimes k}$, the eigenvalues $a_j = \hat{c}_{1\dots k}^{(\Lambda)}$ are determined only by the value of the quadratic Casimir operator in the representations $T^{(\Lambda)}$:

$$c_{(2)}^{(\Lambda)} := \frac{C_{(2)}(\Lambda)}{C_{(2)}(\Lambda_{(\text{ad})}} ,$$

In this case we call the direct sum of all representations $T^{(\Lambda)}$ with the same $\hat{c}_{1\dots k}^{(\Lambda)} = a_j$, that act in the subspace V_{a_j} , as *Casimir representation* (associated with the eigenvalue a_j), and call the spaces V_{a_j} ($j = 1, 2, \dots, M$) as *Casimir subspaces*.

Our method of constructing the universal formulas for simple Lie algebras \mathfrak{g} is based on the possibility to write eigenvalues $a_j = \hat{c}_{1\dots k}^{(\Lambda)}$ in the universal form $a_j(\alpha, \beta, \gamma)$ in terms of the Vogel parameters. In all known examples the eigenvalues $a_j = \hat{c}_{1\dots k}^{(\Lambda)}$ are linear functions of $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$. When considering the expansion of $\mathfrak{g}^{\otimes k}$, it is sufficient to find only universal expressions for the eigenvalues of the quadratic Casimir operators. The set of all representations $T^{(\Lambda)}$ (for all simple Lie algebras \mathfrak{g}) associated with the same universal eigenvalue $a_j(\alpha, \beta, \gamma) = \hat{c}_{1\dots k}^{(\Lambda)}$ is called the *universal multiplet*.

The split Casimir operators and universality in $\mathfrak{ad}^{\otimes k}$

Our method of universal description special representations of simple Lie algebras is based on the extracting of invariant Casimir subspaces in $V_{\mathfrak{ad}}^{\otimes k}$ by means of the projectors which constructed with the help of char. ident. for k -split CO

$$\begin{aligned}(\widehat{C}_{\mathfrak{ad}})_{b_1 b_2 \dots b_k}^{a_1 a_2 \dots a_k} &\equiv (\mathfrak{ad} \otimes \dots \otimes \mathfrak{ad})_{b_1 b_2 \dots b_k}^{a_1 a_2 \dots a_k} \left(\sum_{i < j}^k (X^h)_i (X^h)_j \right) = \\ &= (X^h)_{b_i}^{a_i} (X^h)_{b_j}^{a_j} \delta_{b_1 \dots b_i \dots b_j \dots b_k}^{a_1 \dots a_i \dots a_j \dots a_k}, \quad \delta_{b_1 \dots b_k}^{a_1 \dots a_k} := \delta_{b_1}^{a_1} \dots \delta_{b_k}^{a_k}.\end{aligned}$$

This operator acts in the space $V_{\mathfrak{ad}}^{\otimes k}$, where $V_{\mathfrak{ad}} \simeq \mathfrak{ad} \simeq \mathfrak{g}$ is the space of ad-representation. Our aim is to find char. identity

$$\prod_{i=1}^r (\widehat{C}_{\mathfrak{ad}} - a_i) = 0,$$

for k -split CO $\widehat{C}_{\mathfrak{ad}}$ in the adjoint representation. Then we find r projectors in $\mathfrak{ad}^{\otimes k}$ onto invariant Casimir subspaces of $\widehat{C}_{\mathfrak{ad}}$ with eigenvalues a_j ($j = 1, \dots, r$):

$$P_{(a_j)} = \prod_{i \neq j} \frac{(\widehat{C}_{\mathfrak{ad}} - a_i)}{a_j - a_i} \quad \Rightarrow \quad V_{\mathfrak{ad}}^{\otimes k} = \sum_j P_{(a_j)} \cdot (V_{\mathfrak{ad}}^{\otimes k}).$$

As an example we consider the universal decomposition of $\text{ad}^{\otimes 2}$.

Proposition 1. For all LA of the classical series $A_n = \mathfrak{sl}_{n+1}$, $B_n = \mathfrak{so}_{2n+1}$, $C_n = \mathfrak{sp}_{2n}$, $D_n = \mathfrak{so}_{2n}$ (except \mathfrak{sl}_3 and \mathfrak{so}_8), in ad-representation, \widehat{C} has the universal char identity

$$\widehat{C}(\widehat{C} + \frac{1}{2})(\widehat{C} + 1)(\widehat{C} + \frac{\alpha}{2t})(\widehat{C} + \frac{\beta}{2t})(\widehat{C} + \frac{\gamma}{2t}) = 0, \quad \#6 \quad (14)$$

where $(\frac{\alpha}{2t} + \frac{\beta}{2t} + \frac{\gamma}{2t}) = 1/2 \Rightarrow (t = \alpha + \beta + \gamma)$. The values of the Vogel parameters α, β, γ for LA of the classical series were given in Table 1 before.

Since the identity (14) is of 6-th order, we have 6 projectors $P_{(a_j)}$ ($j = 1, 2, \dots, 6$) which correspond to the eigenvalues

$$a_1 = 0, \quad a_2 = -\frac{1}{2}, \quad a_3 = -1, \quad a_4 = -\frac{\alpha}{2t}, \quad a_5 = -\frac{\beta}{2t}, \quad a_6 = -\frac{\gamma}{2t}.$$

Thus we have the decomposition of $\text{ad}^{\otimes 2}$ on the Casimir subrepresentations

$$\text{ad}^{\otimes 2} = (X_1 + X_2) + (X_0 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma)).$$

where representations of \mathfrak{g} in the subspaces $V_{(0)}$, $V_{(-\frac{1}{2})}$, $V_{(-1)}$, $V_{(-\frac{\alpha}{2t})}$, $V_{(-\frac{\beta}{2t})}$, $V_{(-\frac{\gamma}{2t})}$ were respectively denoted by Vogel as

$$X_1 = Y_1 = \text{ad}, \quad X_2, \quad X_0 = \mathbf{1}, \quad Y_2(\alpha), \quad Y_2(\beta), \quad Y_2(\gamma)$$

Then we calculate the Vogel universal dimensions of these representations (and dimensions of invariant eigenspaces $V_{(a_i)}$) by means of trace formulas:

$$\begin{aligned} \dim X_1 &= \dim V_{(0)} = \mathbf{Tr}(P_{(0)}) = \dim \mathfrak{g}, \\ \dim X_2 &= \dim V_{(-\frac{1}{2})} = \mathbf{Tr}(P_{(-\frac{1}{2})}) = \frac{1}{2} \dim \mathfrak{g}(\dim \mathfrak{g} - 3), \\ \dim X_0 &= \dim V_{(-1)} = \mathbf{Tr} P_{(-1)}^{(+)} = 1, \\ \dim Y_2(\alpha) &\equiv \dim V_{(-\frac{\alpha}{2t})} = \mathbf{Tr} P_{(-\frac{\alpha}{2t})}^{(+)} = -\frac{(3\alpha-2t)(\beta-2t)(\gamma-2t)t(\beta+t)(\gamma+t)}{\alpha^2(\alpha-\beta)\beta(\alpha-\gamma)\gamma}, \\ \dim Y_2(\beta) &\equiv \dim V_{(-\frac{\beta}{2t})} = \mathbf{Tr} P_{(-\frac{\beta}{2t})}^{(+)} = -\frac{(3\beta-2t)(\alpha-2t)(\gamma-2t)t(\alpha+t)(\gamma+t)}{\beta^2(\beta-\alpha)\alpha(\beta-\gamma)\gamma}, \\ \dim Y_2(\gamma) &\equiv \dim V_{(-\frac{\gamma}{2t})} = \mathbf{Tr} P_{(-\frac{\gamma}{2t})}^{(+)} = -\frac{(3\gamma-2t)(\beta-2t)(\alpha-2t)t(\beta+t)(\alpha+t)}{\gamma^2(\gamma-\beta)\beta(\gamma-\alpha)\alpha}. \end{aligned}$$

Thus, **Proposition 1** justify the Vogel statement for LA of classical series.

Theorem. (P.Vogel, 1999)

$$\mathfrak{ad}^{\otimes 2} \equiv \mathfrak{g}^{\otimes 2} = \underbrace{\mathfrak{ad} + X_2}_{\wedge(\mathfrak{g}^{\otimes 2})} + \underbrace{1 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma)}_{\text{sym}(\mathfrak{g}^{\otimes 2})},$$

Universal char identities for \widehat{C} for exceptional Lie algebras in $\text{ad}^{\otimes 2}$.

The antisymmetric \widehat{C}_- and symmetric \widehat{C}_+ parts of the split Casimir operators in the ad-representation for all exceptional Lie algebras \mathfrak{g} obey the universal identities

$$\boxed{\widehat{C}_- \left(\widehat{C}_- + \frac{1}{2} \right) = 0}, \quad \boxed{(\widehat{C}_+ + 1) \left(\widehat{C}_+^2 + \frac{1}{6} \widehat{C}_+ - 2\mu \right) \mathbf{P}_+^{(\text{ad})} = 0}, \quad \#3, \quad (15)$$

where the universal parameter μ is fixed as follows

$$\mu = \frac{5}{6(2 + \dim(\mathfrak{g}))}, \quad (16)$$

and for algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ with dimensions 14, 52, 78, 133, 248 we have respectively $\mu = \frac{5}{96}, \frac{5}{324}, \frac{1}{96}, \frac{1}{162}, \frac{1}{300}$.

Moreover the char identities for \widehat{C}_+ for algebras \mathfrak{sl}_3 and \mathfrak{so}_8 have the same structure (15) with $\mu = \frac{1}{12}$ and $\mu = \frac{1}{36}$.

From (15) we obtain the factorized form of the universal char. identity for \widehat{C}_+

$$(\widehat{C}_+ + 1)(\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu)\mathbf{P}_+^{(\text{ad})} \equiv (\widehat{C}_+ + 1)(\widehat{C}_+ + \frac{\alpha}{2t})(\widehat{C}_+ + \frac{\beta}{2t})\mathbf{P}_+^{(\text{ad})} = 0, \quad (17)$$

where we introduced the notation for two roots of eq. $\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu = 0$:

$$\frac{\alpha}{2t} = \frac{1 - \mu'}{12}, \quad \frac{\beta}{2t} = \frac{1 + \mu'}{12}, \quad \mu' := \sqrt{1 + 288\mu} = \sqrt{\frac{\dim \mathfrak{g} + 242}{\dim \mathfrak{g} + 2}}. \quad (18)$$

These roots are related by $3(\alpha + \beta) = t$, and for $\alpha = -2$ this relation defines the line of the exceptional LA on the Vogel (β, γ) plane (as we discussed above). We note that μ' is a rational number (since $\frac{\alpha}{2t}$ and $\frac{\beta}{2t}$ are rational) only for certain finite sequence of $\dim \mathfrak{g}$:

$$\dim \mathfrak{g} = 3, 8, 14, 28, 47, 52, 78, 96, 119, 133, 190, 248, 287, 336, 484, 603, 782, 1081, 1680, 3479, \quad (19)$$

which includes the dimensions 14, 52, 78, 133, 248 of the exceptional Lie algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, and the dimensions 8 and 28 of $\mathfrak{sl}(3)$ and $\mathfrak{so}(8)$, which are sometimes also referred to as exceptional. Dim. 190 corresponds to $\mathfrak{e}_{7+\frac{1}{2}}$.

Remark. The sequence (19) contains $\dim \mathfrak{g}^* = (10m - 122 + 360/m)$, ($m \in \mathbb{N}$) referring to the adjoint representations of the so-called E_8 family of algebras \mathfrak{g}^* ; see the Cvitanović book. For such dimensions we have relation $\mu' = |(m + 6)/(m - 6)|$. Two numbers 47 and 119 from sequence (19) do not belong to the sequence $\dim \mathfrak{g}^*$. Thus, the interpretation of these two numbers as the dimensions of some algebras is missing. Moreover, for values $\dim \mathfrak{g}$ given in (19), using (18), one can calculate dimensions of the corresponding representations $Y(\alpha)$:

$$\dim V_{(-\frac{\alpha}{2t})} = \left\{ 5, 27, 77, 300, \frac{14553}{17}, 1053, 2430, \frac{48608}{13}, \frac{111078}{19}, 7371, 15504, 27000, \frac{841279}{23}, \frac{862407}{17}, 107892, \frac{2205225}{13}, \frac{578151}{2}, 559911, \frac{42507504}{31}, \frac{363823677}{61} \right\}$$

Since $\dim V_{(-\frac{\alpha}{2t})}$ should be integer, we conclude that there not exist Lie algebras with dimensions 47, 96, 119, 287, 336, 603, 782, 1680, 3479, for which we assume characteristic identity (17) and the trace formulas.

Universal formulas for 3-split Casimir operator in $\mathfrak{ad}^{\otimes 3}$

The matrix $\widehat{C}_{b_1 b_2 b_3}^{a_1 a_2 a_3} := (\widehat{C}_{(3)})_{b_1 b_2 b_3}^{a_1 a_2 a_3}$ of the 3-split Casimir operator is

$$(\widehat{C}_{(3)})_{b_1 b_2 b_3}^{a_1 a_2 a_3} = (\widehat{C}_{12} + \widehat{C}_{13} + \widehat{C}_{23})_{b_1 b_2 b_3}^{a_1 a_2 a_3}, \quad (20)$$

and acts in the space $V_{\mathfrak{ad}}^{\otimes 3}$ of the representation $\mathfrak{ad}^{\otimes 3}$.

$$\left(\widehat{C}_{(3)}\right)_{b_1 b_2 b_3}^{a_1 a_2 a_3} = \begin{array}{c} a_3 \text{ --- } b_3 \\ \text{wavy line} \\ a_2 \text{ --- } b_2 \\ \text{wavy line} \\ a_1 \text{ --- } b_1 \\ \text{wavy line} \end{array} + \begin{array}{c} a_3 \text{ --- } b_3 \\ \text{wavy line} \\ a_2 \text{ --- } b_2 \\ \text{wavy line} \\ a_1 \text{ --- } b_1 \\ \text{wavy line} \end{array} + \begin{array}{c} a_3 \text{ --- } b_3 \\ \text{wavy line} \\ a_2 \text{ --- } b_2 \\ \text{wavy line} \\ a_1 \text{ --- } b_1 \\ \text{wavy line} \end{array}$$

According to

$$\mathfrak{ad}^{\otimes 3} = (P_{[3]} + P_{[2,1]} + P_{[1^3]}) \mathfrak{ad}^{\otimes 3},$$

we have decomposition

$$\widehat{C}_{(3)} = (P_{[3]} + P_{[2,1]} + P_{[1^3]}) \widehat{C}_{(3)} = \widehat{C}_{[3]} + \widehat{C}_{[2,1]} + \widehat{C}_{[1^3]}.$$

All calculations were done with "Mathematica".

Proposition 3. For 3-split Casimirs $\widehat{C}_{[1^3]}$, $\widehat{C}_{[3]}$ and $\widehat{C}_{[2,1]}$ we have the universal char. identities

$$\widehat{C}_{[1^3]} \left(\widehat{C}_{[1^3]} + \frac{1}{2} \right) \left(\widehat{C}_{[1^3]} + \frac{3}{2} \right) \left(\widehat{C}_{[1^3]} + \frac{1}{2} + \hat{\alpha} \right) \left(\widehat{C}_{[1^3]} + \frac{1}{2} + \hat{\beta} \right) \left(\widehat{C}_{[1^3]} + \frac{1}{2} + \hat{\gamma} \right) = 0, \quad \#6$$

$$\begin{aligned} & \left(\widehat{C}_{[3]} + \frac{1}{2} \right) \left(\widehat{C}_{[3]} + 1 \right) \left(\widehat{C}_{[3]} + \frac{1}{2} - \hat{\alpha} \right) \left(\widehat{C}_{[3]} + \frac{1}{2} - \hat{\beta} \right) \left(\widehat{C}_{[3]} + \frac{1}{2} - \hat{\gamma} \right) \times \\ & \left(\widehat{C}_{[3]} + 3\hat{\alpha} \right) \left(\widehat{C}_{[3]} + 3\hat{\beta} \right) \left(\widehat{C}_{[3]} + 3\hat{\gamma} \right) P_{[3]} = 0, \quad \#8 \end{aligned} \quad (21)$$

$$\begin{aligned} & \left(\widehat{C}_{[2,1]} + \frac{1}{2} \right) \left(\widehat{C}_{[2,1]} + 1 \right) \left(\widehat{C}_{[2,1]} + \frac{1}{2} - \hat{\alpha} \right) \left(\widehat{C}_{[2,1]} + \frac{1}{2} - \hat{\beta} \right) \left(\widehat{C}_{[2,1]} + \frac{1}{2} - \hat{\gamma} \right) \times \\ & \left(\widehat{C}_{[2,1]} + \frac{1}{2} + \hat{\alpha} \right) \left(\widehat{C}_{[2,1]} + \frac{1}{2} + \hat{\beta} \right) \left(\widehat{C}_{[2,1]} + \frac{1}{2} + \hat{\gamma} \right) \times \\ & \left(\widehat{C}_{[2,1]} + \frac{3}{2}\hat{\alpha} \right) \left(\widehat{C}_{[2,1]} + \frac{3}{2}\hat{\beta} \right) \left(\widehat{C}_{[2,1]} + \frac{3}{2}\hat{\gamma} \right) P_{[2,1]'} = 0, \quad \#11. \end{aligned}$$

where $\hat{\alpha} = \frac{\alpha}{2t}$, $\hat{\beta} = \frac{\beta}{2t}$, $\hat{\gamma} = \frac{\gamma}{2t}$. All formulas in (21) are homogeneous and symmetric under permutations (α, β, γ) . Our results are in agreement with [P.Vogel (1999), A.M. Cohen and R. de Man (1996)].

The dimensions of irreps corresponding to the eigenvalues of $\widehat{C}_{[2,1]}$

[A.P. Isaev, S.O. Krivonos, A.A. Provorov,

Int.J.Mod.Phys.A 38 (2023) 06n07, 2350037; e-Print: 2212.14761 [math-ph]]

$$\begin{aligned}
 \dim_{-\frac{1}{2}} &= 2X_2 = && 2 \times \frac{1}{2} \dim(g) (\dim(g) - 3), \\
 \dim_{-1} &= 2X_1 = && 2 \times \dim(g), \\
 \dim_{\hat{\alpha}-\frac{1}{2}} &= B = && \frac{(\hat{\alpha}-1)(\hat{\beta}-1)(\hat{\gamma}-1)(2\hat{\alpha}+\hat{\beta})(2\hat{\alpha}+\hat{\gamma})(2\hat{\beta}+1)(2\hat{\gamma}+1)(3\hat{\beta}-1)(3\hat{\gamma}-1)}{8\hat{\alpha}^2(\hat{\alpha}-\hat{\beta})(\hat{\alpha}-\hat{\gamma})(2\hat{\beta}-\hat{\gamma})(2\hat{\gamma}-\hat{\beta})\hat{\beta}^2\hat{\gamma}^2}, \\
 \dim_{\hat{\beta}-\frac{1}{2}} &= B' = && \frac{(\hat{\beta}-1)(\hat{\gamma}-1)(\hat{\alpha}-1)(2\hat{\beta}+\hat{\gamma})(2\hat{\beta}+\hat{\alpha})(2\hat{\gamma}+1)(2\hat{\alpha}+1)(3\hat{\gamma}-1)(3\hat{\alpha}-1)}{8\hat{\beta}^2(\hat{\beta}-\hat{\gamma})(\hat{\beta}-\hat{\alpha})(2\hat{\gamma}-\hat{\alpha})(2\hat{\alpha}-\hat{\gamma})\hat{\gamma}^2\hat{\alpha}^2}, \\
 \dim_{\hat{\gamma}-\frac{1}{2}} &= B'' = && \frac{(\hat{\gamma}-1)(\hat{\alpha}-1)(\hat{\beta}-1)(2\hat{\gamma}+\hat{\alpha})(2\hat{\gamma}+\hat{\beta})(2\hat{\alpha}+1)(2\hat{\beta}+1)(3\hat{\alpha}-1)(3\hat{\beta}-1)}{8\hat{\gamma}^2(\hat{\gamma}-\hat{\alpha})(\hat{\gamma}-\hat{\beta})(2\hat{\alpha}-\hat{\beta})(2\hat{\beta}-\hat{\alpha})\hat{\alpha}^2\hat{\beta}^2}, \\
 \dim_{-\hat{\alpha}-\frac{1}{2}} &= Y_2 = && -\frac{(3\hat{\alpha}-1)(\hat{\beta}-1)(\hat{\gamma}-1)(2\hat{\beta}+1)(2\hat{\gamma}+1)}{8\hat{\alpha}^2(\hat{\alpha}-\hat{\beta})(\hat{\alpha}-\hat{\gamma})\hat{\beta}\hat{\gamma}}, \\
 \dim_{-\hat{\beta}-\frac{1}{2}} &= Y_2' = && -\frac{(3\hat{\beta}-1)(\hat{\gamma}-1)(\hat{\alpha}-1)(2\hat{\gamma}+1)(2\hat{\alpha}+1)}{8\hat{\beta}^2(\hat{\beta}-\hat{\gamma})(\hat{\beta}-\hat{\alpha})\hat{\gamma}\hat{\alpha}}, \\
 \dim_{-\hat{\gamma}-\frac{1}{2}} &= Y_2'' = && -\frac{(3\hat{\gamma}-1)(\hat{\alpha}-1)(\hat{\beta}-1)(2\hat{\alpha}+1)(2\hat{\beta}+1)}{8\hat{\gamma}^2(\hat{\gamma}-\hat{\alpha})(\hat{\gamma}-\hat{\beta})\hat{\alpha}\hat{\beta}}, \\
 \dim_{-\frac{3}{2}\hat{\alpha}} &= C = && -\frac{2}{3} \frac{(1+2\hat{\alpha})(1+2\hat{\beta})(1+2\hat{\gamma})(1-\hat{\beta})(1-\hat{\gamma})(\hat{\beta}+\hat{\gamma})(2\hat{\beta}+\hat{\gamma})(2\hat{\gamma}+\hat{\beta})}{\hat{\alpha}^3\hat{\beta}\hat{\gamma}(\hat{\alpha}-2\hat{\beta})(\hat{\alpha}-2\hat{\gamma})(\hat{\alpha}-\hat{\beta})(\hat{\alpha}-\hat{\gamma})}, \\
 \dim_{-\frac{3}{2}\hat{\beta}} &= C' = && -\frac{2}{3} \frac{(1+2\hat{\beta})(1+2\hat{\gamma})(1+2\hat{\alpha})(1-\hat{\gamma})(1-\hat{\alpha})(\hat{\gamma}+\hat{\alpha})(2\hat{\gamma}+\hat{\alpha})(2\hat{\alpha}+\hat{\gamma})}{\hat{\beta}^3\hat{\gamma}\hat{\alpha}(\hat{\beta}-2\hat{\gamma})(\hat{\beta}-2\hat{\alpha})(\hat{\beta}-\hat{\gamma})(\hat{\beta}-\hat{\alpha})}, \\
 \dim_{-\frac{3}{2}\hat{\gamma}} &= C'' = && -\frac{2}{3} \frac{(1+2\hat{\gamma})(1+2\hat{\alpha})(1+2\hat{\beta})(1-\hat{\alpha})(1-\hat{\beta})(\hat{\alpha}+\hat{\beta})(2\hat{\alpha}+\hat{\beta})(2\hat{\beta}+\hat{\alpha})}{\hat{\gamma}^3\hat{\alpha}\hat{\beta}(\hat{\gamma}-2\hat{\alpha})(\hat{\gamma}-2\hat{\beta})(\hat{\gamma}-\hat{\alpha})(\hat{\gamma}-\hat{\beta})}.
 \end{aligned}$$

All calculations were done with "Mathematica".

Beyond Vogel universality

Consider the split Casimir operator $\hat{C}_{\square \otimes Y_n}$ in the representations $\square \otimes Y_n$ where \square and Y_n – defining and n -th Cartan power of adjoint representations of Lie algebras \mathfrak{g} .

The unified characteristic identity that valid for all simple Lie algebras (except the \mathfrak{e}_8 algebra) is

$$\left(\hat{C}_{\square \otimes Y_n} + \frac{1}{2}(1 + \alpha(1 - n))\right) \left(\hat{C}_{\square \otimes Y_n} + n\frac{\hat{\alpha}}{2}\right) \left(\hat{C}_{\square \otimes Y_n} + \frac{\hat{\beta}}{2}\right) \left(\hat{C}_{\square \otimes Y_n} + \frac{\hat{\gamma}}{2}\right) = 0. \quad (22)$$

The characteristic identity (22) yields four projectors which are written as

$$\begin{aligned}
 P_{a_1}^{(n)} &= \frac{(2 \hat{C}_{\square \otimes Y_n} + n \hat{\alpha})(2 \hat{C}_{\square \otimes Y_n} + \hat{\beta})(2 \hat{C}_{\square \otimes Y_n} + \hat{\gamma})}{(2n \hat{\alpha} - \hat{\alpha} - 1)(\hat{\alpha}n + \hat{\beta} - \hat{\alpha} - 1)(\hat{\alpha}n + \hat{\gamma} - \hat{\alpha} - 1)}, \\
 P_{a_2}^{(n)} &= \frac{\left(\hat{C}_{\square \otimes Y_n} + \frac{1}{2} + \frac{\hat{\alpha}}{2}(1-n)\right)(2 \hat{C}_{\square \otimes Y_n} + \hat{\beta})(2 \hat{C}_{\square \otimes Y_n} + \hat{\gamma})}{\left(\frac{1}{2} + \frac{\hat{\alpha}}{2} - n \hat{\alpha}\right)(\hat{\beta} - n \hat{\alpha})(\hat{\gamma} - n \hat{\alpha})}, \\
 P_{a_3}^{(n)} &= \frac{\left(\hat{C}_{\square \otimes Y_n} + \frac{1}{2} + \frac{\hat{\alpha}}{2}(1-n)\right)(2 \hat{C}_{\square \otimes Y_n} + n \hat{\alpha})(2 \hat{C}_{\square \otimes Y_n} + \hat{\gamma})}{\left(-\frac{\hat{\beta}}{2} + \frac{1}{2} + \frac{\hat{\alpha}}{2}(1-n)\right)(\hat{\beta} - n \hat{\alpha})(\hat{\beta} - \hat{\gamma})}, \\
 P_{a_4}^{(n)} &= \frac{\left(\hat{C}_{\square \otimes Y_n} + \frac{1}{2} + \frac{\hat{\alpha}}{2}(1-n)\right)(2 \hat{C}_{\square \otimes Y_n} + n \hat{\alpha})(2 \hat{C}_{\square \otimes Y_n} + \hat{\beta})}{\left(-\frac{\hat{\gamma}}{2} + \frac{1}{2} + \frac{\hat{\alpha}}{2}(1-n)\right)(\hat{\gamma} - n \hat{\alpha})(\hat{\gamma} - \hat{\beta})},
 \end{aligned} \tag{23}$$

where the eigenvalues are

$$a_1 = -\frac{1}{2}(1 + \hat{\alpha}(1-n)), \quad a_2 = -n \frac{\hat{\alpha}}{2}, \quad a_3 = -\frac{\hat{\beta}}{2}, \quad a_4 = -\frac{\hat{\gamma}}{2}, \tag{24}$$

and we have the universal decomposition

$$\square \otimes Y_n = \Lambda_1^{(n)} + \Lambda_2^{(n)} + \Lambda_3^{(n)} + \Lambda_4^{(n)}. \tag{25}$$

$$\dim V_{a_1} = \frac{\left(4\hat{\alpha}(\hat{\alpha}n - \hat{\alpha} - 1)(n - 1)c_{(2)}^{(\square)} - \hat{\alpha}\hat{\beta}\hat{\gamma} \dim \mathfrak{g}\right) n \dim(Y_n) \dim(\square)}{(\hat{\alpha}n - \hat{\alpha} + \hat{\gamma} - 1)(\hat{\alpha}n - \hat{\alpha} + \hat{\beta} - 1)(1 + \hat{\alpha} - 2\hat{\alpha}n) \dim \mathfrak{g}}, \quad (26)$$

$$\dim V_{a_2} = \frac{(1 - \hat{\alpha}n + \hat{\alpha})\left(4n(1 - \hat{\alpha}n)c_{(2)}^{(\square)} + \hat{\beta}\hat{\gamma} \dim \mathfrak{g}\right) \dim(Y_n) \dim(\square)}{(\hat{\gamma} - \hat{\alpha}n)(\hat{\beta} - \hat{\alpha}n)(1 + \hat{\alpha} - 2\hat{\alpha}n) \dim \mathfrak{g}}, \quad (27)$$

$$\dim V_{a_3} = \frac{(1 - \hat{\alpha}n + \hat{\alpha})\left(4(1 - \hat{\beta})c_{(2)}^{(\square)} + \hat{\alpha}\hat{\gamma} \dim \mathfrak{g}\right) n \dim(Y_n) \dim(\square)}{(\hat{\gamma} - \hat{\beta})(\hat{\alpha}n - \hat{\beta})(1 + \hat{\alpha} - \hat{\alpha}n - \hat{\beta}) \dim \mathfrak{g}}, \quad (28)$$

$$\dim V_{a_4} = \frac{(1 - \hat{\alpha}n + \hat{\alpha})\left(4(1 - \hat{\gamma})c_{(2)}^{(\square)} + \hat{\alpha}\hat{\beta} \dim \mathfrak{g}\right) n \dim(Y_n) \dim(\square)}{(\hat{\beta} - \hat{\gamma})(\hat{\alpha}n - \hat{\gamma})(1 + \hat{\alpha} - \hat{\alpha}n - \hat{\gamma}) \dim \mathfrak{g}}, \quad (29)$$

Expressions for $\dim V_{a_3}$ and $\dim V_{a_4}$ are related to each other by the exchange $\hat{\beta} \leftrightarrow \hat{\gamma}$. One can directly check that we have $\dim V_{a_4} = 0$ and $\dim V_{a_3} = 0$ respectively for the Lie algebras of the classical and exceptional series of the simple Lie algebras.

This defines the universal expressions for the Casimir operator values $c_{(2)}^{(\square)}$ in the case of the Lie algebras of the classical and exceptional series:

$$c_{(2)}^{(\square)} = \frac{\hat{\alpha}\hat{\beta} \dim \mathfrak{g}}{4(\hat{\gamma} - 1)} \quad (\text{for } \mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}), \quad c_{(2)}^{(\square)} = \frac{\hat{\alpha}\hat{\gamma} \dim \mathfrak{g}}{4(\hat{\beta} - 1)} \quad (\text{for } \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7), \quad (30)$$

We note that the first and second formula in (30) are related by the permutation $\hat{\beta} \leftrightarrow \hat{\gamma}$. We can permute the parameters $\hat{\beta} \leftrightarrow \hat{\gamma}$ for the Lie algebras of the classical series and leave the parameters $\hat{\beta}, \hat{\gamma}$ for exceptional algebras untouched. In order not to confuse the new choice of parameters $\hat{\beta}$ and $\hat{\gamma}$ with the old one, we introduce the notation $\hat{\beta}' = \hat{\gamma}, \hat{\gamma}' = \hat{\beta}$ for the Lie algebras of the classical series and $\hat{\beta}' = \hat{\beta}, \hat{\gamma}' = \hat{\gamma}$ for the exceptional Lie algebras. In this case, we have $\hat{\gamma}' = 1/3$ for the Lie algebras \mathfrak{sl}_3 and \mathfrak{so}_8 as for all exceptional Lie algebras, and the second formula for the Casimir operator value in (30):

$$c_{(2)}^{(\square)} = \frac{\hat{\alpha}\hat{\gamma}' \dim \mathfrak{g}}{4(\hat{\beta}' - 1)}, \quad (31)$$

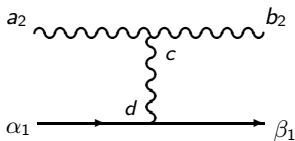
becomes valid (universal) for all simple Lie algebras, **except** \mathfrak{e}_8 .

Remarkably, for $\square \times Y_1$, the characteristic identity (22) is symmetric under all permutations of the Vogel parameters α, β, γ :

$$\left(\hat{C}_{\square \otimes \text{ad}} + \frac{1}{2}\right) \left(\hat{C}_{\square \otimes \text{ad}} + \frac{\hat{\alpha}}{2}\right) \left(\hat{C}_{\square \otimes \text{ad}} + \frac{\hat{\beta}}{2}\right) \left(\hat{C}_{\square \otimes \text{ad}} + \frac{\hat{\gamma}}{2}\right) = 0. \quad (32)$$

One can visualize the split Casimir operator $\hat{C}_{\square \otimes \text{ad}}$ in the representation $\square \otimes \text{ad}$ as the following Feynman diagram:

$$(\hat{C}_{\square \otimes \text{ad}})^{\alpha_1 a_2}_{\beta_1 b_2} = T_{\beta_1}^{\alpha_1} (X_d) X_c^{a_2 b_2} g^{cd} =$$



Formulas for $\dim V_{a_i}$ are drastically simplified:

$$\dim V_{a_1}^{(1)} = \dim \Lambda_2^{(n=1)} = \dim \square, \quad \dim V_{a_3}^{(1)} = \dim \Lambda_4^{(n=1)} = 0,$$

$$\dim V_{a_2}^{(1)} = \dim \Lambda_1^{(n=1)} = \frac{(1 - \hat{\alpha} - \hat{\beta})(\hat{\gamma} - 1)}{\hat{\alpha}\hat{\beta}(\hat{\gamma} - \hat{\alpha})} \dim \square, \quad (33)$$

$$\dim V_{a_4}^{(1)} = \dim \Lambda_3^{(n=1)} = \frac{(1 - \hat{\beta} - \hat{\gamma})(\hat{\alpha} - 1)}{\hat{\gamma}\hat{\beta}(\hat{\alpha} - \hat{\gamma})} \dim \square,$$

where a_i are the eigenvalues of the quadratic split Casimir operator in the representations $\Lambda_k^{(1)}$ (see Table):

$$a_1 = \hat{c}_{(2)}^{(\Lambda_2^{(1)})} = -1/2, \quad a_2 = \hat{c}_{(2)}^{(\Lambda_1^{(1)})} = -\hat{\alpha}/2,$$

$$a_3 = \hat{c}_{(2)}^{(\Lambda_4^{(1)})} = -\hat{\beta}'/2, \quad a_4 = \hat{c}_{(2)}^{(\Lambda_3^{(1)})} = -\hat{\gamma}'/2.$$

So we can summarize the results in Table.

	$\hat{\alpha}$	$\hat{\beta}'$	$\hat{\gamma}'$	$\Lambda_1^{(n=1)}$	$\Lambda_2^{(n=1)}$	$\Lambda_3^{(n=1)}$	$\Lambda_4^{(n=1)}$
\mathfrak{sl}_N	$-\frac{1}{N}$	$\frac{1}{2}$	$\frac{1}{N}$	$([2], [1])$	$([1], [\emptyset])$	$([1^2], [1])$	0
\mathfrak{so}_N	$-\frac{1}{N-2}$	$\frac{N-4}{2N-4}$	$\frac{2}{N-2}$	$[2, 1]$	$[1]$	$[1^3]$	0
$\mathfrak{sp}_{N=2r}$	$-\frac{2}{N+2}$	$\frac{N+4}{2N+4}$	$\frac{1}{N+2}$	$[3]$	$[1]$	$[2, 1]$	0
\mathfrak{g}_2	$-1/4$	$5/12$	$1/3$	$\lambda_{(2)} + \lambda_{(1)}$	$\lambda_{(1)}$	$2\lambda_{(1)}$	0
\mathfrak{f}_4	$-1/9$	$5/18$	$1/3$	$\lambda_{(1)} + \lambda_{(4)}$	$\lambda_{(4)}$	$\lambda_{(3)}$	0
\mathfrak{e}_6	$-1/12$	$1/4$	$1/3$	$\lambda_{(5)} + \lambda_{(1)}$	$\lambda_{(1)}$	$\lambda_{(4)}$	0
\mathfrak{e}_7	$-1/18$	$2/9$	$1/3$	$\lambda_{(7)} + \lambda_{(1)}$	$\lambda_{(1)}$	$\lambda_{(6)}$	0
$\hat{c}_{(2)}$	—	—	—	$a_2 = -\frac{\hat{\alpha}}{2}$	$a_1 = -\frac{1}{2}$	$a_4 = -\frac{\hat{\gamma}'}{2}$	$a_3 = -\frac{\hat{\beta}'}{2}$

Conclusion.

Now we discuss some promising directions of investigations that follow from the results obtained in this paper.

1. First of all, it is clear that there exists a uniform (valid for all simple Lie algebras) decomposition of the tensor product $Y_k \otimes Y_n$. Thus, one can find universal formulas for dimensions of an infinite series of representations arising in the decomposition of $Y_k \otimes Y_n$.
2. It seems that there is a direct generalization (q and t deformations) of all the universal formulas for dimensions and identities following from the decomposition of tensor products $Y_k \otimes Y_n$.

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1. Bulgaria, Peak Musala, 2925 meters, 18.09.1966, 9 a.m.
From left to right: Xian Ding Chang (China),
Ivan Todorov (Bulgaria), Petr Isaev (Dubna, USSR)
(Peak Musala is the highest peak in Bulgaria,
as well as in the entire Balkan Peninsula)

2. Austria, Vienna, September 2004.
Left to right: Alexey Isaev (JINR, Dubna),
David Broadhurst (The Open University, Milton Keynes),
Ivan Todorov (ESI, Wien)



Hecke algebraic properties of dynamical R -Application to related quantum matrix algebras

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(Received 26 February 1998; accepted for publication 5 June 1999)

The quantum dynamical Yang–Baxter (or Gervais–Neveu–Felder) equation defines an R -matrix $\hat{R}(p)$, where p stands for a set of mutually commuting parameters. A family of $SL(n)$ -type solutions of this equation provides a new Hecke algebra. We define quantum antisymmetrizers, introduce the quantum determinant and compute the inverse quantum matrix for matrix type $\hat{R}(p)a_1a_2 = a_1a_2\hat{R}$. It is pointed out that such a quantum matrix is the operator realization of the chiral zero modes of the WZNW model. *American Institute of Physics*. [S0022-2488(99)02801-7]

Quantum matrix algebra for the $SU(n)$ WZNW model

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Received 13 January 2003, in final form 2 April 2003

Published 7 May 2003

Online at stacks.iop.org/JPhysA/36/5497

Abstract

The zero modes of the chiral $SU(n)$ WZNW model give rise to an *intertwining quantum matrix algebra* \mathcal{A} generated by an $n \times n$ matrix $a = (a_{ij}^a)$, $i, \alpha = 1, \dots, n$ (with noncommuting entries) and by rational functions of n commuting elements q^{α} satisfying $\prod_{i=1}^n q^{\alpha_i} = 1$, $q^{\alpha} a_{ij}^a = a_{ij}^a q^{\alpha_i + \alpha_j - \frac{1}{2}}$. We study a generalization of the Fock space (\mathcal{F}) representation of \mathcal{A} for generic q (q not a root of unity) and demonstrate that it gives rise to a *model* of the quantum universal enveloping algebra $U_q = U_q(\mathfrak{sl}_n)$, with each irreducible representation entering \mathcal{F} with multiplicity 1. For an integer $\hat{u}(n)$ height $h = (k + n \geq n)$ the complex parameter q is an even root of unity, $q^h = -1$, and the algebra \mathcal{A} has an ideal \mathcal{I}_h such that the factor algebra $\mathcal{A}_h = \mathcal{A}/\mathcal{I}_h$ is finite dimensional. All physical U_q modules—of shifted weights satisfying $p_{1n} = p_1 - p_n < h$ —appear in the Fock representation of \mathcal{A}_h .